UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL ATOMIC ENERGY AGENCY
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

H4-SMR 1012 - 8

AUTUMN COLLEGE ON PLASMA PHYSICS

13 October - 7 November 1997

DYNAMO GROWTH RATE AND THE CANCELLATION EXPONENT

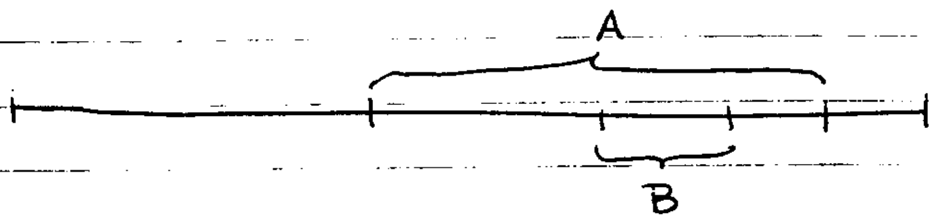
E. OTT

University of Maryland, College Park, U.S.A.

These are lecture notes, intended for distribution to participants.

II. DYNAMO GROWTH RATE AND THE CANCELLATION EXPONENT

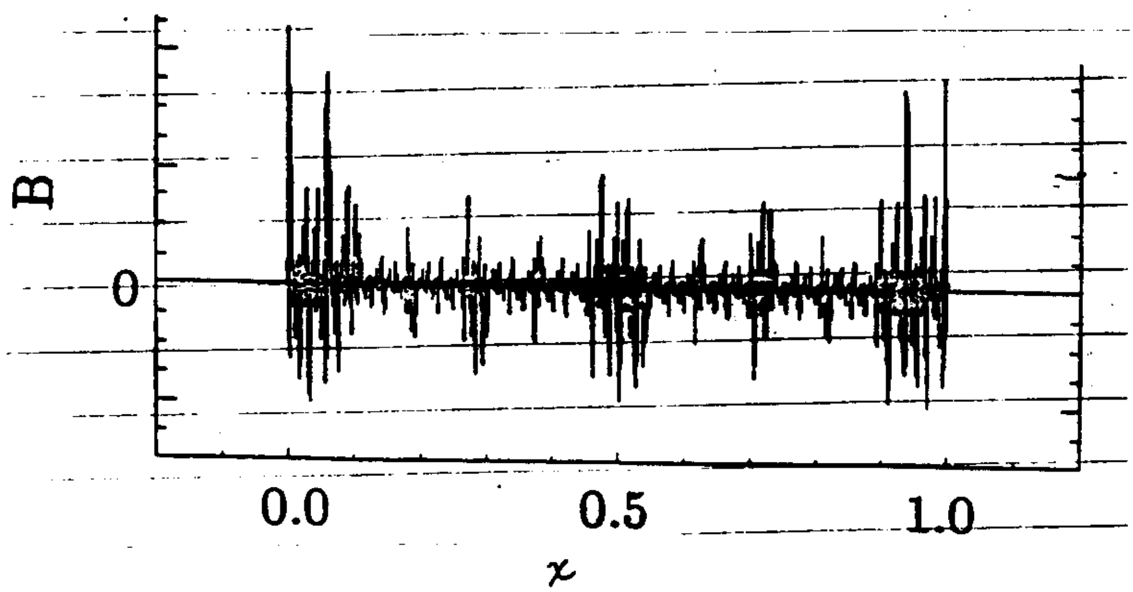
<u>Def.</u> We say that the measure μ is sign-singular, if, for any interval A for which $\mu(A) \neq 0$, there is an interval B contained in A such that $\mu(A)$ and $\mu(B)$ have opposite sign.



$$\mu(A)\mu(B) < 0 .$$

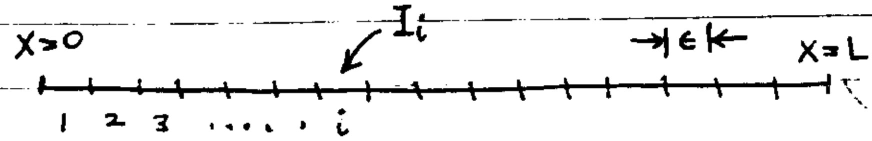
(Here $\mu(A)$ is a "measure": It assigns a number $\mu(A)$ to a set A. In our physical context we think of A as a area  A and $\mu(A)$ as the magnetic flux through A : $\int_A \underline{B} \cdot d\underline{S}$.)

- Sign singularity occurs in magnetic dynamos as R_m gets large.



Sign singular → changes sign on arbitrarily fine scale.

Cancellation exponent



I_i = i^{th} interval of length ϵ

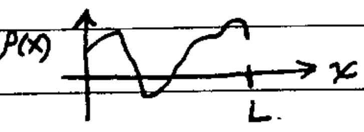
$$\mathcal{H} = \lim_{\epsilon \to 0} \frac{\ln \sum |\mu(I_i)|}{\ln(1/\epsilon)}$$

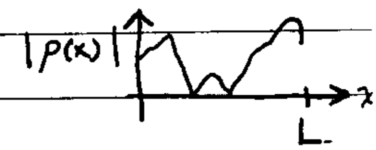
Example: Probability measure $\mu(I_i) \geq 0$, $\sum_i \mu(I_i) = 1$

$$\sum_i |\mu(I_i)| = \sum \mu(I_i) = 1 \Rightarrow \mathcal{H} = 0$$

Example: Measure with a smooth bounded density $\rho(x)$

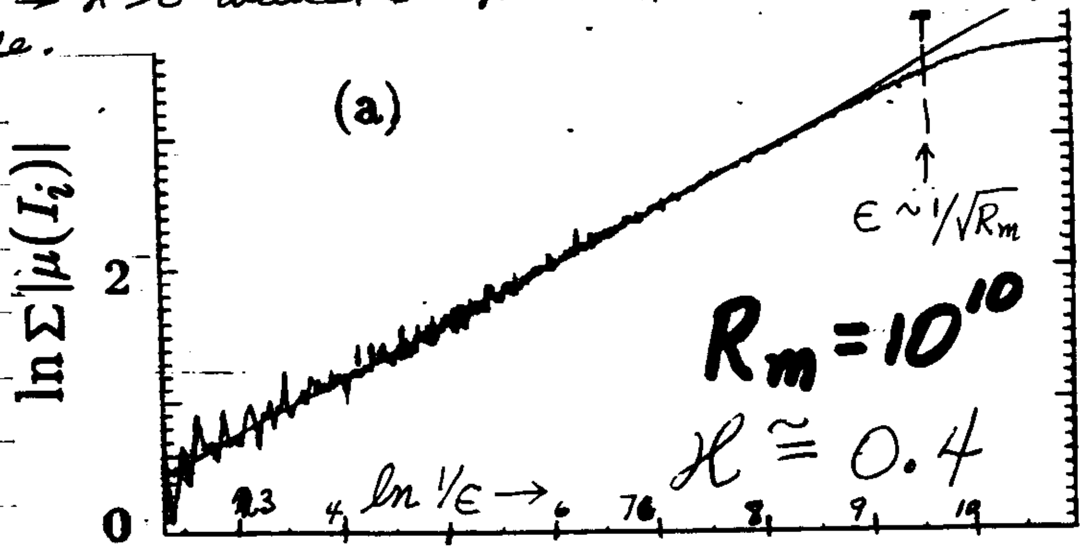
$$\mu(I_i) = \int_{I_i} \rho(x)\, dx$$





$$\lim_{\epsilon \to 0} \sum_i |\mu(I_i)| = \int_0^L |\rho(x)|\, dx < \infty \Rightarrow \mathcal{H} = 0$$

How can $\mathcal{H}$ be nonzero? The sum must increase to ∞ as $\epsilon \to 0$ and this can occur only if cancellation of + and − decreases ⇒ $\mathcal{H} > 0$ indicates sign variations on arbitrarily fine scale.



Four Strip Baker's Map Dynamo Model

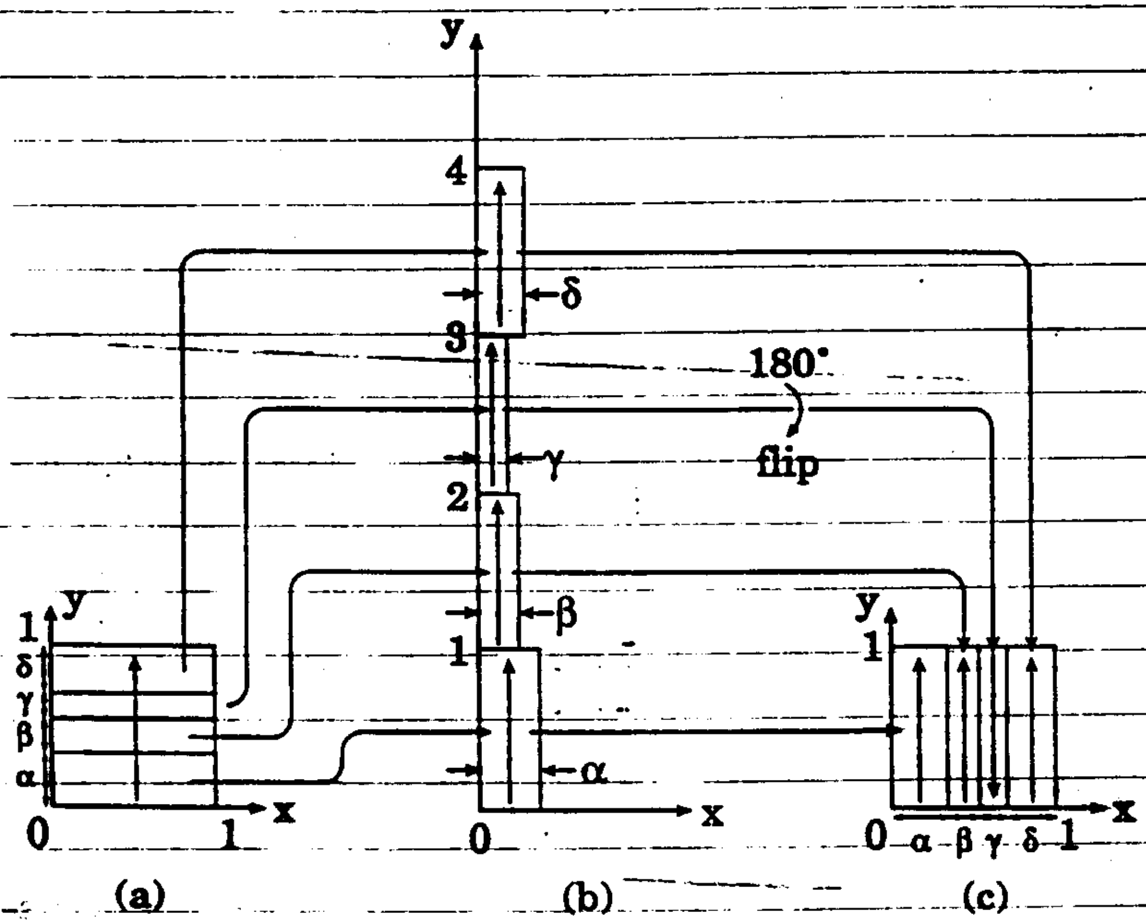


Figure 4: 4-strip baker's map.

Finn & Ott
Phys. Fluids
(1988)

$\alpha = 7/16,\ \beta = \gamma = 1/16,\ \delta = 7/16$

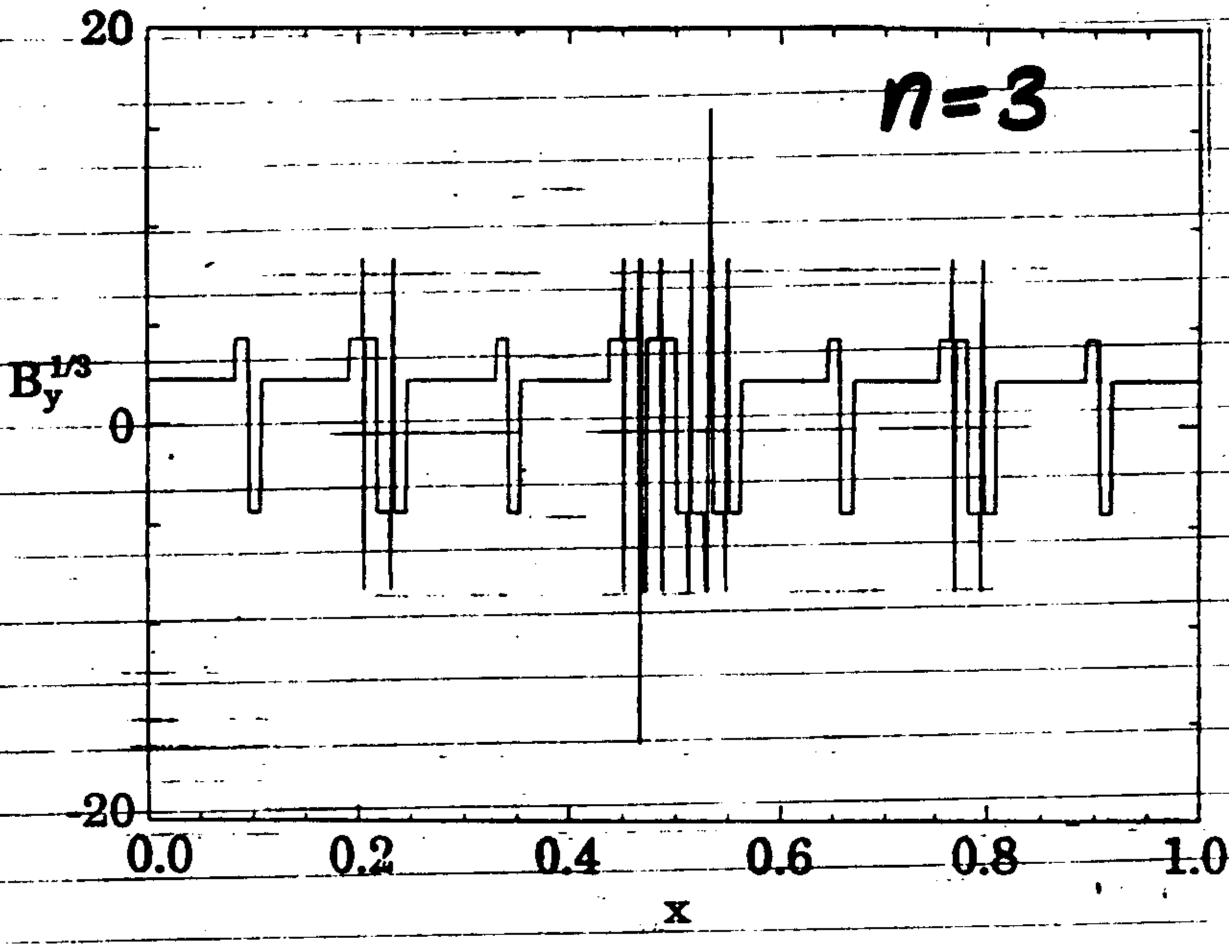


Figure 5: Plots of $\hat{B}_y^{1/3}$ versus x for (a) $n = 3$, (b) $n = 5$, and (c) $n = 7$. The spatial distribution of the magnetic field is calculated from 1000 points which are uniformly distributed over the interval (0, 1), so the plots do not resolve the scales smaller than 10^{-3}.

2.5

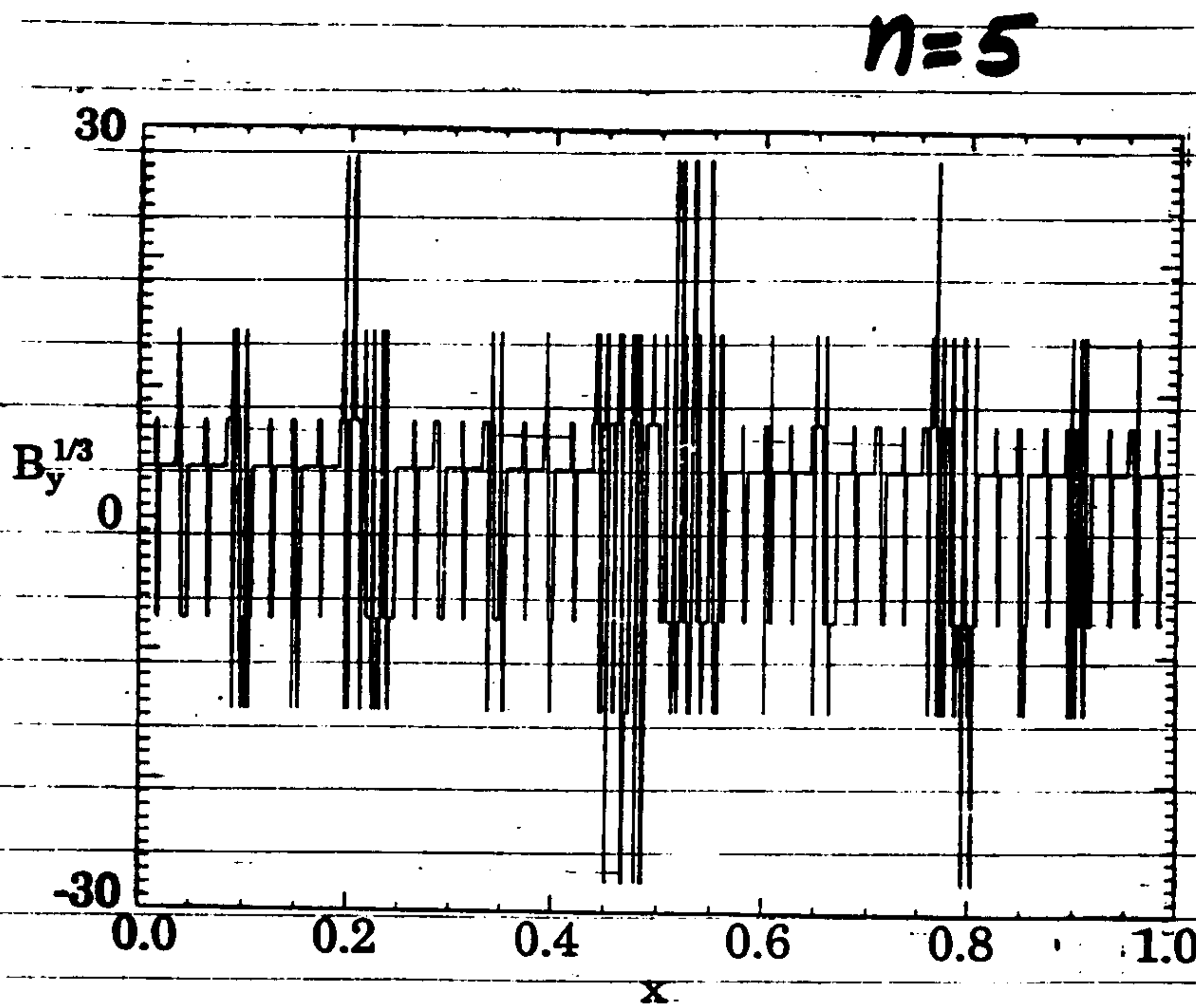


Figure 5: ~~(continuation) Figure 5(b)~~.

2.6

n=7

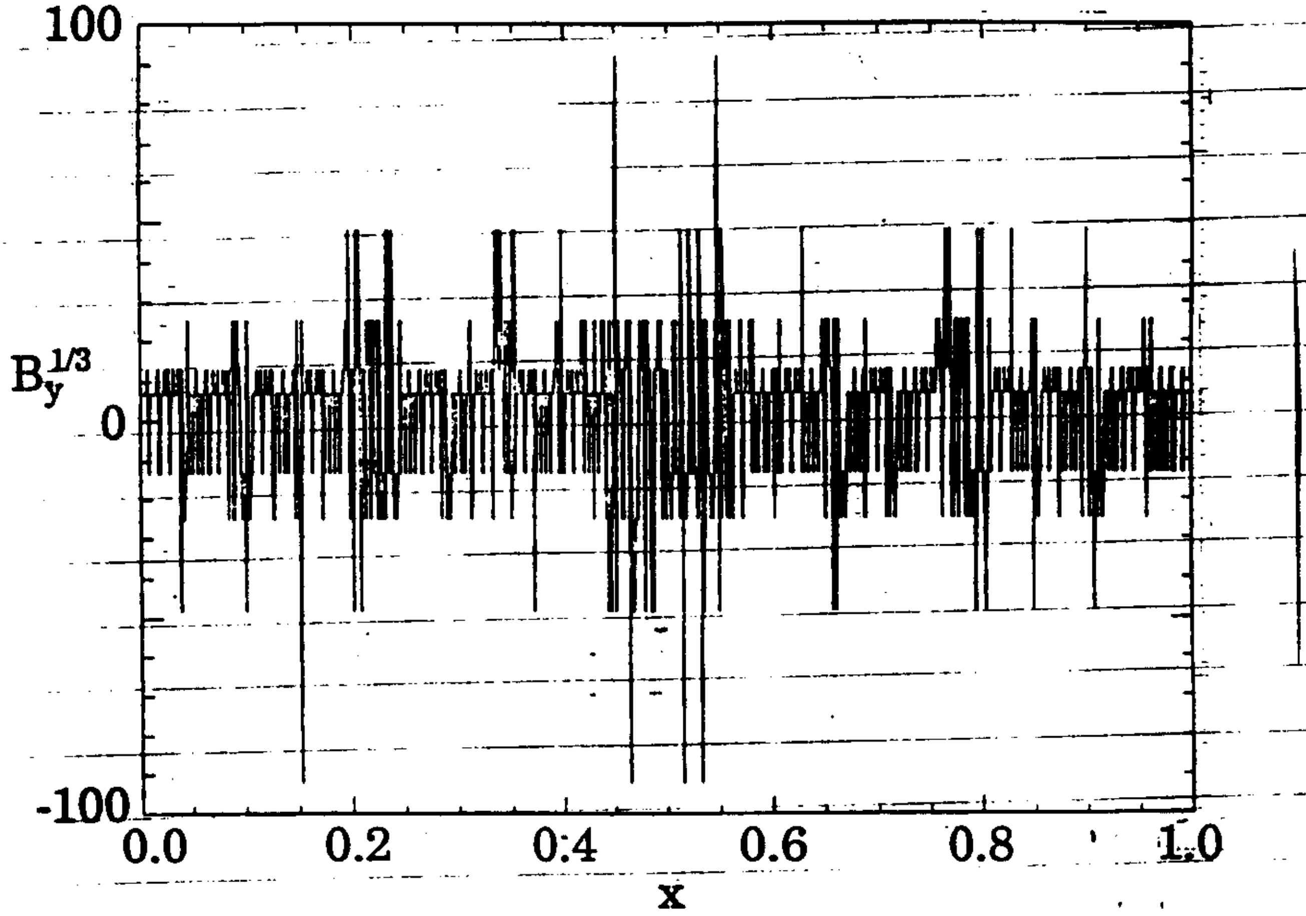


Figure 5: (continuation) Figure 5(c).

Cancellation in this model:

Flux Φ doubles on each iterate

$N_+ = $ # of up strips

$N_- = $ # of down strips

$N_+ - N_- = 2^n$

$N_+ + N_- = 4^n$

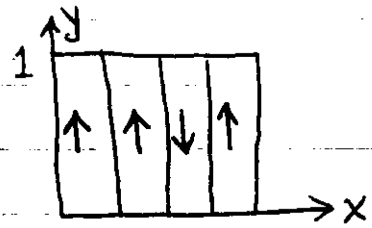
$$\frac{N_+ - N_-}{N_+ + N_-} = \left(\frac{1}{2}\right)^n$$

Even though flux grows exponentially ($\sim 2^n$) the cancellation is growing exponentially as well

E.g., if $\alpha = \beta = \gamma = \delta = 1/4$ then $B_{n+1} = 4 B_n$

mag. field in any strip.

$n=1$

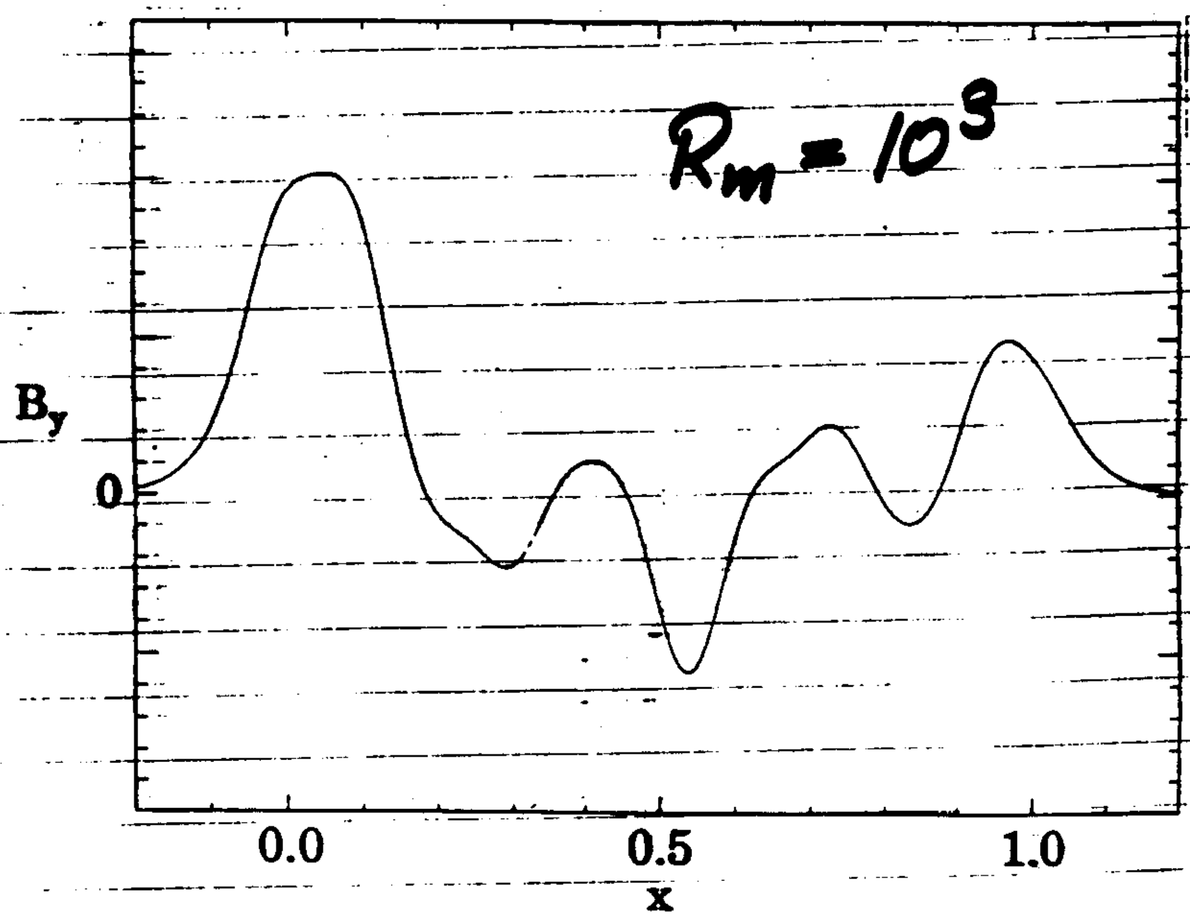


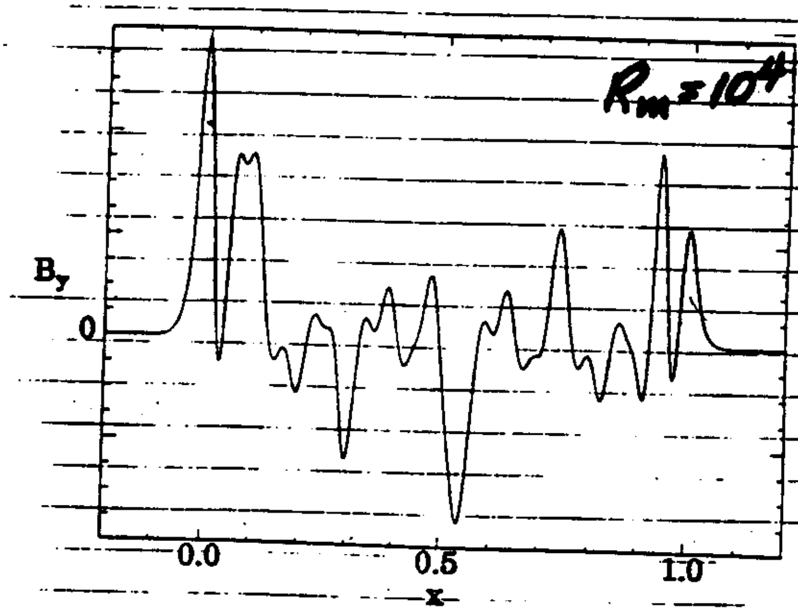
time	# of strips $N_+ + N_-$	$\Phi_+/\Phi_0 = N_+$	$\Phi_-/\Phi_0 = N_-$	Φ/Φ_0
$n=1$	4	3	1	2
$n=2$	16	10	6	4
$n=3$	64	36	28	8
$n=4$	256	136	120	16

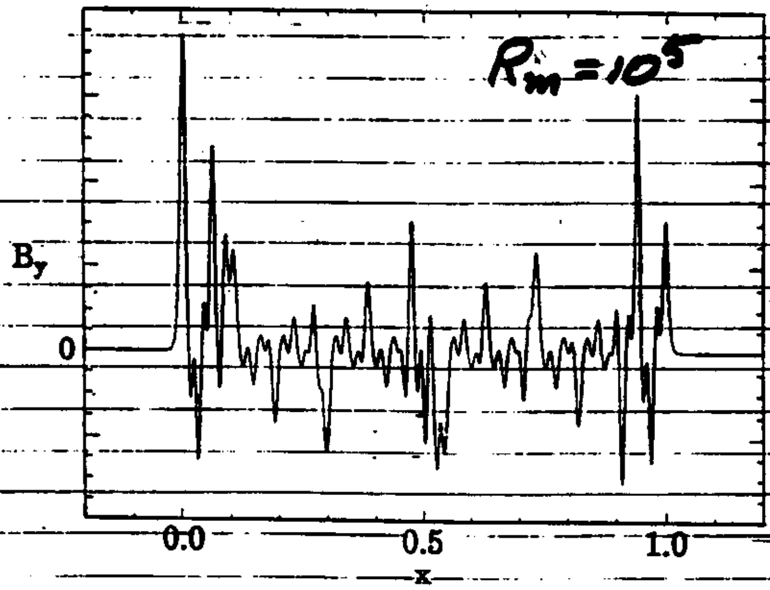
Baker's Map with Diffusion

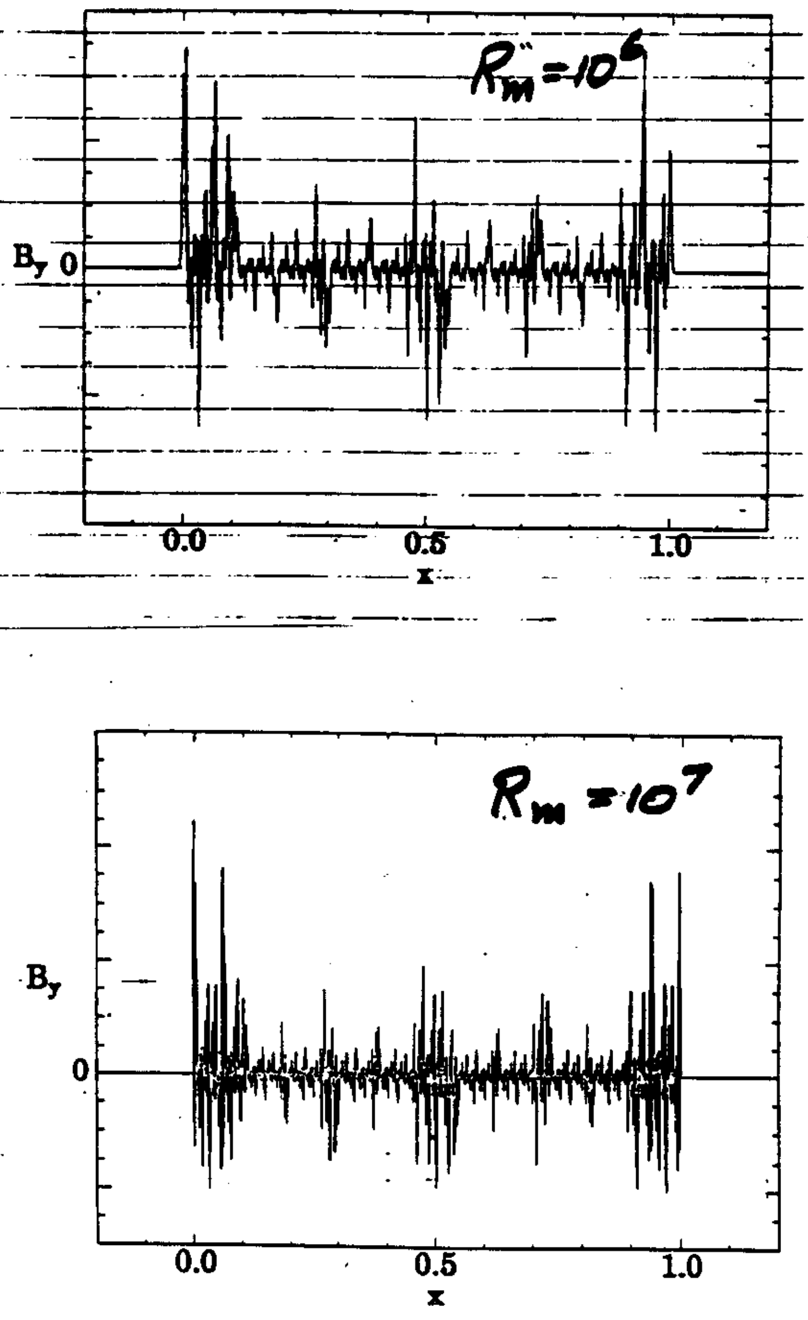
Apply Map impulsively
Diffuse ($\underline{v} = 0$)
Apply Map impulsively
Diffuse ($\underline{v} = 0$)
⋮

Baker's map with diffusion
(Finn & Ott P.F. 1990)




$R_m = 10^4$
B_y
0
0.0
0.5
1.0
x


$R_m = 10^5$
B_y
0
0.0
0.5
1.0
x



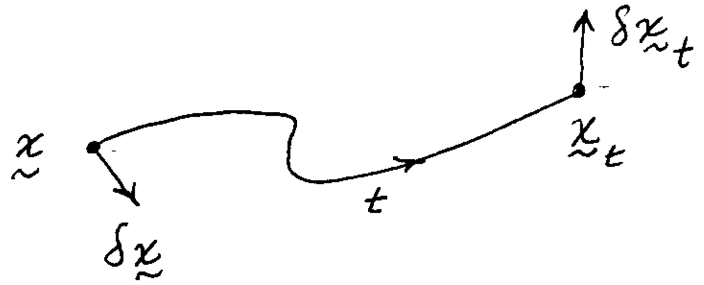
$R_m = 10^6$
B_y
0
0.0
0.5
1.0
x
$R_m = 10^7$
B_y
0
0.0
0.5
1.0
x

Can we relate the dynamo growth rate in the $R_m \to \infty$ limit to $\mathcal{H}$ and purely dynamical quantities determinable from $d\underset{\sim}{x}/dt = \underset{\sim}{v}(\underset{\sim}{x}, t)$?

γ_* = Fast dynamo growth rate

Problem: What is γ_*?

Finite Time Lyapunov Numbers:



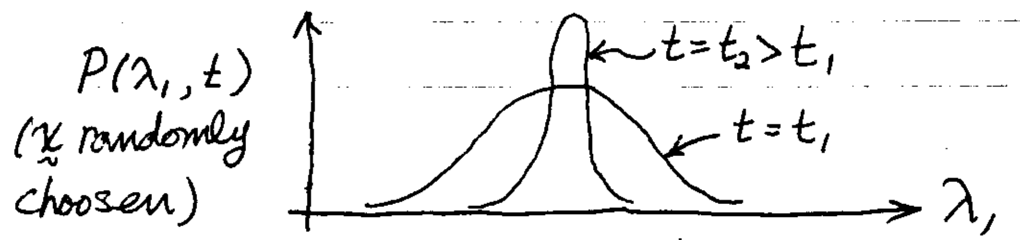
$$\delta \underset{\sim}{x}_t = \underset{\approx}{M}(t, \underset{\sim}{x})\, \delta \underset{\sim}{x}$$

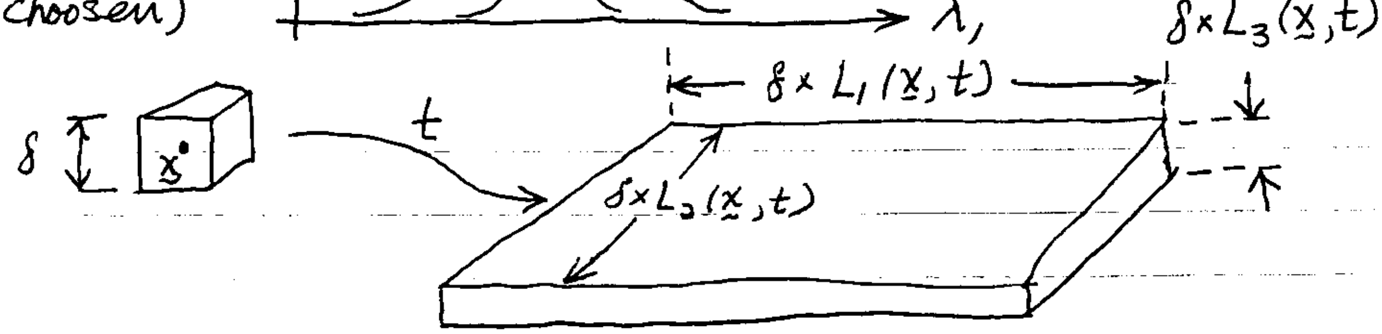
$$L_i(\underset{\sim}{x}, t) = \left[\text{Eigenvalues of } \underset{\approx}{M}^T \underset{\approx}{M} \right]^{1/2}$$

$$L_1(\underset{\sim}{x}, t) \geqslant L_2(\underset{\sim}{x}, t) \geqslant L_3(\underset{\sim}{x}, t)$$

Lyapunov exponents at finite time

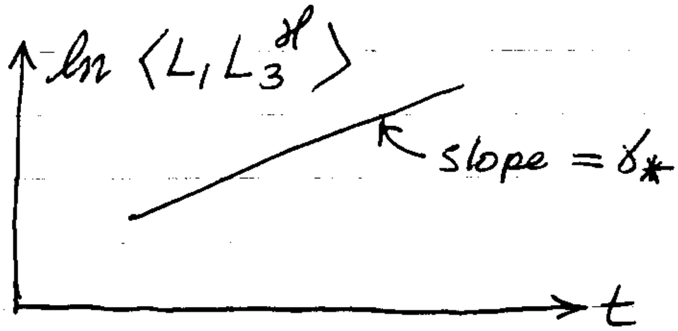
$$\lambda_i(\underset{\sim}{x}, t) = \frac{1}{t} \ln L_i(\underset{\sim}{x}, t)$$





Result: (Du and Ott J. Fluid Mech. [1993])

$$\gamma_* = \lim_{t\to\infty} \frac{1}{t} \ln \langle L_1 L_3^{\mathcal{H}} \rangle$$



Advantage: Solve $d\underline{x}/dt = \underline{v}$ to get γ_* rather than actually solving PDE and letting R_m become large.

No cancellation $\mathcal{H} = 0$ (e.g., stretch-twist-fold)

$$\gamma_* = \lim_{t\to\infty} \frac{1}{t} \ln \langle L_1 \rangle$$ (rate of growth of length of a line)

Numerical experiments solving $\partial \underline{B}/\partial t + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{v} + \frac{1}{R_m} \nabla^2 \underline{B}$ for $R_m = 10^5$ and a smooth time varying $\underline{v}(\underline{x}, t)$ in C. Reyl, T.M. Antonsen and E. Ott, Phys. Fluids 3 (1996).