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Perturbation results for some variational problems without compactness

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Perturbation results for some variational problems without compactness

Kazunaga Tanaka

0. Introduction

In this series of lectures, I would like to talk about some topics in perturbation methods in variational theory. In particular, I would like to deal with some situation where the Palais-Smale compactness condition does not hold.

To express the fundamental idea, first we consider the simplest case: Let us consider a function

$$f : \mathbf{R} \to \mathbf{R}$$

such that

$$\omega_- = \lim_{x \to -\infty} f(x) \in \mathbf{R} \quad \text{and} \quad \omega_+ = \lim_{x \to \infty} f(x) \in \mathbf{R}$$

exist and $\lim_{x \to \pm\infty} f'(x) = 0$.

Any sequence $(x_n) \subset \mathbf{R}$ with $x_n \to \pm\infty$ satisfies

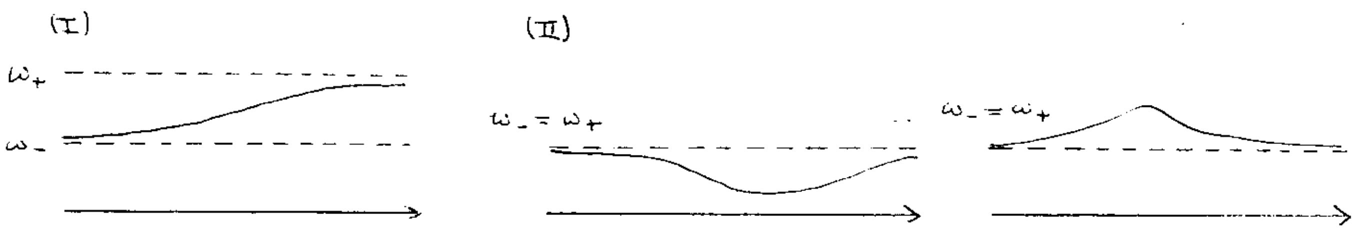
$$f(x_n) \to \omega_\pm \quad \text{and} \quad f'(x_n) \to 0,$$

thus (x_n) is a Palais-Smale sequence at level $\omega_\pm$ but (x_n) does not have convergent subsequences. That is, $f(x)$ does not satisfy the Palais-Smale condition and we cannot use minimization method directly to find critical points.

In this setting, we can easily see that

(I) If $\omega_- \neq \omega_+$, then the function $f(x)$ does not have any critical point in general. Eg. $f(x) = \arctan x$.

(II) If $\omega_- = \omega_+$, then the function $f(x)$ has at least one critical point; Critical point can be found as either a minimum or a maximum of $f(x)$.



Such a situation occurs also for functionals corresponding to nonlinear differential equations. And the main purpose of my lectures is to give such examples and to to explain general ideas to deal with.

I will talk about the following topics:

(a) Nonlinear elliptic equations in $\mathbf{R}^N$. Results in this part is due to Bahri and Li [**BL**]. Here we consider

$$\begin{aligned} &-\Delta u + u = a(x)u^p && \text{in } \mathbf{R}^N, \\ &u > 0 && \text{in } \mathbf{R}^N, \\ &u \in H^1(\mathbf{R}^N), \end{aligned} \tag{0.1}$$

where $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p < \infty$ if $N = 1, 2$ and $a(x) \in C(\mathbf{R}^N, \mathbf{R})$ is a function such that

$$0 < \inf_{x \in \mathbf{R}^N} a(x) \leq \sup_{x \in \mathbf{R}^N} a(x) < \infty$$

(I) We can easily see that if $a(x)$ satisfies for some i

$$\frac{\partial a}{\partial x_i}(x) > 0 \qquad \text{for all } x \in \mathbf{R}^N,$$

then (0.1) does not have any positive solution.

(II) On the other hand, Bahri and Li [**BL**] showed that (0.1) has a positive solution if $a(x)$ satisfies

$$a(x) \to 1 \qquad \text{as } |x| \to \infty$$

under some additional conditions.

(b) Periodic solutions for singular Hamiltonian systems. We consider the existence of a periodic solution of the following Hamiltonian system:

$$\begin{aligned} &\ddot{q} + \nabla V(q) = 0, \\ &\frac{1}{2}|\dot{q}|^2 + V(q) = H, \end{aligned} \tag{0.2}$$

for a given $H \in \mathbf{R}$. This problem is called the prescribed energy problem. We consider a situation related to 2 body problem in celestial mechanics:

$$V(q) \sim -\frac{1}{|q|^\alpha} \qquad \text{near } q = 0.$$

The order $\alpha > 0$ of a singularity 0 plays an important role for the existence of periodic solutions. The case $\alpha > 2$ is called strong force and the case $\alpha \in (0,2)$ is called weak force. Here we study a boundary case $\alpha = 2$. If $V(q) = -\frac{1}{|q|^2}$, we can easily see that (0.2) has a periodic solution if and only if $H = 0$. We try to find a periodic solution of (0.2) with $H = 0$ under the condition:

$$V(q) \sim -\frac{b}{|q|^{\alpha}} \qquad \text{at } |q| \sim \infty$$

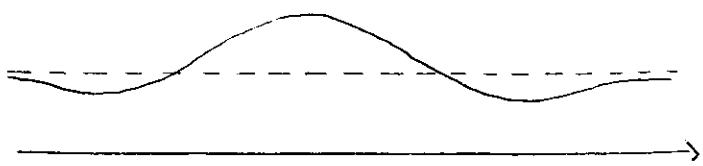
for some $b > 0$. We will show that

(I) If $b \neq 1$, then (0.2) does not have any periodic solution in general.

(II) If $b = 1$, then (0.2) has at least one periodic solution.

We will also discuss a related problem on the existence of closed geodesics on non-compact Riemannian manifolds.

(c) Nonlinear nonhomogenous elliptic problem. Finally we give an example in which we can observe a situation like this picture:



We study nonlinear elliptic equation again and here we deal with equations with nonhomogenous term:

$$\begin{aligned} &-\Delta u + u = a(x)u^p + f(x) && \text{in } \mathbf{R}^N, \\ &u > 0 && \text{in } \mathbf{R}^N, \\ &u \in H^1(\mathbf{R}^N), \end{aligned} \tag{0.3}$$

where $f(x) \in H^{-1}(\mathbf{R}^N)$ satisfies $f(x) \geq 0$ and $f(x) \not\equiv 0$. We show the existence of four positive solutions of (0.3) for small $f(x) \geq 0$ and $f(x) \not\equiv 0$ under the conditions:

$$\begin{aligned} &a(x) \in (0, 1] \qquad \text{for all } x \in \mathbf{R}^N, \\ &a(x) \not\equiv 1, \\ &a(x) \to \infty. \end{aligned}$$

In this situation, we can obtain some additional information about the behavior of the corresponding functional at infinity. Such a information enables us to obtain multiple existence of critical points.

1. Nonlinear elliptic equations in $\mathbf{R}^N$

The result of this section is due to Bahri and Li [**BL**]. Here we consider the existence of a positive solution of the following nonlinear elliptic equations:

$$\begin{aligned} -\Delta u + u &= a(x)u^p && \text{in } \mathbf{R}^N, \\ u &> 0 && \text{in } \mathbf{R}^N, \\ u &\in H^1(\mathbf{R}^N), \end{aligned}$$

where $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < \infty$ if $N = 1, 2$.

1.1. Preliminary

Here we consider the case $N = 1$ for the sake of simplicity. Thus we consider

$$\begin{aligned} -u_{xx} + u &= a(x)u^p && \text{in } \mathbf{R}, \\ u &> 0 && \text{in } \mathbf{R}, \\ u &\in H^1(\mathbf{R}), \end{aligned} \tag{1.1}$$

where $p \in (1, \infty)$ and $a(x) \in C(\mathbf{R}, \mathbf{R})$.

When $a(x) \equiv 1$, we can easily see that (1.1) has a unique (up to translation) solution $\omega(x)$. That is, all possible solution of (1.1) with $a(x) \equiv 1$ can be written $\omega(x - y)$ for some $y \in \mathbf{R}^N$. $\omega(x)$ is characterized as a mountain pass critical point for the functional:

$$I_\infty(u) = \|u\|_{H^1}^2 - \frac{1}{p+1}\int_{\mathbf{R}} u_+^{p+1}\,dx : H^1(\mathbf{R}^N) \to \mathbf{R},$$

where $\|u\|_{H^1} = \left(\int_{\mathbf{R}} |u_x|^2 + |u|^2\,dx\right)^{1/2}$ and $u_+ = \max\{u, 0\}$. We can also say that $\omega(x)$ is corresponding to the minimizer of the following problem:

$$c_\infty = \inf\{J_\infty(v);\ v \in \Sigma_+\},$$

where

$$J_\infty(v) = \max_{t>0} I_\infty(tu) = (\frac{1}{2} - \frac{1}{p+1})\left(\frac{\|u\|_{H^1}^{p+1}}{\int_{\mathbf{R}} u_+^{p+1}\,dx}\right)^{2/p-1},$$

$$\Sigma_+ = \{v \in H^1(\mathbf{R});\ \|v\|_{H^1} = 1,\ v_+ \not\equiv 0\}.$$

1.2. Perturbed problem

We return to the problem (1.1) for general $a(x)$. Solutions of (1.1) can be characterized as critical points of

$$I(u) = \|u\|_{H^1}^2 - \frac{1}{p+1}\int_{\mathbf{R}} a(x)u_+^{p+1}\,dx : H^1(\mathbf{R}^N) \to \mathbf{R}.$$

Equivalently, it is corresponding to critical points of the following constraint problem:

$$J\ \ (v) = \max_{t>0} I\ \ (tu) = (\frac{1}{2} - \frac{1}{p+1}) \left(\frac{\|u\|_{H^1}^{p+1}}{\int_{\mathbf{R}} a u_+^{p+1}\, dx} \right)^{2/p-1} : \Sigma_+ \to \mathbf{R}.$$

From now on we look for critical points of $I(u)$ or $J(u)$ under the assumption: $a(x) \to \omega_\pm$ as $x \to \infty$. First, we show non-existence result.

Theorem 1.1. *Suppose $a'(x) > 0$ for all $x \in \mathbf{R}$. Then* (1.1) *does not have any positive solution.*

Proof. Suppose that $u(x)$ is a positive solution of (1.1). Direct calculation gives us

$$\begin{aligned} I'(u)u_x &= \lim_{h\to 0} \frac{1}{h}(I(u(\cdot + h)) - I(u)) \\ &= -\frac{1}{p+1} \lim_{h\to 0} \int_{\mathbf{R}} a(x)(u(x+h)_+^{p+1} - u(x)^{p+1})\, dx \\ &= -\frac{1}{p+1} \lim_{h\to 0} \int_{\mathbf{R}} (a(x-h) - a(x))u(x)^{p+1}\, dx \\ &= \frac{1}{p+1} \int_{\mathbf{R}} a'(x)u^{p+1}\, dx > 0. \end{aligned}$$

Thus there are no positive solutions of (1.1). ∎

By the above theorem, we see that if $\omega_+ \neq \omega_-$, there are no positive solutions in general. Next we consider the case $\omega_+ = \omega_-$. (We assume $\omega_+ = \omega_- = 1$ without loss of generality.) As a special case of the result of Bahri and Li [**BL**], we have

Theorem 1.2. *Suppose*

$$a(x) > 0 \qquad \text{for all } x \in \mathbf{R},$$

$$a(x) \to 1 \qquad \text{as } x \to \pm\infty,$$

and there exist $\lambda > 2$ and $C > 0$ such that

$$a(x) \geq 1 - Ce^{-\lambda|x|} \qquad \text{for all } x \in \mathbf{R}.$$

Then (1.1) *has at least one positive solution.*

The remainder of this section is devoted to sketch the proof of Theorem 1.2.

1.3. Proof of Theorem 1.2

First we study the break down of Palais-Smale condition for $I(u)$ and $J(u)$.

Proposition 1.3. *Suppose that $(u_j) \in H^1(\mathbf{R})$ satisfies*

$$I(u_j) \to c,$$
$$I'(u_j) \to 0 \qquad \text{as } j \to \infty$$

for some $c \in \mathbf{R}$. Then there exists a subsequence — still denoted by u_j — and a solution $u_0(x)$ of (1.1), an integer $\ell \in \mathbf{N} \cup \{0\}$ and sequences $(y_j^k)_{j=1}^{\infty} \subset \mathbf{R}$ such that

$$u_j \rightharpoonup u_0 \qquad \text{weakly in } H^1(\mathbf{R}),$$
$$\|u_j(x) - (u_0(x) + \sum_{k=1}^{\ell} \omega(x - y_j^k))\|_{H^1} \to 0,$$
$$I(u_j) \to I(u_0) + kI_\infty(\omega) = I(u_0) + kc_\infty,$$
$$|y_j^k| \to \infty, \ |y_j^k - y_j^{k'}| \to \infty \quad (k \neq k')$$

as $j \to \infty$. A similar result holds also for $J(u)$.

Corollary 1.4. *The Palais-Smale condition breaks down only on levels*

$$I(u_0) + kc_\infty,$$

where $k \in \mathbf{N}$ and $u_0 \in H^1(\mathbf{R})$ is a critical point of $I(u)$. In particular, the Palais-Smale condition holds in $(-\infty, c_\infty)$.

To prove Theorem 1.2, the following values play an important role.

$$\underline{b} = \inf\{J(v);\, v \in \Sigma_+\},$$
$$\overline{b} = \inf_{\gamma \in \Gamma} \sup_{t \in \mathbf{R}} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma(t) \in C(\mathbf{R}, \Sigma_+);\, \gamma(t)(x) = \omega(x - t) \text{ for large } |t|\}.$$

It is easily seen that

$$\underline{b} \leq c_\infty \leq \overline{b}. \tag{1.2}$$

Using the interaction phenomena, Bahri and Li [**BL**] showed

Lemma 1.5. $\overline{b} < 2c_\infty$.

I will give an explanation about the interaction phenomena in Section 3.

Proof of Theorem 1.2. From (1.2) we see that one of the following 3 cases take a place

Case 1: $\underline{b} < c_\infty$,

Case 2: $\underline{b} = c_\infty$ and $\bar{b} > c_\infty$,

Case 3: $\underline{b} = \bar{b} = c_\infty$.

Case 1: Since the Palais-Smale condition holds in $(-\infty, c_\infty)$, we see easily that $\underline{b}$ is a critical value of $J(u)$.

Case 2: Under the assumption $\underline{b} = c_\infty$, we can see that the Palais-Smale condition holds in $(-\infty, c_\infty) \cup (c_\infty, 2c_\infty)$. Thus we can see that $\bar{b}$ is a critical value of $J(v)$ by virtue of Lemma 1.5.

Case 3: Under the assumption $\underline{b} = \bar{b} = c_\infty$, we can find a sequence $(v_j) \subset \Sigma_+$ such that

$$\begin{aligned} &J(v_j) \to c_\infty, \\ &J'(v_j) \to 0, \\ &\int_0^\infty |v_j'|^2 + |v_j|^2 \, dx = \frac{1}{2}. \end{aligned} \tag{1.3}$$

If (v_j) does not converges, it follows from Proposition 1.3 that

$$||v_j - \frac{1}{||\omega||_{H^1}} \omega(\cdot - y_j)||_{H^1} \to 0$$

for some $y_j \to \pm\infty$. Thus

$$\int_0^\infty |v_j'|^2 + |v_j|^2 \, dx \to 1 \text{ or } 0 \text{ as } j \to \infty.$$

This contradicts (1.3). Therefore (v_j) converges — after extracting a subsequence — to a critical point of $J(v)$.

2. Periodic solutions of singular Hamiltonian systems

Here we consider the existence of periodic solutions of a singular Hamiltonian system of 2 body type:

$$\ddot{q} + \nabla V(q) = 0 \tag{HS.1}$$

We assume that $V(q)$ satisfies

(V0) $V(q) \in C^2(\mathbf{R}^N \setminus \{0\}, \mathbf{R})$.

(V1) $V(q) < 0$ for all $q \in \mathbf{R}^N \setminus \{0\}$.

(V2) There exists an $\alpha > 0$ such that

$$V(q) \sim -\frac{1}{|q|^\alpha} \qquad \text{near } q = 0;$$

more precisely, for $W(q) = V(q) + \dfrac{1}{|q|^\alpha}$

$$|q|^\alpha W(q),\ |q|^{\alpha+1}\nabla W(q),\ |q|^{\alpha+2}\nabla^2 W(q) \to 0 \qquad \text{as } |q| \to 0.$$

Here we study the prescribed energy problem: For a given $H \in \mathbf{R}$, we try to find a periodic solution of (HS.1) satisfying

$$\frac{1}{2}|\dot{q}|^2 + V(q) = H. \tag{HS.2}$$

Here we study the case $\alpha = 2$. It is a border case between strong force and weak force. For $\alpha = 2$, we can expect the existence of periodic solutions only for $H = 0$. We also make an assumption on the behavior of $V(q)$ at infinity:

(V3) There exists $b > 0$ such that

$$V(q) \sim -\frac{b}{|q|^2} \qquad \text{as } |q| \sim \infty;$$

more precisely, for $\tilde{W}(q) = V(q) + \dfrac{b}{|q|^2}$

$$|q|^2 \tilde{W}(q),\ |q|^3 \nabla \tilde{W}(q),\ |q|^4 \nabla^2 \tilde{W}(q) \to 0 \qquad \text{as } |q| \to \infty.$$

If $b \neq 1$, periodic solutions of (HS.1)–(HS.2) do not exist in general. In fact, we have

Theorem 2.1. *Suppose $\varphi(r) \in C^2([0,\infty),\mathbf{R})$ satisfies*

$$\begin{aligned} &\varphi'(r) \neq 0 \qquad \text{for all } r > 0,\\ &\varphi(r) \to 1 \qquad \text{as } r \to 0,\\ &\varphi(r) \to b > 0 \qquad \text{as } r \to \infty \end{aligned}$$

and let

$$V(q) = -\frac{\varphi(|q|)}{|q|^2}.$$

Then (HS.1)–(HS.2) *with $H = 0$ does not have periodic solutions.*

Conversely, if $b = 1$, we have the following existence result:

Theorem 2.2. *Assume (V0)–(V2) with $\alpha = 2$ and (V3) with $b = 1$. Then* (HS.1)–(HS.2) *with $H = 0$ has at least one periodic solution.*

You may think that the situation of Theorems 2.1 and 2.2 are quite different from that of Theorems 1.1 and 1.2. However you can find similarity in the following way.

Periodic solutions of (HS.1)–(HS.2) are characterized as critical points of

$$E(u) = \frac{1}{2}\int_0^1 (H - V(u))|\dot{u}|^2\,d\tau$$

acting on 1-periodic functions. In unperturbed case $V(q) = -\frac{1}{|q|^2}$, $H = 0$, it becomes

$$E_\infty(u) = \frac{1}{2}\int_0^1 \frac{1}{|u|^2}|\dot{u}|^2\,d\tau.$$

$E_\infty(u)$ has the following invariance:

$$E_\infty(tu) = E_\infty(u) \qquad \text{for all } t > 0 \text{ and } u \tag{2.1}$$

and this invariance is corresponding to the shift-invariance of $I_\infty(u)$:

$$I_\infty(u(\cdot - y)) = I_\infty(u) \qquad \text{for all } y \in \mathbf{R} \text{ and } u \in H^1(\mathbf{R}).$$

To see the scale invariance (2.1) more clearly, we remark that $\mathbf{R}\times S^{N-1}$ and $\mathbf{R}^N\setminus\{0\}$ are diffeomorphic through a mapping

$$\mathbf{R}\times S^{N-1} \to \mathbf{R}^N;\ (s,x) \mapsto e^s x.$$

And we reduce (HS.1)–(HS.2) to the existence problem for closed geodesics on non-compact Riemannian manifold $\mathbf{R}\times S^{N-1}$ with a metric g^V defined by

$$g^V_{(s,x)} = e^{2s}(H - V(e^s x))g^0_{(s,x)}.$$

Here g^0 is the standard product metric on $\mathbf{R} \times S^{N-1}$. Under the assumption (V2) with $\alpha = 2$ and (V3), we have

$$e^{2s}(H - V(e^s x)) \to 1 \qquad \text{as } s \to -\infty,$$
$$e^{2s}(H - V(e^s x)) \to b \qquad \text{as } s \to \infty,$$

For the existence of closed geodesics on $(\mathbf{R} \times S^{N-1}, g)$, we have

Theorem 2.3. *Let g be a Riemannian metric on $\mathbf{R} \times S^{N-1}$ and suppose*

$$g_{(s,x)} \sim g^0 \qquad \text{as } s \sim \pm\infty. \tag{2.2}$$

Then $(\mathbf{R} \times S^{N-1}, g)$ has at least one non-constant closed geodesic.

We can derive our Theorem 2.2 from Theorem 2.3 easily. Closed geodesics on $(\mathbf{R} \times S^{N-1}, g)$ can be characterized as critical points of

$$E(u) = \frac{1}{2} \int_0^1 g_u(\dot{u}, \dot{u})\, d\tau : \Lambda \to \mathbf{R},$$

where

$$\Lambda = \{u \in H^1(0,1; \mathbf{R} \times S^{N-1});\ u(0) = u(1)\}.$$

The following property is important in proving Theorem 2.3.

Proposition 2.4. *Assume (2.2) and suppose that $(u_j)_{j=1}^{\infty} \subset \Lambda$ satisfies for some $c > 0$*

$$E(u_j) \to c,$$
$$\|E'(u_j)\|_{(T_{u_j}\Lambda)^*} \to 0.$$

Then there is a subsequence — we still denote it by u_j — such that one of the following 2 statements holds:

(i) There is a non-constant closed geodesic $u_0 \in \Lambda$ on $(\mathbf{R} \times S^{N-1}, g)$ such that

$$u_j \to u_0 \text{ in } \Lambda.$$

(ii) There is a closed geodesic $x_0(t)$ on the standard sphere S^{N-1} such that if we write $u_j(t) = (s_j(t), x_j(t))$, then

1° $s_j(0) \to \infty$ *or* $s_j(0) \to -\infty$.

2° $\tilde{u}_j(t) \equiv (s_j(t) - s_j(0), x_j(t)) \to (0, x_0(t))$ *in* Λ.

3° $x_0(t)$ *can be written*

$$x_0(t) = (\cos kt, \sin kt, 0, \cdots, 0)$$

for some $k \in \mathbf{N}$ if we take a suitable coordinate.

In particular, we have

Corollary 2.5. *The Palais-Smale condition for $E(u)$ breaks down only the levels $2\pi^2k^2$ ($k \in \mathbf{N}$).*

To find a critical point of $E(u)$, we consider 2 minimax values: For a $(N-2)$ dimensional compact orientable manifold M, we set

$$\Gamma(M) = \{\gamma \in C(M,\Lambda);\ \gamma(M) \text{ is NOT contractible in } \Lambda\}.$$

We consider the following class of compact manifolds:

$$\mathcal{M}_{N-2} = \{M; M \text{ is a } (N-2) \text{ dimensional compact connected manifold such that } \Gamma(M) \neq \emptyset\}$$

and we define

$$b(M) = \inf_{\gamma\in\Gamma(M)} \max_{u\in\gamma(M)} E(u),$$
$$\underline{b} = \inf_{M\in\mathcal{M}_{N-2}} b(M).$$

Next to define another minimax value, we introduce

$$\overline{\Gamma} = \{\gamma \in C(\mathbf{R}\times S^{N-2},\Lambda); \gamma(r,z)(t) = (r,\sigma_0(z)(t)) \quad \text{for sufficiently large } |r|\},$$

where $\sigma_0(z)$ is defined for $z = (z_1,\cdots,z_{N-1}) \in S^{N-2} = \{(z_1,\cdots,z_{N-1});\ z_1^2+\cdots+z_{N-1}^2 = 1\}$ by

$$\sigma_0(z)(t) = \begin{cases} (2z_1,\cdots,2z_{N-2},\sqrt{4z_{N-1}^2-3}\cos 2\pi t, \sqrt{4z_{N-1}^2-3}\sin 2\pi t) & \text{if } |z_{N-1}| \geq \sqrt{3}/2, \\ (\frac{2|z_{N-1}|}{\sqrt{3}}\frac{z_1}{\sqrt{1-z_{N-1}^2}},\cdots,\frac{2|z_{N-1}|}{\sqrt{3}}\frac{z_{N-2}}{\sqrt{1-z_{N-1}^2}},\sqrt{\frac{3-4z_{N-1}^2}{3}},0) & \text{if } |z_{N-1}| < \sqrt{3}/2. \end{cases}$$

Lastly we set

$$\overline{b} = \inf_{\gamma\in\overline{\Gamma}} \sup_{u\in\gamma(\mathbf{R}\times S^{N-2})} E(u).$$

We can see

$$0 < \underline{b} \leq 2\pi^2 \leq \overline{b}.$$

We consider once again 3 cases

Case 1: $\underline{b} < 2\pi^2$,
Case 2: $\underline{b} = 2\pi^2$ and $\overline{b} > 2\pi^2$,
Case 3: $\underline{b} = \overline{b} = 2\pi^2$.

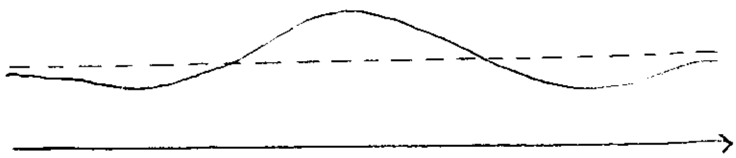
For $E(u)$, we do not have $\overline{b} < 8\pi^2$ and we have to study the case

$$\overline{b} = 2\pi^2k^2 \qquad (k \geq 2)$$

separately. We use notion of Morse indices to deal with this case.

3. Nonhomogeneous nonlinear elliptic problems

Here we give an example in which we can observe the corresponding functional has a property like this picture:



Here we go back to elliptic problems and we study

$$\begin{aligned} -\Delta u + u &= a(x)u^p + f(x) && \text{in } \mathbf{R}^N, \\ u &> 0 && \text{in } \mathbf{R}^N, \\ u &\in H^1(\mathbf{R}^N), \end{aligned} \tag{3.1}$$

where $a(x) \in C(\mathbf{R}^N, \mathbf{R})$, $f(x) \in L^2(\mathbf{R}^N)$, $f(x) \geq 0$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p < \infty$ if $N = 1, 2$.

Here we show

Theorem 3.1. *Assume*

$$0 < a(x) \leq 1 \qquad \text{for all } x \in \mathbf{R}^N, \tag{3.2}$$

$$a(x) \not\equiv 1, \tag{3.3}$$

$$\exists \lambda > 2 \, \exists C > 0 : a(x) \geq 1 - Ce^{-\lambda|x|} \quad \text{for all } x \in \mathbf{R}^N.$$

Then there exists a $\epsilon_0 > 0$ such that if $f \geq 0$, $f \not\equiv 0$, $\|f\|_{H^{-1}} \leq \epsilon_0$, then (3.1) has at least four positive solutions.

To give a proof of Theorem 3.1, we use the following functionals:

$$I_{a,f}(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p+1}\int_{\mathbf{R}^N} a(x)u_+^{p+1}\,dx - \int_{\mathbf{R}^N} a(x)u_+^{p+1}\,dx,$$

$$I_\infty(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p+1}\int_{\mathbf{R}^N} u_+^{p+1}\,dx.$$

We also consider the constraint functional on Σ_+:

$$J_{a,f}(u) = \inf_{t>0} I_{a,f}(tu),$$

$$J_\infty(u) = \inf_{t>0} I_\infty(tu).$$

We remark that if $\|f\|_{H^{-1}}$ is sufficiently small, we can find a positive solution $u_{0f}(x)$ as a perturbation of 0 and other solutions can be found as critical points of $J_{a,f}(u)$ on Σ_+.

As to the break down of the Palais-Smale condition, we have

Proposition 3.2. *The Palais-Smale condition breaks down only at levels*

$$I_{a,f}(u) + kc_\infty,$$

where u is a critical point of $I_{a,f}(u)$, $k \in \mathbf{N}$ and $c_\infty = \inf_{\Sigma_+} J_\infty(u)$. In particular, the Palais-Smale condition holds in $(-\infty, I_{a,f}(u_{0f}) + c_\infty)$.

As in Section 1, we define

$$\underline{b} = \inf_{v \in \Sigma_+} J_{a,f}(v),$$
$$\bar{b} = \inf_{\gamma \in \Gamma} \sup J_{a,f}(\gamma(x)).$$

For small $\|f\|_{H^{-1}}$, it follows from the assumptions (3.2) and (3.3) that

$$c_\infty < \bar{b} < 2c_\infty.$$

Since $I_{a,f}(u_{0f}) \to 0$ as $f \to 0$, we have

$$I_{a,f}(u_{0f}) + c_\infty < \bar{b} < I_{a,f}(u_{0f}) + 2c_\infty.$$

Thus, we can find a critical point corresponding to $\bar{b}$.

To find 2 more solutions, we observe our functional has a property like the above picture. We will show that $\underline{b} < I_{a,f}(u_{0f}) + I_\infty(\omega)$. More precisely, we will show

Proposition 3.3. *For a sufficiently small $\|f\|_{H^{-1}}$, we have*

$$\operatorname{cat}\{v \in \Sigma_+;\ J_{a,f}(v) \le I_{a,f}(u_{0f}) + c_\infty - \delta\} \ge 2$$

for some $\delta > 0$. In particular, we have

$$\{v \in \Sigma_+;\ J_{a,f}(v) \le I_{a,f}(u_{0f}) + c_\infty - \delta\} \ne \emptyset.$$

We remark that when $f \equiv 0$,

$$\{v \in \Sigma_+;\ J_{a,f}(v) \le I_{a,f}(u_{0f}) + c_\infty - \delta\} = \emptyset.$$

To prove Proposition 3.3, we use the following estimate.

Lemma 3.4. *There exists $R_0 \geq 1$ such that for $|y| \geq R_0$*

$$I_{a,f}(u_{0f} + \omega(\cdot - y)) < I_{a,f}(u_{0f}) + I_\infty(\omega).$$

The above estimate describes some interaction phenomena between u_{0f} and $\omega(\cdot - y)$. This kind of estimate (between $\omega(\cdot - y)$ and $\omega(\cdot + y)$) was used successfully in [**BL**] to show the existence of a positive solution of (0.1).

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