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WORKSHOP ON PATTERN RECOGNITION AND ANALYSIS OF SEISMICITY

(5 - 16 December 1983)

CRACK FUSION DYNAMICS: EARTHQUAKES AS A PROBLEM IN STATISTICAL MECHANICS

L. Knopoff

Earthquake statistics

1. High degree of randomness
2. Statistical inter-dependence
   i.e. clustering
3. Self-similarity

What is nature of clustering?

Mechanics: One fracture influences subsequent fractures.

4. stress redistribution (fracture mechanics)
5. Aftershocks ⇒ non-elastic rheology

Mathematics:

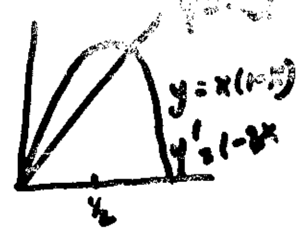
6. Non-linearity
7. We have tried methods of renormalization
   a. Success: tertiary creep experimental evidence is consequence of internal growth and fusion of cracks of various scales
   b. We show periodic seismicity (large scale) is a consequence of (Limit cycle attractor)
      1) tertiary creep rheology
      2) elastic rebound model of large eqs.

For regular (homogeneous) models.

What is influence of irregularity? (inhomogeneity)

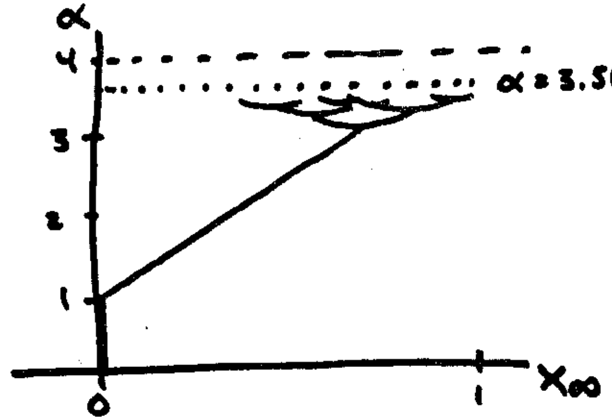
7*. b. Strange attractor.
(stochastic inhomogeneity)

Logistic Equation

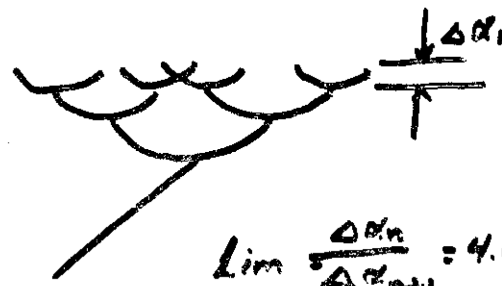


$0 < X_1 < 1$ $\quad X_{n+1} = \alpha X_n (1 - X_n)$

$0 < \alpha \leq 4$

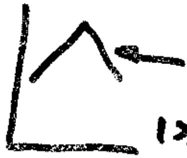


M. J. Feigenbaum
Phys. Letts 1975



$$\lim_{n \to \infty} \frac{\Delta \alpha_n}{\Delta \alpha_{n+1}} = 4.6692016\ldots$$

$X_{n+1} = \lambda \sin \pi X_n$



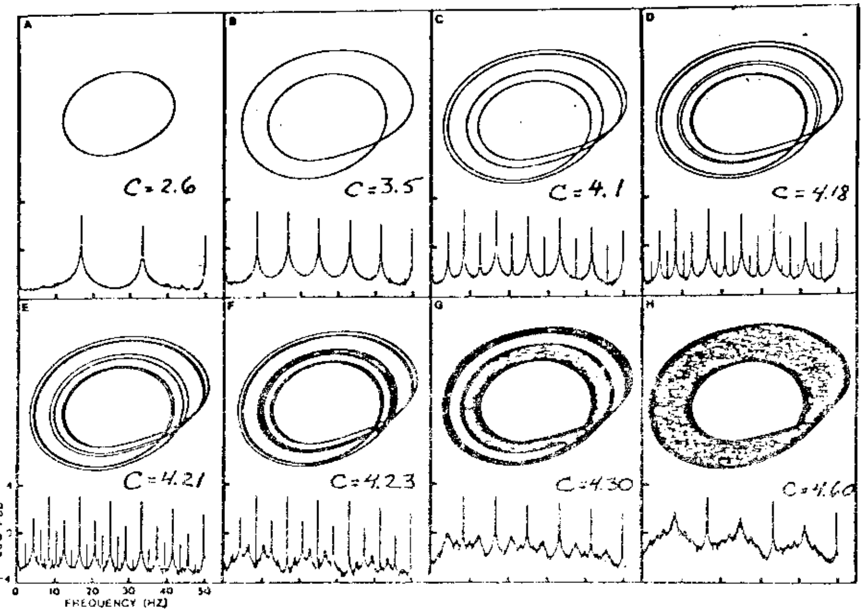
$y = f\left(|x - \tfrac{1}{2}|^{1+\varepsilon}\right)$

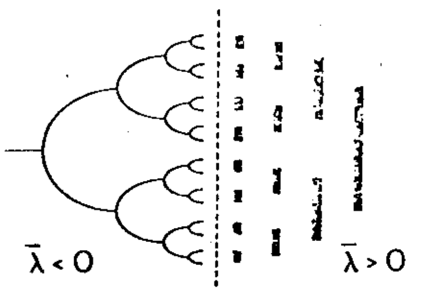
$1 \gg \varepsilon > 0$

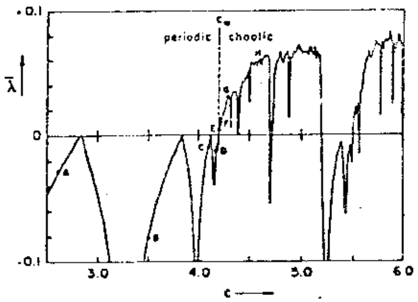
$$\dot{x} = -(y+z)$$
$$\dot{y} = x + 0.2y$$
$$\dot{z} = 0.2 + xz - Cz$$

Rössler (1976)

$$x_{n+1} = x_n^2 - y_n$$
$$y_{n+1} = -.4224 + x_n$$

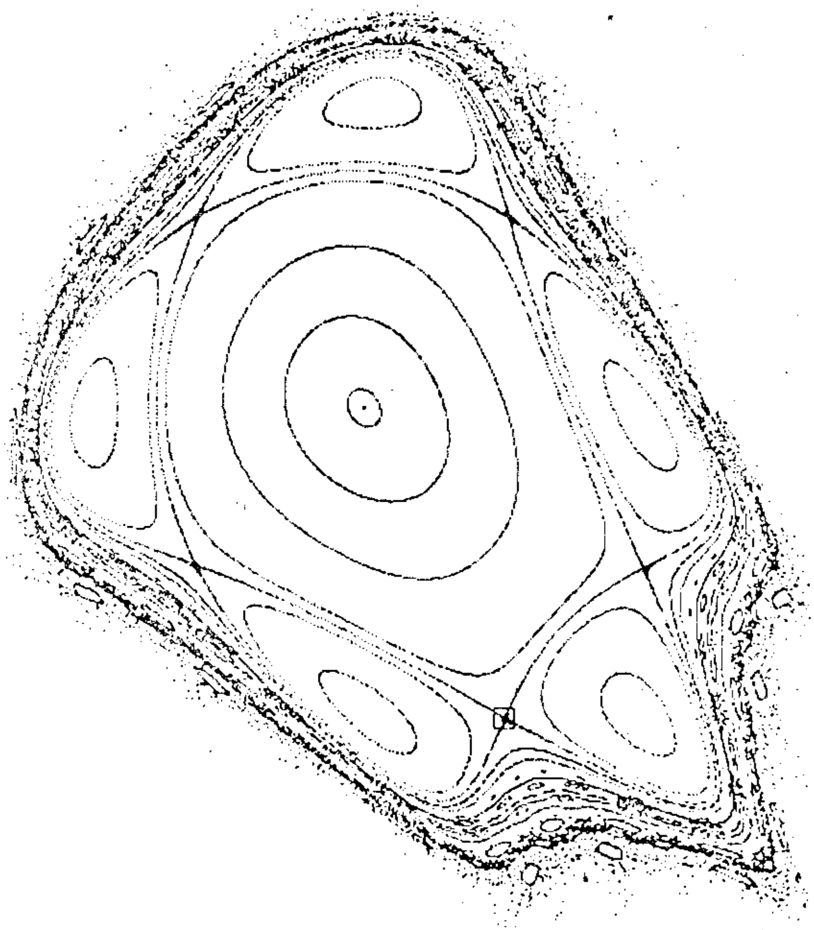






Crutchfield et al. (1980)

$$x_{n+1} = x_n^2 - y_n$$
$$y_{n+1} = -.4224 + x_n$$



...he "continuous" part of the Fourier power spectrum increases by orders ...de, especially near zero-frequency, i.e. for very long periods. The same ...experimental Turbulence spectra. Hence, we speak of the '*Onset of* ...', sharp peaks at the basic frequency remain in Fig. 21H. When the ...olens more than just one circular band, at other parameter values, those ...disappear as well, cf. Fig. 24. These pictures of the Rössler attractor ...ined by the Santa-Cruz (/Eugene, OR) group on an *analogue*-computer [232, ...facilitates parameter searches for interesting attractors. The ...is not high enough therefore to resolve the behavior inside the strange ...the Hénon attractor (3.8) this will be done, with the aid of a digital ...260], in the next section.

...*ssipative* system we have many Feigenbaums, of different basic periods m_i, ...usly present at a given value of μ. There are many interweaving branches ...nt trees coming very close together in infinite hierarchies, cf. sections ...or the *attracting* (repelling) branches of these trees in a dissipative ...he infinite hierarchies of different trees would be difficult to construct ...enough to an attractor, *all* orbits move towards it forever. In a ...system we find Feigenbaums of different basic periods m_i (in $m_i 2^k$) ...fferent μ-regions, one after the other [215,240,235;234], sometimes with ...tween them. This makes it even more difficult to predict where the "last" [240,215] than in a conservative system, cf. section 2.6, and where ...le chaos might set in.

...sequences have been established *rigorously* for mappings of the interval ...to itself which are of the form

$$y_{t+1} = f(|y_t|^{1+\varepsilon}, \mu), \qquad 0 < \varepsilon << 1, \qquad (3.33)$$

...y_{t+1} is smooth, except at f's maximum at $y = 0$ [237], under some ...nical conditions on f [237]. These results involve perturbation expansions are restricted to small ε values, whereas the interesting case is ε = 1 (3.9). ...lytical results have been extrapolated and extended to N-dimensional [227].

...doubling bifurcations are observed in real turbulence experiments as we ...in section 3.4. Below we discuss chaotic, non-periodic, attractors.

...ge Attractors

...descriptive, but less attractive, name for these objects might be ...ic Attractors', since the *motion along the attractor should* be '*ergodic*', ...er 2, *and* '*mixing*' [250-256,272,270,92,90-98], i.e. t-dependent correlation ...should vanish as t→∞. The latter precludes periodic attractors. A *non-attractor* has infinitely many intersection points with a transverse ...icular") surface. Yet, there cannot be any continuous 'curve' in this ...of section' with the attractor passing through *every* point of an 'interval' ...urve; otherwise it would not be an attractor *along* that 'interval' [270]. ...bility remaining is for the Strange Attractor to pass through a '*Cantor* ...nfinite number of points which are not dense on any 'interval' [168]. ...of such attractors have been constructed [250-272,169]. None of the above ...s of a Strange Attractor are easy to test, for a given system of equations ...rent opinions can be heard on the strangeness of particular attractors [272,

It is apparent however that there is a bewildering variety of types of ...s, *in between* Strange Attractors and simple Limit Cycles, cf. [250-279]. ...you that there also is a great variety of conservative systems - in between ...and integrable systems - as we saw in chapter 2. Some attractors [250-302] ...ger than others. An attractor is often called 'Strange' already if it ...consist of a single *periodic* orbit [272]. In that case no chaotic behavior ...ed along the attractor and 'Aperiodic Attractor' might be a more ...ve name [224,239,257,259,252].

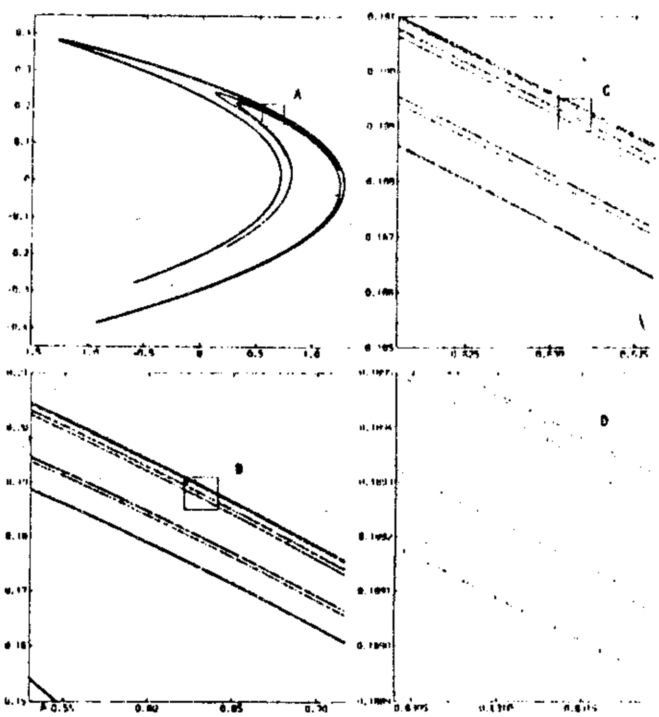


Figure 22
The Hénon-Attractor, at a = 1.4 and b = 0.3 in (3.8). Horizontally plotted is x_t, vertically: bx_{t-1} (taken from [260]). The complete attractor (A) seems simple, yet a 15 fold magnification of the little "box", in A, shows more "curves" (B). A further magnification (x10), of the small box in B, shows several more (C) "curves". A final magnification (x10), of the box in C, again shows new ones (D). This suggests a 'Cantor Set' cross-section. Along the curves points are repelled. Points are "transversally" attracted to the curves. Note the conservative curves in Figs. 6b,c. Text: see next page.

Consider for example Hénon's mapping

$$x_{t+1} - bx_{t-1} = 1 - ax_t^2, \qquad \text{cf. (3.8)},$$

Numerically For $a \leq 1.3$ Feigenbaum (periodic) attractor
$a \geq 1.4$ Strange attractor

Giornale dell' Istituto Italiano degli Attuari 7 (1936) 74-80

SULLA TEORIA DI VOLTERRA DELLA LOTTA PER L'ESISTENZA

A. KOLMOGOROFF.

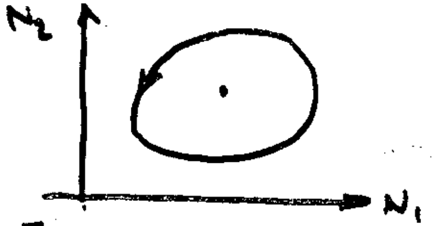
SUNTO. — L'A. studia le equazioni differenziali che si riferiscono alla lotta per l'esistenza, analoghe a quelle già considerate dal Volterra, facendo delle ipotesi di carattere puramente qualitativo sulla forma delle equazioni stesse.

1. La questione delle azioni reciproche di una specie mangiante e di una specie mangiata si riduce nelle ricerche di Vito Volterra[1] alla considerazione delle equazioni differenziali

$$\frac{dN_1}{dt} = (\varepsilon_1 - \gamma_1 N_2) N_1,$$
$$[1_a] \qquad \frac{dN_2}{dt} = (-\varepsilon_2 + \gamma_2 N_1) N_2,$$

dove N_1 e N_2 sono delle quantità di individui rispettivamente della specie mangiata e della specie mangiante che dipendono dal tempo t, ε_1 e $-\varepsilon_2$ i loro coefficienti di accrescimento e γ_1, γ_2 delle costanti. È naturale che le espressioni analitiche scelte dal Volterra per i secondi membri delle equazioni [1_a] possono essere considerate soltanto come prima approssimazione dello stato reale delle cose. Diversi autori hanno proposto altre relazioni per esprimere la dipendenza delle derivate $\frac{dN_1}{dt}$ e $\frac{dN_2}{dt}$ dalle quantità N_1 ed N_2. Rinunciando a queste ipotesi speciali, la cui scelta è del tutto arbitraria, scriviamo le equazioni delle azioni reciproche sotto la forma seguente

1) Cfr. per es. V. VOLTERRA, *Ricerche matematiche nelle associazioni biologiche.* « Giornale dell'Istituto Italiano degli Attuari », Anno II, n. 3, luglio 1931-IX.



Thermal Activation process:

$$T_{fract} = \tau_0 e^{-\frac{\sigma_a V_0}{kT}}$$

Stress Corrosion:

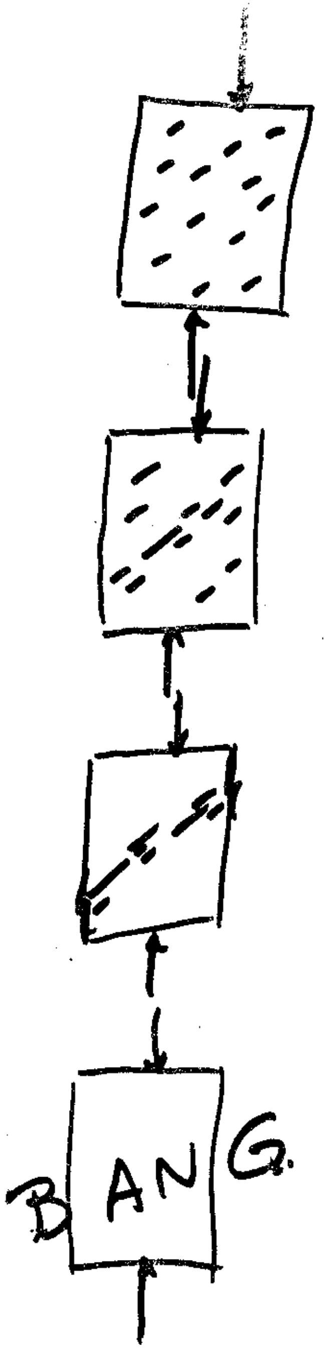
(Usual) Macroscopic model: in a high stress state, the strength of materials decreases (corrodes) sufficiently that fracture takes place at such time that the strength is equal to the stress

Microscopic model: it is a consequence of linkages of a broad spectrum of cracks; we take the spectrum and the linkage/fusion process to be self-similar.

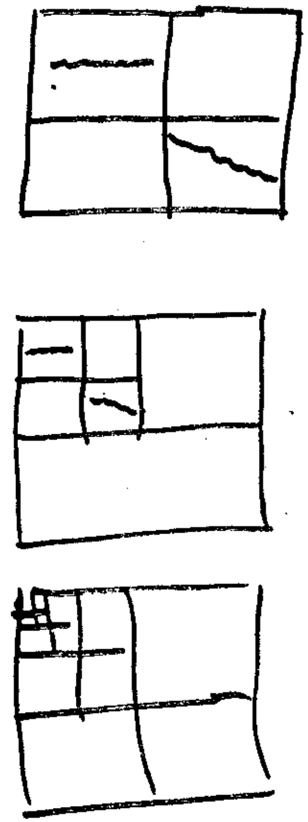
stress varies as $r^{-1/2}$ (r meas. from edge)

If density of cracks is L, then spacing between cracks varies as $1/L$. So stress in barrier between cracks varies as $L^{1/2}$.

$$T_{fract} \sim \tau_0 e^{-c\sqrt{L}}$$

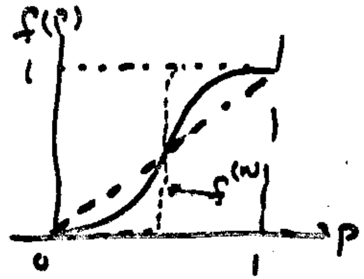


Renormalization model of Fracture



let p_0 be prob. that 0^{th} size box be fractured

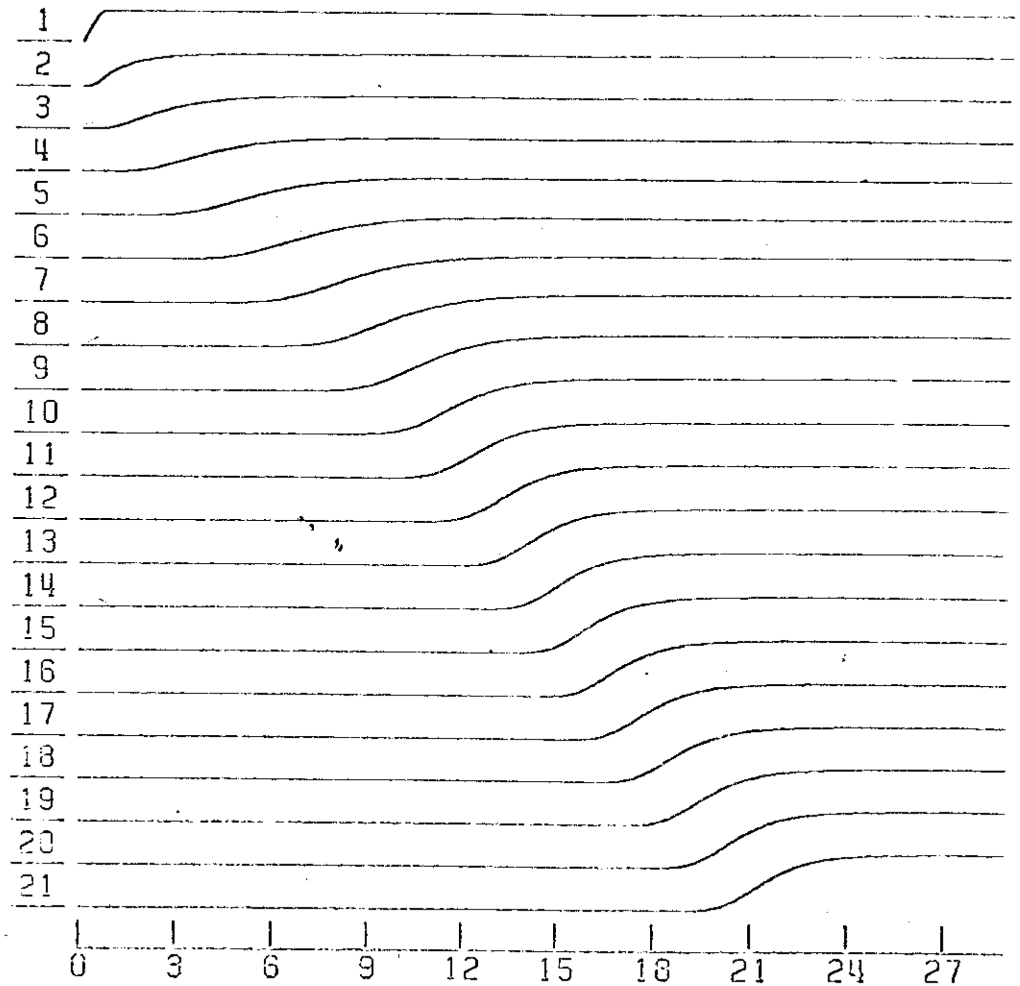
then $p_1 = f(p_0)$ generate prob of 1^{st} box.



$P_n = f^{(n)}(p_0)$

Try
$\dot{x}_n = f(x_{n-1}) - x_n$

(we used)
$f(x) = 3x^2 - 2x^3$



Length:
Let crack sizes be $\underline{2^n}$ $-\infty < n \leq N$

Number:
Let population in category $\underline{n}$ be $\underline{x_n}$

Fusion:
If 2 cracks of size $\underline{n}$ fuse, they form a crack of size $\underline{n+1}$

If 2 cracks of dissimilar sizes $n > m$ fuse, they form a crack of size n.

Aftershocks:
Large shocks have a 'fallout' of small cracks

Creep:
The time at which a large crack is formed is later than the <u>onset</u> of the process of fusion

Goals: <u>Instabilities</u>
Is it possible to produce clustering, especially of large earthquakes from the steady input of shocks originating in external (i.e. plate tectonics) causes?
Clustering to be repetitive (more or less)

For two crack size categories let L & B be the number of cracks at time t. Then we write

$$\frac{dB}{dt} = \gamma \tilde{L}^2 - \alpha B$$

$$\frac{dL}{dt} = \mu(L,B,t) + \eta \tilde{L}^2 - \nu \tilde{L}^2 L - \kappa L B \cdot \gamma L^2$$

—— source terms, positive feedback

—— negative feedback

1 Plate Tectonics

2 Fusion of two L-cracks

3 Healing (Dittrich, 1998) [illegible] log t

4 Aftershocks

5 Fusion of an L & a B crack

$$\tilde{L}(t) = L\left(t - \tau e^{-\varepsilon\sqrt{L(t')}}\right) \approx L\left(t - \tau e^{-\varepsilon\sqrt{L(t)}}\right)$$

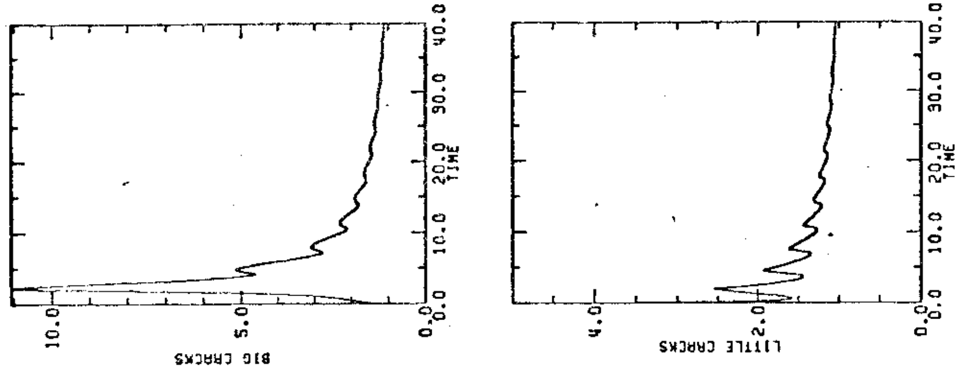
$$t - \tau < t' < t$$

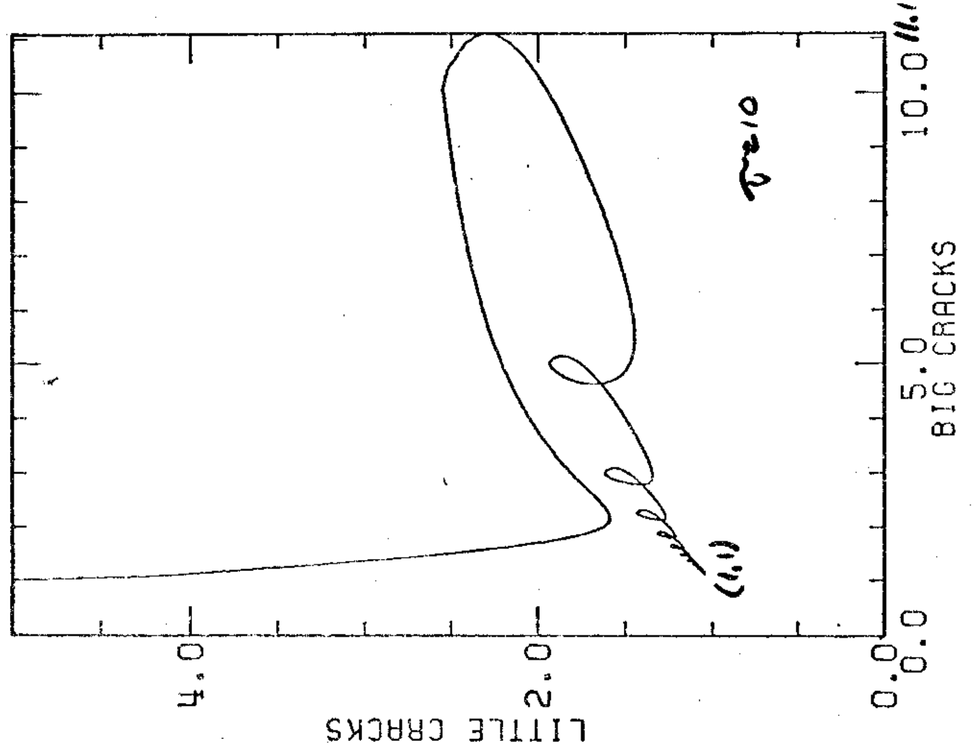
Numerical examples

$\mu = 1$, $\eta = 3$, $\nu = 1$, $\kappa = 2$, $\gamma = 1$, $\alpha = 1$ (equilibrium pt. = $L = 1$, $B = 1$)

$\varepsilon = 1$

Initialize: $L = 1 + 4e^{a\tau t}$, $B = 1$ for $t < 0$





$$\dot B = \gamma \tilde L^2 - \alpha B$$

$$\dot L = \mu_0 + \eta \tilde L^2 - \nu \tilde L^2 L - \kappa L - \gamma L^2$$

$\mu_0 = 1$
$\eta = 3$
$\nu = 1$
$\kappa = 2$
$\gamma = 1$
$\alpha = 1$

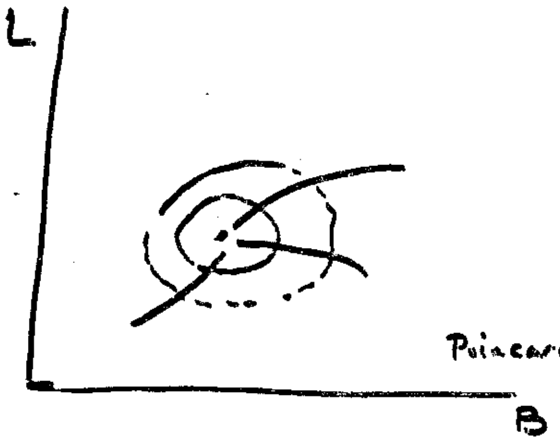
Time delays might destabilize

consider $\dot x = -\alpha x \Rightarrow x = x_0 e^{-\alpha t}$

consider $\dot x(t) = -\alpha x(t-\tau)$

$x = \sin \alpha t, \cos \alpha t$

if $\tau = \frac{\pi}{2\alpha}$



$$\dot B = \gamma \tilde L^2 - \alpha B$$

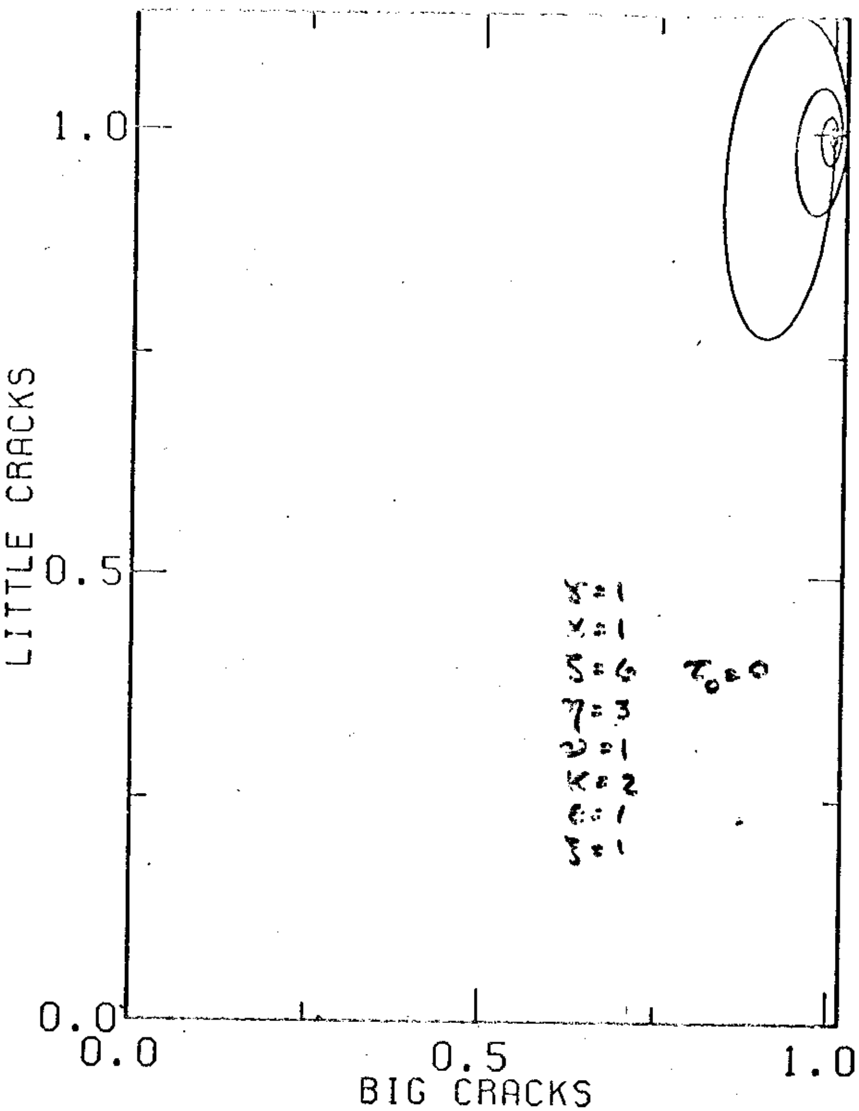
$$\dot L = \mu(t) + \eta \tilde L^2 - \nu \tilde L^2 L - \kappa L B - \gamma L^2$$

$$\dot \mu = \delta (1 - \xi \tilde L^2)$$

$$\xi = \frac{(1+\delta)\mu}{\delta + \mu}$$

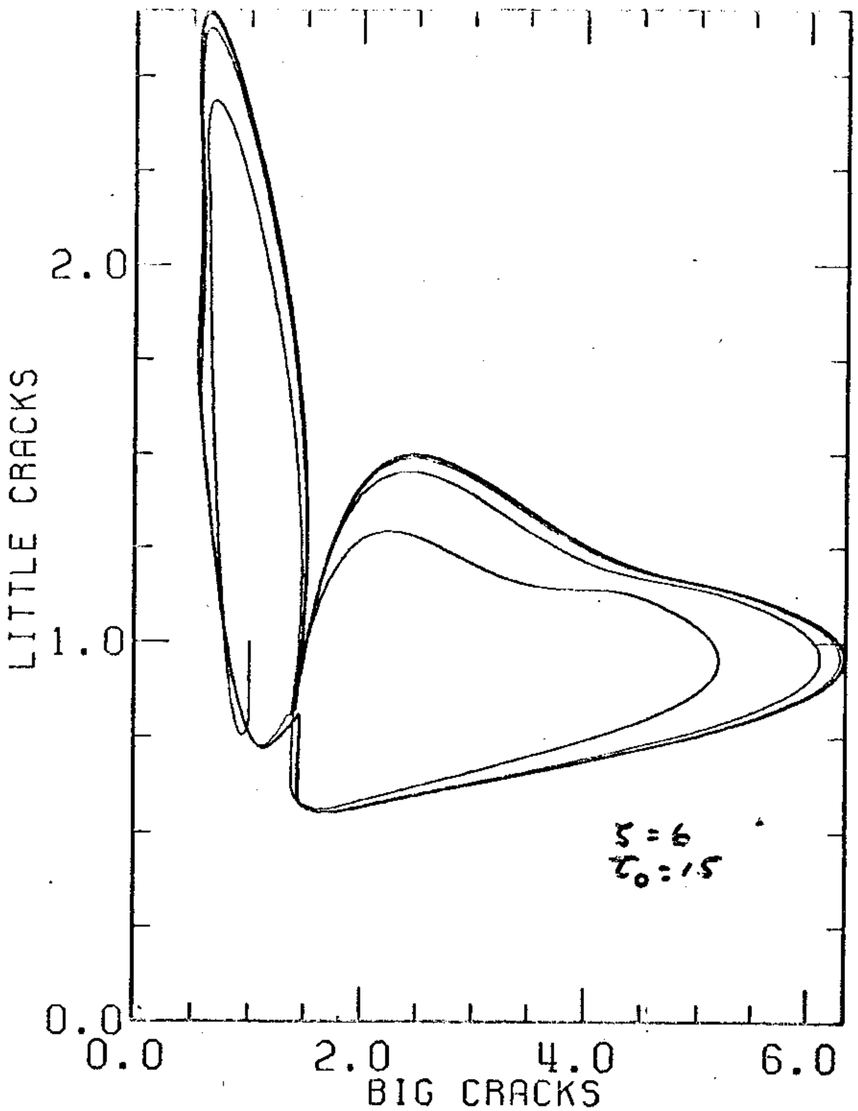
for $\gamma = 1$
$\alpha = 1$
$\delta = 6$
$\eta = 3$
$\nu = 1$
$\kappa = 2$
$\xi = 1$

, Hopf bifurcation at $\tau_0 = .27$



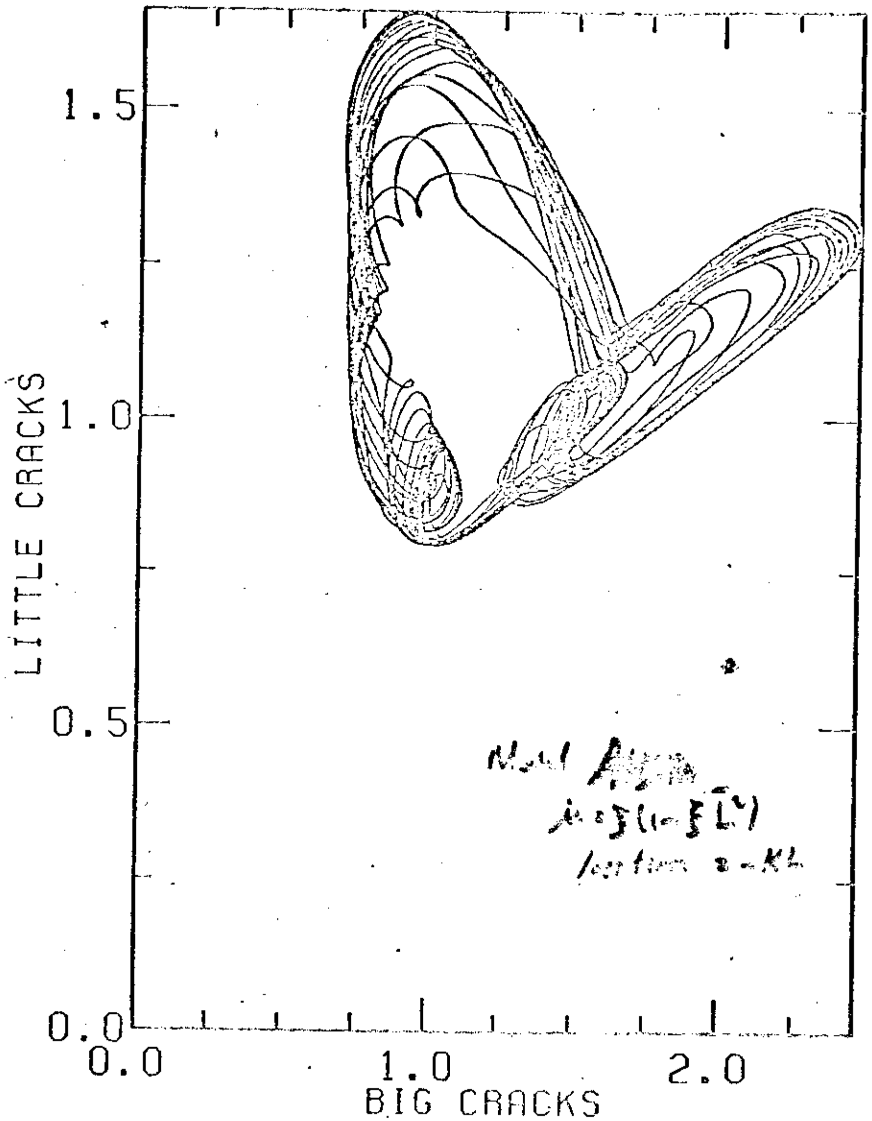
LITTLE CRACKS
BIG CRACKS
1.0
0.5
0.0
0.0
0.5
1.0

-17-



LITTLE CRACKS
BIG CRACKS
2.0
1.0
0.0
0.0
2.0
4.0
6.0

-18-



References V

Feigenbaum sequences, Hénon, Rössler, etc. attractors
Strange attractors, etc.

A good summary with 302 references is
⇒ Self-generated chaotic behavior in nonlinear mechanics by R.H.G. Helleman
in <u>Fundamental Problems in Statistical Mechanics</u>, vol. 5,
E.G.D. Cohen, No. Holland Publ. (1980) p. 165.

Also 2 reprints by Newman & Knopoff attached

CRACK FUSION DYNAMICS: A MODEL FOR LARGE EARTHQUAKES

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Abstract. The physical processes of the fusion of small cracks into larger ones are nonlinear in character. A study of the nonlinear properties of fusion may lead to an understanding of the instabilities that give rise to clustering of large earthquakes. We have investigated the properties of simple versions of fusion processes to see if instabilities culminating in repetitive massive earthquakes are possible. We have taken into account such diverse phenomena as the production of aftershocks, the rapid extension of large cracks to overwhelm and absorb smaller cracks, the influence of anelastic creep-induced time delays, healing, the genesis of "juvenile" cracks due to plate motions, and others. A preliminary conclusion is that the time delays introduced by anelastic creep may be responsible for producing catastrophic instabilities characteristic of large earthquakes as well as aftershock sequences. However, it seems that nonlocal influences, i.e. the spatial diffusion of cracks, may play a dominant role in producing episodes of seismicity and clustering.

Introduction

Observations of seismic gaps and clustering phenomena are indications that more traditional models of earthquake occurrence, such as that of a simple Poisson random process, are inappropriate. There is an indication from laboratory experiments on fracturing that large-scale fractures are the end result of a sequential process of fusion of smaller cracks into larger ones. We study how the fusion of small cracks into larger ones can produce a catastrophic cascade of events which culminates in a massive earthquake as its endpoint. We include in our model such diverse phenomena as the production of aftershocks, the rapid extension of large cracks to overwhelm and absorb smaller cracks, the influence of anelastic creep-induced time delays, and others.

The processes wherein cracks of different sizes can interact are essentially independent of size, so long as the cracks considered are larger than the dimensions characteristic of individual grains or crystals (i.e. a few millimeters) and smaller than the distances between the triple junctions of the major plates. This property of cracks is a profound feature of rock physics and is manifest in the scale invariance of crack size distributions in rocks and in the spectrum or time signature of crack fusion events ranging from microcrack formation to earthquake aftershock sequences (Kagan and Knopoff, 1978 and 1980). What is particularly significant about the observed scale invariance of earth materials is that it may be possible, by employing an approximate description of the rock physics, to renormalize the constitutive equations describing crack fusion events. A relevant example of how scale invariance has led to a dramatic simplification of the description of a complex physical phenomenon is the application to hydrodynamic turbulence. There, in particular, it was possible to obtain a description of the onset of nonlinear fluid behavior and the cascade of energy through a sequence of events wherein eddies of smaller dimensions were formed. We contend that the application of similar techniques to the dynamics of crack fusion can provide a physical description (in contrast with a statistical one) of fusion events and the instabilities that give rise to such phenomena as aftershock sequences, clustering, episodes of seismicity, and seismic gaps.

If we assume that we know the rates of crack fusion, healing, and other processes that may be involved, it is possible to describe the evolution of cracks by an integro-differential equation for the probability of finding a crack with a given length and orientation that is centered at a particular location. The spatial dependence of this formulation of crack fusion dynamics assumes a form commonly encountered in problems of nonlinear diffusion. This is especially significant since nonlinear diffusion processes combined with nonlinear sources and sinks can give rise to wave-like behavior that is spatially confined. [See, for example, Newman's (1981) treatment of certain nonlinear diffusion problems encountered in population genetics and combustion.] In combination with spatial inhomogeneities, nonlinear diffusion may have an important role in producing the properties of seismicity listed above that we think are evidence for instability. Nonlinear diffusion imparts a multidimensional character to the problem.

In this paper, we consider a preliminary version of the problem in order to expose the instabilities that result from the introduction of nonlinearities into a system describing cracks. To do this in the simplest way, we eliminate the spatial dependence of the crack distribution by reconstituting our dynamical equations in terms of the populations of cracks of different sizes, at a fixed, isolated region on a fault, for example. The integral equation character of these expressions can be simplified by categorizing crack lengths into different "bins," partitioned according to the logarithm of their length. In that way, our "renormalization" exploits the scale invariance of crack size distributions. It is the formation of these cracks or their abrupt lengthening due to fusion that we envision as being descriptive of an earthquake event. A particularly simple illustration of this approach, characterized by only two crack size categories is described below. In this context, we will see that aperiodic earthquake sequences are a (possible) direct consequence of time delays introduced by anelastic creep and stress weakening.

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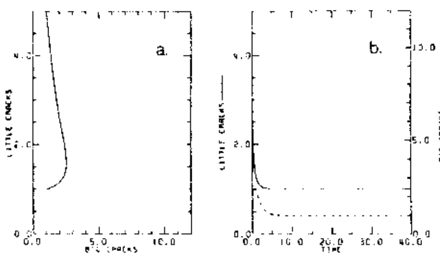

Fig. 1. Evolution of crack populations. Time Delay = 0.0.

Preliminary Model

For an approximate description of earthquake events, we classify cracks at a point in an earthquake zone as being of only two sizes; they are either little or big, the number in the respective populations being denoted by L and B. The model equations are

$$dB/dt = \gamma\Lambda^2 - \alpha B$$
$$dL/dt = \mu + \eta\Lambda^2 - \nu\Lambda^2 L - \kappa L - \gamma L^2 \qquad (1)$$

In these equations, we propose that the fusion of little cracks produces big cracks, but that big cracks themselves do not undergo fusion since there are no cracks of larger size. Thus the process is dissipative for little cracks, and they must ultimately vanish unless otherwise replaced. We envision the process of replenishment as due to plate tectonics. The motion of the plates, steady at large distances from the plate boundary, is restricted by the friction at the boundary; we assume that partial relief of the accumulation of stress is accomplished through the occurrence of small cracks at the plate boundary. (The process is similar to that of the slow opening of a door at a sticky hinge in which a large amount of creaking at the hinge occurs.) This term is given by the coefficient μ in equation (1) and is presumed to be steady.

We assume that the fracture strength of the rocks along a fault is significantly lower immediately after an earthquake than before. The low value of strength promotes the possibility of fusion of a crack that has occurred earlier, with a later one. We further assume that there is a healing process that operates to cause the breaking or yield strength of the fault to return, after some time interval, to a high value once again; we may assume that the recovery proceeds exponentially to a constant with a coefficient equal to the increase in strength. This recovery of strength allows for the occurrence of an independent earthquake on the same segment of fault, much later in time. Thus, a faulted segment has a low enough strength to permit fusion for a certain interval of time after formation; after a certain critical time it will be strong enough to resist fusion. In this model, the faulted segment heals and is removed from the population that participates in the fusion process. In this paper we model the process of healing by assuming that a constant fraction of the population of cracks of a given size undergoes recovery to full strength at any instant. These healing terms are modeled with the coefficients α and κ in equation (1).

The term describing the fusion of small cracks is proportional to L^2; this is similar to the calculation of the number of handshake pairs in a room of L persons, which is $L(L-1)/2$, and is therefore approximately proportional to L^2 for large L. When two small cracks unite, the population of little cracks is reduced and and the population of big cracks is increased; hence a term with negative sign appears in the equation dL/dt and a corresponding term with positive sign appears in the equation for dB/dt, and is therefore a source term for big cracks. [Equations (1) include the crack fusion terms $\gamma\Lambda^2$ and $-\gamma L^2$. Depending upon our precise choice of normalization, a slightly different formulation of the crack fusion terms may be necessary. However, the particular normalization employed has no significant influence on the qualitative character of the computed results.] We assume the transition from little to big cracks is not instantaneous but delayed by anelastic creep. We represent by $\Lambda(t)$ the number of little cracks at some earlier time when fusion began to occur. The time delay τ depends on temperature according to

$$\tau = \tau' \exp(-E/kt) \qquad (2)$$

where E is an activation energy, k is the Boltzmann constant and T the temperature. Since we can associate E with the stress σ, the delay time will have an exponential dependence on σ, a conclusion confirmed by the experiments of Griggs (1940) and others. Since the short range stresses at a crack tip decrease as $r^{-1/2}$ where r is the distance from the edge of a crack and L varies inversely as the mean crack separation, equation (2) becomes

$$\tau = \tau' \exp(-\epsilon\sqrt{L}) \qquad (3)$$

where ε is a temperature-dependent parameter (taken here to be fixed). Formally, the value of Λ used in (3) must be representative of L over the interval (i.e. must be an average of some sort). For simplicity, we use the current value of L, i.e.

$$\Lambda(t) = L\{t - \tau' \exp[-\epsilon\sqrt{L(t)}]\} \qquad (4)$$

It should be emphasized that the terms representing energy released in abrupt earthquakes are those with positive coefficients on the right hand sides of equations (1). Thus the term $\gamma\Lambda^2$ represents the fusion of small cracks into large ones and, therefore, is a measure of the rate of occurrence of major earthquakes.

Two additional idealized processes are

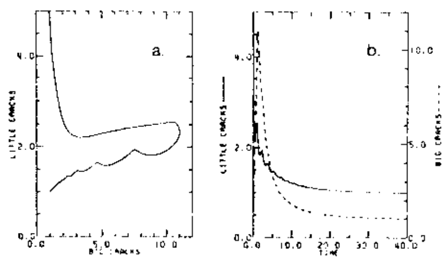

Fig. 2. Evolution of crack populations. Time Delay = 5.0.

included in the calculation. The term $\eta\Lambda^2$, where η is also a positive constant, represents the rate of aftershock fallout of small cracks which is taken to be proportional to the rate of occurrence of large earthquakes. There is, in addition, the loss term $\nu\Lambda^2 L$ which represents the rate of consumption of small cracks by the extension of large ones in large earthquakes.

The form of equations (1) is most frequently associated with predator-prey systems in population dynamics and is often associated with the names of Lotka and Volterra. [See Davis (1962) or Rosen (1970) for a discussion of the character of such autonomous systems of equations.] Our equations differ from the usual predator-prey equations in that the equation for little cracks is completely decoupled from terms involving big cracks.

The purpose of the calculation is to see whether a steady external source term can lead to instabilities; i.e. to non-uniform production of large and small earthquakes, but especially large ones. A particular hypothesis is that oscillatory behavior can be sustained.

In the absence of a time delay (the only case directly amenable to analysis), equations (1) will have only one real-valued stationary (i.e. steady state) solution. When equations (1) are linearized around the stationary solution, we always obtain solutions which decay exponentially fast to the steady state solution. Moreover, we can show that the solutions for L and B, in the nonlinear regime including creep-induced delays, are always positive as required from physical considerations. What is significant to us here is a remarkable theorem due to Poincaré and to Bendixson which has a profound implication to the physical description of this problem. [See Coddington and Levinson (1955) or May (1972) for a discussion of the Poincaré-Bendixson Theorem.] This theorem shows that the solution of equations of the form (1) either:

(a) approaches a closed path, namely a limit cycle (an isolated periodic solution of the autonomous system of equations); or
(b) converges to an equilibrium point.

A limit cycle can emerge only if the equilibrium point of (1) is unstable. Since this is not the case, the evolution of equations (1) cannot describe the behavior of earthquake events including aftershock sequences. The implication, then, is that only a time delay could destabilize the situation, giving rise to oscillations or, possibly, growth. [For example, the equation

$$dy(t)/dt = -y(t) \tag{5}$$

is characterized by solutions of the form exp(-t) whereas the equation

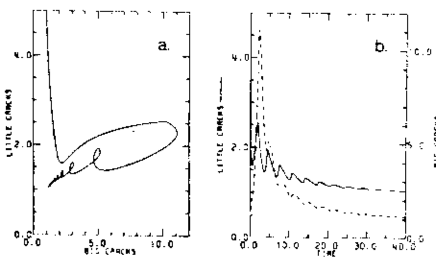

Fig. 3. Evolution of crack populations. Time Delay = 10.0.

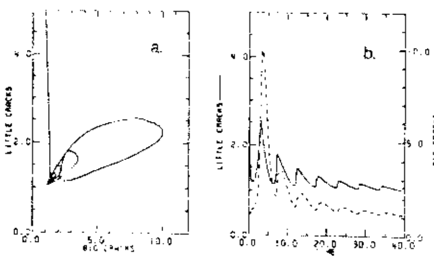

Fig. 4. Evolution of crack populations. Time Delay = 15.0.

$$dy(t)/dt = -y(t-\pi/2) \tag{6}$$

is characterized by solutions of the form sin(t) and cos(t).]

This, then, is the basis of the preliminary model for which we now provide illustrative examples. The model includes the effects of crack healing, plate tectonics, and crack fusion. However, these effects alone do not suffice to describe earthquake activity. Indeed, the critical factor appears to be anelastic time delays which describe such processes as creep or stress corrosion.

Illustrative Examples

For this study, we employed the values

$$\gamma = 1, \quad \lambda = 1, \quad \mu = 1, \quad \eta = 3, \quad \nu = 1,$$
$$\kappa = 2, \quad \text{and } \varepsilon = 1. \tag{7}$$

The stationary solution is

$$L = B = 1 \tag{8}$$

By employing this normalization, we have eliminated two of the adjustable parameters in (7). A third parameter is eliminated by plotting L vs. B, thereby eliminating the explicit time dependence in the relative crack population behavior. (This type of figure is referred to as a "phase diagram" or the "phase trajectory" for the dynamical evolution.) Thus, we have four adjustable parameters remaining plus the time delay constant τ'. It should be emphasized that the evolution of the solution to equations (1) is relatively robust insofar as the choice of parameters is concerned; i.e. changing the numerical values assigned to the parameters does not substantially change the qualitative character of the solution.

In the present illustration, we consider the consequences of varying the time delay constant τ'. In Figures 1-4, we show two plots for each of the values of τ' of 0.0, 5.0, 10.0, and 15.0. Plot a describes the behavior of little cracks as a function of the number of big cracks with time as a parameter along the curve, and plot b describes the evolution of little and big cracks as functions of time. [Although the numbers of little cracks are decoupled from the big crack population, plots a help to demonstrate the competition between little and big crack populations in a time independent manner.]

In this example, we choose as initial conditions one in which there is an injection of small cracks at t = 0; this might be imagined to correspond to the occurrence of little cracks in a region due to a major earthquake in a neighboring region. We chose L = 5 and B = 1 at t = 0 (with L = 1 + 4 exp(0.7t) for t < 0; we set B = 1 in order to describe an equilibrium state for big cracks). With no time delay, we obtain a rapid return of both crack populations to their equilibrium values. The initial behavior of the system is dominated by transient, nonlinear crack fusion events, with healing etc. being initially insignificant. Consequently, the L vs. B relationship is largely linear since the little and big crack populations are depleted and fed, respectively, by crack fusion events. Thereafter, healing and other terms restore the system to equilibrium. It should be mentioned that an equilibrium or steady state solution is not equivalent to the absence of earthquakes. Big earthquakes persist at a rate of $\gamma\Lambda^2$. At equilibrium, the earthquake rate is exactly balanced by the healing rate. Nearly periodic departures from equilibrium, on the other hand, can be associated with aftershocks and, possibly, episodes of seismicity.

When the time delay is increased to 5.0, the little vs. big crack "phase diagram" displays two new behavioral characteristics. First, the initial transient response manifest between little and big cracks is no longer linear. As the little cracks begin to fuse, the number of little cracks is quickly reduced. However, only after about one creep delay time the fusion of little cracks produces big cracks and, hence, earthquakes. Thus, initially, the phase trajectory drops to low values of the numbers of little cracks without much increase in the number of big cracks; this is then followed by an increase in the number of big cracks; the phase trajectory has a sharp bend during its early development. Second, the large growth in the numbers of big cracks causes an aftershock fallout of small cracks which, after an interval of time, themselves fuse to increase momentarily the big crack population. This appears in the form of a ripple in the phase diagram which then re-emerges (in attenuated form) one delay time later. The small ripples are particularly evident in the secular behavior of the little crack population which is beginning to show what might be described as an aftershock sequence.

As the time delay is increased further, the ripples obtained in the former plots evolve into cusps and are observed to appear in the secular variation of both the little and big crack populations. In the third set of results, with a time delay of 10.0, the cusps have been transformed into a set of loops or "epicycles." The time evolution of the big and little crack populations now displays pronounced, damped ramp-like oscillations. In particular, the numbers of little cracks show a sequence of sharp rises and slow decays. (This is especially significant since, as pointed out earlier, $\gamma\Lambda^2$ represents the rate of large earthquake events). For this time delay, the loops in the little vs. big crack phase diagram begin to overlap, a feature which can be associated with aftershocks. The last set of plots, with a time delay of 15.0, exhibits loops which overlap strongly in the L vs. B plot while the ramp-like behavior observed earlier has become especially pronounced in the time variability of both the little and big crack populations. Significantly, both crack populations return to a near-equilibrium value before undergoing each aftershock event.

We have observed that the details of the evolving phase trajectory depend highly upon the initial conditions. From this investigation, we conclude that a critical factor for producing clustering of earthquake sequences is anelastic creep and/or stress weakening, although we have not yet been able to produce repetitive clusters. We plan to investigate nonlocal influences such as nonlinear spatial diffusion and inhomogeneities, as well as increase the number of crack sizes beyond the present level of two, as possible influences in producing episodes of seismicity, seismic gaps, and clustering.

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A MODEL FOR REPETITIVE CYCLES OF LARGE EARTHQUAKES

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Abstract. The theory of the fusion of small cracks into large ones reproduces certain features also observed in the clustering of earthquake sequences. By modifying our earlier model to take into account the stress release associated with the occurrence of large earthquakes, we obtain repetitive periodic cycles of large earthquakes. A preliminary conclusion is that a combination of the stress release or elastic rebound mechanism plus time delays in the fusion process are sufficient to destabilize the crack populations and, ultimately, give rise to repetitive episodes of seismicity.

Introduction

In an earlier paper (Newman and Knopoff, 1982) which we refer to as paper I, we outlined a model for clustering of earthquakes in which large scale fractures were presumed to be the end result of a sequential process whereby small cracks fuse into larger ones; these latter in turn fuse to form even larger cracks and so on. Starting from strong suggestions about the geometry of earthquake epicenters, we imagine the crack fusion process to be largely independent of size scale provided that the cracks we consider are significantly larger than grain or crystal dimensions and smaller than the distances between the triple junctions of the major plates. In paper I, we postulated that this scale invariance would permit us to develop a model which would expose the roles of crack healing, plate tectonics and crack fusion in producing such phenomena as aftershock sequences, clustering, episodes of seismicity, and seismic gaps.

The crack fusion process must ultimately lead to depletion of the population of small cracks, and hence of all cracks unless there is some mechanism for replenishment at the lowest level of crack sizes. Since seismicity releases accumulated stress at a plate boundary, we propose that it is the motion of the plates that revives the population of the cracks at the boundary. Thus, the population of small cracks is a measure of the rate of accumulation of stress at the plate boundary. Processes characterized by a hierarchical sequence of events give rise in a natural way to a time delay (Sparrow, 1980). Thus, we may assume that the cumulative effect of the crack fusion process is to introduce a time delay between the production of cracks of the smallest size or juvenile cracks and the culmination of the process which is a large scale rupture or earthquake along a fault. Laboratory investigations on a macroscopic scale (Griggs, 1940 and others) show that this time delay is associated with anelastic creep or stress corrosion, an intrinsically unstable, nonlinear process.

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The naturally occurring scale invariance of fracture leads us naturally to a renormalization model. It is the postulate of a number of workers in the area of renormalization theory and its application to continuous media (e.g. Lorenz, 1963) that renormalization permits a revision of the usual description of a continuous hierarchy of objects in terms of a discrete set of elements (in this case, the hierarchy of objects is of crack sizes). The partial differential equations necessary for a continuum are replaced by ordinary differential equations. It is a further postulate that we can reduce the system to only two crack sizes, which we envision in some sense as end members of the class of crack sizes, by the introduction of time delays described above as substitutes for the interaction of the missing members of the hierarchical class. The suitability of these postulates is under investigation at this time.

In paper I, we assumed a constant rate of replenishment of microscopic cracks due to plate motions. We found that there was a hint of instability in the form of nonrepetitive clustering which was found to be due to the time delays associated with anelastic creep. However, the overall conclusion of that investigation was that the model of paper I must ultimately stabilize and tend to a steady rate of production and disappearance of cracks of all sizes.

Effects of Stress Release

In the present paper, we seek to modify the former model to produce repetitive sequences of earthquake events. In paper I, we assumed that plate tectonics would serve as a steady source for juvenile cracks. In this paper, we propose that a more realistic expectation would be to assume that this stress at a plate boundary accumulates locally at a uniform rate only to be released by the occurrence of a large earthquake. This is no more than the usual elastic rebound model of earthquake occurrence. Thus, a physically appealing generalization of our earlier model is to allow the rate of production of small cracks to be in direct proportion to the tectonically related stress; the stress, in turn, would grow in response to the relative plate motion and would diminish according to the rate of large earthquake events. With this generalization in mind, we present a modification of the model in I. (For a complete description, see paper I.)

We classify cracks at a point in an earthquake zone as being either little or big, the number in the respective populations being denoted by L and B. The model equations are

$$dB/dt = \gamma\Lambda^2 - \alpha B$$

$$dL/dt = \mu + \eta\Lambda^2 - \nu\Lambda^2 L - \kappa BL - \gamma L^2 \qquad (1)$$

$$d\mu/dt = \begin{cases} \zeta(1 - \xi\Lambda^2) & \mu \geq 0 \\ 0 & \mu < 0 \end{cases}$$

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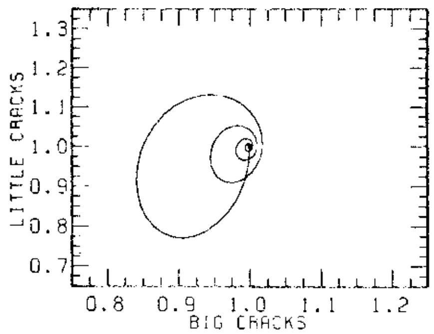

Fig. 1 Crack population evolution. $\tau' = 0.0$. Initial conditions $L = 1$, $B = 1$, $\mu = 0$.

where Λ denotes the time-delayed value of L, i.e.

$$\Lambda(t) = L\{t - \tau'\exp[-\xi\sqrt{L(t)}]\} \qquad (2)$$

The term describing the fusion of small cracks is proportional to L^2, which is the number of possible crack pairings. The terms $-\gamma L^2$ and $\gamma\Lambda^2$ denote the loss of little cracks and the corresponding gain of big cracks associated with crack fusion. Thus Λ^2 is a measure of the rate of occurrence of major earthquakes. We model the healing of big cracks by assuming that a constant fraction of the population of cracks disappears, i.e. that the cracks all are cemented together; this process is represented by the coefficient α. For little cracks, however, the situation is somewhat more complicated. In particular, we assume that little cracks heal if the local stress has been released during the formation of big cracks, otherwise, the little cracks will be held open. In paper I, we did not incorporate this effect and the healing of little cracks was represented by the term $-\kappa L$. Here we will assume that the rate of healing of little cracks varies with both the population of little cracks and the population of big cracks (as an indicator of local stress release); the term $-\kappa BL$ represents the rate of healing of little cracks. The terms $\eta\Lambda^2$ and $-\nu\Lambda^2 L$ represent the production of aftershocks which we envision as a production of small cracks, and the rate of consumption of small cracks by the extension of large fractures in large earthquakes. As in paper I, μ denotes the tectonic stress responsible for creating microscopic cracks near the fault. However, in the absence of large earthquakes (i.e. $\Lambda^2 = 0$), we assume μ will increase at a uniform rate ζ and, in the presence of seismic events, experience a drop in proportion to the number of events, i.e., say $-\zeta\xi\Lambda^2$. (We prevent μ from becoming negative, which would represent a decrease of the crack population in a term associated with plate tectonics.) If ζ were chosen to be zero (i.e. if the tectonic source of microscopic cracks were presumed to be steady), we would recover the model investigated in paper I, apart from our more realistic representation for the healing of little cracks. However, when the tectonic source term μ is allowed to vary, the Poincaré-Bendixson theorem employed in paper I is no longer applicable since we now have three dependent variables and we must employ other methods, both analytic and computational, to understand whether this model can produce repetitive cycles of seismicity and, if such cycles repeat, whether they are periodic. We find that the behavior of a dynamical system which has three interacting elements undergoes a qualitative transition to instability, in contrast with our earlier model. We believe that this is a watershed that is not likely to be reversed upon the construction of a model with a greater number of crack sizes. The issue that requires investigation is the degree to which the instability is chaotic or predictable

Hopf Bifurcation

As before, we used the values

$$\gamma = 1, \alpha = 1, \eta = 3, \nu = 1, \kappa = 2, \text{ and } \epsilon = 1 \qquad (3)$$

which provides as a stationary solution

$$L = B = \mu = 1 \qquad (4)$$

(The existence of a Hopf bifurcation is robust with respect to the parameterization. A realistic choice of the parameters is a subject for further investigation.) In order that (4) be a solution to equations (1), we require that $\xi = 1$. Our model equations (1) cause the little and big crack populations to be strictly positive and have an upper bound. In order to understand the behavior of the two crack populations and the tectonic stress near the stationary solution, we linearize equations (1) and (2) around their steady state solution (4) and seek solutions of the form $\exp(\beta t)$. This procedure leads to a dispersion relation. In the absence of a time delay, we find from the dispersion relation that Re $\beta < 0$ for all $\zeta \geq 0$. Thus, the stress release mechanism without time delays forces the two crack populations to evolve toward a steady state. This is the converse of the conclusion of paper I, which was that time delays in the absence of stress release invariably cause the solution to decay to a steady state. Thus, time delays or stress release alone cannot provide for cyclic seismicity.

Without time delays, our system has three interacting components, as does the Lorenz (1963) model for hydrodynamic turbulence. If we extrapolate from the Lorenz model, our model should be capable in principle of exhibiting much more complex behavior than the two-component model of paper I. Different choices for the physical parameters (3), without a time delay,

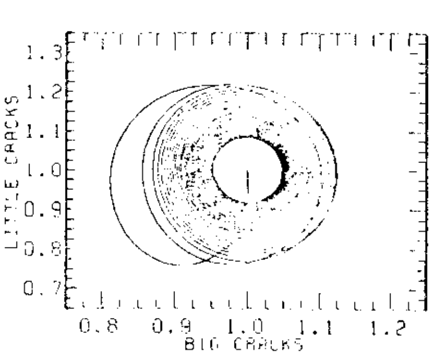

Fig. 2 Crack population evolution. $\tau' = 0.25$. (Semi-stable) Initial conditions as in Fig. 1.

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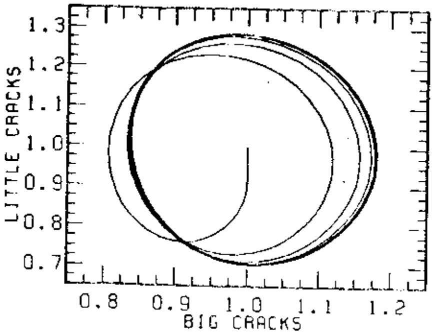

Fig. 3 Crack population evolution. $\tau' = 0.30$. (Supercritical) Initial conditions as in Fig. 1.

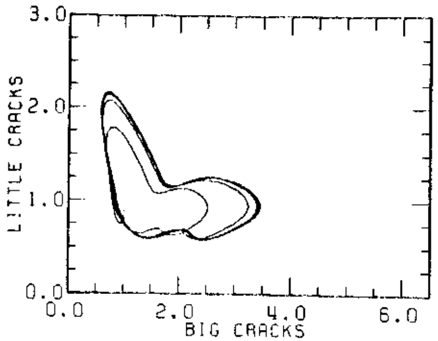

Fig. 5 Crack population evolution. $\tau' = 5.0$. Initial conditions as in Fig. 1.

appear to be incapable of supporting behavior that does not converge to a steady state. However, when we introduce a time delay into our more realistic model, we might expect the three components to respond so out of phase with one another that, through their mutual interaction, they would no longer be strictly dissipative but would instead support purely cyclic behavior. Hence, we seek periodic solutions to the dispersion relation; i.e. for what values of ζ and τ', can β be purely imaginary? We find, for a given value of ζ, that the real part of β (which is negative when $\tau' = 0$) increases as τ' increases and becomes zero at a particular value of τ', which we will call τ_H'. When τ' was increased beyond τ_H', the real part of β becomes positive and the solution becomes unstable and grows exponentially. The transition from stable to periodic to unstable solutions is referred to as a Hopf bifurcation. (In the examples below for $\zeta = 6$, $\tau_H' = 0.27078$ at the bifurcation point.) Thus, we have shown that a combination of the stress release mechanism with anelastic creep associated time delays can destabilize the steady state solution. However, the only other information that we have is that the solutions for L and B are bounded. Will this skeletal version of crack fusion dynamics result in ergodic behavior or intermittency (like the Lorenz model) or will it tend to some repetitive, possibly periodic cycle of activity? In order to resolve these questions of the nonlinear evolution of the crack fusion dynamics model, a computational investigation is required.

Illustrative Example

As in paper I, we consider the consequences of varying the time delay constant τ', for the parameterization (3) and add the parameterization $\zeta = 6$. In Figures 1-6, we plot the phase trajectory for the dynamical evolution in L vs. B for each of the values of τ' of 0.0, 0.25, 0.3, 2.0, 5.0, and 15.0. As noted, the critical or Hopf bifurcation value of τ_H' is 0.27078. Thus, this selection of τ' permits us to explore the evolutionary trend of the model as τ' is varied through and much beyond its bifurcation point.

In this example, we choose as initial conditions for which the two crack populations are (and have always been) at their steady state values but where the tectonic stress has suddenly vanished. With no time delay (Fig 1), the little crack population begins to drop (since there is no tectonic source to replenish their number) and the big crack population begins to

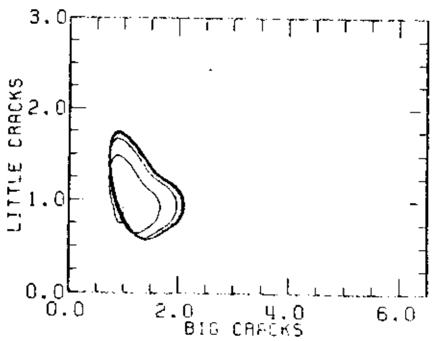

Fig. 4 Crack population evolution. $\tau' = 2.0$. Initial conditions as in Fig 1.

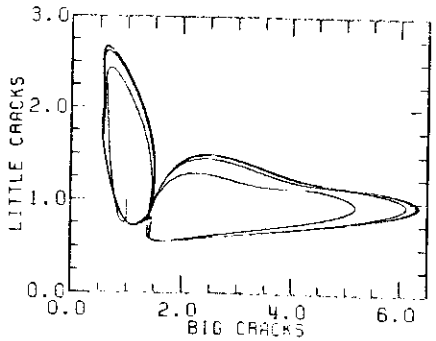

Fig. 6 Crack population evolution. $\tau' = 15.0$. Initial conditions as in Fig. 1.

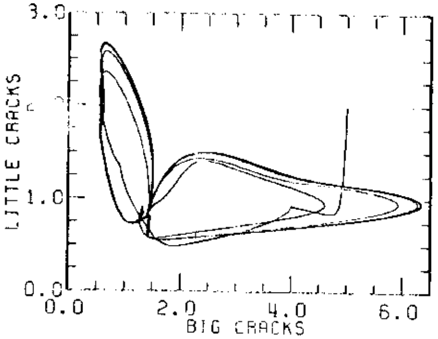

Fig. 7 Crack population evolution. $\tau' = 15.0$. Initial conditions L = 2, B = 5, μ = 2.

drop due to the reduced number of fusion events available to maintain their number. During this time, the tectonic stress begins to accumulate and causes increasing numbers of little cracks to form. These in turn fuse, which in due course results in the occurrence of an episode of seismicity and the stress to drop in consequence. With the diminishing of the little crack population, the cycle repeats itself but with all physical quantitites evolving closer to their equilibrium values than in the previous cycle. In Fig. 2 which is the case $\tau' = 0.25$ (which is just below the critical value of 0.27078), we observe that the cycles decay very slowly to steady state. In Fig. 3, we consider the supercritical case $\tau' = 0.30$ and observe that the B-L phase orbits are now repelled away from the steady state solution and converge asymptotically to a periodic, oscillating solution. In Figs. 4-6, we observe the manner in which the oscillatory solution changes as τ' goes from 2.0 to 5.0 to 15.0. We note that the longer the time delay, the greater the excursion of the orbit in the phase plane from steady state. The period of the phase plane orbit is slightly longer than the time delay τ'. Taken together, these two features reproduce the presumption of many seismologists involved in earthquake forecasting that the greater the elapsed time since a preceding major earthquake, the greater the magnitude of the subsequent event. The cusp-like behavior in the phase plane is symptomatic of a quiescent period where the crack populations are near equilibrium prior to the development of a catastrophic cascade of fusion events that results in a major earthquake. The phase trajectory behavior also exhibits manifestations of precursor and aftershock activity. (With the healing term in (1) set to $-\kappa L$ as in paper I, we have also found a Hopf bifurcation and transition to periodic behavior.) Finally, we found in paper I that the details of the evolution of the phase trajectory depend highly upon the initial conditions in the absence of tectonic stress release. In Fig. 7 as in Fig. 6, we consider $\tau' = 15.0$ but employ as initial conditions L = 2.0, B = 5.0 and $\mu = 2.0$. After the disappearance of the transient, we observe that the phase trajectory is drawn or "attracted" to the same periodic orbit as obtained in Fig. 6, which we characterize as a "limit cycle attractor."

Thus, we conclude for this skeletal model of crack fusion dynamics that the combination of tectonic stress release and time delays (resulting from anelastic creep and/or stress weakening) is critical to the production of repetitive clusters. We plan to investigate nonlocal influences such as nonlinear spatial diffusion and inhomogeneities, as well as increase the number of crack sizes beyond the present level of two, as possible influences on the strictly period episodes of seismicity obtained here and on the occurrence of seismic gaps and clustering.

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