united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR.1071/2

THIRD SCHOOL ON NONLINEAR FUNCTIONAL ANALYSIS AND APPLICATIONS TO DIFFERENTIAL EQUATIONS

(12 - 30 October 1998)

The Maslov-type Index and its Iteration Theory with Applications to Hamiltonian Systems

Yiming Long

Nankai Institute of Mathematics
Nankai University
Tianjin 300071
P.R. China

strada costiera, 11 - 34014 trieste italy - tel. +39 40 2240111 fax +39 40 224163 - sci_info@ictp.trieste.it - www.ictp.trieste.it

The Maslov-type Index and its Iteration Theorey with Applications to Hamiltonian Systems

Yiming Long[1, 2]
Nankai Institute of Mathematics, Nankai University
Tianjin 300071, People's Republic of China

Abstract

In this paper we give an introduction on the Maslov-type index theory for symplectic paths and its iteration theory with applications to existence, multiplicity, minimal period, and stability of periodic solution problems in for nonlinear Hamiltonian systems, and instability of linear Hamiltonian systems, including a survey on recent progresses in this area.

Since the pioneering work of P. Rabinowitz in 1978, topological and variational methods have been widely and deeply applied to the study of nonlinear Hamiltonian systems. On the other hand, as well known Morse theory is a very powerful tool in mathematics. For example, based upon the work [Bo] of R. Bott in 1956 on iteration theory of Morse index, there have been many deep results obtained in the study of closed geodesics on Riemannian manifolds. Therefore in the study of periodic solutions of nonlinear Hamiltonian systems, it is natural to consider applications of the Morse theory. But unfortunately, for functionals on loop space corresponding to Hamiltonian systems, its positive and negative Morse indices are always infinite and usual Morse theory is not directly applicable. For this reason, further understanding and development of possible homotopy invariants for linear Hamiltonian systems as well as for paths in the symplectic matrix group starting from the identity become necessary again. Interests on such invariants started from the earlier works on the stability problems for linear Hamiltonian systems of M. Krein, I. Gelfand, V. Lidskii, J. Moser and others in 1950's (cf. [GL], [Mo], YS]). Since early 1980's, efforts on index theories for Hamiltonian systems have appeared in two different directions. One is the index theory established by I. Ekeland for convex Hamiltonian systems, including its iteration theory with successful applications to various problems on convex Hamiltonian systems (cf. [Ek3] and the reference therein). The other development is the so called Maslov-type index theory for general Hamiltonian systems without any convexity type assumptions, which was defined by C. Conley, E. Zehnder, Y. Long, and C. Viterbo in a sequence of papers [CZ2], [LZ], [Lo1], [Vi2], and [Lo10].

[1] Partially supported by the NNSF and MCSEC of China, and the Qiu Shi Sci. and Tech. Foundation.

[2] Associate Member of the ICTP.

Motivated by the studying of the minimal period, multiplicity, and stability problems of periodic solutions of nonlinear Hamiltonian systems, in recent years we have systematically developed the iteration theory of the Maslov-type index for symplectic paths. This iteration theory unifies the above mentioned iteration theory of Bott and Ekeland, and has turned out to be a powerful tool in the study of various problems of Hamiltonian systems.

In this paper, we give an introduction to this Maslov-type index theory, its iteration theory, and applications to linear and nonlinear Hamiltonian systems, together with a survey on recent progresses in this area.

This paper includes the following parts.

Aknowledgements. This paper is an expanded version of my three lectures given at the "Third School on Nonlinear Functional Analysis and Applications to Differential Euations" held in the ICTP during October 12th to 30th of 1998. It is my pleasure to sincerely thank the organizers of this school, Professors A. Ambrosetti, K. C. Chang, I. Ekeland, and M. S. Narasimhan, for inviting me to give these lectures. I would also like to express my sincere thanks to the ICTP for the supports during my this visit from September to November 1998 as an associate member.

Chapter 1. A Maslov-type index theory for symplectic paths.

Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ be the sets of natural, integral, real, and complex numbers respectively. Let $\mathbf{U}$ be the unit circle in $\mathbf{C}$. As usual for any $n \in \mathbf{N}$, we define the symplectic groups on $\mathbf{R}^{2n}$ by

$$\mathrm{Sp}(2n) = \{M \in \mathcal{L}(\mathbf{R}^{2n}) \,|\, M^T J M = J\},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n denotes the identity matrix on $\mathbf{R}^n$, the subscript n will be omitted when there is no confusion. $\mathcal{L}(\mathbf{R}^{2n})$ is the set of all $2n \times 2n$ real matrices, M^T denotes the transpose of M. The topology of $\mathrm{Sp}(2n)$ is induced from that of $\mathbf{R}^{n^2}$. For $\tau > 0$ and $H \in C^2(S_\tau \times \mathbf{R}^{2n}, \mathbf{R})$ with $S_\tau = \mathbf{R}/(\tau\mathbf{Z})$, we consider the τ-periodic boundary value problem of the following Hamiltonian systems:

$$\dot{x}(t) = JH'(t, x(t)), \tag{1.1}$$

where $H'(t,x)$ denotes the gradient of H with respect to the x variables. Suppose $x = x(t)$ is a τ-periodic solution of (1.1) for some $\tau > 0$. Denote by γ_x the fundamental solution of the linearized Hamiltonian system

$$\dot{y} = JB(t)y, \tag{1.2}$$

where $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$ is defined by $B(t) = H''(t, x(t))$, and $\mathcal{L}_s(\mathbf{R}^{2n})$ is the subset of symmetric matrices in $\mathcal{L}(\mathbf{R}^{2n})$. Then γ_x is a path in $\mathrm{Sp}(2n)$ starting from the idetity matrix I. Based upon the work [AZ] of H. Amann and E. Zehnder in 1980 on the index theory for linear Hamiltonian systems with constant coefficients, C. Conley and E. Zehnder in their celebrated paper [CZ1] of 1984 defined their index theory for nondegenerate paths in the symplectic matrix group $\mathrm{Sp}(2n)$ started from the identity when $n \ge 2$. This index theory was extended to nondegenerate paths in $\mathrm{Sp}(2)$ by the author and E. Zehnder in [LZ] of 1990. Then C. Viterbo in [Vi2] and the author in [Lo1] of 1990 extended this index theory to degenerate symplectic paths which are fundamental solutions of linear Hamiltonian systems with continuous symmetric periodic coefficients. In the work [Lo10], the author further extended this index theory to all continuous degenerate paths in $\mathrm{Sp}(2n)$ for all $n \ge 1$ and gave an axiom characterization of this index theory. We call this index theory the Maslov-type index theory in this paper. The Maslov-type index theory assignes a pair of numerical invariants to the periodic solution x through the associated path γ_x in $\mathrm{Sp}(2n)$ and reflects important properties of the periodic solution x.

§1. Definitions and basic properties.

We start from some notations introduced in [CZ2], [LZ], [Lo1], [Lo10], and [DL]. Define

$$D_1(M) = (-1)^{n-1}\det(M - I), \qquad \forall M \in \mathrm{Sp}(2n).$$

Let

$$\mathrm{Sp}(2n)^{\pm} = \{M \in \mathrm{Sp}(2n) \mid \pm D_1(M) < 0\},$$
$$\mathrm{Sp}(2n)^{*} = \mathrm{Sp}(2n)^{+} \cup \mathrm{Sp}(2n)^{-}, \qquad \mathrm{Sp}(2n)^{0} = \mathrm{Sp}(2n)\backslash\mathrm{Sp}(2n)^{*}.$$

For any two matrices of square block form:

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

the $\diamond$-product of M_1 and M_2 is defined by the $2(i+j)\times 2(i+j)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M\diamond\cdots\diamond M$. Note that the $\diamond$-multiplication is associative, and the $\diamond$-product of any two symplectic matrices is symplectic.

We define $D(a) = \mathrm{diag}(a, a^{-1})$ for $a \in \mathbf{R}\backslash\{0\}$. For θ, λ, and $b \in \mathbf{R}$ we define

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}.$$

Define two $2n \times 2n$ diagonal matrices

$$M_n^{+} = D(2)^{\diamond n}, \qquad M_n^{-} = D(-2)\diamond D(2)^{\diamond(n-1)}.$$

Lemma 1.1. (cf. [CZ2], [LZ], and [SZ2]) 1^o $\mathrm{Sp}(2n)^{*}$ *contains two path connected components* $\mathrm{Sp}(2n)^{+}$ *and* $\mathrm{Sp}(2n)^{-}$, *and there hold* $M_n^{\pm} \in \mathrm{Sp}(2n)^{\pm}$.

2^o *Both of* $\mathrm{Sp}(2n)^{+}$ *and* $\mathrm{Sp}(2n)^{-}$ *are simply connected in* $\mathrm{Sp}(2n)$.

Idea of the proof. Since $D_1(M_n^{+})D_1(M_n^{-}) < 0$, $\mathrm{Sp}(2n)^{*}$ contains at least two path-connected components.

For any given $M \in \mathrm{Sp}(2n)^{*}$, by a small perturbation we can connect M to a matrix M_1 with only simple eigenvalues within $\mathrm{Sp}(2n)^{*}$. Then there holds

$$PMP^{-1} = M_1\diamond\cdots\diamond M_p\diamond N_1\diamond\cdots\diamond N_q = N,$$

where $P \in \mathrm{Sp}(2n)$, $M_i \in \mathrm{Sp}(2)$ for $1 \le i \le p$ and $N_j \in \mathrm{Sp}(4)$ for $1 \le j \le q$, each M_I ahs the form $R(\theta)$ with $\theta \in (0,\pi)\cup(\pi,2\pi)$ or $D(a)$ with $a \in \mathbf{R}\backslash\{0\}$, each N_j has four simple eigenvalues λ_j, $\overline{\lambda_j}$, λ_j^{-1}, and $\overline{\lambda}^{-1}$ out side $\mathbf{R}\cup\mathbf{U}$.

By connecting P to I in $\mathrm{Sp}(2n)$, we get that M can be connected to N within $\mathrm{Sp}(2n)^{2n}$. Then it can be proved that these M_i's and N_j's can be connected to $D(2)$, $D(-2)$, or their $\diamond$-products within $\mathrm{Sp}(2)^{*}$ or $\mathrm{Sp}(4)^{*}$. Note that $D(-2)\diamond D(-2)$ can be connected to $D(2)\diamond D(2)$. This proves that N can be connected to one of M_n^{+} and M_n^{-} within $\mathrm{Sp}(2n)^{*}$. Then $\mathrm{Sp}(2n)^{*}$ contains at most two path connected components, and 1^o is proved.

We refer the readers to [SZ2] for the proof of 2^o. ∎

Fix $\tau > 0$. Let

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I\},$$
$$\mathcal{P}_\tau^*(2n) = \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(\tau) \in \mathrm{Sp}(2n)^*\},$$
$$\mathcal{P}_\tau^0(2n) = \mathcal{P}_\tau(2n) \backslash \mathcal{P}_\tau^*(2n).$$

The topology of $\mathcal{P}_\tau(2n)$ is defined by the $C^0([0,\tau], \mathrm{Sp}(2n))$-topology induced from the topology of $\mathrm{Sp}(2n)$. Note that the following subset of $\mathcal{P}_\tau(2n)$ consists of all foundamental solutions of linear Hamiltonian systems (1.2) with symmetric continuous and τ-periodic coefficients:

$$\hat{\mathcal{P}}_\tau(2n) = \{\gamma \in C^1([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I, \, \dot{\gamma}(1) = \dot{\gamma}(0)\gamma(1)\}.$$

The topology of $\hat{\mathcal{P}}_\tau(2n)$ is defined to be the $C^1([0,\tau], \mathrm{Sp}(2n))$-topology induced from the topology of $\mathrm{Sp}(2n)$.

Definition 1.2. (cf. [Lo1]) *For every $\gamma \in \mathcal{P}_\tau(2n)$, we define*

$$\nu_\tau(\gamma) = \dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) - I).$$

Definition 1.3. (cf. [Lo1]) *Given two paths γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2n)$, if there is a map $\delta \in C([0,1] \times [0,\tau], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot)$, $\delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I$, and $\nu_\tau(\delta(s,\cdot))$ is constant for $0 \le s \le 1$, then γ_0 and γ_1 are* **homotopic on** $[0,\tau]$ **along** $\delta(\cdot,\tau)$ *and we write $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $\delta(\cdot,\tau)$. This homotopy possesses fixed end points if $\delta(s,\tau) = \gamma_0(\tau)$ for all $s \in [0,1]$.*

As well known, every $M \in \mathrm{Sp}(2n)$ has its unique polar decomposition $M = AU$, where $A = (MM^T)^{1/2}$ is symmetric positive definite and symplectic, U is orthogonal and symplectic. Therefore U has the form

$$U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix},$$

where $u = u_1 + \sqrt{-1}u_2 \in \mathcal{L}(\mathbf{C}^n)$ is a unitary matrix. So for every path $\gamma \in \mathcal{P}_\tau(2n)$ we can associate a path $u(t)$ in the unitary group on $\mathbf{C}^n$ to it. If $\Delta(t)$ is any continuous real function satisfying $\det u(t) = \exp(\sqrt{-1}\Delta(t))$, the difference $\Delta(\tau) - \Delta(0)$ depends only on γ but not on the choice of the function $\Delta(t)$. Therefore we may define the mean rotation number of γ on $[0,\tau]$ by

$$\Delta_\tau(\gamma) = \Delta(\tau) - \Delta(0).$$

Lemma 1.4. (cf. [LZ]) *If γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2n)$ possesse common end point $\gamma_0(\tau) = \gamma_1(\tau)$, then $\Delta_\tau(\gamma_0) = \Delta_\tau(\gamma_1)$ if and only if $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ with fixed end points.* ∎

By Lemma 1.1, for every path $\gamma \in \mathcal{P}_\tau^*(2n)$ there exists a path $\beta : [0,\tau] \to \mathrm{Sp}(2n)^*$ such that $\beta(0) = \gamma(\tau)$ and $\beta(\tau) = M_n^+$ or M_n^-. Define the product path $\beta * \gamma$ by

$$\beta * \gamma(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{\tau}{2}, \\ \beta(2t-\tau), & \frac{\tau}{2} < t \le \tau. \end{cases}$$

Then $k \equiv \Delta_\tau(\beta * \gamma)/\pi \in \mathbf{Z}$ and is independent of the choice of the path β by 2^o of Lemma 1.1. In this case we write $\gamma \in \mathcal{P}^*_{\tau,k}(2n)$.

Lemma 1.5. (cf. [LZ]) *These $\mathcal{P}^*_{\tau,k}(2n)$'s give a homotopy classification of $\mathcal{P}^*_\tau(2n)$.*

Definition 1.6. (cf. [CZ2], [LZ]) *If $\gamma \in \mathcal{P}^*_{\tau,k}(2n)$, we define $i_\tau(\gamma) = k$.*

We define the standard non-degenerate symplectic paths by

$$\hat{\alpha}_{1,0,\tau}(t) = D(1 + \frac{t}{\tau}), \qquad \text{for } 0 \le t \le \tau,$$
$$\hat{\alpha}_{1,k,\tau} = (D(2)\phi_{k\pi,\tau}) * \hat{\alpha}_{1,0,\tau}, \quad \forall k \in \mathbf{Z}\backslash\{0\},$$

where $\phi_{\theta,\tau}(t) = R((\theta t/\tau)$. When $n \ge 2$, we define

$$\hat{\alpha}_{n,0,\tau} = (\hat{\alpha}_{1,0,\tau})^{\diamond n},$$
$$\hat{\alpha}_{n,k,\tau} = ((D(2)\phi_{k\pi,\tau}) * \hat{\alpha}_{1,0,\tau})\diamond(\hat{\alpha}_{1,0,\tau})^{\diamond(n-1)}, \quad \forall k \in \mathbf{Z}\backslash\{0\},$$

Then there hold

$$\hat{\alpha}_{n,k,\tau} \in \mathcal{P}^*_{\tau,k}(2n), \qquad \forall k \in \mathbf{Z}.$$

The following lemma is crucial in the study of degenerate symplectic paths.

Lemma 1.7. (cf. [Lo1], [Lo10]) *For any $\gamma \in \mathcal{P}^0_\tau(2n)$, there exists a one parameter family of symplectic paths γ_s with $s \in [-1,1]$ and a $t_0 \in (0,\tau)$ sufficiently close to τ such that*

$$\gamma_0 = \gamma, \quad \gamma_s(t) = \gamma(t) \ \ for \ \ 0 \le t \le t_0, \tag{1.3}$$
$$\gamma_s \in \mathcal{P}^*_\tau(2n) \ \ \forall s \in [-1,1]\backslash\{0\}, \tag{1.4}$$
$$i_\tau(\gamma_s) = i_\tau(\gamma_{s'}), \quad if \ \ ss' > 0, \tag{1.5}$$
$$i_\tau(\gamma_1) - i_\tau(\gamma_{-1}) = \nu_\tau(\gamma), \tag{1.6}$$
$$\gamma_s \to \gamma_0 = \gamma \quad in \ \ \mathcal{P}_\tau(2n) \quad as \ \ s \to 0. \tag{1.7}$$

When $\gamma \in \hat{P}_\tau(2n)$, we also have

$$\gamma_s \in \hat{P}_\tau(2n) \qquad \forall s \in [-1,1], \tag{1.8}$$
$$\gamma_s \to \gamma \quad in \ \ \hat{\mathcal{P}}_\tau(2n) \quad as \ \ s \to 0. \tag{1.9}$$

Idea of the proof. Among these properties of $\{\gamma_s\}$, the most important one is (1.5). The construction of $\{\gamma_s\}$ uses the results on normal forms of symplectic matrices proved in [LD] and [HL]. Here we briefly indicate how this family of paths $\{\gamma_s\}$ is constructed.

For every integer m, $1 \le m \le n$, and $\theta \in \mathbf{R}$, a $2n \times 2n$ rotation matrix $R_m(\theta) = (r_{i,j})$ is defined in [Lo1] and [Lo2] by

$$\begin{cases} r_{m,m} &= & r_{n+m,n+m} & = \cos\theta, \\ r_{n+m,m} &= & -r_{m,n+m} & = \sin\theta, \\ r_{i,i} &= & 1, & \text{if } i \ne m \text{ and } n+m, \\ r_{i,j} &= & 0, & \text{otherwise.} \end{cases}$$

Fix $\gamma \in \mathcal{P}_\tau^0(2n)$. Then there exist an integer q, $1 \le q \le n$, a strictly increasing subsequence $\{m_1, \ldots, m_q\}$ of $\{1, \ldots, n\}$, $\theta_0 \in (0, \frac{\pi}{8n})$ small enough depending on $\gamma(\tau)$, and $P \in \mathrm{Sp}(2n)$ such that for $i = 1, \ldots, q$ the m_i is the least positive integer which satisfies for $0 < |\theta| \le \theta_0$:

$$\begin{aligned} &\dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) P R_{m_1}(\theta) \cdots R_{m_{i-1}}(\theta) P^{-1} - I) \\ &\qquad - \dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) P R_{m_1}(\theta) \cdots R_{m_{i-1}}(\theta) R_{m_i}(\theta) P^{-1} - I) \ge 1, \\ &\dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) P R_{m_1}(\theta) \cdots R_{m_q}(\theta) P^{-1} - I) = 0. \end{aligned}$$

Here we set $R_{m_0}(\theta) \equiv I$. Note that the integers q, m_1, ..., m_q, and P are determined by the normal form of the matrix $\gamma(\tau)$.

For $t_0 \in (0, \tau)$, let $\rho \in C^2([0, \tau], [0, 1])$ such that $\rho(t) = 0$ for $0 \le t \le t_0$, $\dot\rho(t) \ge 0$ for $0 \le t \le \tau$, $\rho(\tau) = 1$, and $\dot\rho(\tau) = 0$. For any $(s, t) \in [-1, 1] \times [0, \tau]$, the path γ_s is defined by

$$\gamma_s(t) = \gamma(t) P R_{m_1}(s\rho(t)\theta_0) \cdots R_{m_q}(s\rho(t)\theta_0) P^{-1}. \tag{1.10}$$

When $t_0 \in (0, \tau)$ is sufficiently close to τ, the properties (1.3) to (1.9) hold. ∎

With lemma 1.7, we can give

Definition 1.8. (cf. [Lo1]) *Define* $i_\tau(\gamma) = i_\tau(\gamma_s)$ *for* $s \in [-1, 0)$.

Definition 1.9. *For every path* $\gamma \in \mathcal{P}_\tau(2n)$*, the definitions 1.2, 1.6 and 1.8 assign a pair of integers*

$$(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbf{Z} \times \{0, \ldots, 2n\}$$

to it. This pair of integers is called the **Maslov-type index** *of* γ*. When* $\gamma = \gamma_x$ *for a solution* x *of (1.1), we also write*

$$(i_\tau(x), \nu_\tau(x)) = (i_\tau(\gamma_x), \nu_\tau(\gamma_x)).$$

The following theorem shows that the Definition 1.8 of $i_\tau(\gamma)$ for $\gamma \in \mathcal{P}_\tau^0(2n)$ is independent from the way which is defined.

Theorem 1.10. (cf. [Lo1], [Lo10]) *For any* $\gamma \in \mathcal{P}_\tau^0(2n)$*, and every* $\beta \in \mathcal{P}_\tau^*(2n)$ *which is sufficiently close to* γ*, there holds*

$$i_\tau(\gamma) = i_\tau(\gamma_{-1}) \le i_\tau(\beta) \le i_\tau(\gamma_1) = i_\tau(\gamma) + \nu_\tau(\gamma). \tag{1.11}$$

Specially we obtain

$$i_\tau(\gamma) = \inf\{i_\tau(\beta) \mid \beta \in \mathcal{P}_\tau^*(2n) \text{ and } \beta \text{ is sufficiently close to } \gamma \text{ in } \mathcal{P}_\tau(2n)\}. \tag{1.12}$$

Idea of the proof of (1.11). Fir stly we reduce the general case to the case of that all the paths in consideration are in $\hat{\mathcal{P}}_\tau(2n)$. Then the later case can be proved by using Theorem 2.1 below and a perturbation argument on the Morse index for finite dimensional symmetric matrices. ∎

The following theorem characterizes the Maslov-type index on any continuous symplectic paths in $\mathcal{P}_\tau(2n)$.

Theorem 1.11.(cf. [Lo10]) *The Maslov-type index* $i_\tau : \cup_{n\in\mathbf{N}}\mathcal{P}_\tau(2n) \to \mathbf{Z}$, *is uniquely determined by the following five axioms:*

1° **(Homotopy invariant)** *For* γ_0 *and* $\gamma_1 \in \mathcal{P}_\tau(2n)$, *if* $\gamma_0 \sim \gamma_1$ *on* $[0,\tau]$, *then*

$$i_\tau(\gamma_0) = i_\tau(\gamma_1). \tag{1.13}$$

2° **(Symplectic additivity)** *For any* $\gamma_i \in \mathcal{P}_\tau(2n_i)$ *with* $i = 0$ *and* 1, *there holds*

$$i_\tau(\gamma_0 \diamond \gamma_1) = i_\tau(\gamma_0) + i_\tau(\gamma_1). \tag{1.14}$$

3° **(Clockwise continuity)** *For any* $\gamma \in \mathcal{P}_\tau^0(2)$ *with* $\gamma(\tau) = N_1(1,b)$ *for* $b = \pm 1$ *or* 0, *there exists a* $\theta_0 > 0$ *such that*

$$i_\tau([\gamma(\tau)\phi_{-\theta,\tau}] * \gamma) = i_\tau(\gamma), \qquad \forall\, 0 < \theta \le \theta_0. \tag{1.15}$$

4° **(Counterclockwise jumping)** *For any* $\gamma \in \mathcal{P}_\tau^0(2)$ *with* $\gamma(\tau) = N_1(1,b)$ *for* $b = \pm 1$, *there exists a* $\theta_0 > 0$ *such that*

$$i_\tau([\gamma(\tau)\phi_{\theta,\tau}] * \gamma) = i_\tau(\gamma) + 1, \qquad \forall\, 0 < \theta \le \theta_0. \tag{1.16}$$

5° **(Normality)** *For the standard path* $\hat{\alpha}_{1,0,\tau}$, *there holds*

$$i_\tau(\hat{\alpha}_{1,0,\tau}) = 0. \tag{1.17}$$

Idea of the proof. Using normal forms and perturbation techniques together with the properties 1° and 2° to reduce the uniqueness to the case of paths in $\mathcal{P}_\tau(2)$. Then it follows from the $\mathbf{R}^3$-cylindrical coordinate representation introduecd in the section 3 below immediately. The proof for the sufficiency can be found in [Lo11]. ∎

The following theorem is very useful in the study of the iteration theory for the Maslov-type index.

Theorem 1.12. (Inverse homotopy invariant) (cf. [Lo10]) *For any two paths* γ_0 *and* $\gamma_1 \in \mathcal{P}_\tau(2n)$ *with* $i_\tau(\gamma_0) = i_\tau(\gamma_1)$, *suppose that there exists a continuous path* $h : [0,1] \to \mathrm{Sp}(2n)$ *such that* $h(0) = \gamma_0(\tau)$, $h(1) = \gamma_1(\tau)$, *and* $\dim\ker(h(s) - I) = \nu_\tau(\gamma_0)$ *for all* $s \in [0,1]$. *Then* $\gamma_0 \sim \gamma_1$ *on* $[0,\tau]$ *along* h.

Idea of the proof. Note that $\gamma_0 \sim (h * \gamma_0)$. Since $(h * \gamma_0) \sim \gamma_1$ and γ_1 have the same end points and index, they must be homotopic. This proves the theorem. ∎

§2. Maslov-type index and Morse index.

Fix $\tau > 0$. Suppose $H \in C^2(S_\tau \times \mathbf{R}^{2n}, \mathbf{R})$ and $\|H\|_{C^2}$ is finite. Recall $S_\tau = \mathbf{R}/(\tau\mathbf{Z})$. The classical direct functional corresponding to the system (1.1) is

$$f(x) = \int_0^\tau (-\frac{1}{2}J\dot{x}\cdot x - H(t,x))dt, \tag{2.1}$$

for $x \in \text{dom}(A) \subset L_\tau \equiv L^2(S_\tau, \mathbf{R}^{2n})$ with $A = -J\frac{d}{dt}$. It is well-known that critical points of f on L_τ and τ-periodic solutions of (1.1) are one-to-one correspondent. The Morse indices of f at its critical point x is defined by those of the following quadratic form on L_τ:

$$\phi(y) = \int_0^\tau (-J\dot{y} \cdot y - B(t)y \cdot y)dt, \tag{2.2}$$

where $B(t) = H''(t, x(t))$. Note that the positive and negative Morse indices of f at its critical point x, i.e. the total multiplicities of positive and negative eigenvalues of the quadratic form (2.2), are always infinite. Using the saddle point reduction method on the space L_τ (cf. [AZ] and [Ch2]), we obtain a finite dimensional subspace $Z \subset L_\tau$ consisting of finite Fourier polynomials with $2d = \dim Z$ being sufficiently large, an injective map $u : Z \to L_\tau$ and a functional $a : Z \to \mathbf{R}$, such that there holds

$$a(z) = f(u(z)), \qquad \forall z \in Z, \tag{2.3}$$

and that the critical points of a and f are one to one correspondent. Note that the following important result holds.

Theorem 2.1. (cf. [CZ2], [LZ], [Lo1], [Lo10]) *Under the above conditions, let z be a critical point of a and $x = u(z)$ be the corresponding solution of the system (1.1). Denote the Morse indices of the functional a at z by $m^*(z)$ for $* = +, 0, -$. Then the Maslov-type index $(i_\tau(x), \nu_\tau(x))$ satisfy*

$$m^-(z) = d + i_\tau(x), \quad m^0(z) = \nu_\tau(x), \quad m^+(z) = d - i_\tau(x) - \nu_\tau(x). \tag{2.4}$$

Idea of the proof. 1^o For the non-degenerate case with $n \geq 2$ or $n = 1$ and $i_\tau(x) \in (2\mathbf{Z} + 1) \cup \{0\}$ as in [CZ2] , it suffices to use the homotopy invariance of the Maslov-type index to reduce the computation of the indices to the case of liner Hamiltonian systems with constant coefficients.

2^o For the non-degenerate case with $n = 1$, we first couple the given linearized Hamiltonian system $\mathcal{H}_0$ with a linear Hamiltonian system $\mathcal{H}_0$ on $\mathbf{R}^2$ with constant coefficeint and Maslov-type index 1 to get a new linear Hamiltonian system $\mathcal{H}_2$ on $\mathbf{R}^4$. Then the index formula (2.4) for $\mathcal{H}_0$ follows from that for $\mathcal{H}_1$ subtract from that of $\mathcal{H}_2$.

3^o For the degenerate case, use the paths γ_s and perturbation techniques to reduce the problem to the comparison of non-degenerate cases of γ_1 and γ_{-1}. ∎

Note that from (2.4), the Maslov-type indices can be viewed as a finite representation of the infinite Morse indices of the direct variational formulations. Note also that for general Hamiltonian H whose second derivative may not be bounded, results similar to Theorem 2.1 was proved via Galerkin approximations in [FQ] by G. Fei and Q. J. Qiu.

Next we consider the periodic problem of the calculus of variation, i.e. finding extremal loops of the following functional

$$F(x) = \int_0^\tau L(t, x, \dot{x})dt, \qquad \forall x \in W_\tau = W^{1,2}(S_\tau, \mathbf{R}^n). \tag{2.5}$$

Here we suppose $\tau > 0$ and $L \in C^2(S_\tau \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R})$ such that $L_{p,p}(t,x,p)$ is symmetric and positive definite, and $L_{x,x}(t,x,p)$ is symmetric. An extremal loop x of F corresponds to a 1-periodic solution of the Lagrangian system

$$\frac{d}{dt}L_p(t,x,\dot{x}) - L_x(t,x,\dot{x}) = 0. \tag{2.6}$$

Fix such an extremal loop x, define

$$P(t) = L_{p,p}(t,x(t),\dot{x}(t)), \quad Q(t) = L_{x,p}(t,x(t),\dot{x}(t)), \quad R(t) = L_{x,x}(t,x(t),\dot{x}(t)). \tag{2.7}$$

The Hessian of F at x corresponds to a linear periodic Sturm system,

$$-(P\dot{y} + Qy)^{\cdot} + Q^T\dot{y} + Ry = 0. \tag{2.8}$$

It corresponds to the linear Hamiltonian system (1.2) with

$$B(t) \equiv B_x(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) \end{pmatrix}. \tag{2.9}$$

Denote by γ_x the fundamental solution of this linearized Hamiltonian system (1.2). The Morse index and nullity of the functional F at an extremal loop x in W_τ are always finite. We denote them by $m^-(x)$ and $m^0(x)$ respectively.

Theorem 2.2. (cf. [Vi2], [LA], [AL]) *Under the above conditions, there hold*

$$m^-(x) = i_\tau(\gamma_x), \qquad m^0(x) = \nu_\tau(\gamma_x). \tag{2.10}$$

Idea of the proof. We apply the index theory of [Du]. Using the homotopy invariance properties of this index theory and the Maslov-type index theory to simple standard cases, then (2.10) is proved by concrete computations on these simple cases. ∎

Remark 2.3. Note that in the sense of Theorems 2.1 and 2.2, our Definition 1.9 of the Maslov-type index is natural.

§3. An intuitive explanation of the Maslov-type index theory for symplectic paths in Sp(2).

At the last part of this section, we give an intuitive interpretation of the Maslov-type index theory in terms of the cylinderical coordinate representation in $\mathbf{R}^3$ of Sp(2) firstly introduced in [Lo2] of 1991 by the author as follows. As well known, $M \in \mathrm{Sp}(2)$ if and only if $\det M = 1$. Via the polar decomposition of each element M in Sp(2),

$$M = \begin{pmatrix} r & z \\ z & (1+z^2)/r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{3.1}$$

we can define a map Φ from the element M in Sp(2) to $(r, \theta, z) \in \mathbf{R}^+ \times S_{2\pi} \times \mathbf{R}$, where $\mathbf{R}^+ = \{r \in \mathbf{R} \,|\, r > 0\}$. This map Φ is a C^∞-diffeomorphism. In the following, for simplicity, we identify elements in Sp(2) and their images in $\mathbf{R}\backslash\{z\text{−axis}\}$ under Φ.

Remark 3.1. Note that a different representation of Sp(2) was given by I. Gelfand and V. Lidsikii in [GL] of 1955 which was based on the hyperbolic functions and which maps Sp(2) into a solid torus.

By this $\mathbf{R}^3$-cylindrical coordinate representation of Sp(2), it is easy to see that Sp(2) is homeomorphic to $S^1 \times \mathbf{R}^2$. This can be generalized to general Sp($2n$) which is homeomorphic to a product of S^1 and a simply connected space. Therefore any path $\gamma \in \mathcal{P}_\tau(2)$ rotates around the deleted z-axis in $\mathbf{R}^3$ in someway. There are infinitely many topologically meaningful ways to count the rotation number of γ. But the key point here is to find a natural way to count this rotation so that the rotation number reflects intrinsically analytical properties of the corresponding Hamiltonian system when $\gamma \in \hat{\mathcal{P}}_\tau(2)$.

Under this $\mathbf{R}^3$-cylindrical coordinate representation we have

$$\begin{aligned}
\mathrm{Sp}(2)^+ &= \{(r,\theta,z) \in \mathbf{R}^+ \times S^1 \times \mathbf{R} \,|\, (r^2+z^2+1)\cos\theta > 2r\}, \\
\mathrm{Sp}(2)^0 &= \{(r,\theta,z) \in \mathbf{R}^+ \times S^1 \times \mathbf{R} \,|\, (r^2+z^2+1)\cos\theta = 2r\}, \\
\mathrm{Sp}(2)^- &= \{(r,\theta,z) \in \mathbf{R}^+ \times S^1 \times \mathbf{R} \,|\, (r^2+z^2+1)\cos\theta < 2r\}, \\
\mathrm{Sp}(2)^0_\pm &= \{(r,\theta,z) \in \mathrm{Sp}(2)^0 \,|\, \pm\sin\theta > 0\} = \left\{PN_1(1,\mp 1)P^{-1} \,|\, P \in \mathrm{Sp}(2)\right\}, \\
\mathrm{Sp}(2)^* &= \mathrm{Sp}(2)^+ \cup \mathrm{Sp}(2)^-, \quad \mathrm{Sp}(2)^0 = \mathcal{M}_2^1 \cup \{I\}, \quad \mathcal{M}_2^1 = \mathrm{Sp}(2)^0_+ \cup \mathrm{Sp}(2)^0_-.
\end{aligned}$$

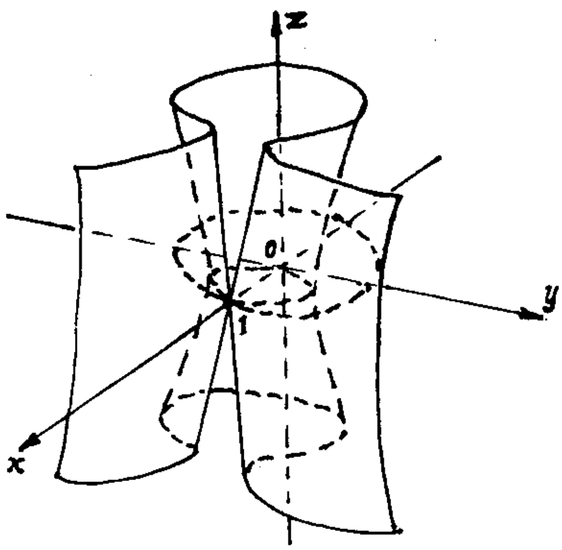

Figure 3.1. $\mathrm{Sp}(2)^0$ in cylindrical coordinates of $\mathbf{R}^3\backslash\{z-\text{axis}\}$.

Note that $\mathrm{Sp}(2)^0$ is a codimension 1 hypersurface in Sp(2), $\mathcal{M}_2^1$ is its regular part. Note that $\mathcal{M}_2^1$ contains two path connected components $\mathrm{Sp}(2)^0_+$ and $\mathrm{Sp}(2)^0_-$, which are two smooth surfaces both diffeomorphic to $\mathbf{R}^2\backslash\{0\}$ as shown in the Figure 3.1. The following Figure 3.2 shows the picture of $\mathrm{Sp}(2)^0 \cap \{z = 0\}$.

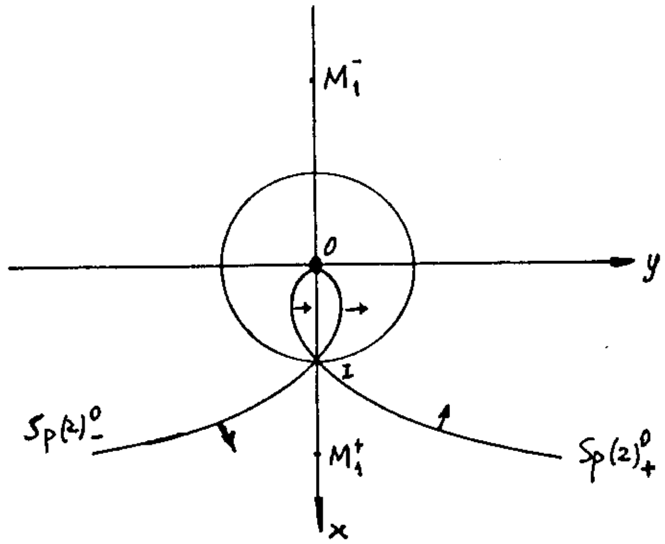

Figure 3.2. The intersection of $\mathrm{Sp}(2)^0$ and the plane $\{z = 0\}$.

Note that for the case of Sp(2), Lemma 1.1 follows from these two pictures immediately.

Now based upon the standard non-degenerate symplectic paths defined in the section 1, from Figures 3.1 and 3.2, it is obvious that for any given $\gamma \in \mathcal{P}_\tau^*(2)$, there exists one and only one $k \in \mathbf{Z}$ such that

$$\gamma \sim \hat{\alpha}_{1,k,\tau}.$$

This proves Lemma 1.5 for the case of $n = 1$, and then makes the Definition 1.6 become meaningful.

Now for $\gamma \in \mathcal{P}_\tau^0(2)$, from Figures 3.1 and 3.2, we immediately obtain the following results:

If $\gamma(\tau) \in \mathrm{Sp}(2)^0 \backslash \{I\}$, all paths $\beta \in \mathcal{P}_\tau^*(2)$ which are C^0-close to γ belong to two homotopy classes, one contains γ_{-1} and the other contains γ_1 defined by (1.10), and there holds

$$i_\tau(\gamma_{-1}) + 1 = i_\tau(\gamma_1).$$

If $\gamma(\tau) = I$, all paths $\beta \in \mathcal{P}_\tau^*(2)$ which are C^0-close to γ belong to three homotopy classes, one contains γ_{-1}, and another one contains γ_1 defined by (1.10). We pick up a path β in the third homotopy class. Then there holds

$$i_\tau(\gamma_{-1}) + 2 = i_\tau(\beta) + 1 = i_\tau(\gamma_1).$$

These results shows that the following definition (1.10) makes sense:

$$i_\tau(\gamma) = \inf\{i_\tau(\beta) \,|\, \beta \in \mathcal{P}_\tau^*(2) \textit{ and } \beta \textit{ is sufficiently close to } \gamma \textit{ in } \mathcal{P}_\tau(2)\}.$$

Note that the definition of the Maslov-type inedx for non-degenerate paths can also be defined via the algebraic topological intersection theory as follows. Let us give an orientation to Sp(2) as shown in Figure 3.2. Suppose $\tau > 0$. Let $\gamma \in \mathcal{P}_\tau^*(2)$ be a C^1-path. We define the positive orientation of γ at $\gamma(t)$ to be the direction of $\dot{\gamma}(t)$. Then it is easy to see that the Maslov-type index of γ can be represented by the following intersection number:

$$i_\tau(\gamma) = [\mathrm{Sp}(2)^0 : (\hat{\alpha}_{1,0,\tau})^{-1} * \gamma], \tag{3.2}$$

where $(\hat{\alpha}_{1,0,\tau})^{-1}(t) = D(2 - t/\tau)$ for $0 \le t \le \tau$. In general, for any $\gamma \in \mathcal{P}_\tau^*(2)$, it is easy to see that all C^1-paths in $\mathcal{P}_\tau^*(2)$ homotopic to γ must possess the same intersection number (3.2). Thus it can be used as the definition of $i_\tau(\gamma)$. The situation is illustrated in Figure 3.3.

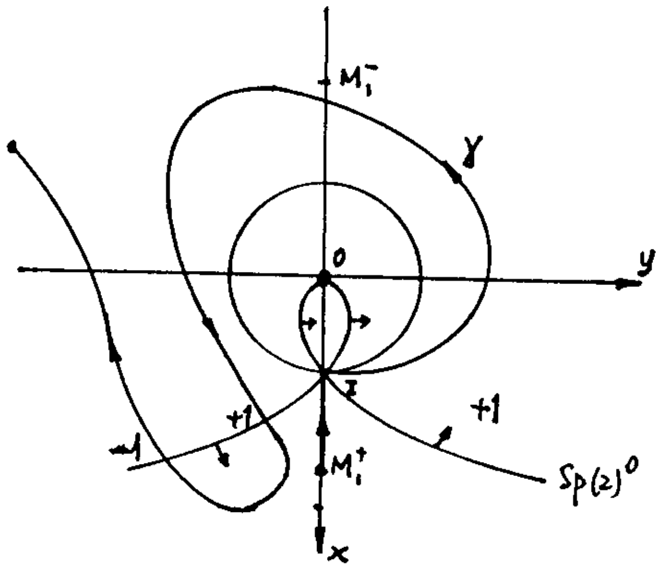

Figure 3.3. Intersection number definition of the Maslov-index theory.

This method can also be used to define the Maslov-type index theory for paths in Sp$(2n)$ starting from the identity with general positive integer n. The key point is to prove that Sp$(2n)^0$ is a codimension 1 cycle in Sp$(2n)$ and possesses a natural orientation which can be used to define the required index theory. This has been done by C. Zhu and the author in [LZh1].

Chapter 2. Iteration Theory of the Maslov-type Index.

For $\tau > 0$ and any $\gamma \in \mathcal{P}_\tau(2n)$, the iteration path $\tilde{\gamma} \in C([0,+\infty), \mathrm{Sp}(2n))$ of γ is defined by

$$\tilde{\gamma}(t) = \gamma(t - j\tau)\gamma(\tau)^j, \qquad \text{for } j\tau \le t \le (j+1)\tau \text{ and } j \in \{0\} \cup \mathbf{N},$$

and $\gamma^m = \tilde{\gamma}|_{[0,m\tau]}$ for all $m \in \mathbf{N}$. Then we can associate to γ through $\tilde{\gamma}$ a sequence of Maslov-type indices

$$\{(i_{m\tau}(\tilde{\gamma}), \nu_{m\tau}(\tilde{\gamma}))\}_{m\in\mathbf{N}}.$$

When $\gamma : [0,+\infty) \to \mathrm{Sp}(2n)$ is the fundamental soluton of (1.2) with $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$, where $\mathcal{L}_s(\mathbf{R}^{2n})$ is the set of symmetric $2n \times 2n$ real matrices, there holds $(\gamma|_{[0,\tau]})^\sim = \gamma$. When x is a τ-periodic solution of (1.1), we define the iterations of x by

$$x^m(t) = x(t - j), \qquad \forall j \le t \le j+1, j = 0, 1, \ldots, m-1.$$

Denotes by

$$(i_{m\tau}(x^m), \nu_{m\tau}(x^m)) = (i_{m\tau}(\tilde{\gamma}_x), \nu_{m\tau}(\tilde{\gamma}_x)).$$

Thus the corresponding index sequence with $\gamma = \gamma_x$ reflects important properties of the τ-periodic solution x of the Hamiltonian system (1.1).

In the celebrated work [Bo] of R. Bott in 1956 as well as [BTZ], the iteration theory of Morse index for closed geodesics is established. In the works of I. Ekeland (cf. [Ek1]-[Ek3]) the iteration theory of his index for convex Hamiltonian systems is established. In [Vi1] of C. Viterbo, the iteration theory for an index theory of nondegenerate star-shaped Hamiltonian systems is established. But for our purpose in the study of general Hamiltonian systems, for example Hamiltonian systems defined on a $2n$-dimensional torus, all these results are not applicable. The only paper we know which studied certain iteration properties of certain Maslov index in such a generality is [CD] of R. Cushman and J. Duistermaat in 1977. But their result is not good enough for our purposes and contains some flaws in certain cases. Specially, Bott-type formulae and mean index of the Maslov-type index theory are still unknown.

Motivated by the study of minimal period problem, multiplicity problem, and stability problems for nonlinear Hamiltonian systems with no any convexity assumptions, we need to establish the Bott-type formulae and certain sharp enough increasing inequalities for the Maslov-type index theory of symplectic paths. This study ends up to our iteration theory of the Maslov-type index established in [DL], [Lo11]-[Lo15], and [LL3] which is introduced in this chapter. Note that in the recent [LZh1], [LZh2], and [ZL], this iteration theory has been established via the spectrul flow method for paths in $\mathrm{Sp}(2n, \mathbf{C})$.

§4. The ω-index theory and splitting numbers.

As we have mentioned in the Section 1, the Maslov-type index theory is defined via the singular hypersurface $\mathrm{Sp}(2n)^0$ in $\mathrm{Sp}(2n)$. This hypersurface is formed by all matrices in $\mathrm{Sp}(2n)$ which possesses 1 as its eigenvalues. In the study of the iteration properties of the Malsov-type index theory, as in [Lo13] for any $\omega \in \mathbf{U}$ it is natural to consider the generalization $D \in C^\infty(\mathbf{U} \times \mathrm{Sp}(2n), \mathbf{R})$ of the determinant function defined by

$$D_\omega(M) = (-1)^{n-1}\omega^{-n}\det(M - \omega I), \quad \forall \omega \in \mathbf{U}, M \in \mathrm{Sp}(2n), \tag{4.1}$$

and the hypersurface

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \mid D_\omega(M) = 0\}, \tag{4.2}$$

which contains all symplectic matrices having ω as an eigenvalue. Similarly for any $\omega \in \mathbf{U}$ we define

$$\begin{aligned}
\mathrm{Sp}(2n)^\pm_\omega &= \{M \in \mathrm{Sp}(2n) \mid \pm D_\omega(M) < 0\}, \\
\mathrm{Sp}(2n)^*_\omega &= \mathrm{Sp}(2n)^+_\omega \cup \mathrm{Sp}(2n)^-_\omega = \mathrm{Sp}(2n)\backslash\mathrm{Sp}(2n)^0_\omega, \\
\mathcal{P}^*_{\tau,\omega}(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \mid \gamma(\tau) \in \mathrm{Sp}(2n)^*_\omega\}, \\
\mathcal{P}^0_{\tau,\omega}(2n) &= \mathcal{P}_\tau(2n)\backslash\mathcal{P}^*_{\tau,\omega}(2n).
\end{aligned}$$

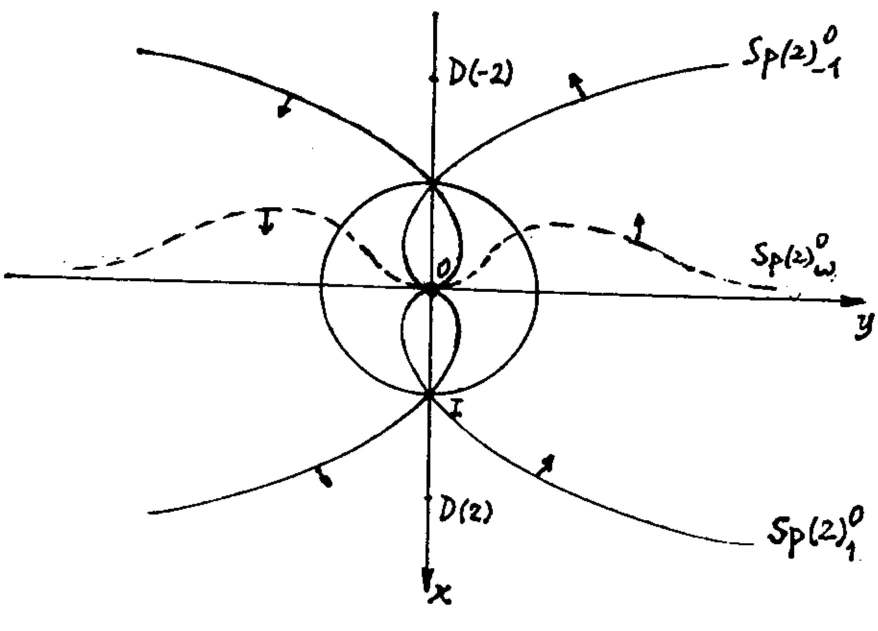

Figure 4.1. Oriented $\mathrm{Sp}(2)^0_\omega$ for $\omega = \pm 1$ and $\omega \in \mathbf{U}\backslash\mathbf{R}$.

In [Lo11], for $\omega \in \mathbf{U}$, the ω-nullity of any symplectic path is defined by

$$\nu_{\tau,\omega}(\gamma) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I), \qquad \forall \gamma \in \mathcal{P}_\tau(2n). \tag{4.3}$$

In [Lo13], the author proved the following result similar to Lemma 1.1.

Lemma 4.1. (cf. [Lo13]) *For any $\omega \in \mathbf{U}$, $\mathrm{Sp}(2n)^*_\omega$ contains two path connected components $\mathrm{Sp}(2n)^+_\omega$ and $\mathrm{Sp}(2n)^-_\omega$, and $M_n^\pm \in \mathrm{Sp}(2n)^\pm_\omega$. Both of these two sets are simply connected in $\mathrm{Sp}(2n)$.*

Based upon this result, the index $i_{\tau,\omega}(\gamma)$ is defined in [Lo11] for any $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$ in a similar way to that used in the Definition 1.6.

Then based upon the results obtained in [Lo13] on the properties of and near $\mathrm{Sp}(2n)^0_\omega$ in $\mathrm{Sp}(2n)$, for any $\omega \in \mathbf{U}$ and $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$ it is defined in [Lo11] that

$$i_{\tau,\omega}(\gamma) = \inf\{i_{\tau,\omega}(\beta) \mid \beta \in \mathcal{P}^*_{\tau,\omega}(2n) \textit{ and } \beta \textit{ is sufficiently close to } \gamma \textit{ in } \mathcal{P}_\tau(2n)\}.$$

In such a way, the ω-index theory assigns a pair of integers to each $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega \in \mathbf{U}$:

$$(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma)) \in \mathbf{Z} \times \{0, \ldots, 2n\}. \tag{4.4}$$

When $\omega = 1$, the ω-index theory coincides with the Maslov-type index theory. Similarly to Theorem 1.11, an axiom characterization of the ω-index theory can be given as in [Lo11].

Now let us fix a path $\gamma \in \mathcal{P}_\tau(2n)$, and move ω on $\mathbf{U}$ from 1 to -1, and study the properties of the ω-index of γ as functions of ω. In [Lo11], the following result is proved.

Lemma 4.2. (cf. [Lo11]) *For fixed $\gamma \in \mathcal{P}_\tau(2n)$, $i_{\tau,\omega}(\gamma)$ as a function of ω is constant on each connected component of $\mathrm{U}\backslash\sigma(\gamma(\tau))$. There holds*

$$\nu_{\tau,\omega}(\gamma) = 0, \qquad \forall \omega \in \mathbf{U}\backslash\sigma(\gamma(\tau)). \tag{4.5}$$

Idea of the proof. It follows from that the index functions are locally constant. ∎

By this lemma, in order to understand the properties of the ω-index as a function of $\omega \in \mathbf{U}$, it is important to study the possible jumps of $i_{\tau,\omega}(\gamma)$ at $\omega \in \mathbf{U}\backslash\sigma(\gamma(\tau))$. These jumps are usually called **splitting numbers**, which play a crucial role in iteration theory of the Maslov-type index theory for symplectic paths. The precise definition of the splitting number is contained in the following result.

Theorem 4.3. (cf. [Lo11]) *For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, choose $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$ with $\gamma(\tau) = M$, and define*

$$S_M^\pm(\omega) = \lim_{\epsilon \to 0+} i_{\tau,\exp(\pm\epsilon\sqrt{-1})\omega}(\gamma) - i_{\tau,\omega}(\gamma). \tag{4.6}$$

Then these two integers are independent of the choice of the path γ. They are called the **splitting numbers** *of M at ω.*

In order to further understand the splitting number, new concepts of the homotopy component of $M \in \mathrm{Sp}(2n)$ and the **ultimate type** of $\omega \in \mathbf{U}$ for $M \in \mathrm{Sp}(2n)$ is introduced by the author in [Lo11] as follows.

Definition 4.4. (cf. [Lo11]) *For any $M \in \mathrm{Sp}(2n)$, define the* **homotopy set** *of M in $\mathrm{Sp}(2n)$ by*

$$\begin{aligned}\Omega(M) = \{N \in \mathrm{Sp}(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U}, \textit{ and} \\ \dim_{\mathbf{C}} \ker_{\mathbf{C}}(N - \lambda I) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \lambda I),\ \forall\, \lambda \in \sigma(M) \cap \mathbf{U}\}.\end{aligned}$$

We denote by $\Omega^0(M)$ *the path connected component of* $\Omega(M)$ *which contains* M*, and call it the* **homotopy component** *of* M *in* $\mathrm{Sp}(2n)$.

For any $M \in \mathrm{Sp}(2n)$, define its conjugate set by

$$[M] = \{N \in \mathrm{Sp}(2n) \mid N = P^{-1}MP \text{ for some } P \in \mathrm{Sp}(2n)\}.$$

Then $[M] \subset \Omega^0(M)$ for all $\omega \in \mathbf{U}$.

Definition 4.5.(cf. [Lo11]) *The following matrices in* $Sp(2n)$ *are called* **basic normal forms** *for eigenvalues on* $\mathbf{U}$*:*

$$\begin{aligned} &N_1(\lambda, b) \quad \text{with } \lambda = \pm 1, \ b = \pm 1, \text{ or } 0, \\ &R(\theta) \ \text{ with } \omega = e^{\theta\sqrt{-1}} \in \mathbf{U}\backslash\mathbf{R}, \\ &N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \text{with } b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathcal{L}(\mathbf{R}^2), \\ &\qquad b_2 - b_3 \neq 0, \quad \text{and } \omega = e^{\theta\sqrt{-1}} \in \mathbf{U}\backslash\mathbf{R}. \end{aligned}$$

A basic normal form matrix M *is* **trivial**, *if for sufficiently small* $a > 0$, $MR((t-1)a)^{\diamond n}$ *possesses no eigenvalue on* $\mathbf{U}$ *for* $t \in [0,1)$*, and is* **nontrivial** *otherwise.*

Note that by direct computations, $N_1(1,-1)$, $N_1(-1,1)$, $N_2(\omega, b)$, and $N_2(\overline{\omega}, b)$ with $\dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I) = 1$, $\omega = \exp(\theta\sqrt{-1}) \in \mathbf{U}\backslash\mathbf{R}$ and $(b_2 - b_3)\sin\theta > 0$ are trivial, and any other basic normal form matrix is nontrivial.

Theorem 4.6.(cf. [Lo11] *For any* $M \in \mathrm{Sp}(2n)$*, there is a path* $f \in C^\infty([0,1], \Omega^0(M))$ *such that* $f(0) = M$ *and*

$$f(1) = M_1 \diamond \cdots \diamond M_k \diamond M_0, \tag{4.7}$$

where the integer $p \in [0, n]$*, each* M_i *is a basic normal form of eigenvalues on* $\mathbf{U}$ *for* $1 \le i \le k$*, and the symplectic matrix* M_0 *satisfies* $\sigma(M_0) \cap \mathbf{U} = \emptyset$.

Idea of the proof. Firstly we connect M within $\Omega^0(M)$ to a product of normal forms via the results of [LD] and [HL]. Then by carefully chosen perturbations and connecting paths, we connect all these normal forms to basic normal forms within $\Omega^0(M)$. ∎

Recall that (cf. Section I.2 of [Ek3] or [YS]) for $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U} \cap \sigma(M)$ being an m-fold eigenvalue, the Hermitian form $\langle\sqrt{-1}J\cdot, \cdot\rangle$, which is called the **Krein form**, is always nondegenerate on the invariant root vector space $E_\omega(M) = \ker_{\mathbf{C}}(M - \omega I)^m$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbf{C}^{2n}$. Then ω is of **Krein type** (p, q) with $p + q = m$ if the restriction of the Krein form on $E_\omega(M)$ has signature (p, q). ω is **Krein positive** if it has Krein type $(p, 0)$, is **Krein negative** if it has Krein type $(0, q)$. If $\omega \in \mathbf{U} \setminus \sigma(M)$, we define the Krein type of ω by $(0, 0)$.

Definition 4.7.(cf. [Lo11]) *For any basic normal form* $M \in \mathrm{Sp}(2n)$ *and* $\omega \in \mathbf{U} \cap \sigma(M)$*, we define the* **ultimate type** (p, q) *of* ω *for* M *to be its usual Krein type if* M *is nontrivial, and to be* $(0,0)$ *if* M *is trivial. For any* $M \in \mathrm{Sp}(2n)$*, we define the ultimate type of* ω *for* M *to be* $(0,0)$ *if* $\omega \in \mathbf{U}\backslash\sigma(M)$*. For any* $M \in \mathrm{Sp}(2n)$*, by Theorem 4.6 there exists a* $\diamond$*-product expansion (4.7) in the homotopy component* $\Omega^0(M)$ *of* M *where each*

M_i is a basic normal form for $1 \le i \le k$ and $\sigma(M_0) \cap \mathbf{U} = \emptyset$. Denote the ultimate type of ω for M_i by (p_i, q_i) for $0 \le i \le k$. Let $p = \sum_{i=0}^{k} p_i$ and $q = \sum_{i=0}^{k} q_i$. We define the **ultimate type** *of ω for M by (p, q).*

It is proved in [Lo11] that the ultimate type of $\omega \in \mathbf{U}$ for M is uniquely determined by ω and M, therefore is well defined. It is constant on $\Omega^0(M)$ for fixed $\omega \in \mathbf{U}$.

Lemma 4.8. (cf. [Lo11]) *For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, denote the Krein type and the ultimate type of ω for M by (P, Q) and (p, q). Then there holds*

$$P - p = Q - q \ge 0. \tag{4.8}$$

The following theorem completely characterizes the splitting numbers.

Theorem 4.9. (cf. [Lo11]) *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, there hold*

$$S_M^+(\omega) = p \quad \text{and} \quad S_M^-(\omega) = q, \tag{4.9}$$

where (p, q) is the ultimate type of ω for M.

Idea of the proof. Use Theorem 4.6 to reduce the proof to the case of basic normal forms. Then carry out the direct computation for each basic normal form. The difficulty part is the computation for $N_2(\omega, b)$'s. We refer to [Lo11] for details. ∎

Corollary 4.10. *If $\omega \in \mathbf{U} \cap \sigma(\gamma(\tau))$ is of Krein type (p, q), there holds*

$$\lim_{\epsilon \to 0^+} \left(i_\tau(e^{\epsilon\sqrt{-1}}\omega) - i_\tau(e^{-\epsilon\sqrt{-1}}\omega) \right) = p - q. \tag{4.10}$$

Corollary 4.11. *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, there holds*

$$0 \le S_M^{\pm}(\omega) \le \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I). \tag{4.11}$$

Remark 4.12. Theorem 4.9 and Corollary 4.10 generalize Theorem IV on p.180 of [Bo], which contains a sign error, and Proposition 9 on p.44 of [Ek3]. Note that there is a sign difference between our J and that in [Ek3]. Note also that the conclusion of our Theorem 4.9 coincides with the Example II on p.181 of [Bo].

§5. Bott-type iteration formulae and the mean index.

Based upon our preparations in the above subsection, next we establish the Bott-type formulae for the Maslov-type index theory.

Fix $\tau > 0$ and $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$. Let $\gamma \in \hat{\mathcal{P}}_\tau(2n)$, i.e. γ is the fundamental solution of (1.2) for some $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$. Fix $k \in \mathbf{N}$. The bilinear form corresponding to the system (1.2) is given by

$$\phi_{k\tau}(x, y) = \frac{1}{2}\langle (A - B)x, y \rangle_{L_\tau}, \qquad \forall x, y \in E_{k\tau} = W^{1,2}(S_{k\tau}, \mathbf{R}^{2n})) \subset L_{k\tau}. \tag{5.1}$$

For $\omega \in \mathbf{U}$ define

$$E_{k\tau}(\tau,\omega) = \{y \in E_{k\tau} \,|\, y(t+\tau) = \omega y(t), \ \forall t\}.$$

For simplicity we identify $E_{k\tau}(\tau,\omega)$ with $E_\tau(\tau,\omega)$. Define $\omega_p = \exp(2p\pi/k\sqrt{-1})$ for $0 \le p \le k$. Then $\omega_p^k = 1$. By direct computation we obtain that $E_{k\tau}(\tau,\omega_p)$ and $E_{k\tau}(\tau,\omega_q)$ is $\phi_{k\tau}$-orthogonal for $0 \le p \ne q \le k$, and there holds

$$E_{k\tau} = \oplus_{\omega^k=1} E_{k\tau}(\tau,\omega). \tag{5.2}$$

Thus we obtain

$$\phi_{k\tau}|_{E_{k\tau}} = \sum_{i=0}^{k-1} \phi_{k\tau}|_{E_{k\tau}(\tau,\omega_i)} = k\sum_{i=0}^{k-1} \phi_\tau|_{E_\tau(\tau,\omega_i)}. \tag{5.3}$$

Now we carry out the saddle point reduction for $\phi_{k\tau}$ on $E_{k\tau}$, and obtain the functional $a_{k\tau} = \phi_{k\tau} \circ u_{k\tau}$ defined on $Z_{k\tau}$. Simultaneously this induces saddle point reductions for ϕ_τ on $E_\tau(\tau,\omega_i)$ for $0 \le i \le k-1$, and yields the functional $a_{\tau,\omega_i} = \phi_\tau \circ u_{\tau,\omega_i}$ defined on $Z_{\tau,\omega}$. By the orthogonality claim (5.3), the Morse index of $\phi_{k\tau}$ on the left hand side of (5.3) splits into the sum of the Morse indices of the functional on the right hand side of (5.3). Note that the dimensions of spaces appeared in (5.3) satisfy

$$d_{k\tau} = \sum_{\omega^k=1} d_{\tau,\omega}. \tag{5.4}$$

Thus by Theorem 2.1, we obtain the following Bott-type formula for $\gamma \in \hat{\mathcal{P}}_\tau(2n)$.

Theorem 5.1. (cf. [Lo11]) *For any $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2n)$, and $m \in \mathbf{N}$, there hold*

$$i_{m\tau}(\tilde{\gamma}) = i_{m\tau,1}(\tilde{\gamma}) = \sum_{\omega^m=1} i_{\tau,\omega}(\gamma), \tag{5.4}$$

$$\nu_{m\tau}(\tilde{\gamma}) = \nu_{m\tau,1}(\tilde{\gamma}) = \sum_{\omega^m=1} \nu_{\tau,\omega}(\gamma). \tag{5.5}$$

Idea of the proof. For the general case of $\gamma \in \mathcal{P}_\tau(2n)$. Choose $\beta \in \hat{P}_\tau(2n)$ such that $\beta(\tau) = \gamma(\tau)$ and $\beta \sim \gamma$. We obtain $i_{\tau,\omega}(\beta) = i_{\tau,\omega}(\gamma)$ for all $\omega \in \mathbf{U}$. From $\beta \sim \gamma$ with fixed end points, this homotopy can be extended to $[0,1] \times [0,k\tau]$. By the inverse homotopy Theorem 1.12, we then obtain $\beta^k \sim \gamma^k$. Thus $i_{k\tau}(\beta^k) = i_{k\tau}(\gamma^k)$ holds. Then the Bott-type formulae (5.4) and (5.5) for β imply those for γ. This completes the proof of Theorem 5.1. ∎

As a direct consequence of Theorem 5.1, we obtain

$$\frac{i_{k\tau}(\gamma^k)}{k} = \frac{1}{2\pi}\sum_{\omega^k=1} i_{\tau,\omega}(\gamma)\frac{2\pi}{k},$$

$$\frac{\nu_{k\tau}(\gamma^k)}{k} = \frac{1}{2\pi}\sum_{\omega^k=1} \nu_{\tau,\omega}(\gamma)\frac{2\pi}{k}.$$

By Lemma 4.2, the function $i_\tau(\omega)$ is locally constant and $\nu_\tau(\omega)$ is locally zero on $\mathbf{U}$ except at finitely many points. Therefore the right hand sides of above equalities are Riemannian sums, and converge to the corresponding integrals as $k \to \infty$. This proves the following result.

Theorem 5.2. (cf. [Lo11]) *For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$ there hold*

$$\hat{i}_\tau(\gamma) \equiv \lim_{k\to+\infty} \frac{i_{k\tau}(\tilde{\gamma})}{k} = \frac{1}{2\pi}\int_{\mathbf{U}} i_{\tau,\omega}(\gamma)d\omega, \tag{5.6}$$

$$\hat{\nu}_\tau(\gamma) \equiv \lim_{k\to+\infty} \frac{\nu_{k\tau}(\tilde{\gamma})}{k} = \frac{1}{2\pi}\int_{\mathbf{U}} \nu_{\tau,\omega}(\gamma)d\omega = 0. \tag{5.7}$$

Specially, $\hat{i}_\tau(\gamma)$ is always a finite real number, and is called the **mean Maslov-type index** *per τ for γ.*

As a direct consequence of Theorem 5.2, for any $\gamma \in \mathcal{P}_\tau(2n)$ we obtain

$$\hat{i}_{k\tau}(\gamma^k) = k\hat{i}_\tau(\gamma), \qquad \forall k \in \mathbf{N}. \tag{5.8}$$

Then through the fundamental solution γ_x of (1.2) with $B(t) = H''(t, x(t))$, the mean index per period τ of a τ-periodic solution x of the nonlinear system (1.1) can be defined by

$$\hat{i}_\tau(x) = \hat{i}_\tau(\gamma_x). \tag{5.9}$$

When τ is the minimal period of x, we denote by $\hat{i}(x) = \hat{i}_\tau(x)$. This yields a new invariant to each periodic solution of the system (1.1).

Remark 5.3. As proved in [Lo11], for a fixed Sturm system (2.8) and the corresponding path $\gamma \in \mathcal{P}_\tau(2n)$ as the fundamental solution of the system (1.2) with coefficient B defined by (2.9), our ω-index pair $(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma))$ and the index functions $\Lambda(\omega)$ and $N(\omega)$ of R. Bott defined in [Bo] satisfy

$$i_{\tau,\omega}(\gamma) = \Lambda(\omega), \quad \nu_{\tau,\omega}(\gamma) = N(\omega), \quad \forall \omega \in \mathbf{U}. \tag{5.10}$$

Note that in [Ek3] the standard symplectic matrix is given by $-J$. For the fundamental solution γ of a fixed linear Hamiltonian system (1.2) with negative definite coefficient $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$, our ω-index and the index functions $j_\tau(\omega)$ and $n_\tau(\omega)$ of I. Ekeland defined in the section I.5 of [Ek3] satisfy

$$\nu_{\tau,\omega}(\gamma) = n_\tau(\omega), \qquad \forall\, \omega \in \mathbf{U}, \tag{5.11}$$

$$i_{\tau,1}(\gamma) + \nu_{\tau,1}(\gamma) = -j_\tau(1) - n, \tag{5.12}$$

$$i_{\tau,\omega}(\gamma) + \nu_{\tau,\omega}(\gamma) = -j_\tau(\omega), \qquad \forall\, \omega \in \mathbf{U}\setminus\{1\}. \tag{5.13}$$

By (5.10) and (5.11)-(5.13), our above theorems generalize the well known Bott formulae (Theorem A of [Bo] with periodic boundary condition) for Morse indices of closed geodesics, and the Bott-type formulae of Ekeland indices (Corollary I.4 of [Ek3]) for convex

Hamiltonian systems, and corresponding result of C. Viterbo in [Vi1] for non-degenerate star-shaped Hamiltonian systems.

§6. Iteration inequalities.

In many of our applications, we need sharp increasing estimates on the iterated Maslov-type index $i_{m\tau}(\gamma^m)$ for $\gamma \in \mathcal{P}_\tau(2n)$. These results are proved in [LL1] based on results in [DL] and in [LL3] based on results in [Lo11].

Theorem 6.1.(cf. [LL1] and [LL3]) 1^o *For any* $\gamma \in \mathcal{P}_\tau(2n)$ *and* $m \in \mathbf{N}$, *there holds*

$$m\hat{i}_\tau(\gamma) - n \le i_{m\tau}(\gamma^m) \le m\hat{i}_\tau(\gamma) + n - \nu_{m\tau}(\gamma^m). \tag{6.1}$$

2^o *The right hand side equality in (6.1) holds for some* $m \in \mathbf{N}$ *if and only if*

$$I_{2p}\diamond N_1(1,-1)^{\diamond(n-p)} \in \Omega^0(\gamma(\tau))$$

for some integer $p \in [0,n]$. *Specifically in this case, all the eigenvalues of* $\gamma(\tau)$ *equal to* 1 *and* $\nu_\tau(\gamma) = n + p \ge n$.

3^o *The left hand side equality in (6.1) holds for some* $m \in \mathbf{N}$ *if and only if*

$$I_{2q}\diamond N_1(1,1)^{\diamond(n-q)} \in \Omega^0(\gamma(\tau))$$

for some integer $q \in [0,n]$. *Specifically in this case, all the eigenvalues of* $\gamma(\tau)$ *equal to* 1 *and* $\nu_\tau(\gamma) = n + q \ge n$.

4^o *Both equalities in (6.1) hold for some* $m = m_1$ *and* $m = m_2 \in \mathbf{N}$ *respectively if and only if* $\gamma(\tau) = I_{2n}$.

Theorem 6.2.(cf. [LL1] and [LL3]) 1^o *For any* $\gamma \in \mathcal{P}_\tau(2n)$ *and* $m \in \mathbf{N}$, *there holds*

$$\begin{aligned} m(i_\tau(\gamma) + \nu_\tau(\gamma) - n) + n - \nu_\tau(\gamma) &\le i_{m\tau}(\gamma^m) \\ &\le m(i_\tau(\gamma) + n) - n - (\nu_{m\tau}(\gamma^m) - \nu_\tau(\gamma)). \end{aligned} \tag{6.2}$$

2^o *The left equality of (6.2) holds for some* $m > 1$ *if and only if there holds*

$$I_{2p}\diamond N_1(1,-1)^{\diamond q}\diamond K \in \Omega^0(\gamma(\tau))$$

for some non-negative integers p *and* q *satisfying* $p + q \le n$ *and some* $K \in \mathrm{Sp}(2(n-p-q))$ *satisfying* $\sigma(K) \subset \mathbf{U}\backslash\mathbf{R}$. *In this case, all eigenvalues of* K *on* $\mathbf{U}^+$ *(on* $\mathbf{U}^-$*) are located on the open arc between* 1 *and* $\exp(2\pi\sqrt{-1}/m)$ *(and* $\exp(-2\pi\sqrt{-1}/m)$*) in* $\mathbf{U}^+$ *(in* $\mathbf{U}^-$*) and are all Krein negative (positive) definite.*

3^o *The right equality of (6.2) holds for some* $m > 1$ *if and only if there holds*

$$I_{2p}\diamond N_1(1,1)^{\diamond r}\diamond K \in \Omega^0(\gamma(\tau))$$

for some non-negative integers p and r satisfying $p+r \le n$ *and some* $K \in \mathrm{Sp}(2(n-p-r))$ *with* $\sigma(K) \subset \mathbf{U}\backslash\mathbf{R}$ *satisfying the following conditions:*

If $m > 2$*, all eigenvalues of K locate within the closed arc between the points* 1 *and* $\exp(2\pi\sqrt{-1}/m)$ *(and* $\exp(-2\pi\sqrt{-1}/m)$*) in* $\mathbf{U}^+\backslash\{1\}$ *(in* $\mathbf{U}^-\backslash\{1\}$*) possess total multiplicity* $n-p-r$*, and are all Krein positive (negative) definite.*

If $m = 2$*, there holds* $(-I_{2s})\diamond N_1(-1,1)^{\diamond t}\diamond H \in \Omega^0(K)$ *for some non-negative integers s and t satisfying* $0 \le s+t \le n-p-r$*, and some* $H \in \mathrm{Sp}(2(n-p-r-s-t))$ *satisfying* $\sigma(H) \subset \mathbf{U}\backslash\mathbf{R}$ *and that all elements in* $\sigma(H)\cap\mathbf{U}^+$ *(or* $\sigma(H)\cap\mathbf{U}^-$*) are all Krein positive (or negative) definite.*

4° *Both equalities of (6.2) hold for some* $m = m_1$ *and* $m = m_2 \in \mathbf{N}$ *respectively if and only if* $\gamma(\tau) = I_{2n}$.

Remark 6.3. 1° Note that there holds $\nu_{m\tau}(\gamma^m) \ge \nu_\tau(\gamma)$ for any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$.

2° By Theorem 2.2, our above theorems also work for the Morse index theory in the calculus of variations and closed geodesics. Note that in particular, our Theorem 6.1 improves the inequality of Morse index theory for closed geodesics

$$|i_{m\tau}(\gamma^m) - m\hat{i}_\tau(\gamma)| \le n,$$

proved by H. Rademacher in [Rd] of 1989.

3° When $\gamma \in \mathcal{P}_\tau(2n)$ is the fundamental solution of a linear Hamiltonian system (1.2) with $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$ being negative definite, the Ekeland index (cf. Section I.4 of [Ek3]) is also defined for γ which we denote by

$$(i_\tau^E(\gamma), \nu_\tau^E(\gamma)) \in (\{0\}\cup\mathbf{N}) \times \{0,1,\ldots,2n\}.$$

By (5.11) and (5.12) as proved in [Br] and [Lo12], the following relation between the Maslov-type index theory and the Ekeland index theory holds:

$$\nu_\tau(\gamma) = \nu_\tau^E(\gamma), \tag{6.3}$$

$$i_\tau(\gamma) + \nu_\tau(\gamma) = -i_\tau^E(\gamma) - n. \tag{6.4}$$

As a direct consequence of (6.3), (6.4), and Theorem 6.2, we obtain the following inequalities for such a γ and any $m \in \mathbf{N}$,

$$\begin{aligned} m(i_\tau^E(\gamma) + \nu_\tau^E(\gamma)) - \nu_\tau^E(\gamma) &\le i_{m\tau}^E(\gamma^m) \\ &\le m(i_\tau^E(\gamma) + 2n) - 2n - (\nu_{m\tau}^E(\gamma^m) - \nu_\tau^E(\gamma)), \end{aligned} \tag{6.5}$$

with the corresponding equality conditions. Here the left hand side inequality in (6.5), which follows from the right hand side inequality of (6.2), recovers Theorem I.5.1 of [Ek3]. Ekeland's this theorem can also be obtained from the left hand side inequality of (6.2).

Our proof of these theorems is based on the following result proved in [LL3]. In particular, this proof uses the properties of the ω-index theory, splitting numbers on homotopy components of symplectic matrices, and mean indices are very crucial in the proofs. Let $\mathbf{U}^+$ and $\mathbf{U}^-$ denote the upper and lower closed unit semi-circle in the complex plane $\mathbf{C}$.

Proposition 6.4. (cf. [LL3]) 1^o *For any* $\gamma \in \mathcal{P}_\tau(2n)$ *and* $\omega \in \mathbf{U}\backslash\{1\}$, *there always holds*

$$i_\tau(\gamma) + \nu_\tau(\gamma) - n \le i_{\tau,\omega}(\gamma) \le i_\tau(\gamma) + n - \nu_{\tau,\omega}(\gamma). \tag{6.6}$$

2^o *The left equality in (6.6) holds for some* $\omega \in \mathbf{U}^+\backslash\{1\}$ *(or* $\mathbf{U}^-\backslash\{1\}$*) if and only if there holds*

$$I_{2p}\diamond N_1(1,-1)^{\diamond q}\diamond K \in \Omega^0(\gamma(\tau))$$

for some non-negative integers p *and* q *satisfying* $0 \le p+q \le n$ *and* $K \in \mathrm{Sp}(2(n-p-q))$ *with* $\sigma(K) \subset \mathbf{U}\backslash\mathbf{R}$ *satisfying that all eigenvalues of* K *located within the open arc between* 1 *and* ω *in* $\mathbf{U}^+$ *(or* $\mathbf{U}^-$*) possess total multiplicity* $n-p-q$ *and are all Krein negative (or positive) definite.*

3^o *The left equality in (6.6) holds for all* $\omega \in \mathbf{U}\backslash\{1\}$ *if and only if*

$$I_{2p}\diamond N_1(1,-1)^{\diamond(n-p)} \in \Omega^0(\gamma(\tau))$$

for some integer $p \in [0,n]$. *Specifically in this case, all the eigenvalues of* $\gamma(\tau)$ *equal to* 1 *and* $\nu_\tau(\gamma) = n+p \ge n$.

4^o *The right equality in (6.6) holds for some* $\omega \in \mathbf{U}^+\backslash\{1\}$ *(or* $\mathbf{U}^-\backslash\{1\}$*) if and only only if there holds*

$$I_{2p}\diamond N_1(1,1)^{\diamond r}\diamond K \in \Omega^0(\gamma(\tau))$$

for some non-negative integers p *and* r *satisfying* $0 \le p+r \le n$ *and* $K \in \mathrm{Sp}(2(n-p-r))$ *with* $\sigma(K) \subset \mathbf{U}\backslash\mathbf{R}$ *satisfying that all eigenvalues of* K *located within the closed arc between* 1 *and* ω *in* $\mathbf{U}^+\backslash\{1\}$ *(or* $\mathbf{U}^-\backslash\{1\}$*) possess total multiplicity* $n-p-r$*; if* $\omega \ne -1$, *all eigenvalues in* $\sigma(K)\cap\mathbf{U}^+$ *(or* $\sigma(K)\cap\mathbf{U}^-$*) are all Krein positive (or negative) definite; if* $\omega = -1$, *there holds*

$$(-I_{2s})\diamond N_1(-1,1)^{\diamond t}\diamond H \in \Omega_0(K)$$

for some non-negative integers s *and* t *satisfying* $0 \le s+t \le n-p-r$, *and some* $H \in \mathrm{Sp}(2(n-p-r-s-t))$ *satisfying* $\sigma(H) \subset \mathbf{U}\backslash\mathbf{R}$ *and that all elements in* $\sigma(H)\cap\mathbf{U}^+$ *(or* $\sigma(H)\cap\mathbf{U}^-$*) are all Krein positive (or negative) definite.*

5^o *The right equality in (6.6) holds for all* $\omega \in \mathbf{U}\backslash\{1\}$ *if and only if*

$$I_{2p}\diamond N_1(1,1)^{\diamond(n-p)} \in \Omega_0(\gamma(\tau))$$

for some integer $p \in [0,n]$. *Specifically in this case, all the eigenvalues of* $\gamma(\tau)$ *must be* 1, *and there holds* $\nu_\tau(\gamma) = n+p \ge n$.

6^o *Both equalities in (6.6) hold for all* $\omega \in \mathbf{U}\backslash\{1\}$ *if and only if* $\gamma(\tau) = I_{2n}$.

Idea of the proof. The proof of (6.6) is based on the estimate of the difference between $i_\tau(\gamma) = i_{\tau,1}(\gamma)$ and $i_{\tau,\omega}(\gamma)$, which is expressed by a sum of splitting numbers when the parameter runs from 1 to ω on $\mathbf{U}$:

$$i_{\tau,\omega}(\gamma) = i_\tau(\gamma) + S^+_{\gamma(\tau)}(1) - \sum_{j=1}^{k}[S^-_{\gamma(\tau)}(\omega_j) - S^+_{\gamma(\tau)}(\omega_j)] - S^-_{\gamma(\tau)}(\omega).$$

Then apply properties of the splitting numbers to get (6.6). For proofs of other parts of the theorem, we refer to [LL3] for details. ∎

Now based on the Proposition 6.4, we can give the proofs of 1^o's of Theorems 6.1 and 6.2 below. The necessary and sufficient conditions of these two theorems follow from the corresponding claims in Proposition 6.4 which we refer to [LL3] for details.

Proof of 1^o of Theorem 6.1. By Theorem 5.2, integrating (6.6) on **U** we obtain

$$i_\tau(\gamma) + \nu_\tau(\gamma) - n \le \hat{i}_\tau(\gamma) \le i_\tau(\gamma) + n. \tag{6.7}$$

Replacing τ by $m\tau$ in (6.7), by (5.8) we obtain (6.1). ∎

Proof of 1^o of Theorem 6.2. By Theorem 5.1, summing (6.6) up over all m-th roots of unit, we obtain

$$\begin{aligned}(m-1)(i_\tau(\gamma) + \nu_\tau(\gamma) - n) + i_\tau(\gamma) &\le i_{m\tau}(\gamma^m) \\ &\le (m-1)(i_\tau(\gamma) + n) + i_\tau(\gamma) - (\nu_{m\tau}(\gamma^m) - \nu_\tau(\gamma)).\end{aligned}$$

This yields (6.2). ∎

The following result follows from Theorems 6.1 and 6.2 immediately.

Lemma 6.5.(cf. [LL3]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, there hold*

$$m \le 1 + \frac{i_{m\tau}(\gamma^m) - i_\tau(\gamma)}{i_\tau(\gamma) + \nu_\tau(\gamma) - n}, \quad \text{if } i_\tau(\gamma) + \nu_\tau(\gamma) - n > 0, \tag{6.8}$$

$$m \le 1 + \frac{i_{m\tau}(\gamma^m) + \nu_{m\tau}(\gamma^m) - \nu_\tau(\gamma) - i_\tau(\gamma)}{i_\tau(\gamma) + n}, \quad \text{if } i_\tau(\gamma) + n < 0, \tag{6.9}$$

$$m \le \frac{i_{m\tau}(\gamma^m) + n}{\hat{i}_\tau(\gamma)}, \quad \text{if } \hat{i}_\tau(\gamma) > 0, \tag{6.10}$$

$$m \le \frac{i_{m\tau}(\gamma^m) + \nu_{m\tau}(\gamma^m) - n}{\hat{i}_\tau(\gamma)}, \quad \text{if } \hat{i}_\tau(\gamma) < 0. \tag{6.11}$$

A direct consequence of these estimates is the following

Corollary 6.6.(cf. [LL3]) *Suppose for $\gamma \in \mathcal{P}_\tau(2n)$ and integers $m, p \in \mathbf{N}$, $q \in \mathbf{Z}$, there hold*

$$i_{m\tau}(\gamma^m) \le n + q, \quad i_\tau(\gamma) + \nu_\tau(\gamma) \ge n + p. \tag{6.12}$$

Then there holds

$$m \le \frac{2n + q}{p}. \tag{6.13}$$

In particular, if $p = q = 1$, we obtain $m \le 2n + 1$.

From Theorem 6.2, we obtain

Theorem 6.7.(cf. [LL3]) 1^o *Suppose for $\gamma \in \mathcal{P}_\tau(2n)$ and integers $m, h \in \mathbf{N}$ and $l, s \in \mathbf{Z}$, there hold*

$$i_{m\tau}(\gamma^m) \le l, \quad i_\tau(\gamma) \ge s, \quad i_\tau(\gamma) + \nu_\tau(\gamma) \ge h \ge n + 1, \quad \left[\frac{l - s}{h - n}\right] \le 1, \tag{6.14}$$

where $[a] = \max\{k \in \mathbf{Z} \mid k \leq a\}$ *for any* $a \in \mathbf{R}$. *Then there holds*

$$1 \leq m \leq 2 \quad \text{and} \quad 0 \leq l - s. \tag{6.15}$$

2^o *Moreover,* $l + n \leq s + h$ *and* $m = 2$ *in (6.14) only if there holds*

$$I_{2p} \diamond N_1(1, -1)^{\diamond q} \diamond K \in \Omega^0(\gamma(\tau)), \tag{6.16}$$

for some non-negative integers p *and* q *satisfying* $0 \leq p + q \leq n$ *and* $K \in \mathrm{Sp}(2(n - p - q))$ *with* $\sigma(K) \subset \mathbf{U} \backslash \mathbf{R}$ *satisfying that all eigenvalues of* K *located on* $\mathbf{U}^+$ *(or* $\mathbf{U}^-$*) possess total multiplicity* $n - p - q$ *and are all Krein negative (or positive) definite. In this case there exists an integer* $k \geq 1$ *such that there hold*

$$i_\tau(\gamma) = 2k + n - 2p - q = s, \tag{6.17}$$

$$\nu_\tau(\gamma) = 2p + q = h - s, \tag{6.18}$$

$$i_{2\tau}(\gamma^2) = 4k + n - 2p - q = l, \tag{6.19}$$

$$h - n = l - s = 2k \geq 2. \tag{6.20}$$

3^o *There exists a path* $\gamma \in \mathcal{P}_\tau(2n)$ *satisfying the conditions in* 2^o *such that (6.14) holds with* $l + n = s + h$ *and* $m = 2$.

A direct consequence of Theorem 6.7 is the following

Corollary 6.8.(cf. [LL3]) *Suppose for* $\gamma \in \mathcal{P}_\tau(2n)$ *and some* $m \in \mathbf{N}$ *and* $p \in \mathbf{Z}$, *there holds*

$$p \geq i_{m\tau}(\gamma^m), \quad i_\tau(\gamma) \geq p - 1, \quad i_\tau(\gamma) + \nu_\tau(\gamma) \geq n + 1. \tag{6.21}$$

Then $m = 1$.

Corollary 6.9.(cf. [DL] and [LL3]) *Suppose for* $\gamma \in \mathcal{P}_\tau(2n)$ *and some* $m \in \mathbf{N}$ *there holds*

$$n + 1 \geq i_{m\tau}(\gamma^m), \quad i_\tau(\gamma) \geq n, \quad \nu_\tau(\gamma) \geq 1. \tag{6.22}$$

Then $m = 1$.

Remark 6.10. Corollary 6.9 was first proved by D. Dong and Y. Long in [DL] by a rather different method. Corollary 6.8 also generalizes the Theorem 3.3 of [WF] which requires $p = n + 1$ in (6.21).

§7. Precise iteration formulae.

The Bott-type formula Theorem 5.1 is a powerful tool to compute and estimate the Maslov-type indices for iterations of paths in $\mathcal{P}_\tau(2n)$. Note that a different method of computing and estimating the Maslov-type indices for such iterated paths have been developed in [DL], [Lo15], and [Lo16]. The main idea is to reduce the computation of the index of a given path to those of paths in Sp(2) and some special paths in Sp(4) ending at the basic normal forms by a sequence of homotopies in the sense of Definition 1.3. But in terms of the cylinderical coordinate representation of Sp(2) in $\mathbf{R}^3$, the computation of the Maslov-type index of any path in Sp(2) starting from I is almost obvious. The cases in Sp(4) can also be reduced to the case of Sp(2). This method yields rather precise information on the Maslov-type indices for iterations with very simple proofs. This method is equivalent to the method of Bott-type formulae. For example, a different proof of the Bott-type formulae (2.17) and (2.18) can be given by computing both sides of (2.17) and (2.18) on paths in $\mathcal{P}_\tau(2) \cup \mathcal{P}_\tau(4)$ ending at basic normal forms in Definition 4.5.

The following is the main result in [Lo16].

Theorem 7.1. (cf. [Lo11]) *For any $M \in \mathrm{Sp}(2n)$, there is an $f \in C([0,1], \Omega^0(M))$ such that $f(0) = M$ and*

$$\begin{aligned} f(1) =& N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \diamond N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1, u_1) \diamond \cdots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ & \diamond N_2(\lambda_1, v_1) \diamond \cdots \diamond N_2(\lambda_{r_0}, v_{r_0}) \diamond M_0, \end{aligned} \tag{7.1}$$

where $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*$, and r_0 are nonnegative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; θ_j, α_j, $\beta_j \in (0,\pi) \cup (\pi, 2\pi)$; $N_2(\omega_j, u_j)$'s are nontrivial and $N_2(\lambda_j, v_j)$'s are trivial basic normal forms; $\sigma(M_0) \cap \mathbf{U} = \emptyset$. We denote by

$$I(m,\theta) \equiv -[[m\theta/(2\pi)] - m\theta/(2\pi)] \in \{0,1\}, \quad \forall m \in \mathbf{N}, \theta \in \mathbf{R}. \tag{7.2}$$

The integers $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*, r_0$, $\sum_{j=1}^{r_*} I(m,\alpha_j) - r_*$, and $\sum_{j=1}^{r_0} I(m,\beta_j) - r_0$, and the real numbers θ_j for $1 \le j \le r$, are uniquely determined by M.

Note that $I(m,\theta) = 0$ if $m\theta = 0 \bmod 2\pi$, and $I(m,\theta) = 1$ otherwise. The following is the main result in this section.

Theorem 7.2. (cf. [Lo16]) *For $\tau > 0$, let $\gamma \in \mathcal{P}_\tau(2n)$. In Theorem 7.1 we let $M = \gamma(\tau)$ and use notations there. Then for any $m \in \mathbf{N}$ there hold*

$$\begin{aligned} i_{m\tau}(\gamma^m) =& m(i_\tau(\gamma) + p_- + p_0 - r) + 2\sum_{j=1}^{r} [m\theta_j/(2\pi)] \\ & - p_- - p_0 - \frac{1+(-1)^m}{2}(q_0 + q_+) + 2\sum_{j=1}^{r} I(m,\theta_j) - r \\ & + 2(\sum_{j=1}^{r_*} I(m,\alpha_j) - r_*), \end{aligned} \tag{7.3}$$

$$\nu_{m\tau}(\gamma^m) = \nu_\tau(\gamma) + \frac{1+(-1)^m}{2}(q_- + 2q_0 + q_+) + 2\varphi(m,\gamma(\tau)), \tag{7.4}$$

where we denote by

$$\varphi(m,\gamma(\tau)) = (r - \sum_{j=1}^{r} I(m,\theta_j)) + (r_* - \sum_{j=1}^{r_*} I(m,\alpha_j)) + (r_0 - \sum_{j=1}^{r_0} I(m,\beta_j)). \tag{7.5}$$

Remark 7.3. Note that using Theorem 7.2, results in the Section 6 can also be proved.

Based upon Theorem 7.1, the proof of Theorem 7.2 is reduced to paths in $\mathcal{P}_\tau(2)$ and $\mathcal{P}_\tau(4)$ with end matrices listed in (7.1). To illustrate the computations of Maslov-type indices for iterated paths, next we give a pictorial proof of the iteration formulae for several most important cases which we shall need in the later sections of our applications. For more details about this computation, we refer to [Lo16] as well as [DL] and [Lo15]. In the following we fix a $\tau > 0$ and use simply γ to denote its iteration path $\tilde{\gamma}$.

Let $\mathrm{Sp}(2)^0_\pm = \Omega^0(N_1(1, \mp 1))$.

Case 1. *$\gamma \in \mathcal{P}^0_\tau(2)$ and $\gamma(\tau) \in \mathrm{Sp}(2)^0_-$.*

In this case we must have $k \equiv i_\tau(\gamma)$ being odd and $\nu_\tau(\gamma) = 1$. From the fact $(\mathrm{Sp}(2)^0_-)^m \subset \mathrm{Sp}(2)^0_-$, we obtain $\nu_{m\tau}(\gamma) = 1$ for all $m \in \mathbf{N}$. From the Figure 7.1 we obtain $i_\tau(\gamma) = (k+1) - 1$, and $i_{m\tau}(\gamma) = m(k+1) - 1$ for all $m \in \mathbf{N}$. Thus in this case we obtain

$$i_{m\tau}(\gamma) = m(i_\tau(\gamma) + 1) - 1, \quad \nu_{m\tau}(\gamma) = 1, \quad \forall m \in \mathbf{N}. \tag{7.6}$$

Note that this formula can also be obtained from the Bott-type formula (2.17).

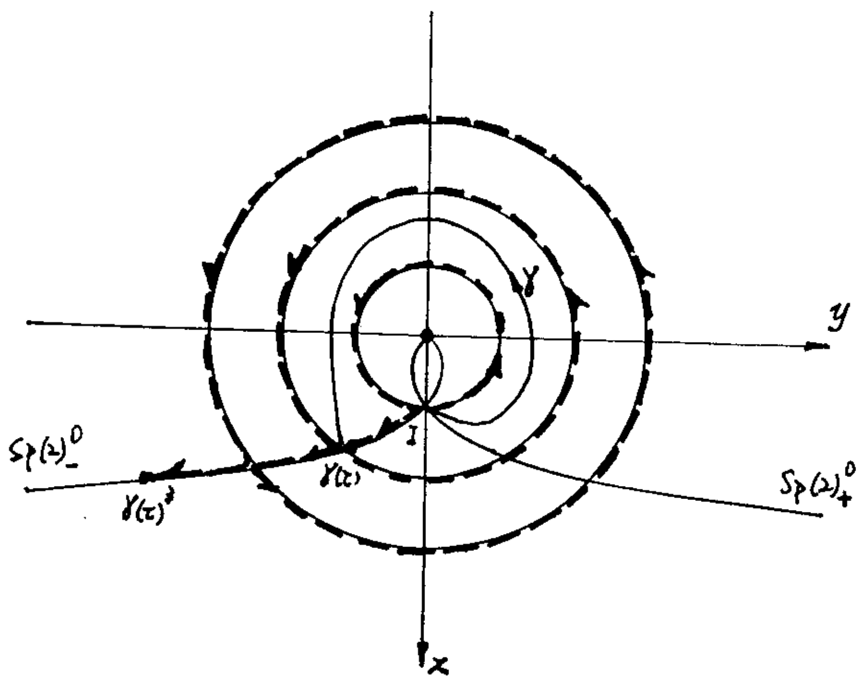

Figure 7.1. Computation of indices for iterations of the path γ in the Case 1.

Case 2. *$\gamma \in \mathcal{P}_\tau(2n)$ and $\gamma(\tau) = I$.*

Similar to the case 1, we must have $i_\tau(\gamma)$ being odd and $\nu_\tau(\gamma) = 2$. In this case we obtain

$$i_{m\tau}(\gamma) = m(i_\tau(\gamma) + 1) - 1, \quad \nu_{m\tau}(\gamma) = 2, \quad \forall m \in \mathbf{N}. \tag{7.7}$$

Case 3. *$\gamma \in \mathcal{P}_\tau(2n)$ and $\gamma(\tau) \in \mathrm{Sp}(2)^0_+$.*

In this case, we must have $i_\tau(\gamma)$ being even and $\nu_\tau(\gamma) = 2$. Similar to the case 1 we obtain

$$i_{m\tau}(\gamma) = mi_\tau(\gamma), \quad \nu_{m\tau}(\gamma) = 1, \quad \forall m \in \mathbf{N}. \tag{7.8}$$

Case 4. $\gamma \in \mathcal{P}_\tau(2n)$ *and* $\sigma(\gamma(\tau)) = \{a, a^{-1}\}$ *with* $a \in \mathbf{R}\backslash\{0, \pm 1\}$.

In this case we have that $i_\tau(\gamma)$ is odd if $a < 0$ and $i_\tau(\gamma)$ is even if $a > 0$. Similar to the case 1 we obtain

$$i_{m\tau}(\gamma) = mi_\tau(\gamma), \quad \nu_{m\tau}(\gamma) = 0, \quad \forall m \in \mathbf{N}. \tag{7.9}$$

Chapter 3. Applications to Hamiltonian Systems.

We have introduced the iteration theory of the Maslov-type index in the previous sections. We believe that this theory is important to many problems related to Hamiltonain systems. In this chapter, we introduce three applications of this iteration theory to the prescribed minimal period problem, stability problem, and multiplicity problem of nonlinear Hamiltonian systems obtained by D. Dong and the author in [DL], by the author in [Lo12] and [Lo14], and one application of the Maslov-type index theory to the instability problem of linear Hamiltonian systems obtained by T. An and the author in [LA].

§8. Rabinowitz' conjecture on prescribed minimal period solutions.

Let us consider the following periodic boundary value problem of the autonomous Hamiltonian systems,

$$\dot{x} = JH'(x), \tag{8.1}$$

$$x(\tau) = x(0). \tag{8.2}$$

Suppose the Hamiltonian function H satisfies the following conditions:

(H1) $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$

(H2) There exist $\mu > 2$ and $r_0 > 0$ such that there hold

$$0 < \mu H(x) \le H'(x) \cdot x, \qquad \forall |x| \ge r_0,$$

(H3) $H(x) = o(|x|^2)$ at $x = 0$.

(H4) $H(x) \ge 0$ for all $x \in \mathbf{R}^{2n}$.

In his pioneering work [Ra1] of 1978, Rabinowitz proved the existence of non-constant period solutions of (8.1)-(8.2) with any prescribed $\tau > 0$ under these conditions. Because a τ/k-periodic function is also a τ-periodic function for every $k \in \mathbf{N}$, Rabinowitz conjectured that this problem possesses a non-constant solution with any prescribed minimal period. Since then, a large amount of contributions on this conjecture have been made by many mathematicians. Among all these results, two kinds of methods are used to determine the minimality of the period of a solution. The first method depends on a priori estimates on the solutions, and is used by many authors (cf. [AM], [CE1], [CE2], [De], [Ek3], [GM1], [GM2], [Lo7]). The second method depends on the dual action principle of convex Hamiltonian systems, the iteration inequality of Morse-Ekeland index theory, Bott's formula, and Hofer's topological characterization of Mountain-Pass points. This method is firstly introduced by Ekeland and Hofer in their celebrated paper [EH], and has been used by many other authors to various convex Hamiltonian systems (cf. [AC], [Ek3], [GM3]).

Different from these two methods, in [DL] D. Dong and the author introduced a new method, the iteration method of Maslov-type index theory, to study this prescribed minimal period solution problem. In this section we apply the iteration formula of the

Maslov-type index theory to this problem and describe two recent results obtained in [DL] via the iteration formula of the Maslov-type index theory.

Example 1. $n = 1$, *i.e the problem on* $\mathbf{R}^2$.

Under suitable conditions on H, using the saddle point theorem, the saddle point reduction method or Galerkin approximation method, one can find a τ-periodic non-constant solution x_0 of (8.1)-(8.2) such that there holds

$$n + 1 - \nu_\tau(x_0) \le i_\tau(x_0) \le n + 1, \quad 1 \le \nu_\tau(x_0) \le 2n. \tag{8.3}$$

Denoting the minimal period of x_0 by $\tau/m = \delta$ for some $m \in \mathbf{N}$, and the restriction of x_0 to the interval $[0, \tau/m]$ by x. Since in our case there holds $n = 1$, we obtain

$$2 - \nu_{m\delta}(x^m) \le i_{m\delta}(x^m) \le 2, \quad 1 \le \nu_{m\delta}(x^m) \le 2. \tag{8.4}$$

Let $\gamma = \gamma_x$. Then we distinct our further study in two cases:

Case 1. $\gamma(\delta) \in \mathrm{Sp}(2)^0_-$ *or* $\gamma(\delta) = I$.

By (7.6) and (7.7) we obtain

$$i_\delta(x) \text{ is odd}, \;\; i_{m\delta}(x^m) = m(i_\delta(x) + 1) - 1, \;\; 1 \le \nu_{m\delta}(x^m) \le 2. \tag{8.5}$$

Plugging (8.5) into (8.4) we obtain $0 \le m(i_\delta(x)+1)-1 \le 2$. This yields $1 \le m(i_\delta(x)+1) \le 3$. Thus $i_\delta(x)+1 \ge 1$. But $i_\delta(x)$ is odd, there must hold $i_\delta(x)+1 \ge 2$. This implies $m = 1$.

Case 2. $\gamma(\delta) \in \mathrm{Sp}(2)^0_-$ *or* $\gamma(\delta) = I$.

By (7.8) we obtain

$$i_\delta(x) \text{ is even}, \;\; i_{m\delta}(x^m) = m i_\delta(x), \;\; \nu_{m\delta}(x^m) = 1. \tag{8.6}$$

Plugging (8.6) into (8.4) we obtain $1 \le m i_\delta(x) \le 2$. This yields $i_\delta(x) \ge 1$. But $i_\delta(x)$ is even, there must hold $i_\delta(x) \ge 2$. This implies $m = 1$.

Therefore we have proved

Theorem 8.1. *For $\tau > 0$ suppose x is a non-constant τ-periodic solution of (8.1)-(8.2) satisfying (8.3) with $n = 1$. Then x possesses τ as its minimal period.*

Note that by Example 11.6 given in [DL], when $n \ge 2$ the condition (8.3) is not sufficient to yield the minimality of τ as the minimal period of x.

Example 2. *Controlling the minimal period via Maslov-type indices.*

The main tool in our following study is the Corollary 2.21.

Theorem 8.2. *Suppose the following condition holds:*

(H1'). $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$.

For $\tau > 0$, let $x \in C^2(S_\tau, \mathbf{R}^{2n})$ be a τ-periodic solution of the problem (8.1)-(8.2) with minimal period τ/k for some $k \in \mathbf{N}$. If the Maslov-type indices of x satisfy the following conditions:

(X1) $i_\tau(x) \le n + 1$.

(X2) $i_{\tau/k}(x|_{[0,\tau/k]}) \ge n$.

Then $k = 1$, i.e. the solution x possesses minimal period τ.

Proof. Note that (8.1) is autonomous, and x is not a constant function. Thus $\dot{x}$ is a nontrivial τ/k-periodic solution of the linear system (1.2) with $B(t) = H''(x(t))$. This proves that the following condition holds:

(X3) $\nu_{tau/k}(x|_{[0,\tau/k]}) \geq 1$.

Now we can apply Corollary 2.21 to the solution x, and conclude that $k = 1$, i.e. x possesses minimal period τ. ∎

Note that the condition (X1) is satisfied by solutions obtained via the saddle point theorem, the condition (X2) is satisfied if H is convex along the orbit of x, and the condition (X3) is satisfied if (8.1) is autonomous and x is not constant. Thus our Theorem 8.2 actually already contains all the results on Rabinowitz' conjecture so far under various convexity conditions mentioned earlier. Specially, Theorem 8.2 points out that the minimality of the given period τ is completely determined by the Maslov-type indices of the solution, and does not depend on the particular method which was used to obtain the solution.

The following result with more accesible condition on H is proved in [DL].

Theorem 8.3.(cf. [DL]) *Suppose the Hamiltonian function H satisfies (H1'), (H3), (H4), and for some $\tau > 0$ the following conditions hold:*

(HT) There exists a positive definite matrix $B \in \mathcal{L}_s(\mathbf{R}^{2n})$ such that

$$H'(x) = Bx + o(|x|) \qquad as \quad |x| \to \infty.$$

with $i_\tau(B) > n$ and $\nu_\tau(B) = 0$.

(H5) $H''(x) \geq 0$ for all $x \in \mathbf{R}^{2n}$.

(H6) The set $D = \{x \in \mathbf{R}^{2n} | H'(x) \neq 0,\ 0 \in \sigma(H''(x))\}$ is hereditarily disconnected, i.e. every connected component of D contains only one point.

Then the problem (8.1) and (8.2) possesses a nonconstant τ-periodic solution with τ as its minimal period.

Idea of the proof. Note that (H1'), (H3), (H4), and (HT) imply the existence of a τ-periodic nonconstant solution x_0 of (8.1)-(8.2) such that (X1) holds. By (H5) and (H6) we obtain (X2). Thus Theorem 8.2 can be applied to conclude that τ is the minimal period of x_0. ∎

Note that using the method of [DL], G. Fei and Q. Qiu proved the following result by the Galerkin approximation method.

Theorem 8.4.(cf. [FQ2]) *Suppose the Hamiltonian function H satisfies (H1'), (H2), (H3), (H4), (H5), and (H6). Then for any $\tau > 0$, the problem (8.1) and (8.2) possesses a nonconstant τ-periodic solution with τ as its minimal period.*

Note that the conditions (H5) and (H6) are weaker than the condition $H''(x) > 0$ for all $x \neq 0$ used in [EH] and [Ek3]. Thus Theorems 8.3 and 8.4 generalize corresponding results of I. Ekeland and H. Hofer. In [LL3], Theorem 8.4 was slightly generalized.

Remark 8.5. Note that recently a new method has been introduced into the study of the prescribed minimal period solution problem for the second order Hamiltonian systems without convexity conditions by [Lo5], [Lo6], and [Lo8]. We consider the existence of non-constant periodic solutions with prescribed minimal period for the following autonomous second order Hamiltonian systems,

$$\ddot{x} + V'(x) = 0, \qquad \forall x \in \mathbf{R}^n, \tag{8.8}$$

Then we have the following result under precisely Rabinowitz' original structure conditions.

Theorem 8.6.(cf. [Lo5] and [Lo8]) *Suppose V satisfies the following conditions:*

(V1) $V \in C^2(\mathbf{R}^n, \mathbf{R})$.

(V2) There exist constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu V(x) \le V'(x) \cdot x, \qquad \forall |x| \ge r_0.$$

(V3) $V(x) \ge 0 \qquad \forall x \in \mathbf{R}^n$.

(V4) $V(x) = o(|x|^2)$, *at* $x = 0$.

Then for every $\tau > 0$, the system (8.8) possesses a non-constant τ-periodic even solution with minimal period τ/k for some integer k satisfying $1 \le k \le n+1$.

Theorem 8.7.(cf. [Lo5] and [Lo8]) *Suppose V satisfies conditions (V1)-(V3) and the following condition,*

(V5) There exist constants $\omega > 0$ and $r_1 > 0$ such that

$$V(x) \le \frac{\omega}{2}|x|^2, \qquad \forall |x| \le r_1.$$

Then for every positive $\tau < \frac{2\pi}{\sqrt{\omega}}$, the conclusion of Theorem 8.6 holds.

Our proofs of these theorems depend on a new iteration inequality of the Morse index theory for the functional corresponding to the system (8.8) defined on even function spaces. We refer readers to [Lo5] and [Lo8] for more details, and to [Lo6] for further results in this spirit.

Remark 8.8. Results for the first order Hamiltonian system (8.1) similar to Theorems 8.6 and 8.7 are still unknown. We suspect that this may reflects a substantial difference between first order and second order Hamiltonian systems.

§9. Hyperbolic closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$.

In this section, we introduce a result proved in [Ek2] and [Lo12] on the stability problem of closed characteristics in prescribed energy surface in $\mathbf{R}^{2n}$, which depends on the precise iteration formula of the Maslov-type indices of hyperbolic periodic solutions of Hamiltonian systems.

Let Σ be a C^2 compact hypersurface in $\mathbf{R}^{2n}$ bounding a convex set C with non-empty interior. We denote the set of all such hypersurfaces in $\mathbf{R}^{2n}$ by $\mathcal{H}(2n)$. For $x \in \Sigma$ let

$N_\Sigma(x)$ be the unit vector on the outward normal to Σ at x. We consider the given energy problem of finding $\tau > 0$ and an absolutely continuous curve $x : [0, \tau] \to \mathbf{R}^{2n}$ such that

$$\begin{cases} \dot{x}(t) = JN_\Sigma(x(t)), & x(t) \in \Sigma, \qquad \forall t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{9.1}$$

A solution (τ, x) of the problem (9.1) with τ being the minimal period of x is called **a closed characteristic** on Σ. Denote by $\mathcal{J}(\Sigma)$ the set of all closed characteristics on Σ. To cast the problem (9.1) into a Hamiltonian version, we follow the Chapter V of [Ek3]. For a given $\Sigma \in \mathcal{H}(2n)$ bounding a convex set C. Without loss of generality we assume that the origin is in the interior of C. Let $j_C : \mathbf{R}^{2n} \to [0, +\infty)$ be the gauge function of C defined by

$$j_C(0) = 0 \qquad \text{and} \qquad j_C(x) = \inf\{\lambda \mid \frac{x}{\lambda} \in C\} \quad \text{for} \quad x \neq 0.$$

Fix a constant α satisfying $1 < \alpha < 2$ in this section. As usual we define the Hamiltonian function $H_\alpha : \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H_\alpha(x) = j_C(x)^\alpha, \qquad \forall x \in \mathbf{R}^{2n}.$$

Then $H_\alpha \in C^1(\mathbf{R}^{2n}, \mathbf{R}) \cap C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (9.1) is equivalent to the following problem

$$\begin{cases} \dot{x}(t) = JH'_\alpha(x(t)), & H_\alpha(x(t)) = 1, \qquad \forall t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \tag{9.2}$$

A solution (τ, x) is **hyperbolic**, if in $\sigma(\gamma_x(\tau))$ except that 1 is a double eigenvalue all the other eigenvalues are not on the unit circle $\mathbf{U}$ in the complex plane $\mathbf{C}$.

Note that if we use our matrix J through out all the discussion in [Ek3], by [Br] and [Lo12] we obtain (6.3) and (6.4). Thus by Theorem V.4 of [Ek3], there exist $\alpha \in (1, 2)$ sufficiently close to 2 and $c_0 > 0$ such that for every $k \in \mathbf{N}$ there are a solution (τ, x) of (9.2) and an $m \in \mathbf{N}$ satisfying

$$\left| \frac{i_\tau(x)}{\tau} \right| \leq c_0, \tag{9.3}$$

$$i_{m\tau}(x^m) \leq 2k - 2 + n \leq i_{m\tau}(x^m) + \nu_{m\tau}(x^m) - 1. \tag{9.4}$$

In this case we call (τ, x) being **(k, m)-variationally visible.**

Next we study the structure of the associated symplectic matrix $\gamma_x(\tau)$ of (τ, x) for any $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ with $1 < \alpha < 2$.

Now by Lemma I.7.3 of [Ek3], the non-zero vectors $\dot{x}(0)$ and $x(0)$ satisfy

$$\begin{cases} \gamma_x(\tau)\dot{x}(0) = \dot{x}(0), \\ \gamma_x(\tau)x(0) = \tau(\alpha - 2)\dot{x}(0) + x(0). \end{cases} \tag{9.5}$$

Define

$$\xi_1 = \tau(\alpha-2)\dot{x}(0), \qquad \xi_2 = x(0). \tag{9.6}$$

Then (9.5) becomes

$$\begin{cases} \gamma_x(\tau)\xi_1 = \xi_1, \\ \gamma_x(\tau)\xi_2 = \xi_1 + \xi_2, \end{cases} \tag{9.7}$$

Lemma 9.1.(cf. [Lo12]) *For $1 < \alpha < 2$ and $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, there exist matrices $P \in \mathrm{Sp}(2n)$ and $M \in \mathrm{Sp}(2n-2)$ such that there holds*

$$\gamma_x(\tau) = P(N_1(1,1)\diamond M)P^{-1}, \tag{9.8}$$

where $N_1(1,1)$ is defined in the Definition 4.5. Note that there holds $N_1(1,1) \in \mathrm{Sp}(2)^0_-$ (cf. The section 3).

Idea of the Proof. Fix $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and define ξ_1 and ξ_2 by (9.6). Firstly, we use the star-shape property of Σ and $1 < \alpha < 2$ to show

$$\xi_1^* J \xi_2 = \tau(\alpha-2)\langle H'_\alpha(x(0)), x(0)\rangle < 0. \tag{9.9}$$

Secondly, we use (9.9) to show $\{\xi_1, \xi_2\}$ form a Jordan block of $\gamma_x(\tau)$. belonging to the eigenvalue 1. Then (9.8) follows from these result by choosing a symplectic coordinate system suitably. ∎

Remark 9.2. Lemma 9.1 holds for $\alpha = 2$ or $\alpha > 2$ with $N_1(1,1)$ in (9.8) being replaced by I_2 or $N_1(1,-1)$ respectively.

Note that Lemma I.7.3 of [Ek3] and the discussions on the pp.407-408 of [Ek2] only proves that the vectors $\dot{x}(0)$ and $x(0)$ satisfy (9.5). Our Lemma 3.2 further proves that they actually form a 2-dimensional invariant subspace of $\gamma_x(\tau)$ belonging to the eigenvalue 1, no matter whether x is hyperbolic or not. Specially our Lemma 9.1 on $\gamma_x(\tau)$ is stronger than (7) of [Ek2].

Suppose (τ, x) is a hyperbolic solution of (9.2). From (9.8), since P can be connected to I in $\mathrm{Sp}(2n)$, there is a path $h : [0, \tau] \to \mathrm{Sp}(2n)$ such that $h(0) = \gamma_x(\tau)$, $\dim\ker(h(s)-I) = 1$ for all $0 \le s \le \tau$, and $h(\tau)$ is of one of the following forms:

$$N_1(1,1)\diamond D(2)\diamond\cdots\diamond D(2)\diamond D(-2), \tag{9.10}$$

$$N_1(1,1)\diamond D(2)\diamond\cdots\diamond D(2)\diamond D(2). \tag{9.11}$$

Then by the homotopy invariance of the Maslov-type index theory described in the Theorem 1.11 for any $m \in \mathbf{N}$ there hold

$$i_{m\tau}(\gamma_x) = i_{m\tau}(h * \gamma_x), \qquad \nu_{m\tau}(\gamma_x) = \nu_{m\tau}(h * \gamma_x). \tag{9.12}$$

Here for notational simplicity, we identify a path γ and its iterations. Therefore from (9.10)-(9.12), (7.6), and (7.9) (or by (7.3) and (7.4)) we obtain

$$i_{m\tau}(x^m) = m(i_\tau^{(1)} + 1) - 1 + m i_\tau^{(2)} + \cdots + m i_\tau^{(n)} \tag{9.13}$$

$$= m(i_\tau(x) + 1) - 1, \qquad \forall m \in \mathbf{N}, \tag{9.14}$$

$$\nu_{m\tau}(x^m) = 1, \qquad \forall m \in \mathbf{N}, \tag{9.15}$$

and $i_\tau(x)$ is even if (9.10) happens, $i_\tau(x)$ is odd if (9.11) happens. Here in (9.13) we use $i_\tau^{(k)}$ to denote the index of the k-th path in $\mathcal{P}_\tau(2)$ corresponding to the k-th 2×2 matrix in (9.10) or (9.11).

Remark 9.3. As proved in [Lo12], similar to (9.14) and (9.15) for hyperbolic $(\tau,x)\in\mathcal{J}(\Sigma,\alpha)$, there hold $i_\tau(x)\ge n$ and

$$\begin{aligned} i_{m\tau}(x^m) &= m(i_\tau(x)+1)-1, \quad \nu_{m\tau}(x^m)=2, \quad \forall m\in\mathbf{N}, \quad \text{if } \alpha=2,\\ i_{m\tau}(x^m) &= mi_\tau(x), \quad \nu_{m\tau}(x^m)=1, \quad \forall m\in\mathbf{N}, \qquad \text{if } \alpha>2. \end{aligned}$$

Note that all these iteration formulae for hyperbolic paths are special cases of Theorem 7.2.

Now from (9.4) and (9.15), every (k,m)-variationally visible hyperbolic solution (τ,x) of (9.2) satisfies

$$i_{m\tau}(x^m)=2k-2+n. \tag{9.16}$$

Definition 9.4. *Define the variationally visible hyperbolic index cover set $\mathcal{I}_h(\Sigma,\alpha)$ of (Σ,α) by*

$$\begin{aligned} \mathcal{I}(\Sigma,\alpha) = \{q\in\mathbf{N}\mid \exists m,k\in\mathbf{N} \text{ and } (\tau,x)\in\mathcal{V}_{m,k}(\Sigma,\alpha) \text{ such that}\\ q\in[i_{m\tau}(x^m), i_{m\tau}(x^m)+\nu_{m\tau}(x^m)-1]\}, \end{aligned} \tag{9.17}$$

and the variationally visible hyperbolic index cover set $\mathcal{I}_h(\Sigma,\alpha)$ of (Σ,α) by

$$\begin{aligned} \mathcal{I}_h(\Sigma,\alpha) = \{q\in\mathbf{N}\mid \exists m,k\in\mathbf{N} \text{ and hyperbolic } (\tau,x)\in\mathcal{V}_{m,k}(\Sigma,\alpha)\\ \text{such that } q=i_{m\tau}(x^m)\}. \end{aligned} \tag{9.18}$$

Now we can describe the main result obtained in [Lo12]:

Theorem 9.5. (cf. [Lo12]) *For any $\Sigma\in\mathcal{H}(2n)$ and $1<\alpha<2$, suppose the minimal periods of all hyperbolic variationally visible solution of (9.2) are uniformally bounded from above. Then there holds*

$$\mathcal{I}(\Sigma,\alpha)\setminus\mathcal{I}_h(\Sigma,\alpha)\ne\emptyset. \tag{9.19}$$

Proof. Suppose (9.19) is not true, we prove it by contradiction. In this case by (9.3) there there holds an integer $q_0>0$ such that there hold

$$\begin{aligned} 2\mathbf{N}-2+n &\subset \mathcal{I}(\Sigma,\alpha)=\mathcal{I}_h(\Sigma,\alpha)\\ &\subset \{m(i_\tau(x)+1)-1 \mid (\tau,x) \text{ is hyperbolic variationally visible}\}. \end{aligned} \tag{9.20}$$

By (9.3) and the uniformly boundedness assumption on the minimal periods, there exists an integer $q_0>0$ such that all (τ,x) appeared in the right hand side of (9.20) satisfies

$$n\le i_\tau(x)\le q_0. \tag{9.21}$$

If n is even, so is $2k-2+n$. By (9.14), (9.15), and (9.21) all hyperbolic $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ with odd $i_\tau(x)$ can not be variationally visible. Therefore there exists $q_0 \in \mathbf{N}$ such that any hyperbolic $(\tau, x) \in \mathcal{V}(\Sigma, \alpha)$ must satisfy $i_\tau(x) = 2q$ with $q \in [n/2, q_0]$ and $i_{m\tau}(x^m) = m(2q+1) - 1$ for all $m \in \mathbf{N}$. This implies that the right hand side of (9.20) must fall into only finitely many possible patterns:

$$\{m(2q+1)-1\}_{m\in\mathbf{N}}, \qquad \text{for} \quad n/2 \le q \le q_0. \tag{9.22}$$

Together with (9.20) there holds

$$2\mathbf{N} - 2 + n \subset \{m(2q+1) - 1 \,|\, n/2 \le q \le q_0\}. \tag{9.23}$$

Now we choose a prime number $p > \max\{n, 2q_0 + 1\}$, and define $k = (p+1-n)/2$. By (9.23) there must exist integers $q \in [n/2, q_0]$ and $m \in \mathbf{N}$ such that there holds

$$p = 2k - 1 + n = m(2q+1). \tag{9.24}$$

Since $p > 2q_0 + 1$, we must have $m > 1$. This contradicts to the choice of p, and proves the Theorem when n is even.

The proof for odd n is similar and can be found in [Lo12], thus is omitted. ∎

A direct consequence of this theorem is:

Theorem 9.6. (cf. [Lo12]) *On every C^2-compact hypersurface Σ in $\mathbf{R}^{2n}$ bounding a convex set with non-empty interior, either there exists a sequence of variationally visible hyperbolic closed characteristics with their minimal periods tending to infinity, or there exists at least one variationally visible nonhyperbolic closed characteristic.*

Remark 9.7. The existence of at least one closed characteristic on any $\Sigma \in \mathcal{H}(2n)$ was first established by P. Rabinowitz [Ra1] (for a much wider class of hypersurfaces) and A. Weinstein [We] independently in 1978. The question of whether there exists nonhyperbolic closed characteristic on any $\Sigma \in \mathcal{H}(2n)$ is a long standing problem in Hamiltonian dynamics as mentioned at the end of [Ek3] by I. Ekeland. We refer the readers to [Del], [DDE], and [Ek3] for further results and references therein. Note that in the recent paper [LL2] of C. Liu and the author, Theorems 9.5 and 9.6 have been generalized to star-shaped compact smooth hypersurfaces in $\mathbf{R}^{2n}$.

§10. Multiple periodic points of the Poincaré map of Lagrangian systems on tori.

In this section, we use the iteration inequality (6.1) together with other techniques to study the multiplicity of periodic solutions of Lagrangian systems on tori.

We consider the following Lagrangian system

$$\frac{d}{dt}L_{\dot{x}}(t, x, \dot{x}) - L_x(t, x, \dot{x}) = 0, \quad x \in \mathbf{R}^n, \tag{10.1}$$

where $L_{\dot{x}}$ and L_x denote the gradients of L with respect to $\dot{x}$ and x respectively. We consider the following conditions on the Lagrangian function L:

(L1) $L(t,x,p) = \frac{1}{2}A(x)p\cdot p + V(t,x)$, *where* $\frac{1}{2}A(x)p\cdot p \geq \lambda|p|^2$ *for all* $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$ *and some fixed constant* $\lambda > 0$.

(L2) $A \in C^3(\mathbf{R}^n, \mathcal{L}(\mathbf{R}^n))$ *is symmetric,* $V \in C^3(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$, *both* A *and* V *are* 1*-periodic in all of their variables.*

Let L satisfy the conditions (L1) and (L2). The system (10.1) can be viewed as defined on the standard torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$. We search for τ-periodic solutions of the system (10.1) with $\tau \in \mathbf{N}$. Solutions $x(t)$ of (10.1) on T^n determines a one parameter family of diffeomorphisms $\Phi_L^t \in \mathrm{Diff}(TT^n)$ satisfying $\Phi_L^t(x(0), \dot{x}(0)) = (x(t), \dot{x}(t))$. We call the time-1-map $\Phi_L = \Phi_L^1$ the **Poincaré map** of the system (10.1) corresponding to the Lagrangian function L. The following Theorems 10.1 and 10.2 are main results in recent [Lo15] of the author and [LLu] of G. Lu and the author.

Theorem 10.1. (cf. [Lo15]) *Suppose the function L satisfies the conditions (L1) and (L2). Then the corresponding Poincaré map Φ_L of the Lagrangian system (10.1) possesses infinitely many periodic points on T^n produced by contractible integer periodic solutions.*

Two solutions x_1 and x_2 of the system (10.1) on T^n are **geometrically distinct**, if their orbits $\mathcal{O}(x_i) = \cup\{x_i(t) \in T^n \mid t \in \mathbf{R}\}$ for $i = 1, 2$ are not the same.

Theorem 10.2. (cf. [LLu]) *Suppose the function L satisfies the conditions (L1), (L2), and is independent of the time t. Then the autonomous Lagrangian system (10.1) possesses infinitely many geometrically distinct contractible integer periodic solution orbits on T^n.*

These theorems can be viewed as the generalizations of the corresponding contractible solution structures of the following simple pendulum equation:

$$\ddot{x} + \lambda \sin x = 0, \tag{10.2}$$

where $\lambda = g/l$, g is the gravitation constant, and l is the length of the pendulum. The flow defined by (10.2) is shown in the Figure 10.1, where the circular orbits indicated in the island are those found by Theorem 10.2.

The study of integer periodic solutions of the system (10.1) under the conditions (L1) and (L2) possesses a very long history. The most standard model is the pendulum type systems with periodic forcing terms (cf. [CLZ]). We refer to [Ra1] and [FW] for further references. Note that in Theorems 10.1 and 10.2 we have no any non-degeneracy restriction on any solutions of (10.1). This is rather different from the many known results for Lagrangian system (10.1) as mentioned in [FW] and those for first order Hamiltonian systems in [CZ2], [CZ3], [LZ], [SZ1], and [SZ2]. Our these two theorems can be viewed as a confirm answer and generalizations to the Lagrangian system analogue of C. Conley's conjecture on periodic points of Hamiltonian diffeomorphisms mentioned on the pages 1304 to 1305 of [SZ2]. We suspect that (10.1) always possesses infinitely many geometrically distinct contractible integer periodic solutions with mutually distinct solution orbits on T^n provided (L1) and (L2) hold.

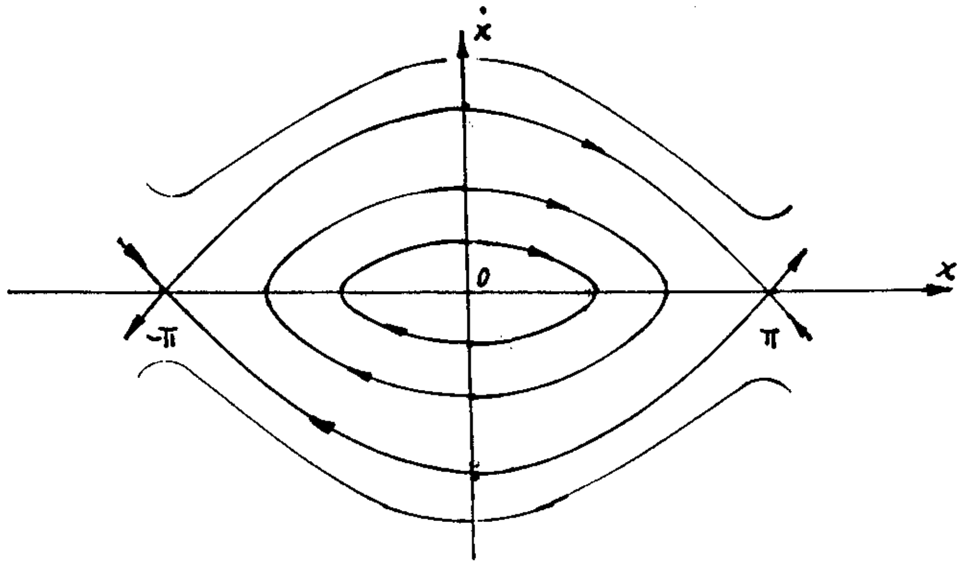

Figure 10.1. The dynamics of a simple pendulum.

In this introduction of these results, we give the idea of the proof of Theorem 10.1. Here special emphasis is given on how the iteration theory of the Maslov-type index plays a role in such a proof. For the details of the proof of Theorem 10.1 and the proof of Theorem 10.2, we refer the readers to [Lo15] and [LLu].

In the following we fix an L satisfying (L1) and (L2). For any $\tau \in \mathbf{N}$, let $S_\tau = \mathbf{R}/(\tau\mathbf{Z})$ and $E_\tau = W^{1,2}(S_\tau, \mathbf{R}^n)$ with the usual inner product and the norm:

$$\langle x, y\rangle = \int_0^\tau (x \cdot y + \dot{x} \cdot \dot{y})dt, \qquad \|x\| = \langle x, x\rangle^{1/2}, \qquad \forall\, x, y \in E_\tau.$$

Define

$$f_\tau(x) = \int_0^\tau L(t, x, \dot{x})dt, \qquad \forall\, x \in E_\tau. \tag{10.3}$$

By the conditions (L1) and (L2), it is well known that $f_\tau \in C^3(E_\tau, \mathbf{R})$ satisfies the Palais-Smale condition and that critical points of f_τ correspond to contractible τ-periodic solutions of (10.1) on T^n. But for distinct $h, k \in \mathbf{N}$, although two critical points $x \in E_h$ of f_h and $y \in E_k$ of f_k are different in this analytical setting, they may produce the same periodic point for Φ_L on T^n. To get over this obstacle, we consider the following analytic concepts of iterations and towers.

Let $x \in E_\tau$ with $\tau \in \mathbf{N}$. For any $m \in \mathbf{N}$, the m-th iteration $x^m \in E_{m\tau}$ of x is defined at the beginning of Chapter 2. We define the iteration map $\psi^m : E_\tau \to E_{m\tau}$ by $\psi^m(x) = x^m$. In this case, we call $\{x^m\}_{m\in\mathbf{N}} \subset \Pi_{m\in\mathbf{N}} E_{m\tau}$ the **tower** based on $x \in E_\tau$. A tower $\{x^p\}$ based on $x \in E_\tau$ is called a subtower of another tower $\{y^q\}$ based on $y \in E_\alpha$, if there exists $k \in \mathbf{N}$ and $j \in \mathbf{Z}^n$ such that $x = y^k + j$. Two towers $\{x^p\}$ based on $x \in E_\tau$ and $\{y^q\}$ based on $y \in E_\alpha$ are called T^n-**distinct** , if there exists no tower $\{z^m\}$ based on $z \in E_\beta$, $h, k \in \mathbf{N}$, and $i, j \in \mathbf{Z}^n$ such that $x = z^h + i$ and $y = z^k + j$. Note that the functional f_τ defined by (10.3) satisfies

$$f_{m\tau}(x^m + j) = mf_\tau(x), \qquad \forall\, x \in E_\tau,\, j \in \mathbf{Z}^n,\, \tau, m \in \mathbf{N}, \tag{10.4}$$

Here the solution tower is an analytic concept, and is not a geometric concept. Using this concept, Theorem 10.1 can be rephrased as the following

Theorem 10.3.(cf. [Lo15]) *Suppose the function L satisfies the conditions (L1) and (L2). Then the system (10.1) possesses infinitely many T^n-distinct solution towers based on integer periodic solutions in $\mathbf{R}^n$.*

We prove Theorem 10.3 indirectly by supposing the following assumption:

(LF) *The system (10.1) possesses only finitely many T^n-distinct solution towers based on contractible integer periodic solutions in $\mathbf{R}^n$.*

Therefore under (LF) we can assume that there exist positive integers τ and p such that (10.1) possesses only finitely many $\mathbf{Z}^n$-translation independent τ-periodic solutions $\{x_1, \ldots, x_p\}$ on $\mathbf{R}^n$ and all the other $m\tau$-periodic solutions of (10.1) on $\mathbf{R}^n$ for $m \in \mathbf{N}$ are $\mathbf{Z}^n$-translations of iterations of these x_k's, i.e. the critical point set $\mathcal{K}(f_{m\tau})$ of $f_{m\tau}$ in $E_{m\tau}$ has the form

$$\mathcal{K}(f_{m\tau}) = \{x_k^m + j \mid 1 \le k \le p,\ j \in \mathbf{Z}^n\}, \tag{10.5}$$

and the critical value set $f_{m\tau}(\mathcal{K}(f_{m\tau})) \subset \mathbf{R}$ is a finite non-empty set.

The critical module of f_τ at its isolated critical point x is defined by

$$C_q(f_\tau, x) = H_q(W, W^-), \qquad \forall q \in \mathbf{Z}. \tag{10.6}$$

where (W, W^-) is a Gromoll-Meyer pair of f_τ at x and W^- is the exit set which are introduced in [GM], and we refer to Definition I.5.1 on p.48 of [Ch2]. In this discussion, we always choose the coefficient field of the homology to be $\mathbf{R}$, and omit it from all the homological notations.

Note that $E_{m\tau}$ possesses an orthogonal decomposition

$$E_{m\tau} = \mathbf{R}^n \oplus W_{m\tau}$$

with $W_{m\tau}$ formed by functions in $E_{m\tau}$ possessing zero mean value. Using Lemma II.5.2 on p.127 of [Ch2] and our (10.4), we obtain that there exist two real numbers $c_m < d_m$ depending on $f_{m\tau}$ such that

$$c_m < f_{m\tau}(x) < d_m, \qquad \forall\, x \in \mathcal{K}(f_{m\tau}), \tag{10.7}$$

and the following isomorphisms hold:

$$H_n((f_{m\tau})_{d_m}, (f_{m\tau})_{c_m}) \cong H_n(E_{m\tau}, (f_{m\tau})_{c_m}) \cong H_n(T^n) \neq 0, \tag{10.8}$$

where $(f_{m\tau})_c \equiv \{z \in E_{m\tau} \mid f_{m\tau}(z) \le c\}$. By the isolatedness of points in $\mathcal{K}(f_{m\tau})$, a slight modification of J. Q. Liu's theorem (Theorem II.1.5 on p.89 of [Ch2]) which makes it work for critical modules defined by (10.6) yields the existence of at least one critical point x of $f_{m\tau}$ such that

$$c_m < f_{m\tau}(x) < d_m, \qquad \text{and} \qquad C_n(f_{m\tau}, x) \neq 0. \tag{10.9}$$

Together with Example 1 on p.33 of [Ch2], this critical point x must be a non-minimal saddle point. From (10.9) and the shifting theorem of [GM] (cf. also p.50 of [Ch2]), we then obtain immediately,

Lemma 10.4.(cf. [Lo15]) *By the assumption (LF), for every $m \in \mathbf{N}$, there exists an $m\tau$-periodic solution y_m of the system (10.1) such that*

$$C_n(f_{m\tau}, y_m) \neq 0 \quad \text{and} \quad n - \nu_{m\tau}(y_m) \le i_{m\tau}(y_m) \le n. \tag{10.10}$$

Next we consider the homological injectivity of the homomorphism induced by the iteration map ψ^m under the assumption (LF).

By (LF) and (10.4), if necessary replacing the Lagrangian function L by $L + b$ with a large enough constant $b > 0$, we always assume in the following that there exists a constant $A_0 > 0$ such that for every $m \in \mathbf{N}$ and every critical point x of $f_{m\tau}$ in $E_{m\tau}$ there holds

$$f_{m\tau}(x) \ge A_0 > 0. \tag{10.11}$$

Note that the following Lemma 10.5 is crucial for our proof of (10.17) below and thus for the homological injection Theorem 10.6. The proof of Lemma 10.5 is the only place in this paper where we need the iteration inequality (6.1) of the Maslov-type indices (i.e. Morse indices here) proved in [LL1].

Lemma 10.5.(cf. [Lo15]) *Suppose* (LF) *holds. Then there exists a constant $A_1 \ge A_0$ such that for every $m \in \mathbf{N}$ and every $m\tau$-periodic solution x of* (10.1) *there holds*

$$C_{n+1}(f_{m\tau}, x) = 0, \qquad \text{whenever} \quad f_{m\tau}(x) > A_1. \tag{10.12}$$

Proof. Note that there holds $x_k^m + j = (x_k + j)^m$. By the shifting theorem and the generalized Morse lemma (cf. [GM]), and the Künneth formula, we obtain

$$\begin{aligned} C_{n+1}(f_{m\tau}, x_k^m + j) &\cong C_{n+1-i_{m\tau}(x_k^m)}(\text{degenerate part}, 0) \otimes C_{i_{m\tau}(x_k^m)}(\text{nondegenrate part}, 0) \\ &\cong C_{n+1-i_{m\tau}(x_k^m)}(\text{degenerate part}, 0) \otimes \mathbf{R} \\ &\cong C_{n+1-i_{m\tau}(x_k^m)}(\text{degenerate part}, 0). \end{aligned} \tag{10.13}$$

Note that the dimension of the domain of the degenerate part is $\nu_{m\tau}(x_k^m)$ which always satisfies $0 \le \nu_{m\tau}(x_k^m) \le 2n$.

If $\hat{i}_\tau(x_k) > 0$, then for sufficiently large $m \in \mathbf{N}$, the right hand side of (10.13) must be killed because of the iteration estimate (6.1). If $\hat{i}_\tau(x_k) = 0$, then for any $m \in \mathbf{N}$, the right hand side of (10.13) is still killed because of the iteration estimate (6.1). This proves the lemma. ∎

By (LF) and Lemma 10.4, we obtain an infinite subsequence Q of $\{2^m \mid m \in \{0\} \cup \mathbf{N}\}$, some $k \in \mathbf{N}$ and an x in $\{x_1, \ldots, x_p\}$ such that for every $m \in Q$ there hold $C_n(f_{k\tau}, x^k) \neq 0$, $i_{mk\tau}(x^{mk}) = i_{k\tau}(x^k)$, and $\nu_{mk\tau}(x^{mk}) = \nu_{k\tau}(x^k)$.

To simplify notations, without loss of generality, from now on we rename $k\tau$ by τ and suppose $1 \in Q$, $x \in \{x_1, \ldots, x_p\}$, and for any $m \in Q$:

$$C_n(f_\tau, x) \neq 0, \quad i_{m\tau}(x^m) = i_\tau(x), \quad \nu_{m\tau}(x^m) = \nu_\tau(x). \tag{10.14}$$

Then it is proved in [Lo15] that the iteration map ψ^m induces isomorphisms

$$0 \neq C_n(f_{k\tau}, x^k) \xrightarrow{\cong} C_n(f_{mk\tau}, x^{mk}), \qquad \forall\, k, mk \in Q.$$

Let $c = f_\tau(x)$.

Note that $\mathcal{K}(f_{m\tau}))$ contains only finitely many points. By Theorem I.3.2 on p.23 and the idea of the proof of Theorem I.4.2 on pp.35-36 of [Ch2], there exists $\epsilon > 0$ sufficiently small such that

$$f_{m\tau}(\mathcal{K}(f_{m\tau})) \cap [m(c - 3\epsilon), m(c + 3\epsilon)] = \{mc\},$$

and the inclusion map

$$\begin{aligned} h_2 \circ h_1 : (W(x^m), W(x^m)^-) &\xrightarrow{h_1} ((f_{m\tau})_{m(c+2\epsilon)}, (f_{m\tau})_{m(c-2\epsilon)}) \\ &\xrightarrow{h_2} ((f_{m\tau})_{m(c+2\epsilon)}, (f_{m\tau})^\circ_{m(c-\epsilon)}), \end{aligned} \tag{10.15}$$

induces a monomorphism on homology modules:

$$(h_2 \circ h_1)_* : C_n(f_{m\tau}, x^m) \to H_n((f_{m\tau})_{m(c+2\epsilon)}, (f_{m\tau})^\circ_{m(c-\epsilon)}), \tag{10.16}$$

where B° denotes the interior of B.

If $H_{n+1}(E_{m\tau}, (f_{m\tau})_{m(c+2\epsilon)}) \neq 0$, using (LF), a slight modification of J. Q. Liu's theorem (Theorem II.1.5 on p.89 of [Ch2]) as in the discussion on (10.9), we would obtain a critical point z of $f_{m\tau}$ such that $C_{n+1}(f_{m\tau}, z) \neq 0$. When m is sufficiently large, this violates Lemma 10.5. Thus there exists $m_0 > 0$ such that

$$H_{n+1}(E_{m\tau}, (f_{m\tau})_{m(c+2\epsilon)}) = 0, \qquad \forall\, m \in Q(m_0). \tag{10.17}$$

Here we denote by $Q(k) = \{m \in Q \mid m \geq k\}$. Then the exact sequence of homology modules for the triple

$$(E_{m\tau},\ (f_{m\tau})_{m(c+2\epsilon)},\ (f_{m\tau})^\circ_{m(c-\epsilon)})$$

shows that the inclusion map

$$h_3 : ((f_{m\tau})_{m(c+2\epsilon)}, (f_{m\tau,s})^\circ_{m(c-\epsilon)}) \to (E_{m\tau}, (f_{m\tau})^\circ_{m(c-\epsilon)}) \tag{10.18}$$

induces a monomorphism on homology modules:

$$(h_3)_*: H_n((f_{m\tau})_{m(c+2\epsilon)}, (f_{m\tau})^\circ_{m(c-\epsilon)}) \to H_n(E_{m\tau}, (f_{m\tau})^\circ_{m(c-\epsilon)}). \tag{10.19}$$

Summarizing our discussion now, there exist a τ-periodic solution of (10.1) with $\tau \in \mathbf{N}$, an integer $m_0 > 0$, an infinite integer set Q containing 1, and a small $\epsilon > 0$ such that for $j_{m\tau} = h_3 \circ h_2 \circ h_1$ and all $m \in Q(m_0) = \{k \in Q \mid k \geq m_0\}$ we obtain the following diagram

$$0 \neq C_n(f_\tau, x) \xrightarrow{\psi_*^m} C_n(f_{m\tau}, x^m) \xrightarrow{(j_{m\tau})_*} H_n(E_{m\tau}, (f_{m\tau})^\circ_{m(c-\epsilon)}) \equiv \mathcal{H}_m. \tag{10.20}$$

where ψ_*^m is an isomorphism, and $(j_{m\tau})_*$ is a monomorphism among the homology modules. Specially let $[\sigma]$ be a generator of $C_n(f_\tau, x)$. Then going through the diagram (10.20), there holds

$$(j_{m\tau})_* \circ \psi_*^m[\sigma] \neq 0, \qquad \text{in} \quad \mathcal{H}_m. \tag{10.21}$$

This completes a half of the proof of Theorem 10.3. Now, we can use a Lagrangian version of a technique of V. Bangert and the topological Lemma 1 of V. Bangert and W. Klingenberg in [BK] to show that the left hand side of (10.21) must vanish provided m is sufficiently great. This gives a contradiction to (10.21) and proves the Theorem 10.3. For details of this argument we refer to [Lo15].

Remark 10.6. The idea in the above proof of Theorem 10.3 comes from [Lo14] of our proof for C. Conley's conjecture, which claims as mentioned in [SZ2] that every Hamiltonian diffeomorphism on T^{2n} generated by smooth periodic Hamiltonian functions always possesses infinitely many periodic points produced by contractible integer periodic solutions of the corresponding Hamiltonian system on T^{2n}.

§11. Indexing domains of instability for Hamiltonian systems.

In this section we introduce our study in recent paper [LA] on the homotopy classification problem for unstable linear Hamiltonian systems via the Maslov-type index theory.

Definition 11.1. *For any matrix $M \in \mathrm{Sp}(2n)$, the* **hyperbolic index** *$\varrho(M)$ of M is defined to be the mod 2 number of the total multiplicity of negative eigenvalues of M which are strictly less than -1.*

Definition 11.2. *A matrix $M \in \mathrm{Sp}(2n)$ is* **hyperbolic,** *if all the eigenvalues of M are not on the unit circle $\mathbf{U}$ in the complex plane $\mathbf{C}$ except two of them which are 1. Denote by $\mathrm{Sp}^h(2n)$ the subset of all hyperbolic matrices in $\mathrm{Sp}(2n)$. A path $\gamma \in \mathcal{P}_1(2n)$ is* **hyperbolic,** *if $\gamma(1) \in \mathrm{Sp}^h(2n)$. Denote by $\mathcal{P}^h(2n)$ the subset of all hyperbolic paths in $\mathcal{P}_1(2n)$. A system (1.2) in $\mathcal{H}(2n)$ is* **hyperbolic,** *if its fundamental solution $\gamma_B \in \mathcal{P}^h(2n)$. Denote by $\mathcal{H}^h(2n)$ the subset of all hyperbolic systems in $\mathcal{H}(2n)$. In this case we also write $B \in \mathcal{H}^h(2n)$.*

Note that this concept of the hyperbolicity comes from the study of periodic solutions of autonomous Hamiltonian systems. As we proved in [LA], $\mathcal{P}^h(2n)$ and $\mathcal{H}^h(2n)$ contain countablly infinitely many path connected components which are called the domains of instability of symplectic paths and linear Hamiltonian systems respectively.

Definition 11.3. *Two paths γ_0 and $\gamma_1 \in \mathcal{P}^h(2n)$* **belong to the same domain of instability,** *if γ_0 and γ_1 can be connected by a continuous one-parameter family of paths $\{\gamma_s\}_{0\le s\le 1}$ in $\mathcal{P}^h(2n)$. In this case we write $\gamma_0 \sim_h \gamma_1$. Two Hamiltonian systems of the form (1.2) with $B_i \in C(S^1, \mathcal{L}_s(\mathbf{R}^{2n})) \cap \mathcal{H}^h(2n)$ for $i = 0, 1$* **belong to the same domain of instability,** *if B_0 and B_1 can be connected by a continuous one-parameter family of $\{B_s\}_{0\le s\le 1}$ in $C(S^1, \mathcal{L}_s(\mathbf{R}^{2n})) \cap \mathcal{H}^h(2n)$. In this case we write $B_0 \sim_h B_1$.*

Note that for γ_0 and $\gamma_1 \in \mathcal{P}^h(2n)$, the facts that $\gamma_0 \sim \gamma_1$ on $[0,1]$ along $\delta(\cdot,1)$ in the sense of Definition 1.3 and that $\delta(\cdot,1)$ being a path in $\mathrm{Sp}^h(2n)$ together imply $\gamma_0 \sim_h \gamma_1$.

Note that by definition there holds $\mathrm{Sp}^h(2) = \mathrm{Sp}(2)^0$, where $\mathrm{Sp}(2)^0$ is defined in the section 1 and is path connected as shown in the Figure 1.1.

Theorem 11.4. *For $n \ge 2$, the set $\mathrm{Sp}^h(2n)$ possesses precisely two path connected components defined by*

$$\mathrm{Sp}_i^h(2n) = \{M \in \mathrm{Sp}^h(2n) \mid \varrho(M) = i\}, \qquad \text{for } i = 0, 1. \tag{11.1}$$

Idea of the proof. It suffices to note that every matrix $M \in \mathrm{Sp}^h(2n)$ can be connected to one of the following two matrices within $\mathrm{Sp}^h(2n)$:

$$I_2 \diamond D(2) \diamond \cdots \diamond D(2) \diamond D(2), \tag{11.2}$$

$$I_2 \diamond D(2) \diamond \cdots \diamond D(2) \diamond D(-2). \tag{11.3}$$

Note that these two matrices possess hyperbolic index 0 and 1 respectively. ∎

Theorem 11.5. *Two paths γ_0 and $\gamma_1 \in \mathcal{P}^h(2n)$ belong to the same domain of instability, i.e. $\gamma_0 \sim_h \gamma_1$, if and only if both of the following two conditions hold:*

$$\varrho(\gamma_0(1)) = \varrho(\gamma_1(1)), \tag{11.4}$$

$$i_1(\gamma_0), i_1(\gamma_1) \in \{2k - 1 + \varrho(\gamma_0(1)), 2k + \varrho(\gamma_0(1))\} \quad \text{for some } k \in \mathbf{Z}. \tag{11.5}$$

Idea of the proof. We give a pictorial proof for the case of $n = 2$. The proof of the general case is just an analytical interpretion of this special case.

Given any $\gamma \in \mathcal{P}^h(4)$, by the basic normal form Theorem 4.6, there exists a path $f \in C([0,1], \Omega^0(\gamma(1)))$, $a = 2$ or -2, and $b = 1$, 0, or -1, such that $f(0) = \gamma(1)$ and

$$f(1) = N_1(1,b) \diamond D(a). \tag{11.6}$$

We only consider the case of $b = 1$. The cases of $b = 0$ and -1 are similar and left to the readers. Using the nondegenerate standard paths in Sp(2) of (3.2), we define

$$\alpha = \hat{\alpha}_{1,0,1}, \qquad \text{if} \quad a = 2, \tag{11.7}$$

$$\alpha = \hat{\alpha}_{1,1,1}, \qquad \text{if} \quad a = -2. \tag{11.8}$$

Let $k = i_1(\gamma)$. Note that we must have

$$k \in 2\mathbf{Z} + 1, \qquad \text{if} \quad a = 2, b = 1, \tag{11.9}$$

$$k \in 2\mathbf{Z}, \qquad \text{if} \quad a = -2, b = 1. \tag{11.10}$$

Choose a path $g \in C([0,1], \mathrm{Sp}(2)^0)$ such that $g(0) = I_2$ and $g(1) = N_1(1,b)$. Using $\phi_{\theta,1}$ in (1.15), we define

$$\beta = g * \phi_{(k+1)\pi,1}, \qquad \text{if} \quad a = 2, b = 1, \tag{11.11}$$

$$\beta = g * \phi_{k\pi,1}, \qquad \text{if} \quad a = -2, b = 1. \tag{11.12}$$

Then we obtain

$$\varrho(\beta \diamond \alpha) = \varrho(\alpha) = \frac{2-a}{4} = \varrho(\gamma), \tag{11.13}$$

$$i_1(\beta \diamond \alpha) = i_1(\beta) + i_1(\alpha) = (k+1-1) + 0 = i_1(\gamma), \quad \text{if } a = 2, b = 1, \tag{11.14}$$

$$i_1(\beta \diamond \alpha) = i_1(\beta) + i_1(\alpha) = (k-1) + 1 = i_1(\gamma), \quad \text{if } a = -2, b = 1, \tag{11.15}$$

Thus by Theorem 1.12, we obtain

$$\gamma \sim \beta \diamond \alpha \quad \text{along} \quad f.$$

By the definition of f this implies

$$\gamma \sim_h \beta \diamond \alpha. \tag{11.16}$$

Now by Definition 11.3, we can move $\beta(1)$ within $\mathrm{Sp}(2)^0$ and still keep (11.16) hold. This shows that the condition (11.4) is necessary for (11.16) to hold, and (11.5) gives the range of $i_1(\gamma)$ which can still keep (11.16) to hold. This is shown in the Figure 11.1.

The sufficiency part is easier and left to the readers. ∎

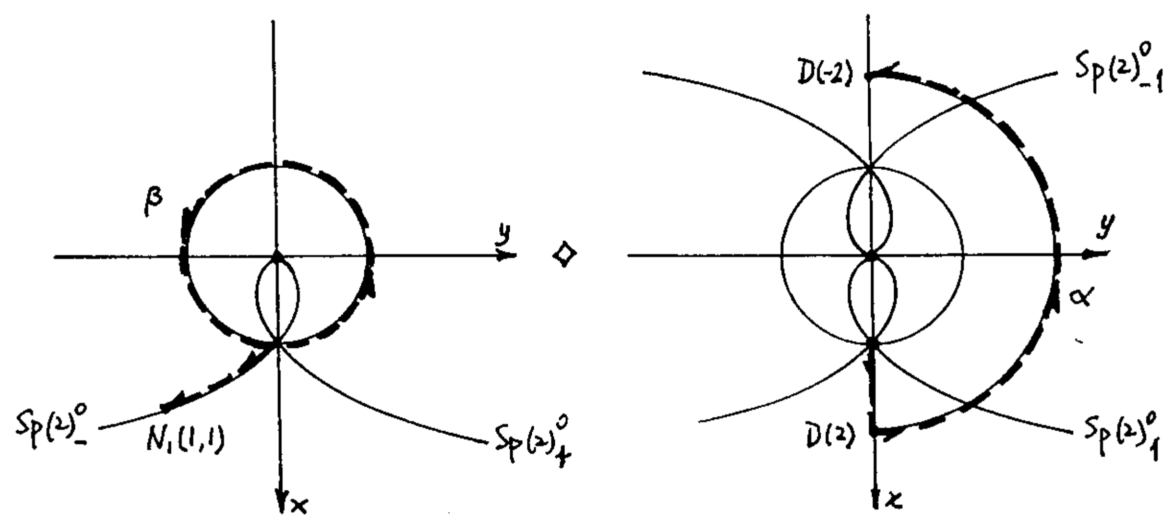

Figure 11.1. The path $\beta \diamond \alpha$ with $b = 1$ and $a = -2$.

Theorem 11.6. *Two Hamiltonian systems of the form (1.2) defined by B_0 and $B_1 \in \mathcal{H}^h(2n)$ belong to the same domain of instability, i.e. $B_0 \sim_h B_1$, if and only if both of the two conditions (11.2) and (11.3) in the Theorem 11.5 hold for $\gamma_i = \gamma_{B_i}$ with $i = 0$ and 1.*

Based upon Theorems 11.5 and 11.6, we define the index of the domains of instability as follows.

Definition 11.7. *The* **hyperbolic index** $\mathrm{ind}_h : \mathcal{P}^h(2n) \to \mathbf{Z}_2 \times \mathbf{Z}$ *for any path $\gamma \in \mathcal{P}^h(2n)$ is defined by*

$$\mathrm{ind}_h(\gamma) = (\varrho(\gamma(1)), k), \qquad \text{if } i_1(\gamma) \in \{2k-1+\varrho(\gamma(1)), 2k+\varrho(\gamma(1))\} \text{ for some } k \in \mathbf{Z}.$$

We index the domains of instability by

$$\begin{aligned} \mathcal{P}^h_{i,k}(2n) &= \{\gamma \in \mathcal{P}^h(2n) \,|\, \mathrm{ind}_h(\gamma) = (i,k)\}, \quad \forall i \in \{0,1\}, \quad k \in \mathbf{Z}, \\ \mathcal{H}^h_{i,k}(2n) &= \{B \in \mathcal{H}^h(2n) \,|\, \mathrm{ind}_h(\gamma_B) = (i,k)\}, \quad \forall i \in \{0,1\}, \quad k \in \mathbf{Z}. \end{aligned}$$

By Theorems 1.5 and 1.6, we obtain that $\mathcal{P}^h(2n)$ and $\mathcal{H}^h(2n)$ are disjoint unions of their path connected components $\mathcal{P}^h_{i,k}(2n)$'s and $\mathcal{H}^h_{i,k}(2n)$'s respectively.

As an application of our index of the domains of instability, in the section 3 of this paper, we consider the periodic problem of the calculus of variation, i.e. finding extremal loops of the following functional

$$F(x) = \int_0^1 L(x,\dot{x})dt, \qquad \forall x \in W_1 = W^{1,2}(S^1, \mathbf{R}^n). \tag{11.17}$$

An extremal loop x of F corresponds to a 1-periodic solution of the Lagrangian system (2.6). In this section we always suppose the Lagrangian function $L \in C^2(\mathbf{R}^{2n}, \mathbf{R})$ and satisfies the Legendre convexity condition

$$L_{\dot{x},\dot{x}}(x,p)z \cdot z > 0, \qquad \forall z \in \mathbf{R}^n \backslash \{0\}, \quad (x,p) \in \mathbf{R}^{2n}. \tag{11.18}$$

Theorem 11.8. *An extremal loop $x \in W$ of the functional F is a nondegenerate local minimum if and only if the corresponding linearized Hamiltonian system (1.2) with B given by (2.9) is hyperbolic and belongs to $\mathcal{H}^h_{0,0}(2n)$ or $\mathcal{H}^h_{1,0}(2n)$.*

Note that V. Bondarchuk [Bn] in 1980 studied the same problem of indexing the domains of instability in $\mathcal{H}^h(2n)$. But our Example 2.10 of [LA] showes that the Lemma 4 in [Bn] and his indexing of domains of instability are incorrect, and therefore the main results in [Bn] are not complete. Note also that Theorem 11.8 generalizes an old result of H. Poincaré in [Po], who studied the problem when $n = 2$.

References

[AZ] H. Amann and E. Zehnder, Nontrivial solutions for a class of non-resonance problems and applications to nonlinear differentialequations. *Ann. Scuola Norm. Sup. Pisa. Cl.Sci. Series* 4. 7.(1980).539-603.

[AC] A. Ambrosetti and V. Coti Zelati, Solutions with minimal period for Hamiltonian systems in a potential well. *Ann. Inst. H. Poincaré, Anal. non linéaire.* 4.(1987).275-296.

[AM] A. Ambrosetti and G. Mancini, Solutions of minimal period for a class of convex Hamiltonian systems. *Math. Ann.* 255.(1981). 405-421.

[AL] T. An and Y. Long, On the index theories of second order Hamiltonian systems. *Nonlinear Analysis, T.M.A.* 34 (1998) 585-592.

[Ar] V. I. Arnold, On a characteristic class entering into conditions of quantization. *Functional Analysis.* 1.(1967). 1-8.

[BTZ] W. Ballmann, G. Thobergsson and W. Ziller, Closed geodesics on positively curved manifolds. *Ann. of Math.* 116 (1982). 213-247.

[Ba] V. Bangert, Closed geodesics on complete surfaces. *Math. Ann.* 251 (1980), 83-96.

[BK] V. Bangert and W. Klingenberg, Homology generated by iterated closed geodesics. *Topology.* 22 (1983), 379-388.

[Bn] V. B. Bondarchuk, A periodic problem in the calculus of variations and deformations of Hamiltonian systems. *Dyna. and Stab. of Mech. Syst.* I. A. Lukovskii ed. Inst. Mat. Akad. Nauk. Ukrain. SSR. Kiev, (1980) 143-155. "*Seven Papers in Appl. Math.*" *Amer. Math. Transl.* (2) 125 (1985), 21-31.

[Bo] R. Bott, On the iteration of closed geodesics and the Sturm intersection theory. *Comm. Pure Appl. Math.* 9.(1956). 171-206.

[Br] V. Brousseau, Espaces de Krein et index des systemes hamiltoniens. *Ann. Inst. H. Poincaré Anal. non linéaire* 7 (1990), 525-560.

[CLE] S. Cappel, R. Lee, and E. Miller, On the Maslov index. *Comm. Pure Appl.Math.* 47.(1994). 121-186.

[Ch1] K. C. Chang, Applications of homology theory to some problems in differential equations. *Proc. Symp. in Pure Math.* F. Browder Ed. Amer. Math. Soc. 45 (1986), 253-262.

[Ch2] K. C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems. Birkhäuser. Boston. (1993).

[CLL] K. C. Chang, J.-Q. Liu, M. Liu, Nontrivial periodic solutions for strong resonance Hamiltonian systems. *Ann. Inst. H. Poincaré Anal. non linéaire.* 14 (1997) 103-117.

[CLZ] K. C. Chang, Y. Long, and E. Zehnder, Forced oscillations for the triple pendulum. *Analysis, et cetera.* P.Rabinowitz and E.Zehnder ed. Academic Press. 177-208.(1990).

[CE1] F. Clarke and I. Ekeland, Hamiltonian trajectories having prescribed minimal period. *Comm. Pure Appl. Math.* 33.(1980).103-116.

[CE2] F. Clarke and I. Ekeland, Nonlinear oscillations and boundary value problems for Hamiltonian systems. *Arch. for Rat. Mech. and Anal.* 78.(1982). 315-333.

[CZ1] C. Conley and E. Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold. *Invent. Math.* 73. (1983), 231-249.

[CZ2] C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure Appl. Math.* 37.(1984). 207-253.

[CZ3] C. Conley and E. Zehnder, Subharmonic solutions and Morse theory. *Physica* 124A. (1984). 649-658.

[CZ4] C. Conley and E. Zehnder, A global fixed point theorem for symplectic maps and subharmonic solutions of Hamiltonian equations on tori. In "Nonlinear Funct. Anal. and Its Appl.", Proc. Symp. in Pure Math. F. Browder Ed. Amer. Math. Soc. 45. (1986). 283-300.

[CD] R. Cushman and J. Duistermaat, The behavior of the index of a periodic linear Hamiltonian system under iteration. *Adv. in Math.* 23. (1977). 1-21.

[Del] G. Dell'Antoio, Variational calculus and stability of periodic solutions of a class of Hamiltonian systems. SISSA Ref. 185/92/FM (Oct. 1992).

[DDE] G. Dell'Antoio, B. D'Onofrio, and I. Ekeland, Les systém hamiltoniens convexes et pairs ne sont pas ergodiques en general. *C. R. Acad. Sci. Paris,* t.315. Series I. (1992). 1413-1415.

[Den] Deng, S., Minimal period solutions of a class of Hamiltonian equation systems. *Acta Math. Sinica.* 27.(1984).664-675.

[DL] D. Dong and Y. Long, The iteration formula of the Maslov-type index theory with applications to nonlinear Hamiltonian systems. Nankai Inst. of Math. Nankai Univ. Preprint. (1993). *ICTP. Report.* No. IC/94/318. (1994). *Trans. Amer. Math. Soc.* 349 (1997) 2619-2661.

[Du] J. J. Duistermaat, On the Morse index in variational calculus. *Adv. in Math.* 21. (1976). 173-195.

[Ek1] I. Ekeland, Une théorie de Morse pour les systémes hamiltoniens convexes. *Ann. Inst. H. Poincaré Anal. Non Linéarire.* 1. (1984). 19-78.

[Ek2] I. Ekeland, An index theory for periodic solutions of convex Hamiltonian systems. In *Nonlinear Funct. Anal. and its Appl.* Proc. Symposia in Pure Math. 45-1. (1986). 395-423.

[Ek3] I. Ekeland, Convexity Methods in Hamiltonian Mechanics. Springer. Berlin. (1990).

[EH] I. Ekeland and H. Hofer, Periodic solutions with prescribed period for convex autonomous Hamiltonian systems. *Invent. Math.* 81 (1985), 155-188.

[FQ1] G. Fei and Q. Qiu, Periodic solutions of asymptotically linear Hamiltonian systems. Preprint (1996), *Chinese Ann. of Math.* to appear.

[FQ2] G. Fei and Q. Qiu, Minimal period solutions of nonlinear Hamiltonian systems. Preprint (1996).

[FW] A. Fonda and M. Willem, Subharmonic oscillations of forced pendulum-type equations. *J. Diff. Equa.* 81. (1989). 215-220.

[GL] I. M. Gelfand and V. B. Lidskii, On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients. *Usp. Mat. Nauk.* 10(1955). 3-40 (Russian); *Transl. Amer. Math. Soc.* 8(2) (1958), 143-181.

[Gh] N. Ghoussoub, Duality and Perturbation Methods in Critical Point Theory. Cambridge Univ. Press. (1993).

[GM1] M. Girardi and M. Matzeu, Some results on solutions of minimal period to superquadratic Hamiltonian equations. *Nonlinear Anal. T.M.A.* 7.(1983). 475-482.

[GM2] M. Girardi and M. Matzeu, Solutions of minimal period for a class of nonconvex Hamiltonian systems and applications to the fixed energy problem. *Nonlinear Anal. T.M.A.* 10.(1986).371-382.

[GM3] M. Girardi and M. Matzeu, Periodic solutions of convex autonomous Hamiltonian systems with a quadratic growth at the origin and superquadratic at infinity. *Ann. Mat. pura ed appl.* 147.(1987).21-72.

[GM] D. Gromoll and W. Meyer, On differentiable functions with isolated critical points. *Topology.* 8. (1969). 361-369.

[HL] J. Han and Y. Long, Normal forms of symplectic matrices. II. Research Report. Nankai Inst. of Math. Nankai Univ. (1996).

[LiL1] S. Li and J. Q. Liu, Morse theory and asymptotic linear Hamiltonian systems. *J. Diff. Equa.* 78 (1989), 53-73.

[LL1] C. Liu and Y. Long, An optimal inceasing estimate for iterated Maslov-type indices. *Chinese Sci. Bull.* 42. (1997) 2275-2277.

[LL2] C. Liu and Y. Long, Hyperbolic characteristics on star-shaped hypersurfaces. Nankai Inst. of Math. Preprint. (1997). *Ann. IHP. Anal. Nonlineaire.* to appear.

[LL3] C. Liu and Y. Long, Iteration inequalities of the Maslov-type index theory with applications. Nankai Inst. of Math. Nankai Univ. Preprint. (1997). Submitted.

[Lo1] Y. Long, Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems. *Science in China (Scientia Sinica). Series* A. 7. (1990). 673-682. (Chinese edition), 33. (1990). 1409-1419. (English edition).

[Lo2] Y. Long, The structure of the singular symplectic matrix set. *Science in China (Scientia Sinica). Series* A. 5. (1991). 457-465. (Chinese edition), 34. (1991). 897-907. (English edition).

[Lo3] Y. Long, Maslov-type index theory and asymptotically linear Hamiltonian systems. In *Dyna. Syst. and Rel. Topics.* K. Shiraiwa ed. World Sci. (1991). 333-341.

[Lo4] Y. Long, The Index Theory of Hamiltonian Systems with Applications. (In Chinese). Science Press. Beijing. (1993).

[Lo5] Y. Long, The minimal period problem for classical Hamiltonian systems with even potentials. *Ann. Inst. H. Poincare. Anal. Non Lineaire.* 10. (1993). 605-626.

[Lo6] Y. Long, The minimal period problem of periodic solutions for autonomous superquadratic second order Hamiltonian systems. *J. Diff. Eqns.* 111. (1994) 147-174.

[Lo7] Y. Long, Nonlinear oscillations for classical Hamiltonian systems with bi-even subquadratic potentials. *Nonlinear Analysis, T. M. A.* 24. (1995). 1665-1671.

[Lo8] Y. Long, On the minimal period for periodic solutions of nonlinear Hamiltonian systems. ICTP Publication (1994). *Chinese Ann. of Math.* Series B. 18. (1997) 481-485.

[Lo9] Y. Long, On C. Conley's conjecture. *Proc. Variational and Local Methods in the Study of Hamiltonian Systems.* ICTP. (1994). A. Ambrosseti and G. F. Dell'Antonio ed. World Sci. Singapore (1995). 77-98.

[Lo10] Y. Long, A Maslov-type index theory for symplectic paths. *Top. Meth. Nonl. Anal.* 10 (1997), 47-78.

[Lo11] Y. Long, Bott formula of the Maslov-type index theory. Research Report. Nankai Inst. of Math. Nankai Univ. (Nov. 1995). Revised. (May 1996). *Pacific J. Math.* to appear.

[Lo12] Y. Long, Hyperbolic closed characteristics on compact convex smooth hypersurfaces in $\mathbf{R}^{2n}$. Nankai Inst. of Math. Nankai Univ. Preprint. (1996). *J. Diff. Equa.* to appear.

[Lo13] Y. Long, The topological structures of ω-subsets of symplectic groups. Nankai Inst. of Math. Nankai Univ. Preprint. (1996). *Acta Math.Sinica.* to appear.

[Lo14] Y. Long, Periodic points of Hamiltonian diffeomorphisms on tori and a conjecture of C. Conley. ETH-Zürich Preprint. (1994). Revised (1998). Submitted.

[Lo15] Y. Long, Multiple periodic points of the Poincaré map of Lagrangian systems on tori. Nankai Inst. of Math. Nankai Univ. Preprint. (1997). Revised 1998. Submitted.

[Lo16] Y. Long, Precise iteration formulae of the Maslov-type index theory for symplectic paths. In preparation.

[LA] Y. Long and T. An, Indexing the domains of instability for Hamiltonian systems. Nankai Inst. of Math. Preprint. (1996). (Revised 1997) *NoDEA*. to appear.

[LD] Y. Long and D. Dong, Normal forms of symplectic matrices. Nankai Inst. of Math. Nankai Univ. Preprint. (1995). *Acta Math. Sinica.* to appear.

[LLu] Y. Long and G. Lu, Geometrically distinct solution orbits of autonomous Lagrangian systems on tori. In preparation.

[LZ] Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems. In *Stoc. Proc. Phys. and Geom.*, S. Albeverio et al. ed. World Sci. (1990). 528-563.

[LZh1] Y. Long and C. Zhu, The differentiable structure of ω-subsets of symplectic groups. In preparation.

[LZh2] Y. Long and C. Zhu, Maslov-type index theory for symplectic paths and spectral flow (II). Nankai Inst. of Math. Nankai Univ. Preprint. (1997). Revised 1998. Submitted.

[Mo] J. K. Moser, New aspects in the theory of stability of Hamiltonian systems. *Comm. Pure Appl. Math.* 11 (1958), 81-114.

[Po] H. Poincaré, Les méthodes nouvelles de la méchanique celeste. Vol III, §347. Gauthier-Villars, Paris, (1899).

[Ra1] P. Rabinowitz, Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* 31. (1978). 157-184.

[Ra2] P. Rabinowitz, Minimax methods in critical point theory with applications to differential equations. *CBMS Regional Conf. Ser. in Math.* no.65. Amer. Math. Soc. (1986).

[Rd] H.-B. Rademacher, On the average indices of closed geodesics. *J. Diff. Geod.* 29. (1989). 65-83.

[RS] J. Robbin and D. Salamon, Maslov index theory for paths. *Topology.* 32. (1993). 827-844.

[SZ1] D. Salamon and E. Zehnder, Floer homology, the Maslov index and periodic orbits of Hamiltonian equations. *Analysis, et cetera.* Ed. by P. Rabinowitz and E. Zehnder. Academic Press. (1990). 573-600.

[SZ2] D. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure and Appl. Math.* 45. (1992). 1303-1360.

[Vi1] C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems. *Trans. Amer. Math. Soc.* 311. (1989). 621-655.

[Vi2] C. Viterbo, A new obstruction to embedding Lagrangian tori. *Invent. Math.* 100. (1990) 301-320.

[WF] T. Wang and G. Fei, Subharmonics for superquadratic Hamiltonian systems via the iteration method of the Maslov-type index theory. Preprint (1996).

[YS] V. Yakubovich and V. Starzhinskii, Linear Differential Equations with Periodic Coefficients. New York, John Wiley & Sons. (1975).

[ZL] C. Zhu and Y. Long, Maslov-type index theory for symplectic paths and spectral flow (I). Nankai Inst. of Math. Nankai Univ. Preprint. (1997). Revised 1998. Submitted.

October 2nd, 1998.