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Fundamentals and Current Problems of Scattered Fields

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Fundamentals and current problems on Scattered fields

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Lectures partly based on textbook: M. N-V: "Scattering and Diffraction in Physical Optics" (J. Wiley, N.Y. 1991).
and further works.

$$\nabla \times \nabla \times \underline{E} - k^2 \underline{E} = \underline{F}e$$

$$\nabla \times \nabla \times \underline{H} - k^2 \underline{H} = Fm$$

VECTOR
E.M.
WAVES

$$\underline{F}e = 4\pi \left[\frac{ik}{c} \underline{j} + k^2 \underline{P} + ik \nabla \times \underline{M} \right]$$

$$\underline{F}m = 4\pi \left[\frac{1}{c} \nabla \times \underline{j} - ik \nabla \times \underline{P} + k^2 \underline{M} \right]$$

$$\nabla \times \nabla \times \underline{\underline{g}} - k^2 \underline{\underline{g}} = 4\pi \delta(\underline{r} - \underline{r}') \underline{\underline{I}}$$

$$\underline{\underline{g}}(\underline{r}, \underline{r}') = \left(\underline{\underline{I}} + \frac{1}{k^2} \nabla\nabla \right) G(\underline{r}, \underline{r}')$$

$$G(\underline{r}, \underline{r}') = \frac{\exp(ik|\underline{r} - \underline{r}'|)}{|\underline{r} - \underline{r}'|}$$

$$\nabla^2 G + k^2 G = -4\pi \delta(\underline{r} - \underline{r}')$$

SCALAR WAVES

$$\nabla^2 u + k^2 u = -4\pi \rho(\underline{r}) \quad (1)$$

Scattering potential $F(\underline{r})$:

$$\rho(\underline{r}) = -\frac{1}{4\pi} F(\underline{r}) u(\underline{r})$$

$$F(\underline{r}) = \begin{cases} -k^2[n^2(\underline{r}) - 1] & , \underline{r} \in V \\ 0 & , \underline{r} \notin V \end{cases}$$

$V \equiv$ scattering volume.

Integral theorems. Green's Identity:

$$\int_V [u \nabla^2 v - v \nabla^2 u] d^3r = \int_S [u \underline{\nabla} v - v \underline{\nabla} u] \cdot \underline{n} \, ds$$

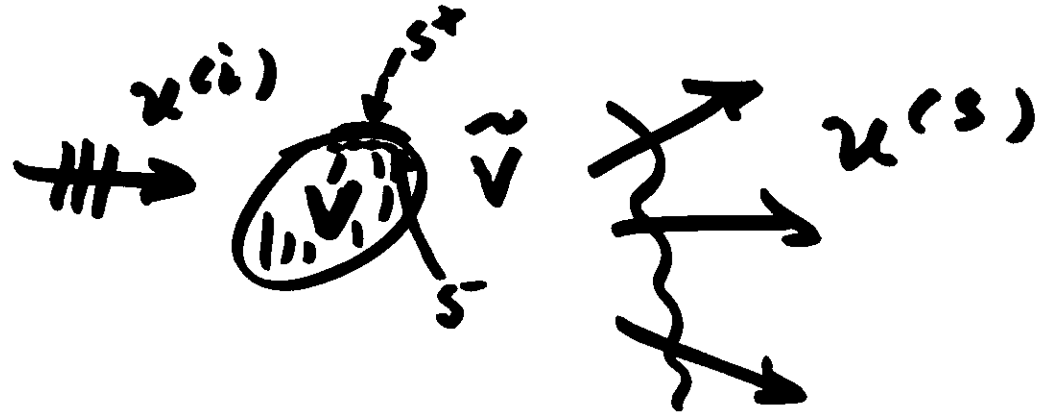
u satisfies (1)

$v \equiv G(\underline{r}, \underline{r}')$

$$\int_V d^3r' \left\{ u\left[-4\pi\delta(\underline{r}-\underline{r}') - k^2 G\right] - G\left[F u - k^2 u\right] \right\} = \Sigma(\underline{r})$$

$$\Sigma(\underline{r}) = \int_S \left[u \underline{\nabla} G - G \underline{\nabla} u\right]\cdot \underline{n}\, ds$$

▶ $\underline{r}, \underline{r}' \in V$ $\qquad$ $r_< \in V$; $\underline{r}_> \in \tilde{V}$



$$u(\underline{r}_<) = -\frac{1}{4\pi}\Sigma_{S^-}(\underline{r}_<) - \frac{1}{4\pi}\int_V d^3\underline{r}'\, G(\underline{r}_<, \underline{r}')\, F(\underline{r}')\, u(\underline{r}') \qquad (1)$$

▶ $\underline{r} \in V$; $\underline{r}' \in \tilde{V}$

$$0 = -\frac{1}{4\pi}\Sigma_{S^+}(\underline{r}_<) + \overbrace{\frac{1}{4\pi}\Sigma_{\infty}(\underline{r}_<)}^{-u^{(i)}} ; \qquad (2)$$

R sphere at ∞.

EXTINCTION THEOREM

▶ $\underline{r} \in \tilde{V}$, $\underline{r}' \in V$

$$0 = -\frac{1}{4\pi}\int_V G(\underline{r}_>, \underline{r}')F(\underline{r}')u(\underline{r}')\,d^3\underline{r}'$$

$$-\frac{1}{4\pi}\Sigma_{S^-}(\underline{r}_>) \qquad (3)$$

▶ $\underline{r} \in \tilde{V}$, $\underline{r}' \in \tilde{V}$

$$u(\underline{r}_>) = \underbrace{-\frac{1}{4\pi}\Sigma_Q(\underline{r}_>)}_{+u^{(i)}(\underline{r}_>)} + \frac{1}{4\pi}\Sigma_{S^+}(\underline{r}_>) \qquad (4)$$

Where continuity conditions:

$$u_2 = u_1$$

$$\nabla u_2 \cdot \underline{n} = \nabla \underline{u}_1 \cdot \underline{n}$$

Then, $\Sigma_{S^+} = \Sigma_{S^-}$

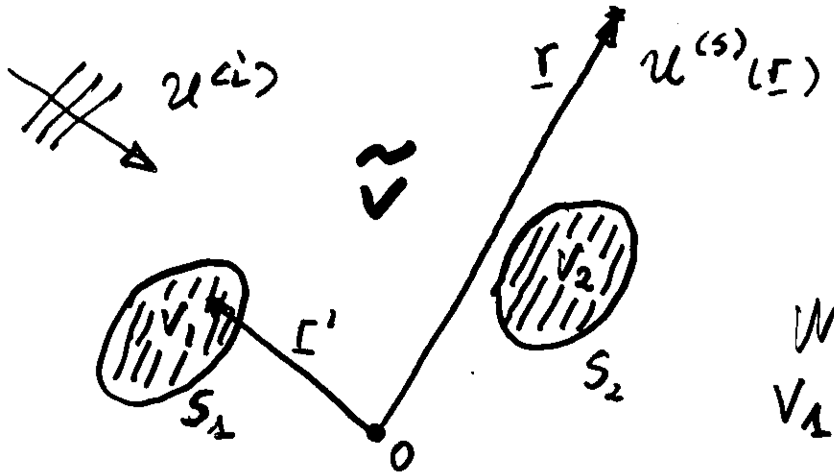
(1) becomes:

$$u(\underline{r}_<) = u^{(i)}(\underline{r}_<) - \frac{1}{4\pi}\int_V d^3r'\,G(\underline{r}_<, \underline{r}')F(\underline{r}')u(\underline{r}')$$

(4) becomes:

$$u(\underline{r}_>) = u^{(i)}(\underline{r}_<) - \frac{1}{4\pi}\int_V d^3r'\,G(\underline{r}_>, \underline{r}')F(\underline{r}')u(\underline{r}')$$

Extinction theorem for a doubly connected domain



With $\underline{r}$ either in V_1 or in V_2

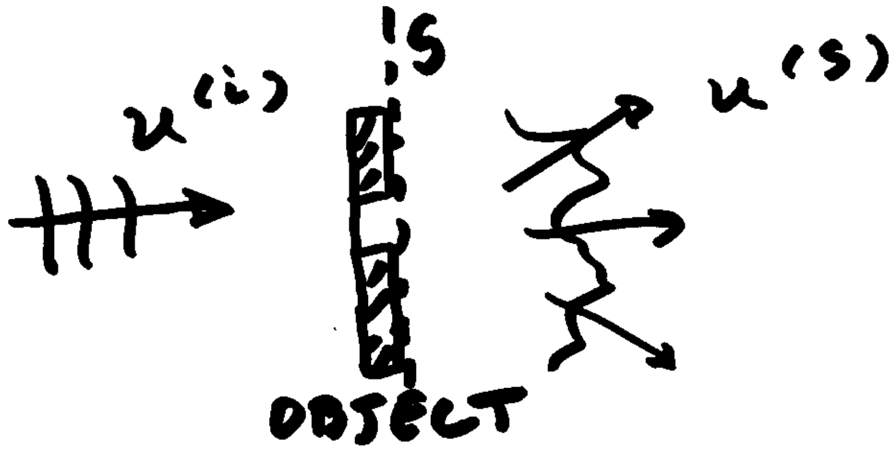
$$\boxed{0 = u^{(i)}(\underline{r}) + \int_{S_1} \left[\underset{(\underline{r}')}{u} \nabla \underset{(\underline{r},\underline{r}')}{G} - \underset{(\underline{r},\underline{r}')}{G} \nabla \underset{(\underline{r}')}{u} \right] \cdot n \, ds(\underline{r}') + \int_{S_2} \left[u \nabla G - G \nabla u \right] \cdot n \, ds(\underline{r}')}$$

Scattered field:

With $\underline{r}$ in $\tilde{V}$

$$-u^{(s)}(\underline{r}) = \frac{1}{4\pi} \int_{S_1} \left[\underset{(\underline{r}')}{u} \nabla \underset{(\underline{r},\underline{r}')}{G} - \underset{(\underline{r},\underline{r}')}{G} \nabla \underset{(\underline{r}')}{u} \right] \cdot \underline{n} \, ds(\underline{r}') + \frac{1}{4\pi} \int_{S_2} \left[u \nabla G - G \nabla u \right] \cdot \underline{n} \, ds(\underline{r}')$$

EXAMPLE: Diffraction by a slit or a transparency



$$u^{(s)} = \frac{1}{4\pi}\oint_S \left[u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right] dS$$

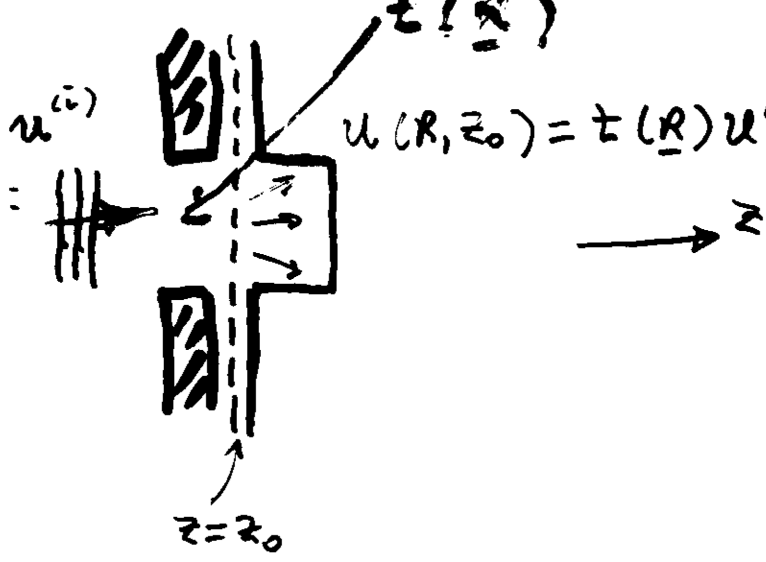
Diffracted field

Boundary values of field at surface S.

Kirchhoff approximation (FOURIER OPTICS)

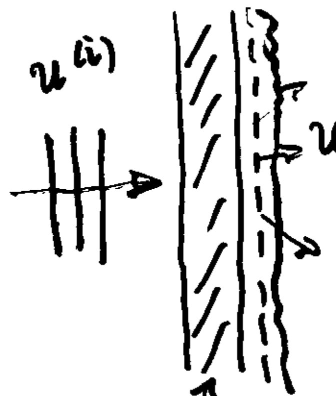
Transmission:

(e.g.) slit



$$u(R,z_0)=t(\underline{R})\,u^{(i)}$$

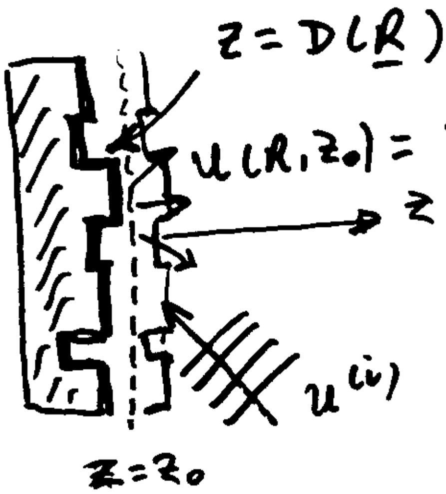
e.g.
Phase screen



$$u(\underline{R},z_0)=t(\underline{R})\,u^{(i)}$$
$$=e^{i\phi(\underline{R})}\,u^{(i)}$$

Transmission: $t(\underline{R})=e^{i\phi(\underline{R})}=e^{ik(n-1)D(\underline{R})}$

Reflection

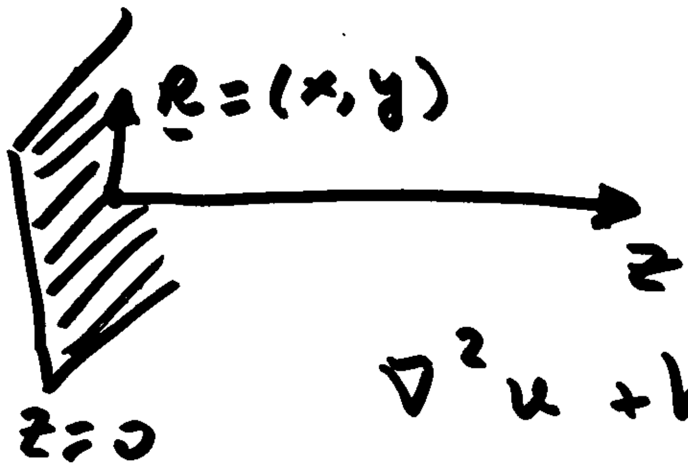


$$u(R,z_0)=u^{(i)}\,e^{ik(z_0-D(R))}$$

Angular spectrum representation

$$u(\underline{R},z)=\int_{-\infty}^{\infty}\hat{u}(\underline{K},z)\,e^{i\underline{K}\cdot\underline{R}}\,d^2\underline{K}$$

$$\underline{k}=(\underline{K},k_z) \qquad \underline{r}=(\underline{R},z)$$



$$\nabla^2 u + k^2 u = 0$$

$$\frac{\partial^2 \hat{u}}{\partial z^2} + k_z^2\,\hat{u} = 0$$

$$\hat{u} = \begin{cases} A(\underline{K})\,e^{ik_z z} & k_z = \sqrt{k^2-K^2} \quad K^2 \le k^2 \\ B(\underline{K})\,e^{-ik_z z} & k_z = i\sqrt{K^2-k^2} \quad K^2 > k^2 \end{cases}$$

$$u(\underline{R},z)=\int_{-\infty}^{\infty} d^2\underline{K}\ \begin{matrix} A(\underline{K}) \\ B(\underline{K}) \end{matrix}\ e^{i\underline{K}\cdot\underline{R}}\,e^{\pm ik_z z}$$

$$u(\underline{R}, z) = \iint_{-\infty}^{\infty} f(\underline{R}') K(\underline{R} - \underline{R}') \, d^2 R'$$

$$f(R') = u(\underline{R}', 0)$$

$$K(\underline{R} - \underline{R}') = \left(\frac{k}{2\pi}\right)^2 \iint_{-\infty}^{\infty} e^{i \underline{K} \cdot (\underline{R} - \underline{R}')} e^{i k_z z} \, d^2 \underline{K}$$

$$= \frac{1}{4\pi} \frac{\partial}{\partial z'} \Bigg[\underbrace{\frac{e^{ik|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|}}_{G(\underline{r}, \underline{r}')} \Bigg]$$

$$u(\underline{R}, z) = \frac{1}{4\pi} \int_S f(\underline{R}') \frac{\partial}{\partial z'} G(\underline{r}, \underline{r}') \, d^2 \underline{R}'$$

1st Rayleigh integral.

$$G(\underline{r}, \underline{r}') = \frac{i}{2\pi} \iint_{-\infty}^{\infty} \frac{d^2 \underline{K}}{k_z} e^{i[\underline{K} \cdot (\underline{R} - \underline{R}') + k_z |z - z'|]}$$

Weyl's representation

Weakly fluctuating random medium

Potential: $F(\underline{r}) = -k^2 [n^2(\underline{r}) - 1]$

$$n = \underbrace{\langle n \rangle}_{1} + \delta n(\underline{r})$$

$$n^2 - 1 \simeq 2\,\delta n \qquad \boxed{F(\underline{r}) = -2k^2\,\delta n(\underline{r})}$$

$$\frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \simeq \frac{e^{ikr}}{r}\, e^{-ik\underline{r}'\cdot\underline{m}} \qquad (kr \to \infty)$$

$$F(\underline{r}')\, u(\underline{r}') = F(r')\, u^{(i)}(\underline{r}')$$

Born approximation

$$\underline{E}(\underline{r}) = -\frac{k^2}{2\pi}\, \underline{m} \times (\underline{m} \times \underline{e}_i) \int_V d^3r'\, e^{-i\underline{K}\cdot r'}\, \delta n(\underline{r}')$$

transversality of far field $\qquad k\underline{m} - \underline{k}^{(i)}$

EXTINCTION THEOREM

$$\underline{E}^{(i)}(\underline{r}_<) + \frac{1}{4\pi}\underline{S}_e(\underline{r}_<) = 0 \qquad \text{(s-waves)}$$

$$\underline{H}^{(i)}(\underline{r}_<) + \frac{1}{4\pi}\underline{S}_h(\underline{r}_<) = 0 \qquad \text{(p-waves)}$$

PERFECT CONDUCTOR:

$$\underline{S}_e(\underline{r}) = \frac{4\pi i k}{c}\int_S \underline{J}(\underline{r}')\,\underline{\underline{G}}(\underline{r},\underline{r}')\,ds'$$

$$\underline{S}_h(\underline{r}) = \frac{4\pi}{c}\int_S \underline{J}(\underline{r}')\cdot\nabla\times\underline{\underline{G}}(\underline{r},\underline{r}')\,ds'$$

$$\underline{\underline{G}}(\underline{r},\underline{r}') = \left(\underline{\underline{U}} + \frac{1}{k^2}\nabla\nabla\right)G_0(\underline{r},\underline{r}')$$

$$G_0(R) = \frac{e^{ikR}}{R}$$

PERFECT CONDUCTOR

Extinction theorem:

$$0 = \underline{E}^{(i)}(\underline{r}) + \frac{ik}{c}\int_S \underline{J}(\underline{r}')\,\underline{\underline{g}}(\underline{r},\underline{r}')\,dS, \quad z \le D(\underline{R})$$

$$\boxed{\text{Dyadic Green's function: } \underline{\underline{g}}(\underline{r},\underline{r}') = \left(I + \frac{1}{k^2}\nabla\nabla\right)G(\underline{r},\underline{r}')} \ ; \quad G(r) = \frac{e^{ikr}}{r}$$

$$\underline{r}' = (\underline{R}', z' = D(\underline{R}')) \ ; \quad \underline{R}' = (x', y')$$

Monte Carlo generated

Discretizing:

$$E_n^{(i)} = -\frac{ik}{c}\sum_{m=1}^{N} g_{nm} J_m$$

$$n, M = 1, \dots, N \ ; \quad (N = 220).$$

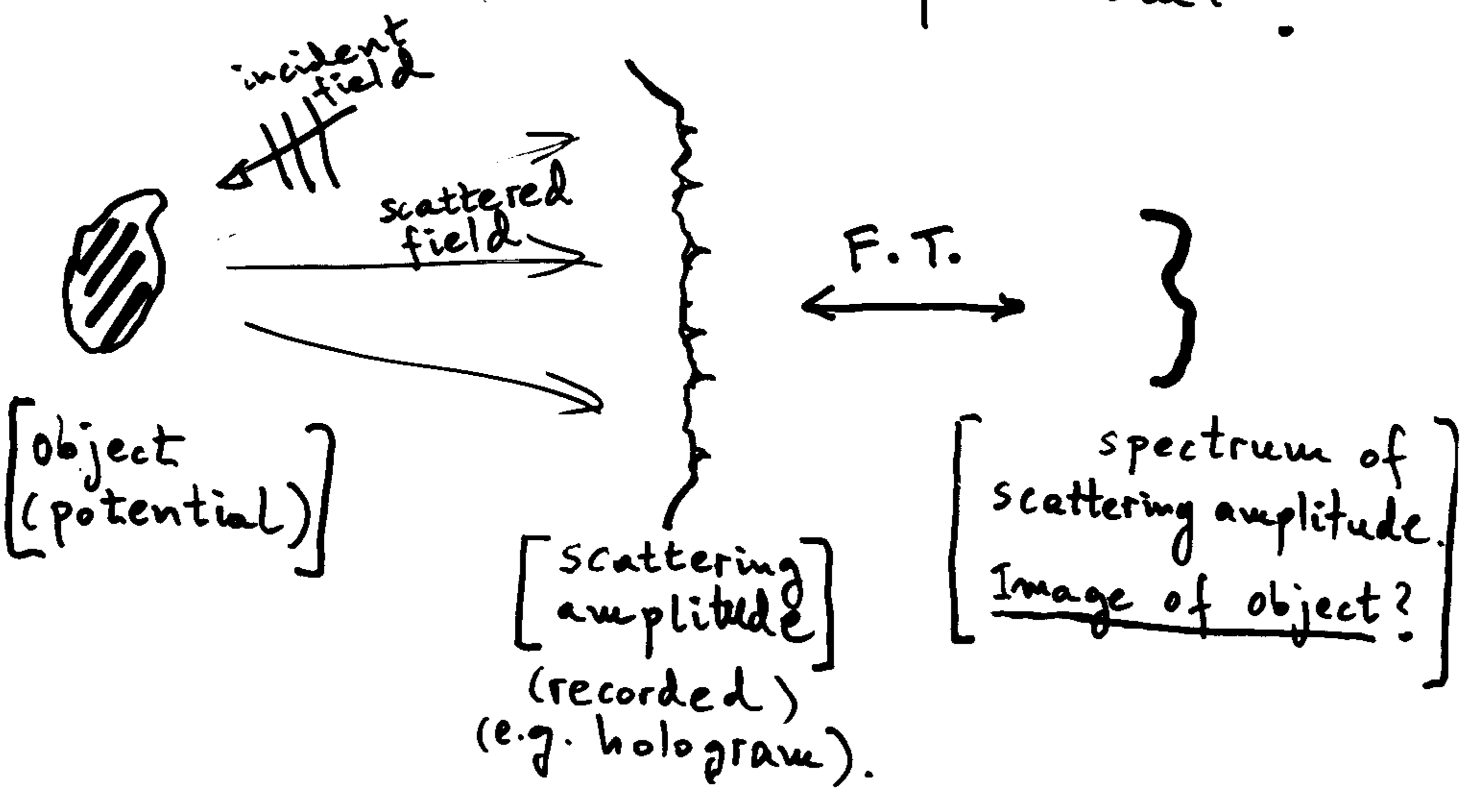
Total field: Reflected + Scattered:

$$E(\underline{r}) = \underline{E}^{(i)}(\underline{r}) + \underbrace{\frac{ik}{c}\int_S \underline{J}(\underline{r}')\,\underline{\underline{g}}(\underline{r},\underline{r}')\,dS}_{\underline{E}^{(s)}(\underline{r}) \equiv \text{scattered field.}}, \quad z > D(\underline{R})$$

QUESTIONS

E. Wolf, Opt. Comm. 1, 153 (1969).

$\boxed{1.}$ What is the relationship between the spectrum of the scattering amplitude and the potential?



$\boxed{2.}$ Are there alternatives to holography when there exists no reference wave?

M.N-V, "Scattering and diffraction in physical optics", (J. Wiley, N.Y. 1991).

E. Wolf, Opt. Comm. 1, 153 (1969)
M. N-V, "Scattering and Diffraction in Physical Optics" (J. Wiley, New York 1991).

$$\nabla^2 u(\underline{r}) + k^2 u(\underline{r}) = F(\underline{r})\, u(\underline{r})$$

Electrons:

$$k^2 = \frac{2m^* E}{\hbar^2}$$

$$F(\underline{r}) = \frac{2m^*}{\hbar^2} \underbrace{V(\underline{r})}_{\text{potential}}$$

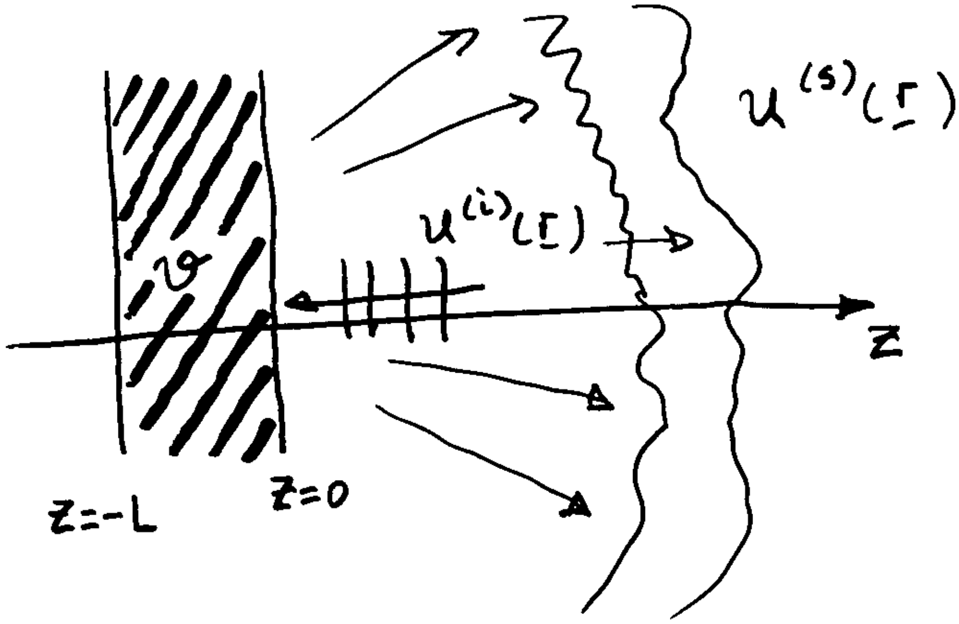
Photons:

$$F(\underline{r}) = -k^2 [\underbrace{n^2(\underline{r})}_{\text{refractive index}} - 1]$$

$$k^2 = \frac{\omega^2}{c^2}$$

$$\nabla^2 \underbrace{\underline{\underline{E}}(\underline{r})}_{\text{electric vector}} + k^2 \underline{\underline{E}}(\underline{r}) = F(\underline{r})\, \underline{\underline{E}}(\underline{r}) + \underbrace{\underline{\nabla}[\underline{\nabla} \cdot \underline{\underline{E}}(\underline{r})]}_{\text{depolarization term}}$$

depolarization term $\simeq 0$ $(T \gg \lambda)$ in the scalar approximation



$$u^{(s)}(\underline{r}) = -\frac{1}{4\pi}\int_{\mathcal{V}} F(\underline{r}')\, u(\underline{r}')\, G(\underline{r},\underline{r}')\, d^3r'$$

$$G(\underline{r},\underline{r}') \triangleq \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{d^2K}{k_z}\, e^{i[\underline{K}\cdot(\underline{R}-\underline{R}')+k_z|z-z'|]}$$

$$\boxed{\begin{aligned} \underline{r} &= (\underline{R}, z) \\ \underline{k} &= (\underline{K}, k_z) \end{aligned}}$$

$$\boxed{\begin{aligned} k_z &= \sqrt{k^2 - K^2}, \quad K^2 \le k^2 \\ k_z &= i\sqrt{K^2 - k^2}, \quad K^2 > k^2 \end{aligned}}$$

$$\underline{u^{(s)}(\underline{r}) = \int_{-\infty}^{\infty} A(\underline{K})\, e^{i(\underline{K}\cdot\underline{R} + k_z z)}\, d^2K}, \quad \boxed{z > 0} \; (!!)$$

$$\underline{A(\underline{K}) = \frac{-i}{8\pi^2 k_z}\int_{\mathcal{V}} d^3r'\, F(\underline{r}')\, u(\underline{r}')\, e^{-i[\underline{K}\cdot\underline{R}' + k_z z']}}$$

$k_z = \sqrt{k^2 - K^2}$, $K^2 \le k^2$, (homogeneous component

$k_z = i\sqrt{K^2 - k^2}$, $K^2 > k^2$, (evanescent components

Due to the Plancherel-Polya theorem, $A(\underline{K})$ is the boundary value on the sphere:

$$|\underline{K}|^2 + k_z^2 = k^2$$

of an <u>entire function of exponential type</u> in the complex space of 3-D: $K_n + iK'_n$, $(n = 1,2,3)$.

Hence:

$U^{(s)}(\underline{r})$ contains <u>both</u> homogeneous and evanescent components, unless it is identically zero.

(E. Wolf, M.N-V, J.O.S.A. <u>A2</u>, 886 (1985).).

<u>Far field</u>:

$$U(\underline{r}) \sim -2\pi i k_z A(\underline{K}) \frac{e^{ikr}}{r} \quad (kr \to \infty)$$

$A(\underline{K})$ unknown in the evanescent region of $\underline{K}$-space. $\Leftarrow \left\{ \begin{array}{l} k_z = \sqrt{k^2 - K^2} \\ K^2 \le k^2 \end{array} \right.$

<u>Resolution limited up to $\lambda/2$.</u>

Far field:

$$\mathcal{U}(\underline{r}) \simeq -2\pi i k_z A(\underline{K}) \frac{e^{ikr}}{r} \quad ; \; (kr \to \infty)$$

A($\underline{K}$) unknown in the evanescent region of K-space. $\Leftarrow \left[\begin{array}{c} k_z = \sqrt{k^2 - K^2} \\ (K^2 \leq k^2) \end{array} \right]$

1. Experiment yields far field, (both in modulus and phase):

Then, $A(\underline{K})$ is known for $K^2 \leq k^2$, (resoluti limited up to $\lambda/2$).

Hence, at any generic plane $z = z_0$:

$$\mathcal{U}^{(s)}(\underline{R}, z_0) = \int_{-\infty}^{\infty} A(\underline{K}) e^{i(\underline{K}\cdot\underline{R} + k_z z_0)} d^2K \; , \; z > 0$$

By F.T. one obtains $\mathcal{U}^{(s)}(\underline{R}, z_0)$ at $z = z_0$ from $A(\underline{K}) e^{ik_z z_0}$. In particular, $\mathcal{U}^{(s)}(\underline{R}, 0)$ at $z = 0$ from $A(\underline{K})$.

What is the connection of $\mathcal{U}^{(s)}(\underline{R}, 0)$ to $F(\underline{R})$? $\mathcal{U}^{(s)}(\underline{R}, 0) = -\frac{1}{4\pi}\int_V d^3r' \, F(\underline{r}')\mathcal{U}(\underline{r}')G(\underline{R}, 0; \underline{r}$

Simplest case: Straight-through propagation inside $\mathcal{V}$.

$\underline{\lambda L \ll T^2}$

(single scattering with rectilinear propagation).

(Eikonal approximation)

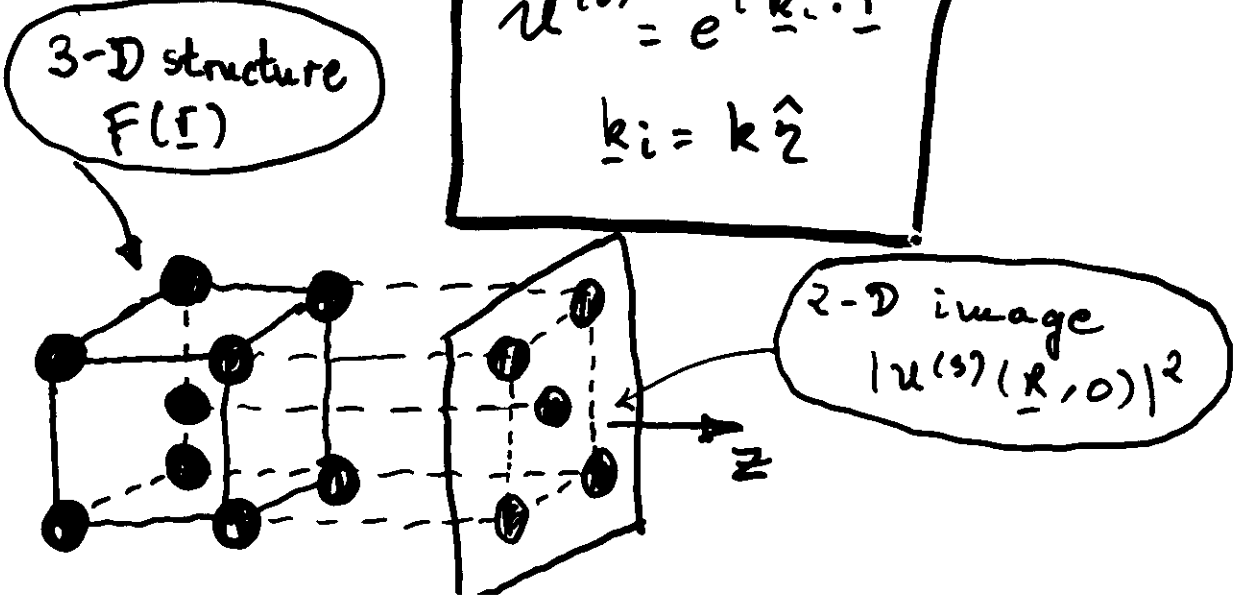
$u^{(i)} = e^{-ikz}$

$$u^{(s)}(\underline{R},0) \simeq \exp\left[ik\int_{-L}^{0} F(\underline{r}')\,dz'\right]$$

<u>Computed tomography</u> yields the 3-D function $F(\underline{r}')$ from its successive projections $\int_a^b F(\underline{r}')\,d\eta$, ($\hat{\eta}$-axis rotated with respect to z-axis).

$\hat{\eta}$-axis chosen as propagation direction of incident wave:

$$\boxed{\begin{aligned} u^{(i)} &= e^{i\underline{k}_i\cdot\underline{r}} \\ \underline{k}_i &= k\hat{\eta} \end{aligned}}$$



2. Experiment yields $u^{(s)}(\underline{R}, z_0)$, (scattered field at plane $z = z_0$).

$$\underbrace{\left[A(\underline{K}) \underbrace{e^{i k_z z_0}}_{\begin{cases} e^{i\sqrt{k^2-K^2}\, z_0}, & K^2 \le k^2 \\ e^{-\sqrt{K^2-k^2}\, z_0}, & K^2 > k^2 \end{cases}} \right]}_{\boxed{\text{low-pass filtered } A(\underline{K})}} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} u^{(s)}(\underline{R}, z_0)\, e^{-i\underline{K}\cdot\underline{R}}\, d^2R$$

This yields again $u^{(s)}(\underline{R}, z_0')$, $z_0' < z_0$; and in particular, $u^{(s)}(\underline{R}, 0)$

Higher resolution if $z_0 \sim \lambda$!!

- <u>Near field optics</u>
- <u>Scanning tunneling microscopy</u>

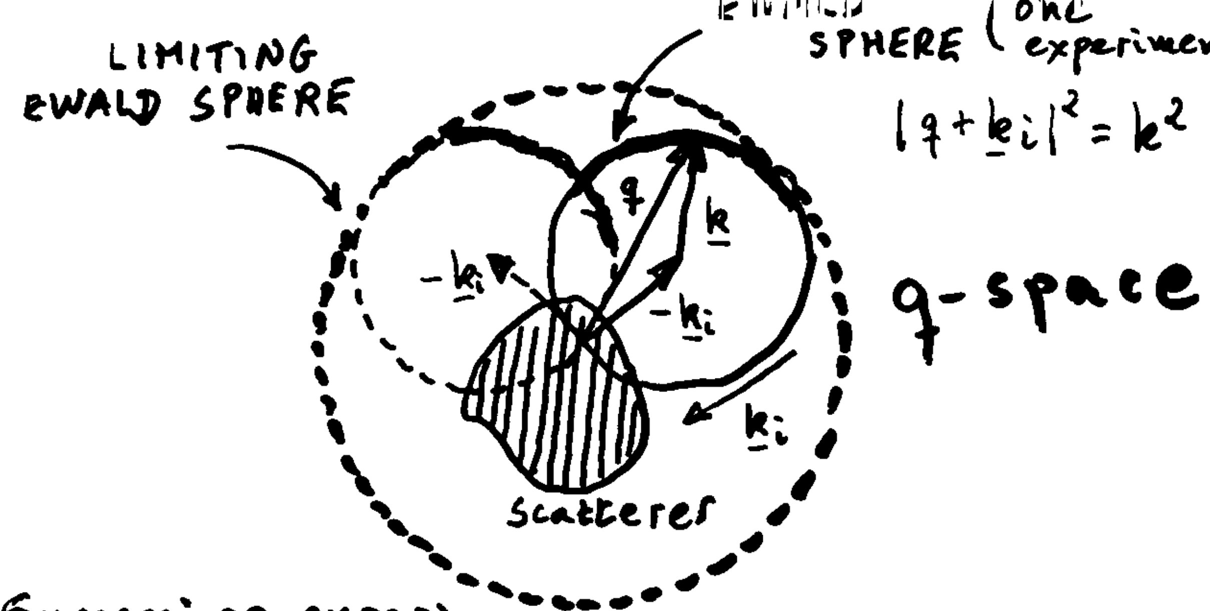
<u>SINGLE SCATTERING</u> $\quad u^{(i)}(\underline{r}) = e^{i\underline{k}^{(i)}\cdot\underline{r}}$

$$A(\underline{K}) = \frac{-i}{8\pi^2 k_z} \int_V d^3r'\, e^{-i\underbrace{[\underline{k} - \underline{k}_i]}_{q}\cdot\underline{r}}\, F(\underline{r}')$$

$$\underline{k} = (\underline{K}, k_z)$$

D F.T. of $F(\underline{r})$ known on the sphere:

$$|q + \underline{k}_i|^2 = k^2 \qquad q = \underline{k} - \underline{k}_i$$



Successive experiments:

Ewald sphere scans the q-space, either by varying its center: $q=-\underline{k}_i$, or by varying its radius $k=\sqrt{2m^*E}/\hbar$, (or $k=\omega/c$).

Lack of uniqueness of the potential:

$$A(\underline{K})=\frac{-i}{8\pi^2 k_z}\int_V d^3r'\, F(\underline{r}')\, u(\underline{r}')\, e^{-i[\underline{K}\cdot\underline{R}'+k_z z']}$$

$$\boxed{|\underline{K}|^2+q_z^2=k^2}$$

Single scattering:

$$A(\underline{Q})=\frac{-i}{8\pi^2 k_z}\int_V d^3r'\, F(\underline{r}')\, e^{-i[\underline{Q}\cdot\underline{R}'+q_z z']}$$

$$q=(\underline{Q},q_z)$$

$$\boxed{|q+\underline{k}_i|^2=k^2}$$

A. J. Devaney, J. Math. Phys. 19, 1526 (1978).

$$\left[A(\underline{Q})\sim\tilde{F}(\underline{Q},q_z)\right]$$

– A. J. Devaney, J. Math. Phys. $\underline{19}$, 1526 (1978)

$\Delta F(\underline{r})$ 1st Born approx.:

$$\widetilde{\Delta F}(\underline{q}) = (|\underline{q}|^2 + 2\underline{k}_i\cdot\underline{q})\,\tilde{\Lambda}(\underline{q})$$

($\tilde{\Lambda}(\underline{q})$ entire function of exponential type)

$\Rightarrow \Delta F(\underline{r})$ non-scatterer.

– M. Nieto-Vesperinas, J. Math. Phys. $\underline{25}$, 2109 (1984)

$A(\underline{Q})$ or $A(\underline{K})$ may possess <u>zero-lines</u>.

R. P. Porter, Prog. Opt. $\underline{27}$, (N. Holland, Amsterdam 1989).

<u>Filtered backpropagation equation</u>:

A. J. Devaney, ('82 – '89)

$$F(\underline{r}) = \int_0^{\pi} d\alpha \int_0^{2\pi} d\beta\; T_{\alpha\beta}(x \sin\alpha \cos\beta + y \sin\alpha \sin\beta + z\cos\alpha)$$

(Diffraction tomography)

Filtered backpropagated field:

$$T_{\alpha\beta}(\xi, \varsigma, \eta) = N_{\alpha\beta}(\xi, \varsigma) * H(\xi, \varsigma, \eta)$$

Filtered projections of $F(\underline{r})$:

$$N_{\alpha\beta}(\xi, \varsigma) = g(\xi, \varsigma) * \mathcal{F}^{-1}\left[\tilde{F}_{\alpha\beta}(\underline{Q})\right]$$

$$\left[g(\xi,\eta) = \iint_{\Omega} d\xi\, d\varsigma\, \exp[-i(Q_1\xi + Q_2\varsigma)] \right] \text{(filter)}$$

Multiple scattering. Inversion methods.

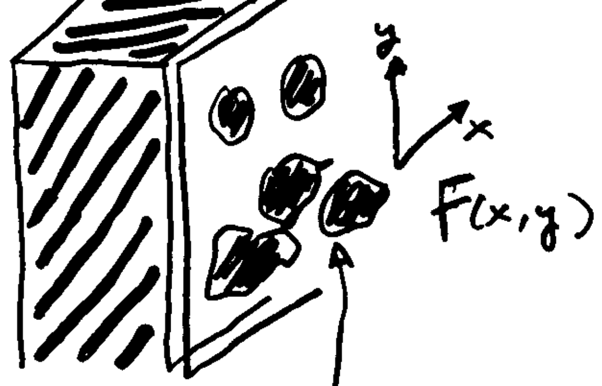
Examples:

Inversion of the Born series (Jost - Kohn algorithm):

R. T. Prosser, J. Math. Phys. 10, 1819 (1969).

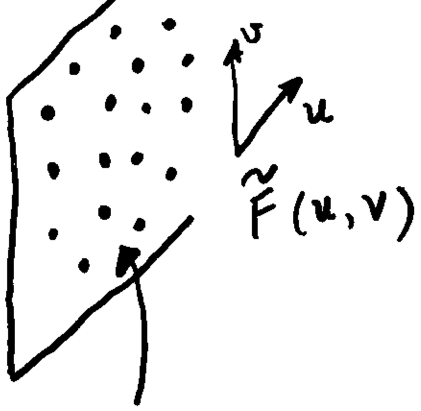
Inversion within the distorted-Wave Born approximation:

A. J. Devaney & M. L. Oristiglio, Phys. Rev. Lett. 51, 237 (1983).



object

Backpropagated field at plane close to object



Far-zone diffracted field

Phase-retrieval problem:

statement:

$$\left[\tilde{F}(u,v) = \int_{-\infty}^{\infty} dx\,dy\; F(x,y)\, e^{i(ux+vy)}\right]$$

$$\tilde{F}(u,v) = |\tilde{F}(u,v)|\, e^{i\phi(u,v)}$$

directly given by experiment, (recorded intensity)

"Unknown", in principle.

Phase retrieval methods:

- Interferometry: holography $\Rightarrow$ $\boxed{\text{Reference wave}}$
- Digital image processing.

Phase retrieval by digital methods

▶ If F(x, y) real and positive, there exist well-established algorithms:

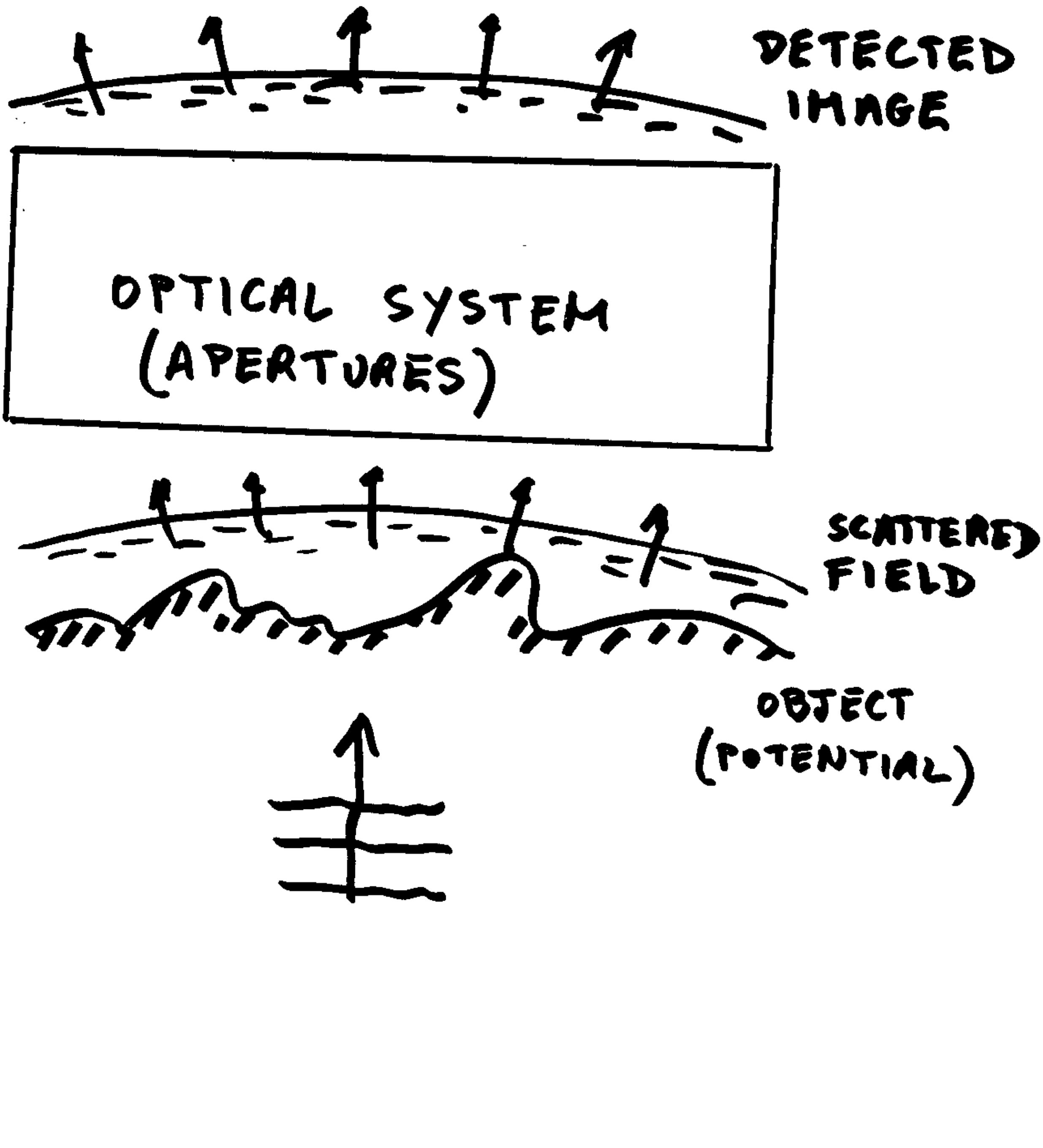
∴ Iterative Fourier transform (optics)
J. R. Fienup, Opt. Lett. 3, 27 (1978)
" Appl. Opt. 21, 2758 (1982)
(and Wackerman), J.O.S.A. A 3, 1897 (1986)

∴ Structure invariants (crystallography)
H. A. Hauptman, Rep. Prog. Phys. 54, 1427 (1991). (review)
H. A. Hauptman + J. Karle ('50's)

∴ Simulated annealing
M. N-V + J. A. Mendez, Opt. Comm. 59, 249 (1986)
(and M. J. Perez Ilzarbe and R. Navarro), J.O.S.A. A 7, 434 (1990)
S. Kirkpatrick, C. D. Gelatt and M. Vecchi, Science 220, 671 (1983)

▶ If F(x, y) complex, algorithms work so far with knowledge of function support.
J. R. Fienup, J.O.S.A. A 4, 118 (1987).

IMAGE FORMATION



CONCEPT OF SUPERRESOLUTION BEYOND λ AND NEAR FIELD

Hertz, Sommerfeld, Bethe.

Bouwkamp, 1954

Synge, 1928

Ash & Nichols, 1972

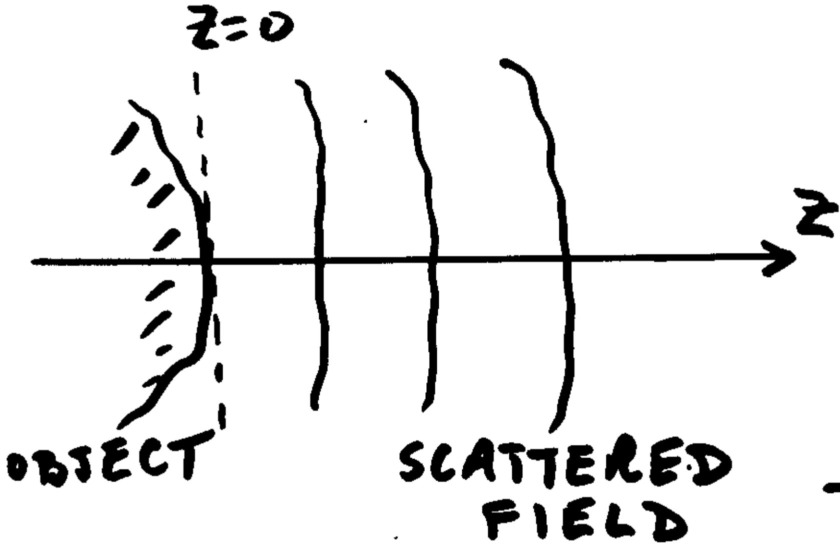
E. Wolf, 1969, A.W. Lohmann, 1978

G. Binnig & H. Rohrer, 1982.

D. W. Pohl, W. Denk & M. Lanz, 1984

Lewis et al., 1986.

Betzig et al., 1992.



$\psi(x, y, z)$

$$\psi(x,y,z) = \int_{-\infty}^{\infty} dp\,dq\, \underbrace{A(p,q)}_{\text{Angular spectrum}} e^{ik(px+qy+mz)}$$

At $z = z_0$ fixed,

$$A(p,q)\, e^{ikmz_0} \rightarrow \text{low pass filter}$$

is given by inverse Fourier transform

$$k = \frac{2\pi}{\lambda}$$

Propagating waves:

$$m = \sqrt{1-p^2-q^2}, \quad p^2+q^2 \leq 1$$

filter: $e^{ik\sqrt{1-p^2-q^2}\,z_0}$

Evanescent components:

$$m = i\sqrt{p^2+q^2-1}, \quad p^2+q^2 > 1$$

filter: $e^{-k\sqrt{p^2+q^2-1}\,z_0}$

NEAR FIELD DETECTION

For instance, at $z_0 = \lambda$

and $p^2+q^2=2$

$$e^{-k\sqrt{p^2+q^2-1}\, z_0} = 1.87\times 10^{-3}$$

Or, to make $e^{-k\sqrt{p^2+q^2-1}\, z_0} = 0.9$

for $p^2+q^2=2$, requires:

$$z_0 = 0.016\,\lambda$$

NEW QUESTIONS

- Bringing the detector close to the object modifies the scattered field. Is the detector passive?

- The image does not resemble the object. Interactions of light with <u>SUBWAVELENGTH</u> structures makes MULTIPLE SCATTERING effects. Classical optics assumptions fail.

BUT, does a scattered field always <u>contain</u> evanescent components?

ANSWER: YES

(E. Wolf & M. Nieto-Vesperinas, 1985)

J. Opt. Soc. Am. **A2**, 886 (1985).

PASSIVE TIPS

Dielectric tip: ($\varepsilon \simeq 2.12$)

diameter < $\underline{0.2\lambda}$

A.M. & M. N-V. J.O.S.A. A $\underline{13}$, 785 (1996)
J.O.S.A. A $\underline{14}$, 618 (1997)
Opt. Lett. $\underline{20}$, 2445 (1995)

Metallic Tip:

diameter < $\underline{0.02\lambda}$

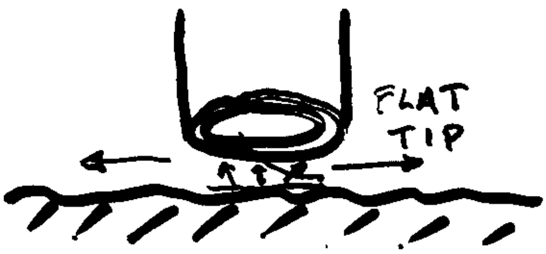
A.M. & M. N.-V. J.O.SA. A (in press)
A.M., R.C., M. N.-V., & J.J. G. J.O.S.A. A (submitted)

(NATO wkshop.) (Kluwer, 1993)

Near-field optical field studies: (hist.)

- G. Mie, Ann. Physik IV 25, 377 (1908)
- H. A. Bethe, Phys. Rev. 66, 163 (1944)
- C. J. Bouwkamp, Philips Res. Rep. 5, 321 (1950)

- E. H. Synge: Phil. Mag. 6, 356 (1928)



- E. A. Ash, G. Nichols, Nature 237, 510 (1972)
 (Microwaves, up to $\lambda/60$, 1-D).
- E.G. Williams, J.D. Maynard, Phys. Rev. Lett. 45, 554 (1980)
 (acoustic wave, $\lambda = 156$ cm, $z_1 = 10$ cm).
- D. W. Pohl, W. Denk, M. Lanz, Appl. Phys. Lett. 44, 651 (198[illegible]
- E. Betzig, M. Isaacson, A. Lewis, Appl. Phys. Lett. 51, 2088 (1987).
 SNOM

G. Binning, H. Rohrer, Ch. Gerber, Phys. Rev. Lett. 49, 57, (1982)
 STM

$z = h\cos\left(\frac{2\pi x}{b}\right)$ $b = 1.25\lambda$
$h = 0.1\lambda$
$\varepsilon_2 = (-17.2, 0.498)$

$z_{\cancel{0}} = d$
$\varepsilon_1 = 2.12$
$\theta_0 = 0°$

p-waves

a: tip radius

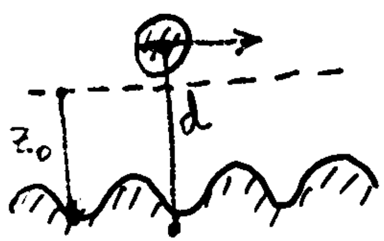
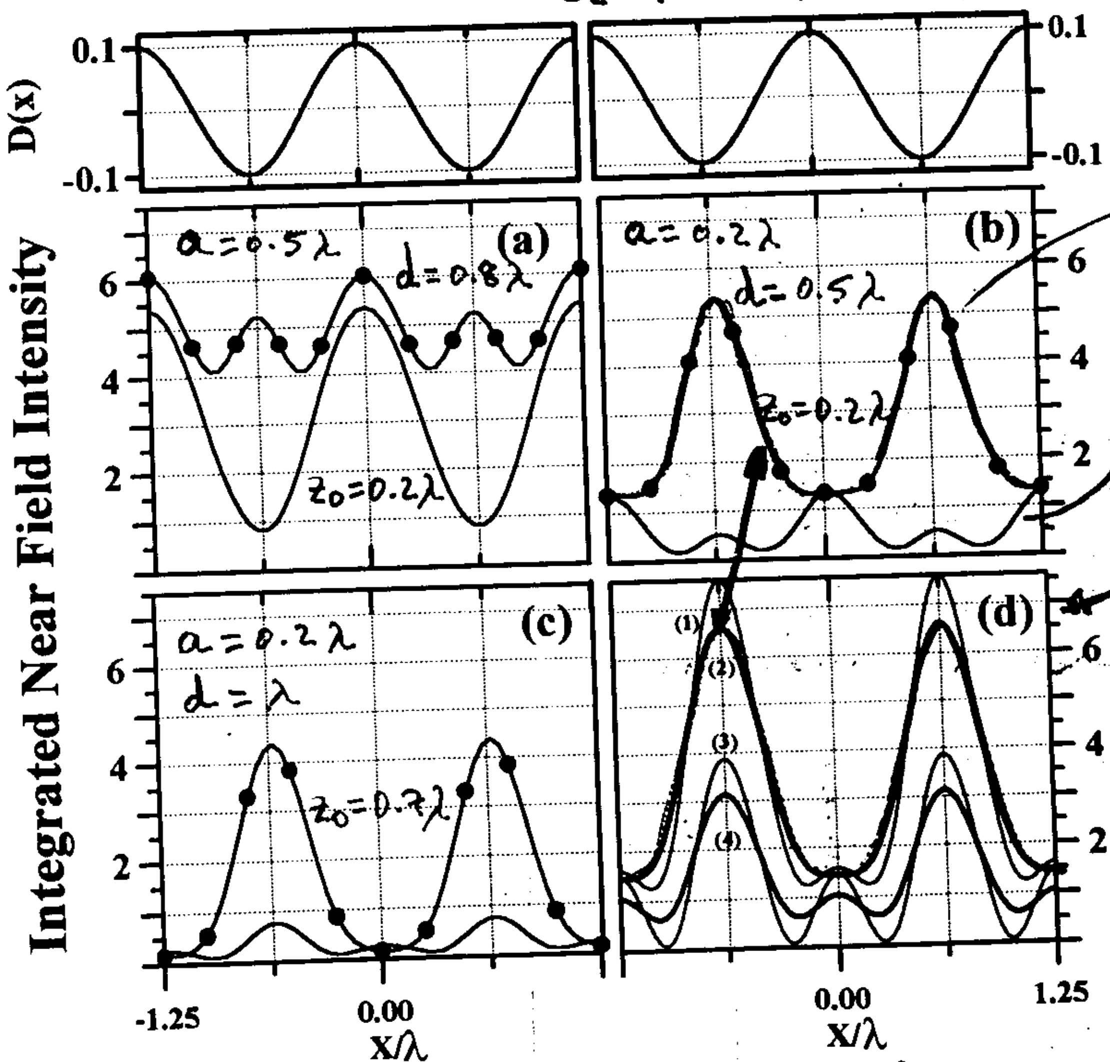


Fig. 3



Integrated Inside ti

Integrated outside at z=

GRATING ALONE

$z_0 = 0.5\lambda$

(1): Total intensit

(2): Total integrate

(3): Scattered out

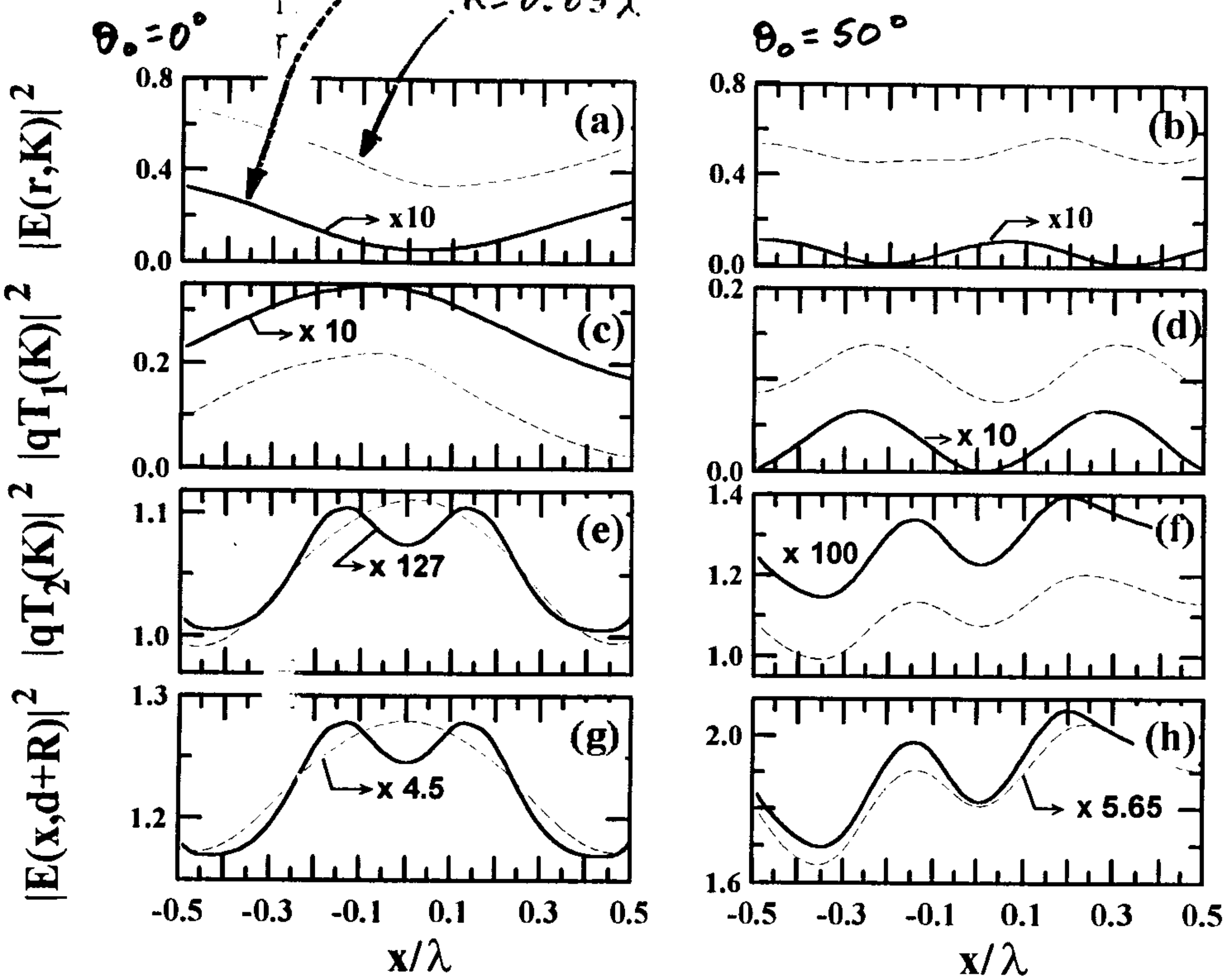
(4): Scatter. integr=

MN-V, Madrazo, JOSA A13 (1996), 785

Tip at $d = 0.055\lambda$

$K = -k\sin\theta_s$

$\theta_s = 50°$



Far field whole system tip + surface

Far field surface - D

Far field tip - C.

Near field inside tip

$h = 0.05\lambda$ $\sigma = 0.06\lambda$ $x_0 = 0.15\lambda$

A. Madrazo, MN-V, JOSA A '97, Vol. 14, P. 2768

Chapter 3

Radiated and Scattered Fields

In this chapter we study time-independent radiated and scattered fields by using the angular spectrum techniques discussed in Chapter 2. This permits to characterize these fields at any point of space, no matter how close it is to the volume containing the sources. Therefore, it is not necessary to use approximations (e.g. Fresnel or Fraunhofer) usually made in other approaches. Once the angular spectrum is known, the field can be computed at any point exterior to the source volume. The usefulness of this method will be seen in this and subsequent chapters. Also, it provides a new way of dealing with certain radiation problems, such as those on fields from moving particles, as shown by R. Asby and E. Wolf [5.1] and E. Lalor and E. Wolf [5.2].

We shall deal in this chapter with *direct source* and *scattering problems*; namely, given the sources, or potentials of the medium, we shall study the field that they produce. *Inverse source* and *scattering problems* will be addressed in Chapter 10.

Even the direct scattering problem is usually an involved subject, requiring the solution of (generally coupled) integro-differential equations. Methods on scattering by volumes have been extensively developed in connection with wave propagation in inhomogeneous (often random) media. We shall see that certain approximations useful in the case of weak, or slightly fluctuating, scatterers permit to obtain analytically simplified solutions. This is the case of the first Born, Rytov and Eikonal approximations. Approaches put forward for dealing with multiple scattering will not be discussed here, as these have been extensively accounted for in other texts and can be found in the references given at the end of this chapter.

3.1 Radiated Fields from a Localized Charge-Current Distribution

Let us consider the monochromatic electromagnetic field radiated into free space by a system of charges and currents specified as follows:

$$q(\mathbf{r}, t) = q(\mathbf{r}) \exp(-i\omega t), \tag{3.1 a}$$

$$\mathrm{j}(\mathbf{r}, t) = \mathrm{j}(\mathbf{r}) \exp(-i\omega t). \tag{3.1 b}$$

We assume that $\mathrm{q}(\mathbf{r})$ and $\mathrm{j}(\mathbf{r})$ are continuous and differentiable functions of position and vanish outside a certain volume V. The real physical magnitudes are obtained, as usual, by taking the real part of these quantities.

We shall discuss the characterization of the field radiated from a finite volume distribution by means of angular spectrum methods, (A.J. Devaney and E. Wolf [5.3], A.T. Friberg and E.Wolf [5.4], and W.H. Carter and E. Wolf [5.5]).

If the medium inside the volume V is non-magnetic, the time independent parts of the vectors **E** and **H** satisfy the inhomogeneous vector equations, (cf.Eqs.(1.8) and (1.10) of Chapter 1):

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = 4\pi i \frac{k}{c} \mathrm{j}(\mathbf{r}), \tag{3.2 a}$$

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k^2\mathbf{H}(\mathbf{r}) = 4\pi \frac{1}{c} \nabla \times \mathrm{j}(\mathbf{r}), \tag{3.2 b}$$

where $k = \omega/c$. And the radiated field *outside* the volume V can be written as (cf. Eqs. (1.46) and (1.47) of Chapter 1):

$$\mathbf{E}(\mathbf{r}_>) = \nabla \times \nabla \times \mathbf{\Pi}(\mathbf{r}_>), \tag{3.3 a}$$

$$\mathbf{H}(\mathbf{r}_>) = -ik\nabla \times \mathbf{\Pi}(\mathbf{r}_>). \tag{3.3 b}$$

where $\mathbf{\Pi}(\mathbf{r})$ is the electric Hertz vector for the radiated field (cf. Eq.(1.42)):

$$\mathbf{\Pi}(\mathbf{r}_>) = \frac{i}{kc}\int_V \mathbf{j}(\mathbf{r}')G(\mathbf{r}_>,\mathbf{r}')d^3r', \tag{3.4}$$

$G(\mathbf{r},\mathbf{r}')$ being the scalar outgoing Green function.

Note that in this case, according to Eq.(1.43), the magnetic Hertz vector is zero. For this reason we do not need to use now the subindices e and m.

3.2 Angular Spectrum Representation of Radiated Fields

Suppose that the radiating volume V is situated within a strip $0 < z < L$, (Fig . 3.1). Let us introduce into (3.4) the angular spectrum representation (2.51) of the outgoing Green function $G(\mathbf{r},\mathbf{r}')$; after interchanging the order of integration, one obtains the following plane wave expansion for the electric Hertz vector:

$$\mathbf{\Pi}(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} \mathbf{a}^{(\pm)}(\mathbf{K}) \exp[i(\mathbf{K}\cdot\mathbf{R}_> \pm k_z z_>)]d^2K, \tag{3.5}$$

where:

$$\mathbf{a}^{(\pm)}(\mathbf{K}) = -\frac{1}{2\pi kck_z}\int_V \mathbf{j}(\mathbf{r}') \exp[-i(\mathbf{K}\cdot\mathbf{R}'_> \pm k_z z')]d^3r'. \tag{3.6}$$

In Eqs.(3.5) and (3.6) $\mathbf{R}$ and $\mathbf{R}$' are two dimensional vectors: $\mathbf{R} = (x,y)$, $\mathbf{R}' = (x',y')$; $\mathbf{r} = (\mathbf{R},z)$, $\mathbf{r}' = (\mathbf{R}',z')$. Also, $\mathbf{K} = k(p,q)$, $k_z = km$. Where p, q and m are the cosine directors satisfying Eqs.(2.9).

The signs *plus* or *minus* in the superscript of $\mathbf{a}(\mathbf{K})$, and in the exponent of the integrands in Eqs.(3.5) and (3.6), are chosen according to whether the point $\mathbf{r}_>$ belongs to the source-free half spaces $\mathcal{R}^+$ or $\mathcal{R}^-$, respectively, (see Fig. 3.1). In addition, we have used the fact that $|z_> - z'| = z_> - z'$ when $\mathbf{r}_>$ is in $\mathcal{R}^+$ and $|z_> - z'| = z' - z_>$ when $\mathbf{r}_>$ is in $\mathcal{R}^-$.

On substituting from (3.5) into (3.3a) and (3.3b), we also obtain the angular spectrum representation of the radiated electromagnetic field in the two half spaces $\mathcal{R}^+$ and

$\mathcal{R}^-$:

$$\mathbf{E}(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} \mathbf{e}^{(\pm)}(\mathbf{K}) \exp[i(\mathbf{K}\cdot\mathbf{R}_> \pm k_z z_>)] d^2K. \tag{3.7}$$

$$\mathbf{H}(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} \mathbf{h}^{(\pm)}(\mathbf{K}) \exp[i(\mathbf{K}\cdot\mathbf{R}_> \pm k_z z_>)] d^2K. \tag{3.8}$$

Where:

$$\mathbf{e}^{(\pm)}(\mathbf{K}) = -\mathbf{k}^{(\pm)} \times (\mathbf{k}^{(\pm)} \times \mathbf{a}^{(\pm)}(\mathbf{K})). \tag{3.9}$$

$$\mathbf{h}^{(\pm)}(\mathbf{K}) = k[\mathbf{k}^{(\pm)} \times \mathbf{a}^{(\pm)}(\mathbf{K})]. \tag{3.10}$$

$\mathbf{k}^{(\pm)}$ being: $\mathbf{k}^{(\pm)} = (\mathbf{K}, \pm k_z)$. On the other hand, the spectral amplitudes are related by: $\mathbf{h}^{(\pm)}(\mathbf{K}) = 1/k[\mathbf{k}^{(\pm)} \times \mathbf{e}^{(\pm)}(\mathbf{K})]$.

Eqs.(3.7)-(3.10) permit the determination of the radiated field at *any* point, either in $\mathcal{R}^+$ or $\mathcal{R}^-$. Note the preferential role played by the z-axis. When the radiating volume is finite and has a shape with no particular symmetry, the z-axis can be arbitrarily chosen. Then, if this axis is varied, the lines $z = 0, 0"$,..., etc, and $z = L, L', L"$,..., etc, conform a convex domain Ω that encloses the volume V, and outside of which Eqs.(3.5), (3.7) and (3.8) are valid with the appropiate rotation of coordinates, (see Fig.3.2).

3.3 The Field and the Intensity Radiated in the Far Zone

Let the point P of observation be situated in the far zone ,(Fig. 3.3). The radiation volume V being finite. Denote the position vector of P by $\mathbf{r}_> = r\mathbf{n}$, $\mathbf{n} = (n_x, n_y, n_z)$ being a unit vector. According to Section 2.12, as $kr \to \infty$ the field in a fixed direction specified by $\mathbf{n}$ is given as:

$$\mathbf{E}(r\mathbf{n}) \sim \mp 2\pi i k n_z \mathbf{e}^{(\pm)}(kn_x, kn_y) \frac{exp(ikr)}{r}, \tag{3.11}$$

$$\mathbf{H}(r\mathbf{n}) \sim \mp 2\pi i k n_z \mathbf{h}^{(\pm)}(kn_x, kn_y) \frac{exp(ikr)}{r}. \tag{3.12}$$

Alternatively, the far zone expressions can be also obtained by making the approximation (Fig. 3.3):

$$|\mathbf{r} - \mathbf{r}'| \sim r - \mathbf{r}' \cdot \mathbf{n}. \tag{3.13}$$

So that the Green function becomes:

$$G(\mathbf{r},\mathbf{r}') \sim \frac{\exp(ikr)}{r} \exp(-ik\mathbf{r}' \cdot \mathbf{n}). \tag{3.14}$$

Substitution of Eq.(3.14) directly into Eqs.(3.3a), (3.3b) and (3.4) yields again Eqs.(3.11) and (3.12) after making use of (3.6), (3.9) and (3.10).

Eqs.(3.11) and (3.12) show that the far field at a given point $r\mathbf{n}$ is proportional to the amplitude of only one plane wave component. Also, it has the envelope of a divergent spherical wave. If the volume V were constituted by sinks rather than by sources, the envelope of the far field would be a convergent spherical wave.

The radiated intensity in the far zone is given by the magnitude of the time averaged Poynting vector, which as seen in Eq.(1.18), is:

$$S^{(f)} = \frac{c}{8\pi} Re[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})]. \tag{3.15}$$

Substitution of Eqs.(3.11) and (3.12) into (3.15), together with the use of (3.6), (3.9) and (3.10), leads to an expression of the radiated intensity in terms of the three dimensional Fourier transform of the current density $\mathbf{j}(\mathbf{r})$. We leave its obtention as an exercice for the reader.

3.4 Scalar Theory of Radiated Wavefields

A scalar wavefield $U(\mathbf{r})$ radiated by a time harmonic source density $\rho(\mathbf{r})$ localized in a volume V, (Fig.3.1), satisfies the inhomogeneous Helmholtz equation, (A.J. Devaney and E. Wolf [5.3]):

$$\nabla^2 U(\mathbf{r}) + k^2 U(\mathbf{r}) = -4\pi\rho(\mathbf{r}), \tag{3.16}$$

whose solution outside the volume V is known from Section 1.6.3 to be:

$$U(\mathbf{r}_>) = \int_V \rho(\mathbf{r}')G(\mathbf{r}_>,\mathbf{r}')d^3r'. \tag{3.17}$$

By introducing the plane wave representation (2.51) for $G(\mathbf{r},\mathbf{r}')$ into (3.17) and using the same notation as in Section 3.2, one gets the following angular spectrum representation for $U(\mathbf{r})$, either in $\mathcal{R}^+$ or $\mathcal{R}^-$:

$$U(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} A^{(\pm)}(\mathbf{K}) \exp[i(\mathbf{K} \cdot \mathbf{R}_> \pm k_z z_>)] d^2K, \tag{3.18}$$

with the angular spectrum being:

$$A^{(\pm)}(\mathbf{K}) = \frac{i}{2\pi k_z} \int_V \rho(\mathbf{r}') \exp[-i(\mathbf{K} \cdot \mathbf{R}' \pm k_z z')] d^3r'. \tag{3.19}$$

Once again the signs *plus* or *minus* are taken according to whether the point $\mathbf{r}_>$ is in $\mathcal{R}^+$ or $\mathcal{R}^-$, respectively.

And the radiated field in the far zone is:

$$U(r\mathbf{s}) \sim (2\pi)^3 \tilde{\rho}(\mathbf{K}) \frac{\exp(ikr)}{r}. \tag{3.20}$$

$\tilde{\rho}(\mathbf{K})$ being the three dimensional Fourier transform of $\rho(\mathbf{r})$ at values $\mathbf{k} = (\mathbf{K}, k_z)$ of its argument.

3.5 Examples of Radiation Fields: Charged Particle with Uniform Two Dimensional Motion

The techniques based on the angular spectrum representation of fields allow the analysis of radiated fields in a new way. As an example we shall study next the electromagnetic field due to a charged particle. The role of homogeneous and evanescent components appear in this example at the root of the threshold condition marking the existence or absence of radiation. This method was first proposed by G. Toraldo di Francia [5.6] and later developed by R. Asby and E. Wolf [5.1] and by E. Lalor and E. Wolf [5.2].

3.5.1 Field due to a Charged Particle Moving in Vacuum

Let us assume a particle with charge e moving on a trajectory characterized by the position vector $r_e(t)$. Then the electric current density $\mathbf{j}(\mathbf{r}, t)$ and the charge density $\rho(\mathbf{r}, t)$ are:

$$\mathbf{j}(\mathbf{r}, t) = ev(t)\delta[\mathbf{r} - \mathbf{r}_e(t)], \tag{3.21}$$

$$\rho(\mathbf{r},t) = e\delta[\mathbf{r} - \mathbf{r}_e(t)]. \tag{3.22}$$

where $v(t) = d\mathbf{r}_e/dt$ is the instantaneous velocity of the particle.

Let us represent the source distributions $\mathbf{j}(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$ in terms of their Fourier components $\tilde{\mathbf{j}}(\mathbf{k},\omega)$ and $\tilde{\rho}(\mathbf{k},\omega)$:

$$\tilde{\mathbf{j}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} dt' \exp(i\omega t')\int d^3r \exp(-i\mathbf{k}\cdot\mathbf{r})\mathbf{j}(\mathbf{r},t') \tag{3.23}$$

$$\tilde{\rho}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} dt' \exp(i\omega t')\int d^3r \exp(-i\mathbf{k}\cdot\mathbf{r})\rho(\mathbf{r},t), \tag{3.24}$$

where the $\mathbf{r}$-integral is extended to the infinite space. By introducing Eqs.(3.21) and (3.22) into (3.23) and (3.24) respectively, we obtain:

$$\tilde{\mathbf{j}}(\mathbf{k},\omega) = \frac{e}{(2\pi)^4}\int_{-\infty}^{\infty} \exp[i(\omega t' - \mathbf{k}\cdot\mathbf{r}_e(t'))]v(t')dt', \tag{3.25}$$

$$\tilde{\rho}(\mathbf{k},\omega) = \frac{e}{(2\pi)^4}\int_{-\infty}^{\infty} \exp[i(\omega t' - \mathbf{k}\cdot\mathbf{r}_e(t'))]dt'. \tag{3.26}$$

Let us represent the electric and magnetic vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ as Fourier integrals:

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},\omega)\exp(-i\omega t)d\omega, \tag{3.27}$$

$$\mathbf{H}(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathbf{H}(\mathbf{r},\omega)\exp(-i\omega t)d\omega. \tag{3.28}$$

Therefore, if $\mathbf{\Pi}(\mathbf{r},\omega)$ is the Fourier component, for each frequency ω, of the Hertz vector $\mathbf{\Pi}(\mathbf{r},t)$, we have, analogously to Eqs.(3.3):

$$\mathbf{E}(\mathbf{r}_>,\omega) = \nabla\times\nabla\times\mathbf{\Pi}(\mathbf{r}_>,\omega). \tag{3.29 a}$$

$$\mathbf{H}(\mathbf{r}_>,\omega) = -ik\nabla\times\mathbf{\Pi}(\mathbf{r}_>,\omega). \tag{3.29 b}$$

Then all the analysis of previous sections applies to the Fourier components at a given frequency ω. Hence, $\mathbf{\Pi}(\mathbf{r},\omega)$ can be represented by means of its angular spectrum $\mathbf{a}^{(\pm)}(\mathbf{K},\omega)$ like in Eq.(3.5):

$$\mathbf{\Pi}(\mathbf{r}_>,\omega) = \int\int_{-\infty}^{\infty} \mathbf{a}^{(\pm)}(\mathbf{K},\omega)\exp[i(\mathbf{K}\cdot\mathbf{R}_> \pm k_z z_>)]d^2K, \tag{3.30}$$

with $\mathbf{a}^{(\pm)}(\mathbf{K},\omega)$ given according to (3.6), namely:

$$\mathbf{a}^{(\pm)}(\mathbf{K},\omega) = -\frac{(2\pi)^2}{kck_z}\tilde{\mathbf{j}}(\mathbf{k}^{\pm},\omega). \tag{3.31}$$

The upper sign *plus* and lower sign *minus* are chosen according to whether the point $\mathbf{r}_>$ is in the half space to the right, $\mathcal{R}^+$, or to the left, $\mathcal{R}^-$, of the strip $0 < z < L$ assumed to contain the trajectory $\mathbf{r} = \mathbf{r}_e(t)$, (cf. Fig. 3.4).

On introducing (3.25) into (3.31) we obtain:

$$\mathbf{a}^{(\pm)}(\mathbf{K},\omega) = -\frac{e}{(2\pi)^2 kck_z}\int\int_{-\infty}^{\infty} \exp[i(\omega t' - \mathbf{k}^{(\pm)}\cdot\mathbf{r}_e(t'))]v(t')dt', \tag{3.32}$$

where, as before, $\mathbf{k}^{(\pm)} = (\mathbf{K}, \pm k_z)$.

Also, from Eqs.(3.6), (3.9), (3.10) and (3.25) we get for the angular spectra $\mathbf{e}^{(\pm)}(\mathbf{K},\omega)$ and $\mathbf{h}^{(\pm)}(\mathbf{K},\omega)$ of each component at frequency ω:

$$\mathbf{e}^{(\pm)}(\mathbf{K},\omega) = \frac{e}{(2\pi)^2 kck_z}\mathbf{k}^{(\pm)}\times[\mathbf{k}^{(\pm)}$$

$$\times\int_{-\infty}^{\infty} \exp[i(\omega t' - \mathbf{k}^{(\pm)}\cdot\mathbf{r}_e(t'))]v(t')dt'], \tag{3.33 a}$$

$$\mathbf{h}^{(\pm)}(\mathbf{K},\omega) = -\frac{e}{(2\pi)^2 ck_z}\mathbf{k}^{(\pm)}$$

$$\times\int_{-\infty}^{\infty} \exp[i(\omega t' - \mathbf{k}^{(\pm)}\cdot\mathbf{r}_e(t'))]v(t')dt']. \tag{3.33 b}$$

3.5.2 Particle Moving Uniformly in Vacuum

Let the particle move with constant velocity $\mathbf{v}_0$. We shall choose the trajectory along the x-axis so that its parametric equation is:

$$x = x_e(t) = v_0 t, \tag{3.34 a}$$

$$y = z = 0. \tag{3.34 b}$$

On substituting from Eqs.(3.34) into Eqs.(3.32) and (3.33) we get:

$$\mathbf{a}^{(\pm)}(\mathbf{K},\omega) = -\frac{e\mathbf{v}_0}{2\pi kck_z}\delta(\omega - K_x v_0), \tag{3.35}$$

where $\mathbf{v}_0 = (v_0, 0, 0)$ and we have taken into account that:

$$\delta(\omega - K_x v_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i(\omega - K_x v_0)t']dt'. \tag{3.36}$$

If the relations $\delta(x) = \delta(-x)$ and $\delta(ax) = \delta(x)/|a|$ are used (a being any real constant), Eq.(3.35) may be written in the form:

$$\mathbf{a}^{(\pm)}(\mathbf{K}, \omega) = -\frac{e}{2\pi k c k_z}\delta(K_x - k\frac{c}{v_0})\hat{\mathbf{x}}, \tag{3.37}$$

$\hat{\mathbf{x}}$ denoting the unit vector in the x-direction in which the particle moves.

Similarly, the angular spectra of the fields $\mathbf{E}(\mathbf{r}, \omega)$ and $\mathbf{H}(\mathbf{r}, \omega)$ at frequency ω are:

$$\mathbf{e}^{(\pm)}(\mathbf{K}, \omega) = \frac{e}{2\pi k c k_z}\mathbf{k}^{(\pm)} \times [\mathbf{k}^{(\pm)} \times \hat{\mathbf{x}}]\delta(K_x - k\frac{c}{v_0}), \tag{3.38 a}$$

$$\mathbf{h}^{(\pm)}(\mathbf{K}, \omega) = -\frac{e}{2\pi c k_z}[\mathbf{k}^{(\pm)} \times \hat{\mathbf{x}}]\delta(K_x - k\frac{c}{v_0}). \tag{3.38 b}$$

Eqs.(3.38) imply that the field due to a charged particle moving with constant velocity v_0 in the x-direction, consists of plane waves such that the x-component K_x of their wavevector are:

$$K_x = K_x^{(0)} \equiv k\frac{c}{v_0} \tag{3.39}$$

Since the speed v_0 of the particle is necessarily smaller than the velocity of light in vacuum, $K_x^{(0)}$ is greater than k and hence, according to (2.9b), k_z is purely imaginary. *The field created by a particle in uniform movement consists of evanescent waves only.* With this value of the transversal component $|\mathbf{K}|$ of the wavevector, the contribution of these plane waves to the far field is nill when $kr \to \infty$, (cf. Eqs. (3.11) and (3.12)). The field is static. This agrees with the well known fact that a uniformly moving particle produces non-radiating fields (se e.g. Refs.3.7, 3.8, or 3.9); namely, there is no radiated power, as can be seen by evaluating the Poynting vector. (Compare with the result of Problem 2.5). In fact, by introducing (3.37) into (3.30) one obtains for the Hertz vector at frequency ω (see Problem 3.6):

$$\mathbf{\Pi}(\mathbf{r}, \omega) = \frac{ie}{\pi k c}\exp[ik\frac{c}{v_0}x]K_0(k\gamma d)\hat{\mathbf{x}}, \tag{3.40}$$

where:

$$\gamma = \sqrt{(\frac{c}{v_0})^2 - 1} \tag{3.41}$$

$$d = \sqrt{y^2 + z^2} \tag{3.42}$$

and, (see Problem 3.5):

$$\int_{-\infty}^{\infty} \frac{1}{k_z} \exp[i(K_y y \pm k_z z)] dK_y = -2iK_0(k\gamma d), \tag{3.43}$$

where $K_0(k\gamma d)$ is the modified Hankel function of argument $k\gamma d$ and zero order.

As $kd \to \infty$ the function $K_0(k\gamma d)$ has the asymptotic behavior:

$$K_0(k\gamma d) \sim \frac{\pi}{2}(\frac{2}{\pi k\gamma d})^{\frac{1}{2}} \exp(-k\gamma d),$$

showing that $\mathbf{\Pi}(\mathbf{r},\omega)$ (and thus the field) decays exponentially with increasing distance d from the line of motion of the particle.

3.5.3 Cherenkov Radiation

Let us consider now a charged particle moving with constant velocity v_0 in the positive x-direction in a homogeneous, isotropic, non magnetic medium, characterized by a refractive index $n(\omega)$. In a way equivalent to that leading to Eqs.(3.37) and (3.38), we obtain the angular spectrum amplitudes:

$$\mathbf{a}^{(\pm)}(\mathbf{K},\omega) = -\frac{e}{2\pi k(\omega) c k_z} \delta(K_x - k(\omega)\frac{v(\omega)}{v_0})\hat{\mathbf{x}}, \tag{3.44}$$

and:

$$\mathbf{e}^{(\pm)}(\mathbf{K},\omega) = \frac{e}{2\pi k(\omega) c k_z} \mathbf{k}^{(\pm)} \times [\mathbf{k}^{(\pm)} \times \hat{\mathbf{x}}] \delta(K_x - k(\omega)\frac{v(\omega)}{v_0}), \tag{3.45 a}$$

$$\mathbf{h}^{(\pm)}(\mathbf{K},\omega) = -\frac{e}{2\pi c k_z} [\mathbf{k}^{(\pm)} \times \hat{\mathbf{x}}] \delta(K_x - k(\omega)\frac{v(\omega)}{v_0}), \tag{3.45 b}$$

where:

$$v(\omega) = \frac{c}{n(\omega)} \tag{3.46}$$

is the phase velocity in the medium for plane waves at frequency ω. Also, in this case:

$$k(\omega) = n(\omega)\frac{\omega}{c} = \frac{\omega}{v(\omega)}. \tag{3.47}$$

Eqs.(3.44) and (3.45) show that the angular spectrum of the field at frequency ω consists of plane waves such that the transversal component of their wavevector has the value:

$$K_x = K_x^{(0)} \equiv k(\omega)\frac{v(\omega)}{v_0}. \tag{3.48}$$

Two cases can now be distinguished, (R. Asby and E. Wolf [5.1]):

a. The Speed of the Particle v_0 is Smaller than the Phase Velocity $v(\omega)$.

In this case $K_x > k$, (namely, $p > 1$), and hence all waves at frequency ω are evanescent. Like in the case of uniform movement in vacuum, the particle *does not radiate* at frequency ω. In Fig.3.4 (a) the point $(K_x^{(0)} = kv(\omega)/c, K_y)$ lies outside the circle $K^2 = k^2$.

b. The Speed of the Particle v_0 is Greater than the Phase Velocity $v(\omega)$.

In this case $K_x < k$, (which means $p < 1$), and the point $(K_x^{(0)} = kv(\omega)/v_0, K_y)$ in Fig.3.4 (b) is on the line that intersects the circle $K^2 = k^2$ at points A and B. These two points limit the range of values of K_y for which $K_x^2 + K_y^2 \leq k^2$; namely, for which the waves are homogeneous (cf. Eqs.(2.9)). I.e., for values:

$$-k(\omega)[1 - (\frac{v(\omega)}{v_0})^2]^{\frac{1}{2}} \leq K_y \leq k(\omega)[1 - (\frac{v(\omega)}{v_0})^2]^{\frac{1}{2}} \tag{3.49}$$

the waves are homogeneous; whereas for values:

$$k(\omega)[1 - (\frac{v(\omega)}{v_0})^2]^{\frac{1}{2}} \;<\; K_y < \infty, \tag{3.50 a}$$

$$-\infty < K_y \;<\; k(\omega)[1 - (\frac{v(\omega)}{v_0})^2]^{\frac{1}{2}} \tag{3.50 b}$$

the waves are evanescent.

The directions of propagation of all homogeneous waves form a circular cone about the x-axis of movement of the particle. The semiangle of this cone is (Fig.3.5):

$$\theta_0(\omega) = \arccos(\frac{v(\omega)}{v_0}). \tag{3.51}$$

On introducing (3.44) into (3.30) the reader can verify that the Hertz vector at frequency ω is given by the following expression:

$$\Pi(\mathbf{r},\omega) = -\frac{e}{2kc}\exp[ik(\omega)\frac{v(\omega)}{v_0}x]H_0^{(1)}(k(\omega)\sigma_\omega d)\hat{\mathbf{x}}, \tag{3.52}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zero order. And:

$$\sigma_\omega = \sqrt{1-(\frac{v(\omega)}{v_0})^2} = \sin\theta_0, \tag{3.53}$$

$$d = \sqrt{y^2+z^2}. \tag{3.54}$$

Eq.(3.52) represents a conical wave with its axis along the line of motion of the particle, (namely, the x-axis).

Asymptotically, ($kd \to \infty$), it can be seen (Problem 5.5) that the Hertz vector decays as $1/(\omega d/c)$. This indicates that the particle now gives rise to a radiated field which is known as *Cherenkov radiation* (see Refs. 3.9 and 3.10).

Other studies, (Refs. 3.2, 3.6 and 3.11), show that the evanescent components of the field due to a uniformly moving charge in vacuum can be converted into homogeneous waves after interaction with a dielectric when $v_0 > v(\omega)$; namely, when the speed of the particle exceeds the phase velocity in the medium, hence producing *Cherenkov* radiation.

3.6 Integro-Differential Equations for the Scattered Electromagnetic Field in a Time Independent Medium. Angular Spectrum Representation Outside the Strip $0 < z < L$

The response of a material to an incident field is described by the *constitutive relations* for the *induced polarization* $\mathbf{P}(\mathbf{r})$ and the *induced magnetization* $\mathbf{M}(\mathbf{r})$. The representation of the scattered field in terms of these quantities is formally equivalent to that used for radiated fields. As a matter of fact, this field can be considered as the field radiated from sources induced in the material. These sources being characterized by the quantities $\mathbf{P}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$. The *direct scattering problem* consists of finding these sources from knowledge of both the incident field and the dielectric and magnetic susceptibilities of the scattering medium.

The angular spectrum representation of scattered fields discussed in this section was established by A.T. Friberg and E. Wolf [5.4], and is analogous to the formalism of Refs. 3.3 and 3.5 discussed in Section 3.2 in connection with radiated fields.

Let $\mathbf{E}^{(i)}(\mathbf{r})$ and $\mathbf{H}^{(i)}(\mathbf{r})$ represent the electric and magnetic vectors, respectively, of a monochromatic field incident on a generally inhomogeneous medium characterized by a refractive index $n(\mathbf{r})$, $\mathbf{r}$ being a position vector. Upon interaction with the medium, a new field $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ is created. We assume that this medium occupies a volume that always in practice is finite, (see Fig. 3.1). By omitting the time dependent part $\exp(-i\omega t)$ of the complex amplitudes, it is customary to express the new field as the sum of the incident and the scattered field:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(i)}(\mathbf{r}) + \mathbf{E}^{(s)}(\mathbf{r}), \tag{3.55 a}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}^{(i)}(\mathbf{r}) + \mathbf{H}^{(s)}(\mathbf{r}), \tag{3.55 b}$$

$\mathbf{E}^{(s)}$ and $\mathbf{H}^{(s)}$ being the scattered electric and magnetic vectors. It should be reminded that only the real part of the product of these functions by their corresponding harmonic time dependent factor represent real physical quantities.

As seen in Section 1.2, the vectors $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ satisfy the following equations:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = 4\pi k[k\mathbf{P}(\mathbf{r}) + i\nabla \times \mathbf{M}(\mathbf{r})], \tag{3.56 a}$$

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k^2\mathbf{H}(\mathbf{r}) = 4\pi k[-i\nabla \times \mathbf{P}(\mathbf{r}) + k\mathbf{M}(\mathbf{r})]. \tag{3.56 b}$$

Eqs.(3.56) are analogous to Eqs.(3.2), the only difference being the source term in the right hand side. In addition, as seen in Section 1.6.2, the scattered electric and magnetic vectors are given at points inside the scattering volume by:

$$\mathbf{E}^{(s)}(\mathbf{r}_<) = \nabla \times \nabla \times \mathbf{\Pi}_e(\mathbf{r}_<) + ik\nabla \times \mathbf{\Pi}_m(\mathbf{r}_<) - 4\pi\mathbf{P}(\mathbf{r}_<), \tag{3.57 a}$$

$$\mathbf{H}^{(s)}(\mathbf{r}_<) = \nabla \times \nabla \times \mathbf{\Pi}_m(\mathbf{r}_<) - ik\nabla \times \mathbf{\Pi}_e(\mathbf{r}_<) - 4\pi\mathbf{M}(\mathbf{r}_<). \tag{3.57 b}$$

At points outside this volume V the fields are given by expressions identical to Eqs.(3.57) with $\mathbf{P}(\mathbf{r}_>) = \mathbf{M}(\mathbf{r}_>) = 0$.

The electric and magnetic Hertz vectors $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ satisfy the integro differential equations similar to (3.4):

$$\mathbf{\Pi}_e(\mathbf{r}) = \int_V \mathbf{P}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3r', \tag{3.58 a}$$

$$\mathbf{\Pi}_m(\mathbf{r}) = \int_V \mathbf{M}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3r'. \tag{3.58 b}$$

On comparing Eqs.(3.56)-(3.58) with Eqs.(3.2)-(3.4) one can straightforwardly obtain the angular spectrum representation for the scattered vectors in the two half spaces $\mathcal{R}^+$ and $\mathcal{R}^-$ in the form given by Eqs.(3.7) and (3.8), where now the spectral amplitudes $\mathbf{e}^{(\pm)}(\mathbf{K})$ and $\mathbf{h}^{(\pm)}(\mathbf{K})$ are expressed by:

$$\begin{aligned} \mathbf{e}^{(\pm)}(\mathbf{K}) = -i\frac{(2\pi)^2}{k_z}[\mathbf{k}^{(\pm)} \times [\mathbf{k}^{(\pm)}\times\tilde{\mathbf{P}}(\mathbf{k}^{(\pm)})] \\ +k[\mathbf{k}^{(\pm)}\times\tilde{\mathbf{M}}(\mathbf{k}^{(\pm)})]], \end{aligned} \tag{3.59 a}$$

$$\begin{aligned} \mathbf{h}^{(\pm)}(\mathbf{K}) = -i\frac{(2\pi)^2}{k_z}[\mathbf{k}^{(\pm)} \times [\mathbf{k}^{(\pm)}\times\tilde{\mathbf{M}}(\mathbf{k}^{(\pm)})] \\ -k[\mathbf{k}^{(\pm)}\times\tilde{\mathbf{P}}(\mathbf{k}^{(\pm)})]]. \end{aligned} \tag{3.59 b}$$

Where $\tilde{\mathbf{P}}(\mathbf{k}^{(\pm)})$ and $\tilde{\mathbf{M}}(\mathbf{k}^{(\pm)})$ are the three dimensional Fourier transforms of $\mathbf{P}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$, respectively. Note that since the components of $\mathbf{k}^{(\pm)}$ satisfy: $K^2 + k_z^2 = k^2$, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{M}}$ are ultimately functions of $\mathbf{K}$.

Eqs.(3.59 a) and (3.59 b) characterize the scattered field at any point either in $\mathcal{R}^+$ or $\mathcal{R}^-$, no matter how close to the scattering volume is. Since this volume V is finite, by varying the orientation of the z-axis, a convex domain Ω is built by a procedure similar to that of Section 3.2, (see Fig. 3.2), outside of which this form of representation of the scattered field is valid.

The fields and intensity in the far zone can be easily obtained from Eqs. (3.59) in an identical way as in Section 3.3.

3.7 Angular Spectrum Representation of the Scattered Electromagnetic Field Inside the Strip $0 < z < L$

Let us refer again to Fig.3.1 and address the obtention of the scattered field at points situated inside the strip $0 < z < L$. Two cases will be considered depending on whether the point is outside or inside the volume V.

3.7.1 Scattered Field Outside the Scattering Volume

When the point $\mathbf{r}$ is inside the strip $0 < z < L$, but outside the scattering volume V, we introduce the plane wave expansion for $G(\mathbf{r}_>, \mathbf{r}')$, (2.51), into Eqs.(3.58); then the electric and magnetic vectors are evaluated from Eqs.(3.57). Let us see first the result of this operation for the Hertz vectors. Due to the existence of the factor $|z_> - z'|$ in the exponent of the integrand when Eq.(2.51) is used for $G(\mathbf{r}_>, \mathbf{r}')$, the strip $0 < z < L$ has to be divided into two regions: $z < z'$ and $z > z'$, (see Fig. 5.9), so that:

$$|z_> - z'| = z_> - z', \quad \text{when} \quad z_> > z', \tag{3.60 a}$$

$$|z_> - z'| = z' - z_>, \quad \text{when} \quad z_> < z'. \tag{3.60 b}$$

Let us denote by $V^-(z_>)$ the portion of V that contains points $\mathbf{r}'$ with $z' < z_>$, and by $V^+(z_>)$ the part of V with points $\mathbf{r}'$ for which $z' > z_>$, (Fig. 3.7). Obviously, these two volumes are both functions of the z-component, $z_>$, of the point $\mathbf{r}_> = (\mathbf{R}_>, z_>)$ at which the scattered field is being considered. In addition, it is evident that the sum of these two volumes equals V, namely:

$$V = V^-(z_>) + V^+(z_>). \tag{3.61}$$

On introducing the expansion (2.51) for $G(\mathbf{r}_>, \mathbf{r}')$ into (3.58) ,and taking (3.60) and (3.61) into account, we obtain:

$$\mathbf{\Pi}_e(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} \mathbf{a}_e^{(+)}(\mathbf{K}, z_>) \exp[i\mathbf{k}^{(+)} \cdot \mathbf{r}_>] d^2K$$

$$+\int\int_{-\infty}^{\infty} \mathbf{a}_e^{(-)}(\mathbf{K}, z_>) \exp[i\mathbf{k}^{(-)} \cdot \mathbf{r}_>] d^2K, \tag{3.62 a}$$

$$\mathbf{\Pi}_m(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} \mathbf{a}_m^{(+)}(\mathbf{K}, z_>) \exp[i\mathbf{k}^{(+)} \cdot \mathbf{r}_>] d^2K$$
$$+\int\int_{-\infty}^{\infty} \mathbf{a}_m^{(-)}(\mathbf{K}, z_>) \exp[i\mathbf{k}^{(-)} \cdot \mathbf{r}_>] d^2K, \tag{3.62 b}$$

where:

$$\mathbf{a}_e^{(\pm)}(\mathbf{K}, z_>) = \frac{i}{2\pi k_z} \int_{V\mp(z_>)} \mathbf{P}(\mathbf{r}') \exp[-i\mathbf{k}^{(\pm)} \cdot \mathbf{r}'] d^3r', \tag{3.63 a}$$

$$\mathbf{a}_m^{(\pm)}(\mathbf{K}, z_>) = \frac{i}{2\pi k_z} \int_{V\mp(z_>)} \mathbf{M}(\mathbf{r}') \exp[-i\mathbf{k}^{(\pm)} \cdot \mathbf{r}'] d^3r'. \tag{3.63 b}$$

Let us introduce now the two dimensional Fourier transform of the induced polarization $\mathbf{P}(\mathbf{r})$ and magnetization $\mathbf{M}(\mathbf{r})$:

$$\tilde{\mathbf{P}}(\mathbf{K}, z) = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} P(\mathbf{r}) \exp(-i\mathbf{K} \cdot \mathbf{R}) d^2R, \tag{3.64 a}$$

$$\tilde{\mathbf{M}}(\mathbf{K}, z) = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} M(\mathbf{r}) \exp(-i\mathbf{K} \cdot \mathbf{R}) d^2R. \tag{3.64 b}$$

By means of Eqs.(3.64) we easily obtain from (3.63):

$$\mathbf{a}_e^{(+)}(\mathbf{K}, z_>) = \frac{2\pi i}{k_z} \int_0^{z_>} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(-ik_z z') dz', \tag{3.65 a}$$
$$z_> > z',$$

$$\mathbf{a}_e^{(-)}(\mathbf{K}, z_>) = \frac{2\pi i}{k_z} \int_{z_>}^{L} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(ik_z z') dz', \tag{3.65 b}$$
$$z_> < z',$$

$$\mathbf{a}_m^{(+)}(\mathbf{K}, z_>) = \frac{2\pi i}{k_z} \int_0^{z_>} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(-ik_z z') dz', \tag{3.66 a}$$
$$z_> > z',$$

$$\mathbf{a}_m^{(-)}(\mathbf{K}, z_>) = \frac{2\pi i}{k_z} \int_{z_>}^{L} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(ik_z z') dz', \tag{3.66 b}$$
$$z_> < z',$$

Eqs.(3.62), together with Eqs.(3.65) and (3.66) show that now the Hertz vectors $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ contain both forward propagating plane waves with spectral amplitudes

$\mathbf{a}_e^{(+)}(\mathbf{K}, z_>)$ and $\mathbf{a}_m^{(+)}(\mathbf{K}, z_>)$, respectively, and backward plane waves with spectral amplitudes $\mathbf{a}_e^{(-)}(\mathbf{K}, z_>)$ and $\mathbf{a}_m^{(-)}(\mathbf{K}, z_>)$, respectively. In this case, these amplitudes depend on the z-component $z_>$ of the point $\mathbf{r}_>$ in which the field is being evaluated.

When Eqs.(3.62), (3.64), (3.65) and (3.66) are introduced into (3.57), the following expression for the electromagnetic field is obtained:

$$\begin{aligned}\mathbf{E}^{(s)}(\mathbf{r}_>) = &\int\int_{-\infty}^{\infty} \mathbf{e}^{(+)}(\mathbf{K}, z_>) \exp(i\mathbf{k}^{(+)} \cdot \mathbf{r}_>) d^2K \\ &+ \int\int_{-\infty}^{\infty} \mathbf{e}^{(-)}(\mathbf{K}, z_>) \exp(i\mathbf{k}^{(-)} \cdot \mathbf{r}_>) d^2K,\end{aligned} \tag{3.67 a}$$

$$\begin{aligned}\mathbf{H}^{(s)}(\mathbf{r}_>) = &\int\int_{-\infty}^{\infty} \mathbf{h}^{(+)}(\mathbf{K}, z_>) \exp(i\mathbf{k}^{(+)} \cdot \mathbf{r}_>) d^2K \\ &+ \int\int_{-\infty}^{\infty} \mathbf{h}^{(-)}(\mathbf{K}, z_>) \exp(i\mathbf{k}^{(-)} \cdot \mathbf{r}_>) d^2K,\end{aligned} \tag{3.67 b}$$

where:

$$\begin{aligned}\mathbf{e}^{(+)}(\mathbf{K}, z_>) = &-i\frac{2\pi}{k_z}[\mathbf{k}^{(+)}\times[\mathbf{k}^{(+)} \\ &\times \int_0^{z_>} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(-ik_z z') dz'] \\ &+k[\mathbf{k}^{(+)}\times \int_0^{z_>} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(-ik_z z') dz']],\end{aligned} \tag{3.68 a}$$

$$\begin{aligned}\mathbf{e}^{(-)}(\mathbf{K}, z_>) = &-i\frac{2\pi}{k_z}[\mathbf{k}^{(-)}\times[\mathbf{k}^{(-)} \\ &\times \int_{z_>}^{L} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(ik_z z') dz'] \\ &+k[\mathbf{k}^{(-)}\times \int_{z_>}^{L} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(ik_z z') dz']],\end{aligned} \tag{3.68 b}$$

$$\begin{aligned}\mathbf{h}^{(+)}(\mathbf{K}, z_>) = &-i\frac{2\pi}{k_z}[\mathbf{k}^{(+)}\times[\mathbf{k}^{(+)} \\ &\times \int_0^{z_>} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(-ik_z z') dz'] \\ &-k[\mathbf{k}^{(+)}\times \int_0^{z_>} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(-ik_z z') dz']],\end{aligned} \tag{3.69 a}$$

$$\begin{aligned}\mathbf{h}^{(-)}(\mathbf{K}, z_>) = &-i\frac{2\pi}{k_z}[\mathbf{k}^{(-)}\times[\mathbf{k}^{(-)} \\ &\times \int_{z_>}^{L} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(ik_z z') dz'] \\ &-k[\mathbf{k}^{(-)}\times \int_{z_>}^{L} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(ik_z z') dz']].\end{aligned} \tag{3.69 b}$$

Eqs.(3.67)-(3.69) show that the electric and magnetic vectors contain both forward and backward plane wave components. Note that when the point at which the field is being considered moves to $\mathcal{R}^+$ or $\mathcal{R}^-$ these equations coincide with those of Section 3.6.

3.7.2 Scattered Field Inside the Scattering Volume. The Slowly Varying Amplitude Approximation

If the point at which the scattered field is considered is inside the volume V of the scatterer, the expressions of Section 3.7.1 for the Hertz vectors $\mathbf{\Pi}_e(\mathbf{r})$ and $\mathbf{\Pi}_m(\mathbf{r})$ are still valid, with $\mathbf{r}_>$ being now replaced by $\mathbf{r}_<$. Namely:

$$\begin{aligned}\mathbf{\Pi}_e(\mathbf{r}_<) &= \int\int_{-\infty}^{\infty} \mathbf{a}_e^{(+)}(\mathbf{K}, z_<) \exp[i\mathbf{k}^{(+)} \cdot \mathbf{r}_<] d^2K \\ &+ \int\int_{-\infty}^{\infty} \mathbf{a}_e^{(-)}(\mathbf{K}, z_<) \exp[i\mathbf{k}^{(-)} \cdot \mathbf{r}_<] d^2K, \end{aligned} \tag{3.70 a}$$

$$\begin{aligned}\mathbf{\Pi}_m(\mathbf{r}_<) &= \int\int_{-\infty}^{\infty} \mathbf{a}_m^{(+)}(\mathbf{K}, z_<) \exp[i\mathbf{k}^{(+)} \cdot \mathbf{r}_<] d^2K \\ &+ \int\int_{-\infty}^{\infty} \mathbf{a}_m^{(-)}(\mathbf{K}, z_<) \exp[i\mathbf{k}^{(-)} \cdot \mathbf{r}_<] d^2K, \end{aligned} \tag{3.70 b}$$

where:

$$\mathbf{a}_e^{(+)}(\mathbf{K}, z_<) = \frac{2\pi i}{k_z} \int_0^{z_<} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(-ik_z z') dz', \qquad z_< > z', \tag{3.71 a}$$

$$\mathbf{a}_e^{(-)}(\mathbf{K}, z_<) = \frac{2\pi i}{k_z} \int_{z_<}^{L} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(ik_z z') dz', \qquad z_< < z', \tag{3.71 b}$$

$$\mathbf{a}_m^{(+)}(\mathbf{K}, z_<) = \frac{2\pi i}{k_z} \int_0^{z_<} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(-ik_z z') dz', \qquad z_< > z', \tag{3.72 a}$$

$$\mathbf{a}_m^{(-)}(\mathbf{K}, z_<) = \frac{2\pi i}{k_z} \int_{z_<}^{L} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(ik_z z') dz', \qquad z_< < z', \tag{3.72 b}$$

On introducing Eqs.(3.70)-(3.72) into Eqs.(3.57) we get:

$$\mathbf{E}^{(s)}(\mathbf{r}_<) = \int\int_{-\infty}^{\infty} \mathbf{e}^{(+)}(\mathbf{K}, z_<) \exp(i\mathbf{k}^{(+)} \cdot \mathbf{r}_<) d^2K$$

$$+\int\int_{-\infty}^{\infty} \mathbf{e}^{(-)}(\mathbf{K}, z_<) \exp(i\mathbf{k}^{(-)} \cdot \mathbf{r}_<) d^2K, \qquad (3.73\text{ a})$$

$$\mathbf{H}^{(s)}(\mathbf{r}_<) = \int\int_{-\infty}^{\infty} \mathbf{h}^{(+)}(\mathbf{K}, z_<) \exp(i\mathbf{k}^{(+)} \cdot \mathbf{r}_<) d^2K$$
$$+\int\int_{-\infty}^{\infty} \mathbf{h}^{(-)}(\mathbf{K}, z_<) \exp(i\mathbf{k}^{(-)} \cdot \mathbf{r}_<) d^2K, \qquad (3.73\text{ b})$$

where:

$$\mathbf{e}^{(+)}(\mathbf{K}, z_<) = -i\frac{2\pi}{k_z}[\mathbf{k}^{(+)} \times [\mathbf{k}^{(+)}$$
$$\times \int_0^{z_<} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(-ik_z z') dz']$$
$$+k[\mathbf{k}^{(+)} \times \int_0^{z_<} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(-ik_z z') dz']]$$
$$-4\pi\tilde{\mathbf{P}}(\mathbf{K}, z_<) \exp(-ik_z z_<), \qquad (3.74\text{ a})$$
$$\mathbf{e}^{(-)}(\mathbf{K}, z_<) = -i\frac{2\pi}{k_z}[\mathbf{k}^{(-)} \times [\mathbf{k}^{(-)}$$
$$\times \int_{z_<}^{L} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(ik_z z') dz']$$
$$+k[\mathbf{k}^{(-)} \times \int_{z_<}^{L} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(ik_z z') dz']]$$
$$-4\pi\tilde{\mathbf{P}}(\mathbf{K}, z_<) \exp(ik_z z_<), \qquad (3.74\text{ b})$$

$$\mathbf{h}^{(+)}(\mathbf{K}, z_<) = -i\frac{2\pi}{k_z}[\mathbf{k}^{(+)} \times [\mathbf{k}^{(+)}$$
$$\times \int_0^{z_<} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(-ik_z z') dz']$$
$$-k[\mathbf{k}^{(+)} \times \int_0^{z_<} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(-ik_z z') dz']]$$
$$-4\pi\tilde{\mathbf{M}}(\mathbf{K}, z_<) \exp(-ik_z z_<), \qquad (3.75\text{ a})$$
$$\mathbf{h}^{(-)}(\mathbf{K}, z_<) = -i\frac{2\pi}{k_z}[\mathbf{k}^{(-)} \times [\mathbf{k}^{(-)}$$
$$\times \int_{z_<}^{L} \tilde{\mathbf{M}}(\mathbf{K}, z') \exp(ik_z z') dz']$$
$$-k[\mathbf{k}^{(-)} \times \int_{z_<}^{L} \tilde{\mathbf{P}}(\mathbf{K}, z') \exp(ik_z z') dz']]$$
$$-4\pi\tilde{\mathbf{M}}(\mathbf{K}, z_<) \exp(ik_z z_<). \qquad (3.75\text{ b})$$

Eqs.(3.71)-(3.75) can be used to calculate the field inside the medium upon interaction of the incident field with the material. The integral equations (3.71) and (3.72) can easily be written in the alternative form:

$$\frac{\partial \mathbf{a}_e^{(\pm)}(\mathbf{K}, z_<)}{\partial z_<} = \pm \frac{2\pi i}{k_z} \tilde{\mathbf{P}}(\mathbf{K}, z_<) \exp(\mp i k_z z_<), \tag{3.76 a}$$

$$\frac{\partial \mathbf{a}_m^{(\pm)}(\mathbf{K}, z_<)}{\partial z_<} = \pm \frac{2\pi i}{k_z} \tilde{\mathbf{M}}(\mathbf{K}, z_<) \exp(\mp i k_z z_<). \tag{3.76 b}$$

As an example, let us assume that the response of the medium to the incident field is isotropic, linear and spatially non dispersive, then according to Section 1.1, the field inside the volume V is related to the induced polarization $\mathbf{P}(\mathbf{r})$ and magnetization $\mathbf{M}(\mathbf{r})$ by the following constitutive relations:

$$\mathbf{P}(\mathbf{r}) = \chi(\mathbf{r})\mathbf{E}(\mathbf{r}) = \chi(\mathbf{r})[\mathbf{E}^{(i)}(\mathbf{r}) + \mathbf{E}^{(s)}(\mathbf{r})], \quad \text{when } \mathbf{r} \text{ belongs to } V. \tag{3.77 a}$$

$$\mathbf{P}(\mathbf{r}) = 0, \quad \text{when } \mathbf{r} \text{ does not belong to } V.$$

$$\mathbf{M}(\mathbf{r}) = \eta(\mathbf{r})\mathbf{H}(\mathbf{r}) = \eta(\mathbf{r})[\mathbf{H}^{(i)}(\mathbf{r}) + \mathbf{H}^{(s)}(\mathbf{r})], \quad \text{when } \mathbf{r} \text{ belongs to } V. \tag{3.77 b}$$

$$\mathbf{M}(\mathbf{r}) = 0, \quad \text{when } \mathbf{r} \text{ does not belong to } V.$$

where $\chi(\mathbf{r})$ and $\eta(\mathbf{r})$ are the *dielectric* and *magnetic susceptibilities*, respectively.

By defining the two dimensional Fourier transforms of $\chi(\mathbf{r})$ and $\eta(\mathbf{r})$ as:

$$\tilde{\chi}(\mathbf{K}, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \chi(\mathbf{r}) \exp(-i\mathbf{K} \cdot \mathbf{R}) d^2R, \tag{3.78 a}$$

$$\tilde{\eta}(\mathbf{K}, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \eta(\mathbf{r}) \exp(-i\mathbf{K} \cdot \mathbf{R}) d^2R. \tag{3.78 b}$$

And introducing Eqs.(3.73) and (3.78), together with the corresponding angular spectrum representation for the incident field, into (3.77), and the result into Eq.(3.76), and taking the definitions (3.64) into account, one obtains straightforwardly a differential equation for $\mathbf{a}_e^{(\pm)}(\mathbf{K}, z)$ and $\mathbf{a}_m^{(\pm)}(\mathbf{K}, z)$ that requires to be iteratively solved. The existence of both *forward* and *backward* waves simultaneously, complicates the calculation.

Sometimes, however, backward waves can be neglected; this happens when the *slowly varying amplitude approximation*, (see, e.g., Ref. 3.12), can be made for $\mathbf{a}_e^{(\pm)}(\mathbf{K}, z)$ and

$\mathbf{a}_m^{(\pm)}(\mathbf{K}, z)$. This approximation amounts to assuming that the interaction of the field with the material is significant only after the wave has travelled a large distance in the medium compared with the wavelength. In this case one can consider that $\mathbf{a}_e^{(+)}(\mathbf{K}, z_<)$ and $\mathbf{a}_e^{(+)}(\mathbf{K}, z_<)$ vary so slowly with $z_<$ that the following conditions hold:

$$|\frac{\partial^2 \mathbf{a}_e^{(+)}(\mathbf{K}, z_<)}{\partial z_<^2}| << k|\frac{\partial \mathbf{a}_e^{(+)}(\mathbf{K}, z_<)}{\partial z_<}|, \tag{3.79 a}$$

$$|\frac{\partial^2 \mathbf{a}_m^{(+)}(\mathbf{K}, z_<)}{\partial z_<^2}| << k|\frac{\partial \mathbf{a}_m^{(+)}(\mathbf{K}, z_<)}{\partial z_<}|, \tag{3.79 b}$$

and:

$$\mathbf{a}_e^{(-)}(\mathbf{K}, z_<) = \mathbf{a}_m^{(-)}(\mathbf{K}, z_<) = 0. \tag{3.80}$$

When the slowly varying amplitude approximation is valid, Eqs.(3.76) can be iteratively solved by using numerical methods like the *finite difference* procedures which use the *Adams-Bashforth algorithm* as a *predictor* and the *Adams-Moulton algorithm* as a *corrector* , (see, e.g., Ref. 3.13).

It should be emphasized that Eqs.(3.76) hold for any kind of medium, and irrespective of whether it is linear or non linear. When the interaction is non linear, the restrictions imposed by the requirements of phase matching between the interacting fields, make the slowly varying amplitude approximation accurate, even in presence of strong interaction, providing that there are no reflections at the boundary surface enclosing the volume V. This is due to the negligible contribution of the backward propagating components because of their phase mismatch. Examples of these calculations for non linear wave interactions in both isotropic and anisotropic materials can be found in Refs. 3.14-3.16.

In general, the iterative solution of Eqs.(3.76) for the spectral components of the Hertz vectors leads to the values of $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ inside the medium; and then by means of the constitutive relations, to the computation of the source terms represented by the induced polarization $\mathbf{P}(\mathbf{r})$ and the induced magnetization $\mathbf{M}(\mathbf{r})$. The fields outside V can be subsequently determined according to the analysis of Sections 3.6 or

3.7.1.

3.8 The First Born Approximation

A well known method of solving the integral equations that result from the combination of Eqs.(3.57), (3.58) and (3.77), and whose solutions yield the source terms $\mathbf{P}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$, is based on the expansion into a *Neumann series*. This series is also known in scattering theory as the *Born series*.

When the scatterer is sufficiently weak and thin, the interaction may be described by the *first Born approximation*, which correspons to the first term of the Neumann series.

Let us substitute Eqs.(3.77) into (3.58), thus obtaining:

$$\mathbf{\Pi}_e(\mathbf{r}) = \int_V \chi(\mathbf{r}')\mathbf{E}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3r', \tag{3.81 a}$$

$$\mathbf{\Pi}_m(\mathbf{r}) = \int_V \eta(\mathbf{r}')\mathbf{H}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3r'. \tag{3.81 b}$$

Eqs.(3.57) and (3.81) constitute a complete system of integral equations for the electric and magnetic vectors. In many cases of practical interest, however, the medium can be assumed non magnetic so that $\eta(\mathbf{r}) = 0$, namely, $\mathbf{\Pi}_m(\mathbf{r}) = 0$. In this case, the vector Helmholtz equation (3.56a) becomes:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = 4\pi k^2\chi(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{3.82}$$

which contains the *potential* $4\pi k^2\chi(\mathbf{r})$ in analogy with the scalar Helmholtz equation for the time independent potential scattering problem in quantum mechanics, (see, e.g., Refs. 3.17 or 3.18). Note that in general, when $\mathbf{\Pi}_m(\mathbf{r}) \neq 0$, the two *potentials* of the Helmholtz equations (3.56a) and (3.56b) are the *electric potential* $4\pi k^2\chi(\mathbf{r})$ and the *magnetic potential* $4\pi k^2\eta(\mathbf{r})$.

The Neumann (or Born) series consists of iteratively solving for the field inside V the integral equations obtained from (3.57) and (3.81), the first term of this iterative series is obtained by substituting $\mathbf{E}$ and $\mathbf{H}$ by the corresponding values $\mathbf{E}^{(i)}$ and $\mathbf{H}^{(i)}$, respectively, in the integrals of Eqs.(3.81). As quoted above, this is the *first Born approximation*. It is seen from (3.77) that it amounts to considering the interaction between the incident field and the medium so weak that this field remains unperturbed inside the volume V.

There exist no accurate criteria of validity of the first Born approximation. Roughly speaking, the perturbation in the medium upon the incident field will be small if the energy of this field is high and the fluctuations of $\chi(\mathbf{r})$ and $\eta(\mathbf{r})$ are not large. Then the energy associated to the scattered field should be much smaller than that of the incident field. Also the volume of interaction should be small. A sufficient criterium that covers this is:

$$kF_0L << 1, \tag{3.83}$$

F_0 being the ***strength*** of the fluctuation of either the electric or magnetic potential and L representing the maximum linear dimension of the scattering volume. For low angle scattering, the validity of the first Born approximation is less restrictive than Eq.(3.83) imposes.

The fields outside the scatterer adopt a very simple form under this approximation. Let us assume that the incident field is a homogeneous plane wave:

$$\mathbf{E}^{(i)}(\mathbf{r}) = \mathbf{e}_i \exp(i\mathbf{k}_i \cdot \mathbf{r}), \tag{3.84 a}$$

$$\mathbf{H}^{(i)}(\mathbf{r}) = \mathbf{h}_i \exp(i\mathbf{k}_i \cdot \mathbf{r}), \tag{3.84 b}$$

where the incident wavevector k_i has magnitude ω/c and $\mathbf{e}_i$ and $\mathbf{h}_i$ are constant amplitude vectors that satisfy the relations: $\mathbf{h}_i = \mathbf{n}_i \times \mathbf{e}_i$, $\mathbf{n}_i = \mathbf{k}_i/k$. On substituting from Eqs.(3.84) into Eqs.(3.77) one readily obtains the following expressions for the three dimensional Fourier transforms of $\mathbf{P}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$:

$$\tilde{\mathbf{P}}(\mathbf{k}) = \tilde{\chi}(\mathbf{k} - \mathbf{k}_i)\mathbf{e}_i, \tag{3.85 a}$$

$$\tilde{\mathbf{M}}(\mathbf{k}) = \tilde{\eta}(\mathbf{k} - \mathbf{k}_i)\mathbf{h}_i. \tag{3.85 b}$$

$\tilde{\chi}(\mathbf{k})$ and $\tilde{\eta}(\mathbf{k})$ are the three dimensional Fourier transforms of $\chi(\mathbf{r})$ and $\eta(\mathbf{r})$, respectively. On substituting Eqs.(3.85) into (3.59) one gets the spectral amplitudes of the electromagnetic field in the half spaces $\mathcal{R}^+$ or $\mathcal{R}^-$ in terms of the Fourier transforms of the electric and magnetic susceptibilities within the first Born approximation, (A.T. Friberg and E. Wolf [5.4]):

$$\mathbf{e}_B^{(\pm)}(\mathbf{K}) = -i\frac{(2\pi)^2}{k_z}[[\mathbf{k}^{(\pm)}\times[\mathbf{k}^{(\pm)}\times\mathbf{e}_i]\tilde{\chi}(\mathbf{k}^{\pm}-\mathbf{k}_i) + k[\mathbf{k}^{(\pm)}\times\mathbf{h}_i]\tilde{\eta}(\mathbf{k}^{(\pm)}-\mathbf{k}_i)], \tag{3.86 a}$$

$$\mathbf{h}_B^{(\pm)}(\mathbf{K}) = -i\frac{(2\pi)^2}{k_z}[[\mathbf{k}^{(\pm)}\times[\mathbf{k}^{(\pm)}\times\mathbf{h}_i]]\tilde{\eta}(\mathbf{k}^{\pm}-\mathbf{k}_i) + k[\mathbf{k}^{(\pm)}\times\mathbf{e}_i]\tilde{\chi}(\mathbf{k}^{(\pm)}-\mathbf{k}_i)]. \tag{3.86 b}$$

Eqs.(3.86) show that these spectral amplitudes are expressed in terms of the three dimensional Fourier transforms of χ and η evaluated at the *wavevector transfer* $\mathsf{K} = \mathbf{k}^{(\pm)} - \mathbf{k}_i$.

The far fields are easily obtained from Eqs.(3.86) in a manner similar to that used in Section 3.3.

3.9 Scattering from a Weakly Fluctuating Random Medium

In this section we shall consider scattering by non magnetic, isotropic, linear, spatially non dispersive, time independent *random continuum media*. These are systems characterized by a dielectric permittivity $\epsilon(\mathbf{r})$ which is a continuous random function of position $\mathbf{r}$. Fundamental accounts on the theory of wave scattering and propagation in random media, as well as on the mathematical grounds of the theory of random variables, can be found e.g. in the works by V.I. Tatarskii [5.19], L.A. Chernov [5.20], A. Ishimaru [4.21], J.L. Doob [4.22] and A. Papoulis [4.23]. An excellent summary of statistical concepts in connection to this problem can be also found in the work by G. Ross [4.24].

We shall consider media that are *statistically homogeneous*, by this it is meant that the ensemble average $< \epsilon >$ of the permittivity over different realizations of the material, (namely, over different samples),is a constant independent of the position. Without loss of generality, this quantity be normalized to unity. Then we shall write $\epsilon(\mathbf{r})$ as the sum of the average and the fluctuating part:

$$\epsilon(\mathbf{r}) = 1 + \delta\epsilon(\mathbf{r}) \tag{3.87}$$

where $\delta\epsilon(\mathbf{r})$ represent the random fluctuating part of $\epsilon(\mathbf{r})$.

But since:

$$\epsilon(\mathbf{r}) = 1 + 4\pi\chi(\mathbf{r}), \tag{3.88}$$

we obtain from (3.87) and (3.88):

$$4\pi\chi(\mathbf{r}) = \delta\epsilon(\mathbf{r}). \tag{3.89}$$

On the other hand, since $\epsilon(\mathbf{r}) = n^2(\mathbf{r})$, $n(\mathbf{r})$ being the refractive index, if the fluctuations are weak, by using (3.89) one can make the following approximation:

$$\delta\epsilon(\mathbf{r}) \sim 2\delta n(\mathbf{r}). \tag{3.90}$$

The *extent* of the fluctuation is characterized by the *covariance* $C_n(\mathbf{r}_1, \mathbf{r}_2)$ (Refs. 3.19 - 3.24):

$$C_n(\mathbf{r}_1, \mathbf{r}_2) = < \delta n(\mathbf{r}_1)\delta n(\mathbf{r}_2) >, \tag{3.91}$$

where, once again, the bracket $< . >$ means ensemble average over many realizations of the medium. A consequence of the statistical homogeneity of ϵ is that:

$$C_n(\mathbf{r}_1, \mathbf{r}_2) = C_n(\boldsymbol{\rho}), \tag{3.92}$$

$$\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2,$$

In addition, if the medium is *statistically isotropic* [5.19]- [4.24]

$$C_n(\boldsymbol{\rho}) = C_n(\rho), \quad \rho = |\mathbf{r}_1 - \mathbf{r}_2|. \tag{3.93}$$

The covariance function $C_n(\rho)$ is an even, monotonically decreasing function of ρ; more details on its properties can be found for instance in Refs. 3.22 and 3.23. The effective width T of $C_n(\rho)$ constitutes the *scale of the fluctuation* and is called the *correlation length*. On the other hand, its height: $\sigma_n^2 = C_n(0) =< (\delta n)^2 >$ is the variance, or *strength* of the fluctuation. The function $B_n(\rho) = [C_n(0)]^{-1}C_n(\rho)$ is the *autocorrelation function* of the fluctuation. If $n(\mathbf{r})$ follows a Gaussian statistics, then the second order moment $C_n(\rho)$ is sufficient to completely characterize it, [4.21], [4.22].

Since the strength of the fluctuation is assumed small, we can use the first Born approximation, as indicated by Eq.(3.83). Then according to Section 3.3, Eqs.(3.11), (3.86), (3.89) and (3.90), the electric vector of the scattered field in the far zone is given by:

$$\mathbf{E}_B^{(s)}(r\mathbf{n}) \sim -\frac{k^2}{2\pi}\mathbf{n}\times(\mathbf{n}\times\mathbf{e}_i)\int_V \exp(-i\mathsf{K}\cdot\mathbf{r})\delta n(\mathbf{r})d^3r, \tag{3.94}$$

with K being the wavevector transfer: $k\mathbf{n} - \mathbf{k}_i$; and the scattered magnetic vector being determined from Eq.(3.94) through:

$$\mathbf{H}_B^{(s)}(r\mathbf{n}) = \mathbf{n}\times\mathbf{E}_B^{(s)}(r\mathbf{n}). \tag{3.95}$$

By introducing (3.94) and (3.95) into (3.15) one gets the ensemble average of the time averaged Poynting vector:

$$< \bar{S}^{(s)}(r\mathbf{n}) >= \frac{ck^4}{64\pi^3 r^2} |\mathbf{n}\times(\mathbf{n}\times\mathbf{e}_i)|^2$$
$$\int_V \int_V \exp(-i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')) < \delta n(\mathbf{r})\delta n(\mathbf{r}') > d^3r d^3r'. \tag{3.96}$$

Let us write:

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{e}_i) = \mathbf{n}\times\mathbf{t}\sin\Theta = \mathbf{e_n}\sin\Theta, \tag{3.97}$$

Θ being the angle between the direction of polarization $\mathbf{e}_i$ of the incident wave and the direction of observation $\mathbf{n}$. $\mathbf{t}$ and $\mathbf{e_n}$ are unit vectors such that:

$$\mathbf{t}\sin\Theta = \mathbf{n}\times\mathbf{e}_i, \tag{3.98}$$

and:

$$\mathbf{e_n} = \mathbf{n}\times\mathbf{t}. \tag{3.99}$$

Also, let us transform the variables $\mathbf{r}$ and $\mathbf{r}'$ of the integrand in Eq. (3.96) into the difference $\boldsymbol{\rho}$ and average $\mathbf{r}_a$:

$$\boldsymbol{\rho} = \mathbf{r}-\mathbf{r}', \tag{3.100 a}$$
$$\mathbf{r}_a = \frac{1}{2}(\mathbf{r}+\mathbf{r}'). \tag{3.100 b}$$

On introducing (3.97) into (3.96), taking the definition (3.91) and Eqs. (3.92) and (3.93) into account, and using the transformation (3.100), we easily obtain from (3.96) :

$$< \bar{S}^{(s)}(r\mathbf{n}) >= \mathbf{n}\frac{ck^4}{64\pi^3 r^2}\sin^2\Theta \int_V d^3r_a \int d^3\rho \exp(-i\mathbf{K}\cdot\rho)C_n(\rho). \tag{3.101}$$

where the integral in $\boldsymbol{\rho}$ is extended to all space, whereas the integral in $\mathbf{r}_a$ is done over the scattering volume V. In establishing these limits of integration we have made use of the fact that the correlation distance T of the fluctuation is much smaller than the linear dimension L of the scattering volume, and therefore $C_n(\rho)$ is negligibly small for $r_a >> T$.

The integral over $\mathbf{r}_a$ is equal to V. On the other hand, the integral in ρ represents the three dimensional Fourier transform of the covariance $C_n(\rho)$. This is the *spectral density* Φ_n:

$$\Phi_n(\mathsf{K}) = \frac{1}{(2\pi)^3}\int d^3\rho \exp(-i\mathsf{K}\cdot\rho)C_n(\rho). \tag{3.102}$$

Sometimes the spectral density is defined as the Fourier transform of the correlation function $B_n(\rho)$ [4.23]-[4.24]. In this case the only difference with Eq. (3.102) is the normalization factor $C_n(0)$.

With the above considerations, and using the definition (3.102), we can express (3.101) as:

$$< \bar{S}^{(s)}(r\mathbf{n}) >= \mathbf{n}\frac{cVk^4}{8r^2}\sin^2\Theta\Phi_n(\mathsf{K}). \tag{3.103}$$

Therefore, the mean scattered intensity in the far zone, (cf. Problem 2.5), per unit volume and per unit of solid angle is:

$$\frac{1}{V}\frac{d < I^{(s)}(r\mathbf{n}) >}{d\Omega} = \frac{ck^4}{8}\sin^2\Theta\Phi_n(\mathsf{K}). \tag{3.104}$$

Note that for a statistically homogeneous and isotropic fluctuation the spectral density can be expressed after performing the angle integration in Eq. (3.102):

$$\Phi_n(\mathsf{K}) = \frac{4\pi}{|\mathsf{K}|}\int_0^\infty C_n(\rho)\sin(|\mathsf{K}|\rho)\rho d\rho. \tag{3.105}$$

It is easy to see that:

$$|\mathsf{K}| = 2k\sin\frac{\theta}{2}, \tag{3.106}$$

where θ is the angle between the direction of observation $\mathbf{n}$ and the direction of propagation $\mathbf{k}_i$ of the incident field. (Fig. 5.8).

Eq.(3.104) shows that the mean scattered intensity is proportional to the spectral density of the refractive index fluctuation, has the $\sin^2\Theta$ dependence and, due to a well known property of Fourier transforms, is more concentrated about the direction of incidence the larger the correlation distance T of the refractive index fluctuation is.

3.10 Scalar Approach to Scattered Scalar Wavefields

Let us assume that the scatterer is non magnetic, namely $\eta = 0$, so that $\mathbf{M}(\mathbf{r})$=0. Then Eqs.(3.56) become:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = 4\pi k^2\mathbf{P}(\mathbf{r}), \tag{3.107 a}$$

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k^2\mathbf{H}(\mathbf{r}) = -4\pi ik\nabla \times \mathbf{P}(\mathbf{r}), \tag{3.107 b}$$

with:

$$\mathbf{P}(\mathbf{r}) = \chi(\mathbf{r})\mathbf{E}(\mathbf{r}). \tag{3.108}$$

Taking into account the vector identity:

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}, \tag{3.109}$$

and introducing (3.109) into (3.107a) we obtain the following differential equation equation for the electric vector:

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = -k^2[n^2(\mathbf{r}) - 1]\mathbf{E}(\mathbf{r}) + \nabla[\nabla \cdot \mathbf{E}(\mathbf{r})], \tag{3.110}$$

where we have made use of the expression:

$$4\pi\chi(\mathbf{r}) = \begin{cases} n^2(\mathbf{r}) - 1 & \text{if } \mathbf{r} \text{ belongs to V} \\ 0 & \text{if } \mathbf{r} \text{ is outside V} \end{cases} \tag{3.111}$$

$n(\mathbf{r})$ denoting the refractive index of the scatterer.

Eq.(3.110) shows that the change in polarization of the electric vector, as a result of the scattering described by the left hand side, is due to the *source term* $\nabla[\nabla \cdot \mathbf{E}(\mathbf{r})]$. When the scale over which $n(\mathbf{r})$ varies is much larger than the wavelength λ, this term, and hence the depolarization, may be neglected, (see also e.g. J.W. Strohben [4.25]). One is then left with a differential equation that for each Cartesian component of the electric vector has the form:

$$\nabla^2U(\mathbf{r}) + k^2U(\mathbf{r}) = F(\mathbf{r})U(\mathbf{r}), \tag{3.112}$$

where the *potential* $F(\mathbf{r})$ is:

$$F(\mathbf{r}) = \begin{cases} -k^2[n^2(\mathbf{r}) - 1] & \text{if } \mathbf{r} \text{ belongs to V} \\ 0 & \text{if } \mathbf{r} \text{ is outside V} \end{cases} \tag{3.113}$$

Note that Eq.(3.112) is formally identical to the time independent Helmholtz equation of *potential scattering* of quantum mechanics, (Refs. 3.17, 3.18). Its integral form was discussed in Section 1.6.3., Eq. (3.112), is also similar to Eq.(3.16), where the *source distribution* $\rho(\mathbf{r})$ is given by Eq.(1.49) and, hence, it contains the wavefunction $U(\mathbf{r})$.

The solution of (3.112) can be written at points outside the scattering volume (cf. Section 1.6.3):

$$U(\mathbf{r}_>) = U^{(i)}(\mathbf{r}_>) + U^{(s)}(\mathbf{r}_>). \tag{3.114}$$

And for points outside V:

$$U(\mathbf{r}_<) = U^{(i)}(\mathbf{r}_<) + U^{(s)}(\mathbf{r}_<). \tag{3.115}$$

Where the scattered field is:

$$U^{(s)}(\mathbf{r}) = -\frac{1}{4\pi}\int_V F(\mathbf{r}')U(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d^3r'. \tag{3.116}$$

Eq.(3.116) constitutes a Fredholm integral equation of the first kind for the field $\mathbf{E}(\mathbf{r}_<)$ inside the scattering volume. Once this field has been found, the field outside the scatterer can be obtained by means of Eq.(3.116) substituting $\mathbf{r}$ by $\mathbf{r}_>$.

With reference to Fig. 3.1, the fields inside the strip $0 < z < L$, (either inside or outside V), and the fields in $\mathcal{R}^+$ or $\mathcal{R}^-$, can be found from Eqs.(3.114)-(3.116) in an way identical to that developed in Sections 3.6 and 3.7. This is left as an exercice for the reader.

It should be remarked however, that although Eq.(3.112) represents an adequate way of describing the strong multiple interaction of a scalar wavefield with a scattering medium, (like e.g. an acoustic wave); when (3.112) is applied to the electromagnetic field, it is nevertheless meaningful only within the domain of validity of the first Born approximation, or the Rytov or the eikonal approximations to be discussed next. Namely, it is only with these simplifications when one can assume lack of depolarization of the field. Those three approximations belong to the range of *small fluctuations*, (Refs. 3.19-3.21, 3.25). Due to its importance in many problems, we shall derive explicitely the expressions for scattered scalar wavefields within the first Born approximation. Then the Rytov and the eikonal approximations will be discussed.

3.10.1 The First Born Approximation for Scalar Wavefields

Let the incident field be a plane monochromatic wave of unit amplitude and wavevector $\mathbf{k}_i$:

$$U^{(i)}(\mathbf{r}) = \exp(i\mathbf{k}_i \cdot \mathbf{r}). \tag{3.117}$$

Then, the first Born approximation assumes that $U(\mathbf{r})$ in the integrand of Eq.(3.116) can be replaced by $U^{(i)}(\mathbf{r})$. As mentioned before, this constitutes the first term of the Neumann series for the integral equation (3.114). When one makes this operation, and Eq.(2.51) is introduced into (3.116), the following expression for the scattered field is obtained, (E. Wolf, [4.26]):

$$U_B^{(s)}(\mathbf{r}_>) = \int\int_{-\infty}^{\infty} A_B^{(\pm)}(\mathbf{K}) \exp[i(\mathbf{K} \cdot \mathbf{R}_> \pm k_z z_>)] d^2K, \tag{3.118}$$

with the angular spectrum being given by:

$$A_B^{(\pm)}(\mathbf{K}) = -i\frac{\pi}{k_z}\tilde{F}(\mathbf{k}^{(\pm)} - \mathbf{k}_i), \tag{3.119}$$

where, as before, $\mathbf{k}^{(\pm)} = (\mathbf{K}, \pm k_z)$, $\mathbf{r}_> = (R_>, z_>)$. And the sign *plus* or *minus* is considered according to whether the point $\mathbf{r}_>$ is assumed in $\mathcal{R}^+$ or $\mathcal{R}^-$, respectively.

Eqs.(3.118) and (3.119) permit to find the scattered field either in $\mathcal{R}^+$ or $\mathcal{R}^-$, no matter how close the point $\mathbf{r}_>$ be to the scattering volume V. Eq. (3.119) shows that the angular spectrum of this scattered field is given by the three dimensional Fourier transform of the *potential* $F(\mathbf{r})$ evaluated at the wavevector transfer $\mathsf{K}^{(\pm)} = \mathbf{k}^{(\pm)} - \mathbf{k}_i$.

On the other hand, the scattered field in the far zone is:

$$U(r\mathbf{n}) \sim -2\pi^2 \tilde{F}(\mathsf{K}^{(\pm)}) \frac{\exp(ikr)}{r}. \tag{3.120}$$

3.10.2 The Rytov Approximation

Another approximation useful in the range of ***small fluctuations*** of the refractive index $n(\mathbf{r})$ is the ***Rytov approximation***, (Refs. 3.19 - 3.21). It is often used in problems of optical and microwave propagation in turbulence, (Ref. 3.25), optical tomography and acoustic wave propagation in biological media. The scale of the fluctuation must be large compared with the wavelength. This means that the field is scattered in a cone of small angle about the direction of incidence; and therefore there is no backscattering into the region $\mathcal{R}^+$ of Fig. 3.1. A discussion on the relative advantages of the Rytov approximation over the first Born approximation may be found e.g. in Refs. 3.19- 3.20.

The Rytov method consists of writing the wavefield $U(\mathbf{r})$ as:

$$U(\mathbf{r}) = \exp[\phi(\mathbf{r})], \tag{3.121}$$

then developing a series solution for $\phi(\mathbf{r})$.

Using Eq. (3.121) we obtain:

$$\nabla^2 U = U[|\nabla\phi|^2 + \nabla^2\phi]. \tag{3.122}$$

Introducing (3.122) into (3.112) one is led to:

$$\nabla^2\phi(\mathbf{r}) + |\nabla\phi(\mathbf{r})|^2 + k^2 = F(\mathbf{r}). \tag{3.123}$$

On the other hand, the incident field may be written as:

$$U^{(i)}(\mathbf{r}) = \exp[\phi^{(i)}(\mathbf{r})], \tag{3.124}$$

where $\phi^{(i)}(\mathbf{r})$ satisfies the homogeneous equation:

$$\nabla^2\phi^{(i)}(\mathbf{r}) + |\nabla\phi^{(i)}(\mathbf{r})|^2 + k^2 = 0. \tag{3.125}$$

Let us substract (3.125) from (3.123) and write:

$$\phi(\mathbf{r}) = \phi^{(i)}(\mathbf{r}) + \phi^{(1)}(\mathbf{r}). \tag{3.126}$$

Then we obtain:

$$\nabla^2\phi^{(1)}(\mathbf{r}) + 2\nabla\phi^{(i)}(\mathbf{r})\cdot\nabla\phi^{(1)}(\mathbf{r}) = -|\nabla\phi^{(1)}(\mathbf{r})|^2 + F(\mathbf{r}). \tag{3.127}$$

Using the identity:

$$\nabla^2(U^{(i)}\phi^{(1)}) = (\nabla^2 U^{(i)})\phi^{(1)} + 2U^{(i)}\nabla\phi^{(i)}\cdot\nabla\phi^{(1)} + U^{(i)}\nabla^2\phi^{(1)}. \tag{3.128}$$

And taking into account that:

$$\nabla^2 U^{(i)} + k^2 U^{(i)} = 0, \tag{3.129}$$

we can write Eq.(3.127) in the following form:

$$(\nabla^2 + k^2)(U^{(i)}\phi^{(1)}) = -[|\nabla\phi^{(1)}|^2 - F(\mathbf{r})]U^{(i)}. \tag{3.130}$$

Eq.(3.130) is of the form (3.112) and, hence, the solution $\phi^{(1)}$ can be expressed by means of (3.116) as:

$$\phi^{(1)}(\mathbf{r}) = \frac{1}{4\pi}\frac{1}{U^{(i)}(\mathbf{r})}\int_V G(\mathbf{r},\mathbf{r}')[|\nabla\phi^{(1)}(\mathbf{r}')|^2 - F(\mathbf{r}')]U^{(i)}(\mathbf{r}')d^3r'. \tag{3.131}$$

Eq.(3.131) can be iteratively solved like Eq.(3.116). The first iteration $\phi^{(1)}_{(0)}$ consists of setting $\phi^{(1)} = 0$ in the integrand of (3.131). Then, using (3.126), we obtain:

$$U(\mathbf{r}) = \exp[\phi^{(i)}(\mathbf{r}) + \phi^{(1)}_{(0)}(\mathbf{r})], \tag{3.132}$$

where, according to (3.131):

$$\phi^{(1)}_{(0)}(\mathbf{r}) = -\frac{1}{4\pi}\frac{1}{U^{(i)}(\mathbf{r})}\int_V G(\mathbf{r},\mathbf{r}')F(\mathbf{r}')U^{(i)}(\mathbf{r}')d^3r'. \tag{3.133}$$

Therefore, from (3.124) the first Rytov approximation may be written as:

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r})\exp[\phi^{(1)}_{(0)}(\mathbf{r})], \tag{3.134}$$

with $\phi^{(1)}_{(0)}(\mathbf{r})$ given by Eq.(3.133).

When the exponential of Eq.(3.134) is expanded into a series, the first two terms are:

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r})[1 + \phi^{(1)}_{(0)}(\mathbf{r})]. \tag{3.135}$$

By taking (3.133) into account, we see that Eq.(3.135) constitutes the first Born approximation for $U(\mathbf{r})$. Therefore, the first Born approximation is included in the first Rytov approximation, thus being adequate when, in addition to λ being very small compared with the scale of the refractive index fluctuation, the series (3.134) converges so fast that only the first terms contribute. This happens when the scattered intensity is small.

In problems dealing with propagation in random media, it is customary to write:

$$\phi(\mathbf{r}) = \chi(\mathbf{r}) + iS(\mathbf{r}). \tag{3.136}$$

The real part of $\phi(\mathbf{r})$, represented by $\chi(\mathbf{r})$, constitutes the logarithm of the amplitude of $U(\mathbf{r})$ and is usually called the *log amplitude fluctuation.* The imaginary part of $\phi(\mathbf{r})$, denoted by $S(\mathbf{r})$, represents the *phase* of $U(\mathbf{r})$. The reason of using the representation (3.136) is that in many practical situations the log amplitude fluctuation follows normal statistics. This is the so called *log normal model.* Then, as the field propagates in the random medium, the fluctuations of $U(\mathbf{r})$ can be easily accounted for analytically; (see discussions on this subject matter in e.g. Refs. 3.21, 3.25, 3.27 and 3.28). (See Problem 3.14).

3.10.3 The Eikonal Approximation

For weak scatterers, the geometrical optics limit (Refs. 3.20, 3.21) can be employed when the total length L of the scattering volume traversed by the passing radiation is such that $\lambda L << T^2$, (see Fig. 3.8), T being the scale of fluctuation of $n(\mathbf{r})$. (A full discussion on the foundations and validity of this limit may be found for instance in Chapter 3 of Ref. 3.29). In this approximation the wavefield inside the medium is expressed as :

$$U(\mathbf{r}) = A(\mathbf{r}) \exp[iS(\mathbf{r})], \tag{3.137}$$

where $A(\mathbf{r})$is the amplitude and $S(\mathbf{r})$ is the phase, *optical path* or *eikonal* [4.29]. The function $S(\mathbf{r})$ of the ray that can be associated in this case, and which describes a path between two points $\mathbf{r}_1$ and $\mathbf{r}_2$, is given by:

$$S(\mathbf{r}_1) - S(\mathbf{r}_2) = k \int_{\mathbf{r}_1}^{\mathbf{r}_2} n(\mathbf{r}) ds, \tag{3.138}$$

ds being the path element.

However, in order to evaluate (3.138) the ray path has to be known. In a weak scatterer this path can be approximated by a straight line along the direction of incidence. Let this direction be along the z-axis, then Eq.(3.138) may be written as:

$$S(\mathbf{R}, L) - S(\mathbf{R}, 0) = k \int_0^L n(\mathbf{R}, z) dz, \quad \mathbf{R} = (x, y), \tag{3.139}$$

The geometric optics limit can be straightforwardly derived from the the first Rytov approximation, (3.134), for a weak scatterer when $\lambda L << T^2$. In this case one can assume that the angle θ of scattering is very small, (see Fig. 3.8). We shall use the angular spectrum representation of the wavefield $U(\mathbf{r})$.

Let the incident field $U^{(i)}(\mathbf{r})$ be a plane wave of unit amplitude, propagating along the z-axis:

$$U^{(i)}(\mathbf{r}) = \exp(ikz). \tag{3.140}$$

Since the propagation is now rectilinear inside V, $\phi^{(1)}$, (Eqs. (3.131) and (3.140)), will be non zero only at points inside the geometrical projection S of the scatterer over a plane $z = constant$, (Fig. 3.9).

From (3.133) and using (2.51) and (3.140), one has at points $\mathbf{r}_>$ in $\mathcal{R}^+$ lying in the projection S:

$$\begin{aligned} \phi^{(1)}_{(0)}(\mathbf{r}_>) = & -\frac{i}{8\pi^2} \frac{1}{\exp(ikz_>)} \int\int_{-\infty}^{\infty} d^2K \exp[i(\mathbf{K} \cdot \mathbf{R}_> + k_z z_>)] \\ & \times \frac{1}{k_z} \int_V \exp[-i(\mathbf{K} \cdot \mathbf{R}' + k_z z')] \exp(ikz') F(\mathbf{r}') d^3r'. \end{aligned} \tag{3.141}$$

The geometric optics limit amounts to assuming that the angular spectrum:

$$A(\mathbf{K}) = \frac{1}{k_z}\int_V \exp[-i(\mathbf{K}\cdot\mathbf{R}' + k_z z')]\exp(ikz')F(\mathbf{r}')d^3r' \tag{3.142}$$

is different from zero only at spatial frequencies $\mathbf{K}$ very nearly zero. That is, such that:

$$k_z = \sqrt{k^2 - K^2} \sim k. \tag{3.143}$$

Then Eq. (3.141) may be written as:

$$\phi^{(1)}_{(0)}(\mathbf{r}_>) = -\frac{i}{8\pi^2 k}\int\int_{-\infty}^{\infty} d^2K \exp(i\mathbf{K}\cdot\mathbf{R}_>)\int_V \exp(-i\mathbf{K}\cdot\mathbf{R}')F(\mathbf{R}', z')d^2R'dz'. \tag{3.144}$$

Rearranging the order of integration in (3.144) and recalling that:

$$\int\int_{-\infty}^{\infty} \exp[-i\mathbf{K}\cdot(\mathbf{R} - \mathbf{R}')]d^2K = (2\pi)^2\delta(\mathbf{K} - \mathbf{K}'). \tag{3.145}$$

From (3.144) one gets:

$$\phi^{(1)}_{(0)}(\mathbf{r}_>) = -\frac{i}{2k}\int_0^L F(\mathbf{R}, z)dz, \tag{3.146}$$
$$\mathbf{r}_> = (\mathbf{R}_>, z_>),,$$

$\mathbf{R}$ being in $\mathbf{S}$.

Let us write:

$$n(\mathbf{r}) = 1 + \delta n(\mathbf{r}) \tag{3.147}$$

where $\delta n(\mathbf{r})$ represents the fluctuation of the refractive index. One has that:

$$n^2(\mathbf{r}) - 1 = 2\delta n(\mathbf{r}) + [\delta n(\mathbf{r})]^2. \tag{3.148}$$

For a weakly fluctuating random medium the second term of (3.148) can be neglected, (cf. Eqs.(3.87) and (3.90)), so that:

$$n^2(\mathbf{r}) - 1 \sim 2\delta n(\mathbf{r}). \tag{3.149}$$

Thus, by introducing (3.149) into (3.113), and the result into (3.146), we obtain finally:

$$\phi^{(1)}_{(0)}(\mathbf{r}_>) = ik\int_0^L \delta n(\mathbf{r})dz. \tag{3.150}$$

Eq.(3.150) constitutes the *eikonal approximation* for $\phi_{(0)}^{(1)}(\mathbf{r})$. From Eqs.(3.150), (3.140) and (3.134) we obtain the result:

$$U(\mathbf{r}_>) = \exp[ik(z_> + \int_0^L \delta n(\mathbf{r})dz)]. \tag{3.151}$$

Note that if $U(\mathbf{r})$ were evaluated at a point $\mathbf{r}_<$ inside the volume V of the scattering medium, the upper limit L of the integral in Eq.(3.151) should be replaced by $z_<$.

Eq.(3.151) constitutes the basis of a wide class of computer tomography methods (see a review of these procedures for example in Ref. 3.30). It is also of wide application in techniques on wave propagation in random media (see e.g. Refs.3. 21 and 3.25).

3.11 Multiple Scattering Theories

The approximations discussed in this chapter are useful to describing field interactions with weak scatterers; and in particular, with weakly fluctuating media. However, they become inadequate as the strength of the interaction, or the refractive index fluctuations, become larger. As a matter of fact, the range of validity of the Rytov approximation, for example, has been found to be less broad than expected from the theoretical predictions, (see Ref. 3.31). Hence, for multiple scattering resulting from stronger interactions, other methods are required. These have been largely developed for wave propagation in random media. They are not studied here, but the reader is referred to specific references quoted below.

One group of such methods contains a phenomenological approach and is based on the *radiative transfer equation*, (see e.g. Refs.(3.32, 3.33). This model also gives rise to the well known *diffusion approximation* (Ref. 3.21) for very dense random media. The connection of this rather heuristic model with the theory based on Maxwell's equations encounters several difficulties that have generated a vast literature, (see for example Refs. 3.34-3.37 and references therein).

Another group of methods contains rigorous diagrammatic procedures that solve iteratively the wave equation for the total field using Feynman graphs, (Refs. 3.38, 3.39

and 3.40). These lead to the *Dyson equation*, (Ref. 3.41), for the mean field, and to the *Bethe - Salpeter equation*, (Ref. 3.42), for its correlation function. These techniques are extensively used. Excellent reviews can be found in Refs. 3.21, 3.25, 3.43, 3.44 and 3.45.. Reviews and references on light scattering by dense media and the associated effect of *photon localization* manifested by the phenomenon of *enhanced backscattering* are given in Refs. 3.46 and 3.47.

3.12 Problems

3.1 A Hertzian dipole consists of two opposite charges, $+e$ and $-e$, separated a distance l and oscillating with time harmonic frequency as $\exp(-i\omega t)$. The current I and the dipole moment of this system are $I = -i\omega e$ and $P = Il$, respectively. Assuming the dipole at the origin and the charges on the z-axis, the electric current density is:

$$\mathbf{j}(\mathbf{r}) = Il\delta(\mathbf{r})\hat{\mathbf{z}}$$

$\hat{\mathbf{z}}$ being the unitary vector along the z-axis.

Find the angular spectra for both the electromagnetic field radiated by this dipole and the time averaged Poynting vector.

3.2 Making use of Section 3.3 show that:

$$\lim_{r\to\infty} r(\nabla \times \mathbf{E} - ik\hat{\mathbf{r}} \times \mathbf{E}) = 0,$$
$$\lim_{r\to\infty} r(\nabla \times \mathbf{H} - ik\hat{\mathbf{r}} \times \mathbf{H}) = 0.$$

These are the radiation conditions for electromagnetic fields (Refs. 3.48 , 3.49)

3.3 Assuming fields with time dependence $\exp(-i\omega t)$, Maxwell's equations imply the possibility of introducing a *vector potential* $\mathbf{A}(\mathbf{r})$, such that:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}),$$

as well as a *scalar potential* $\phi(\mathbf{r})$, defining:

$$\mathbf{E}(\mathbf{r}) = ik\mathbf{A}(\mathbf{r}) - \nabla\phi(\mathbf{r}).$$

Show that these potentials are related by the *Lorentz's condition*:

$$\nabla \cdot \mathbf{A} - ik\phi = 0,$$

and both satisfy the inhomogeneous Helmholtz equations:

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{j},$$

$$\nabla^2\phi + k^2\phi = -4\pi\rho.$$

$\mathbf{A}$ and ϕ are not uniquely defined since the same fields $\mathbf{E}$ and $\mathbf{H}$ are obtained from any other potentials $\mathbf{A}'$ and ϕ' that are given by the *gauge transformation*:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi,$$

$$\phi' = \phi + ik\chi.$$

χ being an arbitrary function.

Show that the Lorentz's condition imposes on χ to satisfy the homogeneous Helmholtz equation:

$$\nabla^2\chi + k^2\chi = 0.$$

3.4 Find the angular spectra of the scalar and vector potentials for a charge moving uniformly in vacuum.

3.5 Show that:

$$\int_{-\infty}^{\infty}\frac{1}{k_z}\exp[i(K_y y \pm k_z z)]dK_y = -2iK_0(k\gamma d),$$

$$\gamma = \sqrt{(\frac{c}{v_0})^2 - 1}, \quad d = \sqrt{y^2 + z^2}, \quad k_z = i\sqrt{K_y^2 + k^2\gamma^2}.$$

$K_0(k\gamma d)$ being the modified Hankel function of zero order and argument $k\gamma d$.

3.6 Prove Eq.(3.40) for the Hertz vector $\mathbf{\Pi}(\mathbf{r},\omega)$ of radiation at frequency ω from a charge with uniform movement in vacuum.

3.7 Consider a spherical shell of radius R with a source density distribution:

$$\rho(\mathbf{r}) = \frac{1}{4\pi r^2}e\delta(r - R), \tag{3.152}$$

where e is a constant.

Show that the angular spectrum of the field radiated by this distribution is zero if $kR = l\pi$, l being a positive integer; (G.A. Schott [3.50], see also Ref. 3.51).

3.8 Find the angular spectrum and far field within the first Born approximation for a plane monochromatic wave scattered by a homogeneous sphere of radius a.

3.9 Find the angular spectrum, far field and Poynting vector, within the first Born approximation, for the scattering of a plane wave by a non magnetic periodic structure of spacing a with permittivity:

$$\epsilon(\mathbf{r}) = \sum_{j=1}^{\infty} f_j \delta(\mathbf{r} - \mathbf{r}_j),$$

where the vector $\mathbf{r}_j$ is: $\mathbf{r}_j = (l, m, n)\mathbf{a}$, ($l$, m and n being integers and f_j representing positive constants.

3.10 A non absorbing *phase screen* is a layer of dielectric material that introduces changes in the phase of the incident wavefield $U^{(i)}$ so that the wave emerging from the screen in a plane inmediately behind it, is:

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r}) \exp[i\varphi(\mathbf{r})],$$

where the real function $\varphi(\mathbf{r})$ represents the phase added to that of the incident field after traversing the screen.

Assume $\varphi(\mathbf{r})$ being a homogeneous and isotropic random function with normal statistics, zero mean, and correlation function $B_\varphi(\rho)$:

$$B_\varphi(\rho) = \frac{< \varphi(\mathbf{r})\varphi(\mathbf{r} + \rho) >}{\sigma_\varphi^2},$$

where σ_φ^2 is the variance of $\varphi(\mathbf{r})$. The width of $B_\varphi(\rho)$ is negligible compared with the linear dimension of the illuminated area. Show that the mean scattered intensity in the far zone , per unit of solid angle and per unit of illuminated area, is, (cf. Problem 2.4):

$$\frac{dI^{(s)}(r\mathbf{n})}{d\Omega} = c\pi\cos^2\theta < |A(\mathbf{s}_\perp)|^2 > .$$

Where, $\mathbf{n}$ is a unit vector characterizing the scattering direction: $\mathbf{n} = (n_x, n_y, n_z)$, $U^{(i)}$ is a plane wave with wavevector: $k\mathbf{n}^{(i)} = k(n_x^{(i)}, n_y^{(i)}, n_z^{(i)})$, and the average of the square modulus of the spectral amplitude $A(\mathbf{s}_\perp)$ is, (cf. Eq. (2.16) and Problem 2.4):

$$< |A(\mathbf{s}_\perp)|^2 >= \frac{ck^2}{4\pi}\cos^2\theta \int_0^\infty d^2\rho \exp(-ik\mathbf{s}_\perp\cdot\rho)$$
$$\exp[-\sigma_\varphi^2(1 - B_\varphi(\rho))].$$

$\mathbf{s}_\perp$ being the vector transfer: $\mathbf{s}_\perp = (n_x - n_x^{(i)}, n_y - n_y^{(i)})$. $\mathbf{n}^{(i)}$ is assumed along the z-axis so that $n_z = \cos\theta$.

Show that if $B_\varphi(\rho) = \exp[-\rho^2/T^2]$ the scattered intensity in the far zonedefined above is:

$$\frac{dI^{(s)}(r\mathbf{n})}{d\Omega} = \frac{ck^2}{4\pi}\cos^2\theta \exp(-\sigma_\varphi^2)[\frac{(2\pi)^2}{k^2}\delta(s_\perp)$$
$$+\pi T^2 \sum_{m=1}^{\infty} \frac{\sigma_\varphi^{2m}}{m!} \exp[-k^2T^2s_\perp^2/4m].$$

The δ-function in the first term represents the *straight-through* component of the scattered intensity, whereas the second term involving the series expansion describes the *diffuse* part scattered at all angles. (This result was originally derived by E.N. Bramley [3.52]).

Find the limits of the above expression when: (a) $\sigma_\varphi^2 << 1$, and (b) $\sigma_\varphi^2 >> 1$.

3.11 Consider a slab of ground glass of uniform refractive index n. One of its surfaces is rough with a random profile: $z = D(x,y)$, (see Fig. 3.10). Show by means of the eikonal approximation that this slab behaves as a phase screen, so that if $U^{(i)}(\mathbf{r})$ denotes an incident wave, the wavefield emerging from the screen in a plane inmediately behind it, is:

$$U(\mathbf{r}) = U^{(i)}(\mathbf{r}) A \exp[ik(n-1)D(x,y)],$$

A being a complex constant. Hence, the variance σ_φ of the phase φ due to the screen is: $\sigma_\varphi = k(n-1)\sigma_D$, σ_D being the the variance of $D(x,y)$.

Apply to this case the results of Problem 3.10.

3.12 A *Goldfischer model* [3.53] of phase sreen is that for which the random phases φ are uncorrelated: $B_\varphi(\rho) = \delta(\rho)$. Suppose that this phase screen is superimposed to a non random planar object of finite extent that emits with an *irradiance distribution* $o(x,y)$, ($o(x,y)$ being therefore a non random real function). Then the field emerging from the screen in a plane inmediately behind it, is, apart from a complex constant that cam be omitted:

$$U(\mathbf{r}) = o(x,y)\exp[i\varphi(x,y)],$$

Show that the mean scattered intensity distribution behind the screen is:

$$\frac{dI^{(s)}(r\mathbf{n})}{d\Omega} = \frac{ck^2}{4\pi}\cos^2\theta \exp(-\sigma_\varphi^2)[(2\pi)^4|\tilde{o}(\mathbf{s}_\perp)|^2 + K],$$

where $\tilde{o}(\mathbf{s}_\perp)$ is the two dimensional Fourier transform of $o(x,y)$, and the d.c. constant K is:

$$K = AC(0), \quad A = \sum_{m=1}^{\infty} \frac{\sigma_\varphi^{2m}}{m!},$$

$C(\varsigma)$ being the (non random) autocorrelation of the irradiance distribution $o(x,y)$:

$$C(\varsigma) = \int\int_{-\infty}^{\infty} o(\mathbf{R})o(\mathbf{R}+\varsigma)d^2R, \quad \mathbf{R} = (x,y)$$

The above result shows that the mean scattered intensity in the far zone is equal to the power spectrum of the object irradiance plus a d.c. level. This has several applications in image synthesis, (Ref. 3.54).

3.13 Show that the mean scattered intensity from an inhomogeneous medium with covariance:

$$B_n(\rho) = < n^2 > \exp(-\frac{\rho}{T}),$$

($< n^2 >$ being the variance and T denoting the correlation distance of the refractive index fluctuations), is given by:

$$\frac{d < I^{(s)}(\theta) >}{d\Omega} = \frac{Vc}{8\pi^2} \frac{k^4 T^3 sin^2\Theta < n^2 >}{(1 + 4k^2 T^2 \sin^2(\theta/2))^2}.$$

Show also that:

a. When the correlation distance l of the refractive index fluctuations is much smaller than the wavelength, one has:

$$\frac{d < I^{(s)}(\theta) >}{d\Omega} = \frac{Vc}{8\pi^2} k^4 T^3 \sin^2\Theta < n^2 >,$$

so that the scattering is isotropic and proportional to λ^{-4}. This is known as *Rayleigh scattering* and describes the blue color of the sky.

b. When T is much larger than the wavelength the scattering is concentrated about the direction of incidence, and one has:

$$\frac{d < I^{(s)}(\theta) >}{d\Omega} = \frac{Vc}{128\pi^2} \frac{\sin^2\Theta < n^2 >}{T \sin^4(\theta/2)}.$$

3.14 The perturbation of a turbulent medium on a wavefield $U(\mathbf{r}) = \exp[\phi(\mathbf{r})]$ is often characterized in terms of the *structure function* D_ϕ, (V.I. Tatarskii, [5.19]):

$$D_\phi(\mathbf{r}, \mathbf{r}') = < [\phi(\mathbf{r}) - \phi(\mathbf{r}')]^2 >,$$

On expressing $\phi(\mathbf{r})$ by Eq. (3.136), prove that:

$$D_\phi = D_\chi + D_S.$$

Where D_χ and D_S are the *log* and *phase* structure functions, respectively.

Assuming homogeneous and isotropic Gaussian statistics for ϕ, find the relationship between D_ϕ, σ_ϕ and $B_\phi(\rho)$.

In a *Kolmogorov model* ([5.19], [5.20], [4.21]), the structure function is: $D_\phi(\rho) = Ar^{5/3}$. Find the mean intensity in the far zone for a field that, emerging from a turbulent layer, and expressed in the form (121), is characterized by such an structure function.

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