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FRACTALS IN FLUIDS

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These are lecture notes, intended for distribution to participants.

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III: FRACTALS IN FLUIDS

Subject What can chaotic dynamics say about fractal properties of scalar fields transported by smooth Lagrangian chaotic velocity fields?

Smooth = wavenumber power spectrum of the velocity field is peaked at low wavenumber k and decays at least as fast as exponentially with increasing k. I.e., $\underset{\sim}{v}$ itself has no fractal properties

Passive = convected with flow, not influencing flow.

$$\partial\phi/\partial t + \nabla\cdot(\underset{\sim}{v}\phi) = D\nabla^2\phi + S$$

ϕ = passive scalar field.
$\underset{\sim}{v}$ = fluid velocity
D = diffusion coefficient of passive scalar
S = source of passive scalar.

Fractals in chaotic dynamics

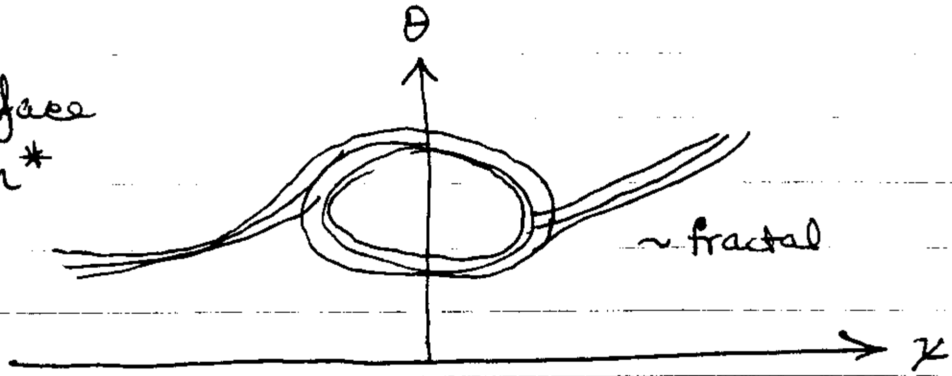
- Strange attractors (e.g., Hénon attractor)

$D_1 \cong 1.26$ for Hénon

- Chaotic scattering

$N(t) \sim e^{-t/\tau}$
τ = decay time

$y=0$ surface of section*



Dimension Formulas

2D Map or surface of section*

Lyapunov exponents*: $h_1 > 0 > h_2$

Area preserving*: $h_1 = -h_2$

- Formula for strange attractor

$$D_1 = 1 + \frac{h_1}{(-h_2)}$$ Kaplan & Yorke (1979)

(Area preserving $\Rightarrow$ $D_1 = 2$)

- Formula for transient chaos

Dimension of stuff that hangs around for a long time

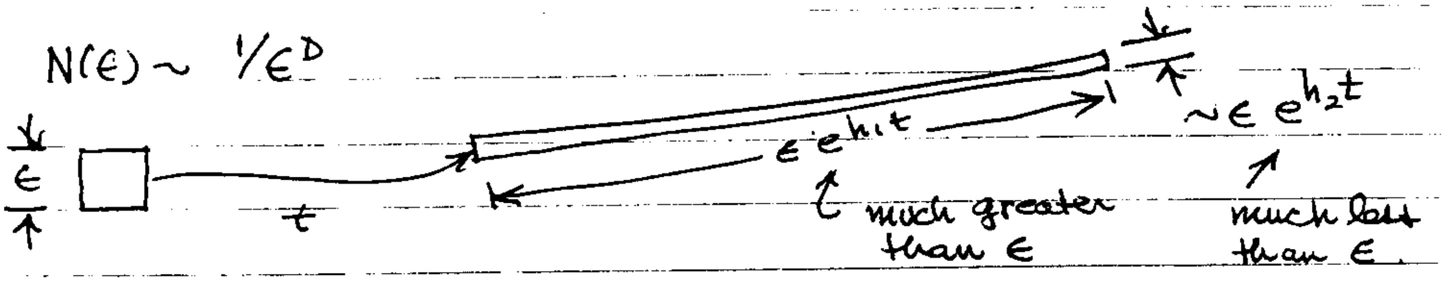
$$D_1 = 1 + \frac{h_1 - (1/\tau)}{(-h_2)}$$ Kantz and Grassberger

Hsu, Ott and Grebogi

(Area preserving $\Rightarrow$ $D_1 = 2 - \frac{1}{h_1 \tau}$)

Derivation for strange attractor

$N(\epsilon) \sim 1/\epsilon^D$



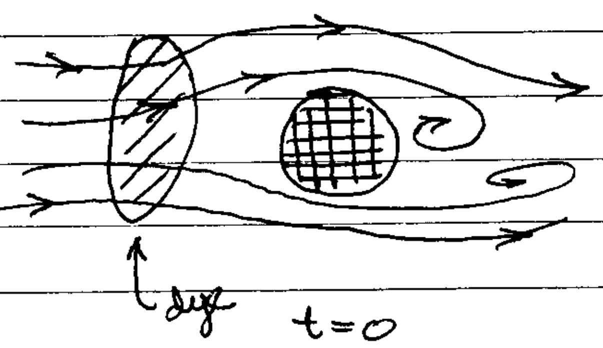
Let $\epsilon' = \epsilon e^{h_2 t} \ll \epsilon$

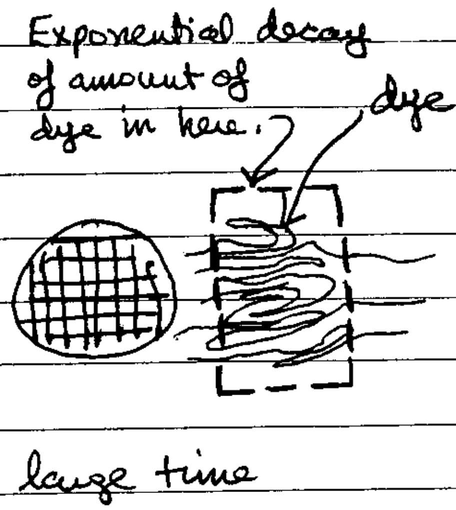
$$N(\epsilon') \sim \frac{1}{(\epsilon')^D} \sim N(\epsilon)\,\frac{\epsilon e^{h_1 t}}{\epsilon e^{h_2 t}}$$

$$\frac{1}{\epsilon e^{h_2 t}} \sim \frac{1}{\epsilon^D} e^{(h_1 - h_2)t} \quad \Rightarrow D = 1 + \frac{h_1}{(-h_2)} \quad \text{for } h_1 + h_2 \leq 0$$

Three Situations Yielding Fractals in Fluid Convection of a Passive Scalar

1. Flow through a chaotic region





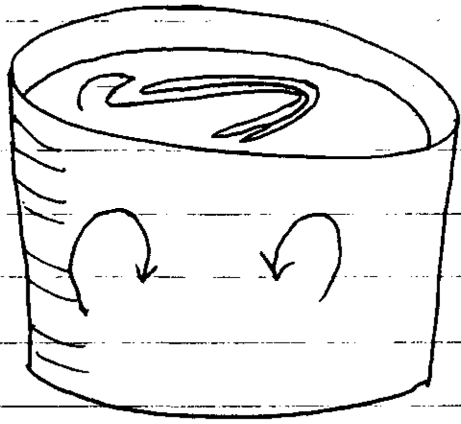
$$\underset{\sim}{v}_{dye} = \underset{\sim}{v}_{fluid} \Rightarrow \nabla \cdot \underset{\sim}{v}_{dye} = 0$$

incompressible

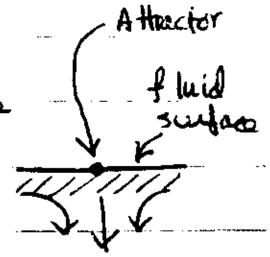
Experiment: J.C. Sommerer et al. Phys. Rev. Lett. (1996).

Verify $D_1 = 1 + (h_1 - 1/\tau)/(-h_2)$

2. Floating particles



$$\underset{\sim}{v}_{\text{floaters}} = \underset{\sim}{v}_{\text{fluid}} \big|_{\text{surface}}$$



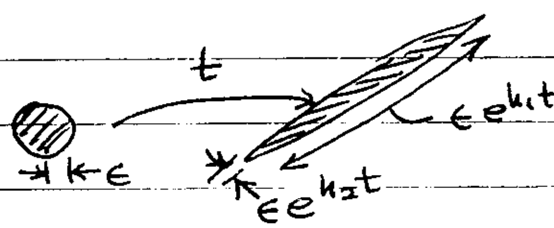
$$\nabla_{\text{surface}} \cdot \underset{\sim}{v}_{\text{floaters}} \neq 0$$

$\Rightarrow$ Strange attractor in the fluid surface.

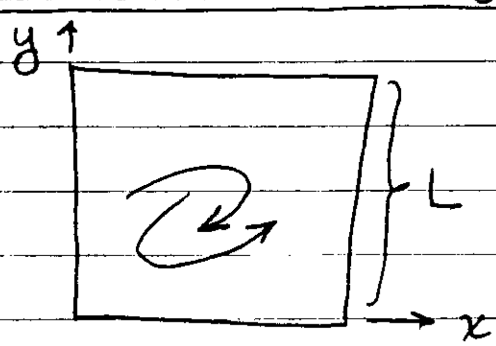
~~Experiment:~~ J. C. Sommerer and E. Ott, Science (1993).

Verify $D_1 = 1 + h_1/(-h_2)$

Measured h_1 and h_2 by



3. Passive Scalar in a Bounded Incompressible 2D flow



2D incompressible flow in (x, y)

$$\nabla_{x,y} \cdot \underset{\sim}{v} = 0$$

$$\underset{\sim}{v} = v_x \underset{\sim}{x}_0 + v_y \underset{\sim}{y}_0$$

$$d\phi/dt = \partial\phi/\partial t + \underset{\sim}{v} \cdot \nabla\phi = D \nabla^2 \phi \qquad \begin{pmatrix} \text{no source} \\ \nabla \cdot \underset{\sim}{v} = 0 \end{pmatrix}$$

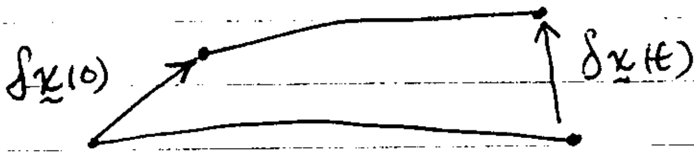
Consider $D = 0$

$\Rightarrow$ Fractal formulas give Dimension of set where ϕ is located $= 2$, ~~[illegible]~~ $\Rightarrow$ not fractal.

$\Rightarrow$ We will look at region where $|\nabla\phi|$ concentrates.

Lagrangian Chaotic Fluid Flow

$$\frac{d\underset{\sim}{x}(t)}{dt} = \underset{\sim}{v}(\underset{\sim}{x}(t), t)$$



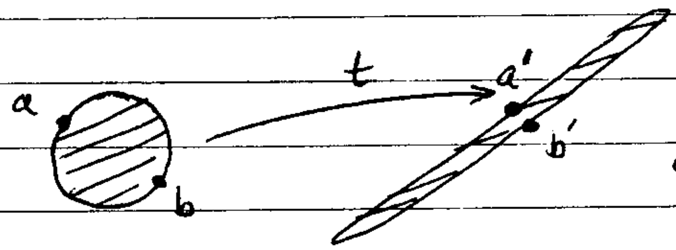
chaos means that

$|\delta\underset{\sim}{x}(t)| \,/\, |\delta\underset{\sim}{x}(0)|$ grows exponentially in time for typical initial conditions

Growth of Gradient

$$\frac{\partial\phi}{\partial t} + \underset{\sim}{v}\cdot\nabla\phi = D\,\nabla^2\phi \qquad \nabla\cdot\underset{\sim}{v} = 0$$

$D = 0 \Rightarrow \frac{d\phi}{dt} = 0$ following fluid elements.



$$\phi(a') = \phi(a)$$
$$\phi(b') = \phi(b)$$

- Large stretch $\longrightarrow$ large gradient
- $|\nabla\phi|$ will concentrate on fractal due to extreme nonuniformity of stretching inherent in chaotic situations.

Diffusionless Approximation

Time is large but not too large

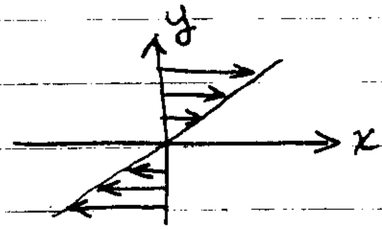
$$\omega_0 \sim |\underset{\sim}{v}|/L$$

scale over which $\underset{\sim}{v}$ varies

$$\sqrt{\frac{D}{\omega_0}} \ll \left(\begin{array}{c}\text{Scale over which}\\ \phi \text{ varies at time}\\ t\end{array}\right) \ll L$$

Numerical Experiments on a Model 2D Incompressible Flow with Irregular Time Dependence

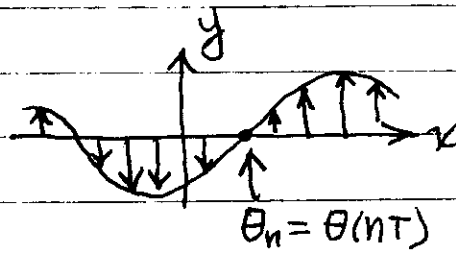
$\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2$



$\underset{\sim}{v}_1$: steady linear shear flow $\underset{\sim}{v}_1 = y\, \underset{\sim}{x}_0$

$\underset{\sim}{v}_2$: pulsed sinusoidal shear flow in y.

$$\underset{\sim}{v}_2 = k \sin(x+\theta(t)) \sum_n \delta(t-n\underset{\tau_1}{T})\, \underset{\sim}{y}_0$$



Integrate $d\underset{\sim}{x}/dt = \underset{\sim}{v}(\underset{\sim}{x}(t), t)$ from $t = nT - 0^+$ to time $t = (n+1)T - 0^+$

$$x_{n+1} = (x_n + y_n) \bmod 2\pi$$

$$y_{n+1} = y_n + k \sin(x_n + y_n + \theta_n)$$

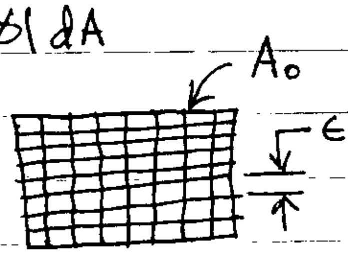
θ_n regarded as random in $[0, 2\pi]$

Fractal Dimension Characterization of $|\nabla\phi|$

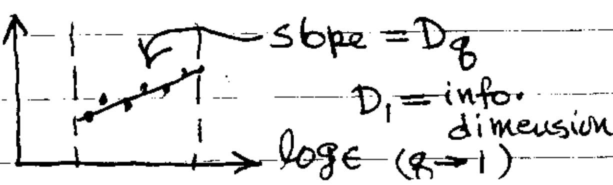
"Gradient measure": $\mu(S, t) = \dfrac{\int_S |\nabla\phi|\, dA}{\int_{A_0} |\nabla\phi|\, dA}$

A_0 = region of interest

S = subset of A_0

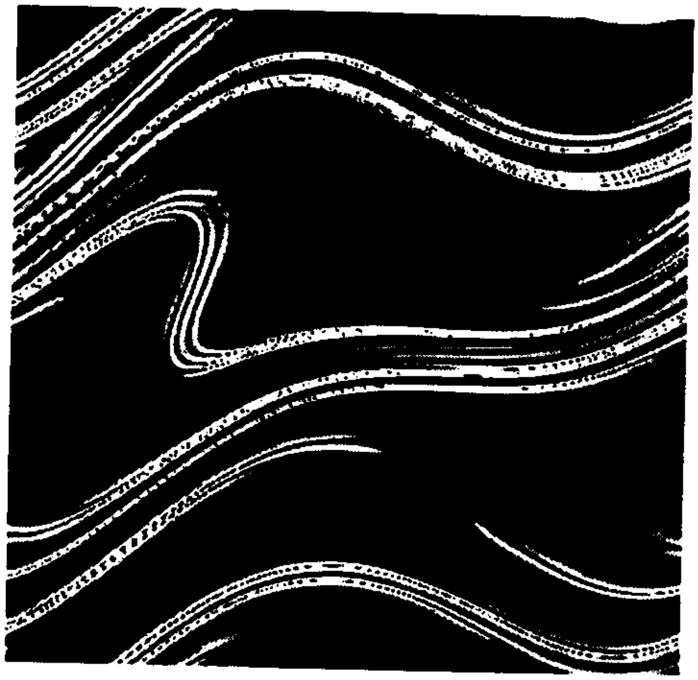


- Break A_0 up into grid of ϵ boxes
- μ_i = measure of box i
- $\sum_i \mu_i^q \sim \epsilon^{(q-1)D_q}$

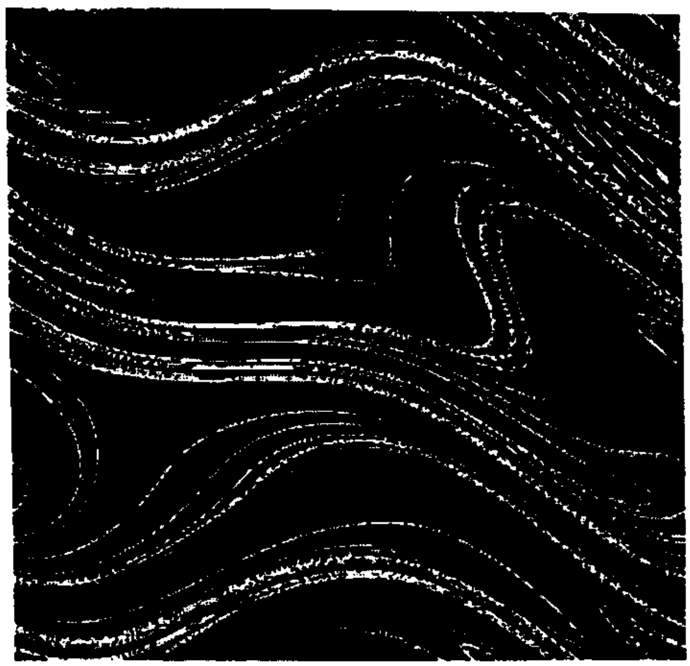


The light colored regions is where $|\nabla\phi|$ is largest. As t increases this region shrinks, contracting on to a fractal set.

t = 5

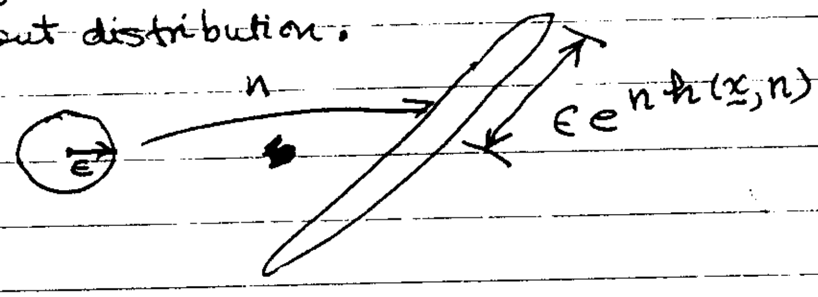


t = 20



Theory for D_q involves finite Time Lyapunov exponents

Similar in spirit to derivation of $D_1 = 1 + \frac{h_1}{(-h_2)}$ for strange attractor, but involves finite time Lyapunov exponent distribution.



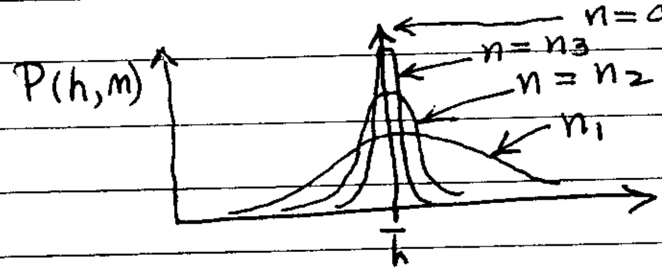
n = time

$h(\underline{x}, n)$ = finite time Lyapunov exponent

$P(h,n)\,dh$ = prob. that exponent lies between h & $h + dh$ given that $\underline{x}$ is choosen randomly.

Large time scaling:

$$P(h,n) \cong \sqrt{\frac{n G''(\bar{h})}{2\pi}}\, e^{-nG(h)}$$



$n = \infty \Rightarrow P$ is a delta function at the usual infinite time Lyapunov exponent $\bar{h}$.

Can estimate $G(h)$ numerically from

$$G(h) = -\frac{1}{n} \ln\left(P(h,n)\right) + \frac{\ln n}{2n} + O\left(\frac{1}{n}\right)$$

Reference: Ott and Antonsen Phys. Rev. A (1989).

D_q from $G(h)$:

$$\boxed{\begin{array}{l} \min\limits_{h} \left[G(h) - \sigma h\right] = q \min\limits_{h} \left[G(h) - h\right] \\ \text{where } \sigma = (q-1)(D_q - 2) + q \end{array}}$$

<u>Experiment</u>: Ramshakar & Gollub Physics of Fluids 1991

<u>Conclusion</u>: Considerations from chaotic dynamics are useful in understanding and analizing a variety of situations in which fractal patterns form in fluids and plasmas.