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Some singular perturbation elliptic problems

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SOME SINGULAR PERTURBATION ELLIPTIC PROBLEMS

by

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Summary

In the first part of these notes we describe and prove in detail results contained in the work [12] on the asymptotic behavior of (global) least energy solutions of an elliptic equation involving a small parameter, respectively under Neumann and Dirichlet boundary conditions. These solutions are known to develop a so-called spike-layer pattern. as the parameter goes to zero. In the second part we summarize, without providing details, some results concerning single and multiple spike-layer solutions which are not of a globally least-energy nature.

Part I.- Least-enegy spike-layer solutions in a degenerate setting

The material contained in the first part of these lectures is included in the work [12]. Here we deal with the study of solutions to a class of nonlinear singularly perturbed elliptic problems of the form

$$\varepsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega, \tag{0.1}$$

where the simplest model for the nonlinearity f is given by $f(t) = t^p$ with $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, and $1 < p < +\infty$ otherwise. Here Ω is a smooth bounded domain, $\varepsilon > 0$ is a small parameter and we are interested in positive solutions to this equation satisfying zero Dirichlet or Neumann boundary conditions on $\partial\Omega$.

The study of solutions to this and related equations has received considerable attention in recent years. A very interesting feature of (0.1) is the appearance of solutions exhibiting a "spike-layer pattern" as $\varepsilon \to 0$. To motivate informally the meaning of this type of concentration phenomena we observe that if u_ε solves equation (0.1) in Ω and x_ε is a point in $\bar{\Omega}$ where u_ε maximizes, then the function $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$ maximizes at the origin and satisfies

$$\Delta v - v + f(v) = 0 \tag{0.2}$$

in the expanding domain $\varepsilon^{-1}\{\Omega - x_\varepsilon\}$, which as $\varepsilon \to 0$ becomes the entire space $\mathbb{R}^N$ or a half-space. Now, when e.g. $f(t) = t^p$, equation (0.2) possesses a least energy solution w (ground state) in entire $\mathbb{R}^N$, maximizing at zero and vanishing exponentially at infinity. When restricted to a half-space passing through the origin, this function is a least energy solution on that domain under Neumann boundary conditions.

More generally, a least energy solution of (0.2) in $\mathbb{R}^N$ exists if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following *structure assumptions.*

(f1) $f(t) \equiv 0$ for $t \leq 0$ and $f(t) = o(t)$ near $t = 0$.

(f2) $f(t) = O(t^s)$ for some $1 < s < \frac{N+2}{N-2}$ if $N \geq 3$, and $s > 1$ if $N = 1, 2$.

(f3) There exists a constant $\theta > 2$ such that $\theta F(t) \leq tf(t)$ for $t \geq 0$, in which

$$F(t) = \int_0^t f(s)ds. \tag{0.3}$$

(f4) The function $t \to f(t)/t$ is strictly increasing.

More precisely, the *energy functional* defined as

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + v^2 - \int_{\mathbb{R}^N} F(v), \quad v \in H^1(\mathbb{R}^N), \tag{0.4}$$

has a least positive critical value c_* characterized as

$$c_* = \inf_{v\neq 0}\sup_{t>0} I(tv). \tag{0.5}$$

An associated critical point w actually solves equation (0.2) and is called a least energy solution. It also decays exponentially at infinity.

If it happened that the scaled solution v_ε converged to one of these w's as $\varepsilon \to 0$ in, say, the H^1-sense, then the actual look of u_ε would be that of a very sharp spike, centered at the point x_ε, while approximating zero at an exponential rate in $1/\varepsilon$ away from it.

A natural question is that of finding solutions exhibiting this type of notable behavior as well as locating their asymptotic spikes. A number of works have appeared in the literature in this subject in recent years. Among the most remarkable results in this line are the works by Ni and Takagi [21], [22] for equation

$$\begin{cases} \varepsilon^2\Delta u - u + f(u) = 0 & \text{in} \quad \Omega \\ u > 0 \ \text{ in } \Omega, \ \partial u/\partial n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.6}$$

and that by Ni and Wei [23] for the Dirichlet problem

$$\begin{cases} \varepsilon^2\Delta u - u + f(u) = 0 & \text{in} \quad \Omega \\ u > 0 \ \text{ in } \Omega, \ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{0.7}$$

In those papers the behavior of a least-energy solution to the problems, for a class of nonlinearities including the subcritical power u^p, has been well understood. Those solutions are characterized by means of the mountain pass value of the associated energy functional.

In more precise terms, associated to (0.6) (resp.(0.7)) we have the "energy" functional

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \int_\Omega F(u), \tag{0.8}$$

whose nontrivial critical points in the space $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) represent solutions of (0.6) (resp.(0.7)). The structure assumptions (f1)-(f4) guarantee the validity of the P.S. condition for this functional, so that the mountain pass theorem applies providing a positive critical value characterized as

$$c_\varepsilon = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\varepsilon(\gamma(t)) \tag{0.9}$$

where $\gamma \in \Gamma$ if and only if $\gamma \in C([0,1], H)$ and $\gamma(0) = 0$, $I_\varepsilon(\gamma(1)) \leq 0$. It was observed in [14] that this number can be further characterized as

$$c_\varepsilon = \inf_{u\neq 0}\sup_{t>0} I_\varepsilon(tu), \tag{0.10}$$

which can be shown to be the least among all nonzero critical values of I_ε. Here H corresponds to $H^1(\Omega)$ or $H_0^1(\Omega)$.

In [21], [22], it is shown that a solution u_ε at this least energy level for the Neumann problem possesses just one local maximum point, which lies on the boundary, and concentrates (up to subsequences) around a point where mean curvature maximizes.

On the other hand, Ni and Wei [23] show that a least energy solution of the Dirichlet problem (0.7) necessarily concentrates around a "most centered point" of the domain, namely around a point of maximum distance to the boundary.

In both problems the method employed consists of a combination of the variational characterization of the solutions u_ε and exact estimates of the value of the energy functional based on a precise asymptotic analysis of u_ε. This process is technically delicate. It amounts to finding sharp estimates of the error commited when approximating the scaled function v_ε by its limit w. In order to carry out this process an extra assumption is needed. It is assumed that the solution to the *limiting problem*

$$\begin{cases} \Delta w - w + f(w) = 0 \quad \text{in } \mathbb{R}^N \\ w > 0, \quad w(0) = \max w, \\ \lim_{|x|\to\infty} w(x) = 0, \end{cases} \tag{0.11}$$

is unique. Moreover, it is assumed that w is *nondegenerate*, in the sense that the linearized equation

$$\Delta\phi - \phi + f'(w)\phi = 0 \tag{0.12}$$

does not have nontrivial solutions which tend to zero at infinity other than linear combinations of the functions $\frac{\partial w}{\partial x_i}$, $i = 1, \ldots, N$.

These facts are indeed nontrivial. They are known to hold true for $f(s) = s^p$ as well as for other nonlinearities like sum of powers, but they are not yet known for, say, a C^1 nonlinearity satisfying (f1)-(f4).

Our main purpose in this part is to show that Ni & Takagi and Ni & Wei results hold without this delicate technical nondegeneracy-uniqueness assumption, thus enlarging considerably the class of nonlinearities for which they are known. In fact, not even differentiability of the nonlinearity will be needed so that a linearized problem may not even make sense. Only the structure assumptions (f1)-(f4) will be required.

On the other hand, the proofs we present here are relatively short and elementary. They rely on a slight but somewhat fundamental change of point of view with respect to the method in the above mentioned works, which simplifies the main step of estimating from below the energy of a least energy solution. Rather than working out a precise asymptotic estimate on ε of the solution itself, step relying on a linearization process where nondegeneracy is used in essential way, we analize further the mountain-pass variational characterization in order to obtain appropriate lower bounds on the energy of the solution. To do this only rough "zero-th order" information on the concentration phenomena will be required.

We would like to emphasize the fact that the nondegeneracy assumption on the limiting equation has played a basic role in the study of point concentration phenomena associated to this kind of equations. It seems to appear first in the work by Floer and Weinstein [15] on concentration on nonlinear Schrödinger equations in one space dimension, extended by Oh [24] to higher dimensions. More recently it appears in works on study of these phenomena in Ambrosetti-Badiale-Cingolani [1] and in recent works by Li [17], [18] and Li & Nirenberg [19]. Essentially this assumption allows in those works the local transformation of the original problem into a finite dimensional one via a Lyapunov-Schmidt reduction. Instead, the direct use of variational methods, relying on more general topological features than the splitting of the space into a direct sum of good invariant subspaces for the linearized operator, permits to obtain good localization results under relatively minimal assumptions in situations where the finite dimensional reduction does not seem possible. In this direction we may also quote the works by the authors [4], [5] and [8] on nonlinear Schrödinger equations, which we will briefly describe in the second part of these lectures.

Next we state our main results in this part of the lectures, which recover the main results in [21], [22] and [23]. The first of them concerns the behavior of a least energy solution of the Neumann problem. We will state a special case under an extra assumption on the nonlinearity while in Theorem 1.1 in §1 the general version will be provided.

We assume:

(f5) The function f is locally Lipschitz in $\mathbb{R}$.

Under this hypothesis problem (0.11) possesses only radially symmetric solutions, as it follows from the classical result of [16].

Theorem 0.1 *Under the hypotheses (f1)-(f5), let u_ε be a least energy solution of (0.6) and $x_\varepsilon \in \Omega$ a point where u_ε reaches its maximum value. Then for sufficiently small ε, x_ε lies on $\partial\Omega$ and*

(1)

$$H(x_\varepsilon) \to \max_{x\in\partial\Omega} H(x),$$

where H denotes mean curvature of the boundary.

(2) The associated critical value can be estimated as

$$c_\varepsilon = \varepsilon^N\{\frac{c_*}{2} - \gamma\varepsilon H(x_\varepsilon) + o(\varepsilon)\} \tag{0.13}$$

where γ is a constant depending on f and N and c_* is given by (0.5). (0.11).

For the Dirichlet problem, our result is the following.

Theorem 0.2 *Assume (f1)-(f4) hold. Let u_ε be a least energy solution of (0.7) and $x_\varepsilon \in \Omega$ a point where u_ε reaches its maximum value. Then*

(1) $\operatorname{dist}(x_\varepsilon, \partial\Omega) \to \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$.

(2) The associated critical value can be estimated as

$$c_\varepsilon = \varepsilon^N\{c_* + \exp[-\frac{2}{\varepsilon}\psi_\varepsilon]\}, \tag{0.14}$$

where $\psi_\varepsilon = \operatorname{dist}(x_\varepsilon, \partial\Omega) + o(1)$ *with* $o(1) \to 0$ *as* $\varepsilon \to 0$. *(0.11).*

We will devote the rest of this first part these results. In §1 we prove Theorem 0.1, while in §2 we prove Theorem 0.2.

1 The Neumann Problem

This section is devoted to the study of the concentration phenomenon in the case of the Neumann problem. We will first state our general result, from which Theorem 0.1 follows as a corollary. We need some preliminaries. In what follows we will always assume f satisfies (f1)-(f4) and that Ω is smooth.

Since we are not assuming hypothesis (f5), solutions of (0.11) may not be radially symmetric. Then, it is necessary to consider explicitely the limiting problem in the half space $S = \mathbb{R}^{N-1} \times \mathbb{R}_+$, that is

$$\begin{cases} \Delta w - w + f(w) = 0 \quad \text{in } S \\ w > 0 \text{ in } S, \quad w(0) = \max_{\partial S} w, \\ \lim_{|x|\to\infty} w(x) = 0, \ \frac{\partial w}{\partial n} = 0 \text{ on } \partial S. \end{cases} \tag{1.1}$$

We observe that given a solution w of (1.1), we can construct a solution of (0.11) by reflexion with respect to S (except for the condition $w(0) = \max w$).

The solutions of (1.1) can be characterized as critical points of the functional

$$I_S(v) = \frac{1}{2}\int_S |\nabla v|^2 + v^2 - \int_S F(v)$$

over the space $H^1(S)$. We are interested in least energy solutions of (1.1), that is solutions with critical value given by

$$c_*^S = \inf_{v\in H^1(S), v\neq 0} \sup_{t>0} I_S(tv). \tag{1.2}$$

We denote by $\mathcal{S}$ the set of all such least energy solutions. Using Schwartz symmetrization techniques, it is not hard to see that $\mathcal{S}$ is a non empty compact subset of $H^1(S)$ and that $c_*^S = c^*/2$ where c_* is given in (0.5).

Consider next the description of Ω on the boundary. Given a point $z \in \partial\Omega$, there is a neighborhood V of z so that $\Omega \cap V$ is as the epigraph of a smooth function. More precisely, after an appropriate rotation and translation we may assume $z = 0$ and we can find a smooth function $G : B \to I\!R$, where B is a ball in R^{N-1}, such that $G(0) = 0$, $G'(0) = 0$ and $\Omega \cap V = \{(y', y_N) \in V \,/\, y_N > G(y')\}$.

Given a solution w of (1.1) we define its restricted energy density as

$$E(w, y') = (\frac{1}{2}|\nabla w|^2 + \frac{1}{2}w^2 - F(w))(y', 0)$$

for $y' \in I\!R^{N-1}$. Then we define the *generalized curvature* at z as the following number

$$\mathcal{H}(z) = \max_{w \in \mathcal{S}} \frac{1}{2} \int_{I\!R^{N-1}} (y', G''(0)y') E(w, y') dy', \tag{1.3}$$

where $(\cdot,\cdot)$ denotes the usual inner product in R^{N-1} and $G''(0)$ denotes the Hessian matrix of G at 0. Since the set $\mathcal{S}$ is compact, the function $\mathcal{H}(z)$ is well defined. One can check that $\mathcal{H}(z)$ does not depend on the particular choice of G, but only on z.

In case $\mathcal{S}$ consists only of radially symmetric functions, as would be if (f5) holds, then

$$\int_{I\!R^{N-1}} (y', G''(0)y') E(w, y') dy' = H(z)(N-1) \int_0^\infty E(w, r) r^N dr, \tag{1.4}$$

where $E(w, r) = (1/2|w'|^2 + \frac{1}{2}w^2 - F(w))(r)$, and H denotes the usual mean curvature. Thus in this case

$$\mathcal{H}(z) = \gamma H(z),$$

where

$$\gamma = \frac{N-1}{2} \max_{w \in \mathcal{S}} \int_0^\infty E(w, r) r^N dr.$$

Now we can state our general concentration theorem for the case of equation (0.6)

Theorem 1.1 *Under the hypotheses (f1)-(f4), let u_ε be a least energy solution of (0.6) and $x_\varepsilon \in \Omega$ a point where u_ε reaches its maximum value. Then $x_\varepsilon \to \bar{x} \in \partial\Omega$, after passing to a subsequence, and*

(1)

$$\mathcal{H}(\bar{x}) = \max_{x\in\partial\Omega} \mathcal{H}(x),$$

where $\mathcal{H}$ denotes the generalized curvature defined above.

(2) The associated critical value can be estimated as

$$c_\varepsilon = \varepsilon^N\{\frac{c_*}{2} - \varepsilon\mathcal{H}(\bar{x}) + o(\varepsilon)\}, \tag{1.5}$$

where c_* is given by (0.5). (0.11).

Proof. Let c_ε be the least energy critical value of the functional I_ε in $H^1(\Omega)$, given by (0.10). Actually one has

$$c_\varepsilon = \inf_{u\in M_\varepsilon} I_\varepsilon(u), \tag{1.6}$$

where

$$M_\varepsilon = \{u \in H^1(\Omega)/u \geq 0, u \neq 0, \int_\Omega \varepsilon^2|\nabla u|^2 + u^2 = \int_\Omega f(u)u\}.$$

as observed in [14], Proposition 2.14. First we obtain a rough upper estimate of c_ε. Let us consider $w_\varepsilon^z(x) = w((x-z)/\varepsilon)$, where $z \in \partial\Omega$ and w is a least energy solution of equation (1.1), properly reflected with respect to S, if necessary. Then, by definition of c_ε and simple calculations, we find

$$c_\varepsilon \leq \sup_{t>0} I_\varepsilon(tw_\varepsilon^z) \leq \varepsilon^N\{\frac{c_*}{2} + o(1)\}. \tag{1.7}$$

Let u_ε be a least energy solution of (0.6), namely a critical point of I_ε at the level c_ε. If x_ε is a point where u_ε reaches its maximum value, then, passing to a subsequence, we may assume $x_\varepsilon \to \bar{x} \in \bar{\Omega}$. Actually $\bar{x}$ must lie on $\partial\Omega$ and moreover $\text{dist}(x_\varepsilon, \partial\Omega)/\varepsilon$ must remain bounded, for otherwise v_ε defined as $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$ would converge in the H^1-sense, passing to a subsequence, to a nontrivial solution of

$$\Delta v - v + f(v) = 0$$

in entire space. This is not possible, for $I(v)$ would be at least c_* in (0.5), and (1.7) implies $I(v) \leq c_*/2$.

Thus $\bar{x} \in \partial\Omega$ and v_ε converges in the H^1-sense to w, a least energy solution on the half-space. Moreover, for certain positive constants a and b we have $v_\varepsilon(y) \leq ae^{-b|y|}$. Let $\tilde{x}_\varepsilon$ be the closest point to x_ε in $\partial\Omega$. After a rotation and a translation ε-dependent we may also assume that $\tilde{x}_\varepsilon = 0$ and that Ω can be described in a fixed neighborhood V of $\bar{x}$ as the set $\{(x', x_N) \mid x_N > G_\varepsilon(x')\}$ where G_ε is smooth, $G_\varepsilon(0) = 0$ and $G'_\varepsilon(0) = 0$. Further, we may also assume that G_ε converges locally in a C^2-sense to a G, a corresponding parametrization at $\bar{x}$.

For an open set Λ, we denote

$$I_\Lambda(v) = \frac{1}{2}\int_\Lambda |\nabla v|^2 + v^2 - \int_\Lambda F(v).$$

Let us also set $\Omega_\varepsilon = \varepsilon^{-1}(\Omega - \tilde{x}_\varepsilon)$. From the variational characterization of $c_\varepsilon = I_\varepsilon(u_\varepsilon)$ in (0.10) we have that

$$\varepsilon^{-N} I_\varepsilon(u_\varepsilon) \geq \varepsilon^{-N} I_\varepsilon(tu_\varepsilon) = I_{\Omega_\varepsilon}(tv_\varepsilon),$$

for all $t > 0$. Let us define the function $\tilde{v}_\varepsilon$ on $S \cap V$ as $\tilde{v}_\varepsilon(y', y_N) = v_\varepsilon(y', y_N)$ if $G_\varepsilon(\varepsilon y') > 0$ and $\tilde{v}_\varepsilon(y', y_N) = v_\varepsilon(y', G_\varepsilon(\varepsilon y'))$ if $G_\varepsilon(\varepsilon y') \leq 0$.

Then

$$I_{\Omega_\varepsilon}(tv_\varepsilon) \geq I_{S\cap V_\varepsilon}(t\tilde{v}_\varepsilon) + I_{(\Omega_\varepsilon \cap V_\varepsilon)\setminus S}(tv_\varepsilon) - I_{(S\cap V_\varepsilon)\setminus \Omega_\varepsilon}(t\tilde{v}_\varepsilon).$$

Let us choose $t = t_\varepsilon$ so that $I_{S\cap V_\varepsilon}(t\tilde{v}_\varepsilon)$ maximizes in t. Then, by definition of the number $c_*^S = c_*/2$ in (1.2) and the exponential decay of v_ε one gets that

$$I_{S\cap V_\varepsilon}(t_\varepsilon \tilde{v}_\varepsilon) \geq \frac{c_*}{2} + O(e^{-2\alpha/\varepsilon}), \tag{1.8}$$

for some $\alpha > 0$. Using again the exponential decay of v_ε we obtain

$$I_1 = -I_{(\Omega_\varepsilon \cap V_\varepsilon)\setminus S}(t_\varepsilon v_\varepsilon) = \tag{1.9}$$
$$-\int_{B_\varepsilon} dy' \int_{G_\varepsilon(\varepsilon y')_-/\varepsilon}^{0} (\frac{1}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2}v_\varepsilon^2 - F(v_\varepsilon))(y', y_N))dy_N + O(e^{-2\alpha/\varepsilon}),$$

where $B_\varepsilon = \{|y'| < \delta/\varepsilon\}$. Similarly, we find that

$$I_2 = I_{(S\cap V_\varepsilon)\setminus \Omega_\varepsilon}(t_\varepsilon \tilde{v}_\varepsilon) = \tag{1.10}$$
$$\int_{B_\varepsilon} dy' \int_0^{G_\varepsilon(\varepsilon y')_+/\varepsilon} (\frac{1}{2}|\nabla v_\varepsilon|^2 + \frac{1}{2}v_\varepsilon^2 - F(v_\varepsilon))(y', G_\varepsilon(\varepsilon y'))dy_N + O(e^{-2\alpha/\varepsilon}).$$

Here we have denoted $a_+ = \max\{a, 0\}$, $a_- = \min\{a, 0\}$. Now we note that $v_\varepsilon \to w$ C^1-locally with uniform exponential decay. Then since $G_\varepsilon(0) = 0$ and $G'_\varepsilon(0) = 0$ and G_ε converges in a C^2 local sense to G, an application of dominated convergence yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(I_1 + I_2) &= \frac{1}{2}\sum_{ij}^{N-1}\int_{\mathbb{R}^{N-1}} G_{ij}(0)y'_i y'_j(\frac{1}{2}|\nabla w|^2 + \frac{1}{2}w^2 - F(w))(y', 0)dy' \\ &= \frac{1}{2}\int_{\mathbb{R}^{N-1}}(y', G''(0)y')E(w, y')dy' \le \mathcal{H}(\bar{x}). \qquad (1.11)\end{aligned}$$

Thus we conclude that

$$c_\varepsilon \ge \frac{c_*}{2} - \varepsilon\mathcal{H}(\bar{x}) + o(\varepsilon).$$

On the other hand, a direct computation along the same lines yields

$$c_\varepsilon \le \sup_{t>0} I(tw^z_\varepsilon) = \frac{c_*}{2} - \varepsilon\mathcal{H}(z) + o(\varepsilon),$$

for any $z \in \partial\Omega$. Here w^z_ε a the function in $\mathcal{S}$ that realizes the maximum in (1.3).

Combining these two estimates directly provides assertions (1) and (2) of the theorem, since in particular we conclude $\mathcal{H}(\bar{x}) \ge \mathcal{H}(z)$ for all $z \in \partial\Omega$. □

Now we can easily derive Theorem 0.1.

Proof of Theorem 0.1. By an argument given in [21] one can prove that the point x_ε, where the function u_ε reaches its maximum lies on $\partial\Omega$. In fact, if this is not the case, using that $\bar{x} \in \partial\Omega$ we find that the limit v of the rescaled function v_ε has a degenerate maximum. This is impossible since v is radial and the radial equation has local uniqueness property because of assumption (f5). The result thus follows from the previous theorem and the observation after equation (1.4). □

Remark. We observe that the above computations still apply in cases where the domain is not of class C^2. In fact, let us say for instance that the domain is smooth everywhere except at 0, where it is described locally as the epigraph of $G(x') = |x'|^{1+\sigma}$ with $\sigma \in (0, 1)$. Then a similar computation provides that the least energy value can be estimated as

$$c_\varepsilon = \varepsilon^N\{\frac{C^*}{2} - \kappa\varepsilon^\sigma + o(\varepsilon^\sigma)\},$$

for some positive constant κ.

Remark. The method described here applies to other problems with simple nonlinearities but without fine structure information in the *limiting equation*. Let us consider for instance the problem of estimating the best constant and extremals in the subcritical sobolev trace embedding

$$S_\lambda \|u\|_{L^{p+1}(\partial\lambda\Omega)} \leq \|u\|_{H^1(\lambda\Omega)},$$

with $1 < p < \frac{N}{N-2}$, where Ω is a bounded, smooth domain and λ a large parameter. It turns out that similarly to the result considered in this section, $S_\lambda \to S_\infty$ where S_∞ is the corresponding constant for the half space. Besides,

$$S_\lambda = S_\infty - \lambda^{-1}\gamma \max H + o(\lambda^{-1})$$

where H is mean curvature and γ a universal positive constant. Moreover, in the scale $1/\lambda$ the extremal constitutes a sharp spike around a point of maximum mean curvature. Here the relevant limiting equation is

$$\Delta w - w = 0 \quad \text{in } H$$

$$\frac{\partial w}{\partial n} = w^p. \quad \text{on } \partial H$$

These results are part of the work [13]. A positive least energy solution in this case *is not* radially symmetric, hence it seems very difficult to prove uniqueness or nondegeneracy as in Kwong's result.

2 The Dirichlet Problem

In this section we study the behavior of a least energy solutions of the nonlinear elliptic equation (0.7) and prove Theorem 0.2.

The least critical value c_ε associated to I_ε in $H_0^1(\Omega)$ can be characterized as

$$c_\varepsilon = \inf_{u \in M_\varepsilon} I_\varepsilon(u) \tag{2.1}$$

where $M_\varepsilon = \{u \in H_0^1(\Omega)/u \geq 0, u \neq 0, \int_\Omega \varepsilon^2|\nabla u|^2 + u^2 = \int_\Omega f(u)u\}$. We observe that thanks to hypothesis (f4), given $u \in H_0^1(\Omega)$, $u \geq 0$ and $u \neq 0$, there exists exactly one $t > 0$ such that $tu \in M_\varepsilon$. See for example [14], Proposition 2.14. At this t one has $I_\varepsilon(tu) = \max_{\tau>0} I_\varepsilon(\tau u)$.

Our proof of Theorem 0.2 is based on energy comparisons with the corresponding problem in a ball $B_\rho = \{x \in I\!R^N / |x| \le \rho\}$, where $\rho > 0$. Thus we consider the equation

$$\begin{cases} \Delta u - u + f(u) = 0 & \text{in} \quad B_\rho \\ u > 0 \ \text{ in } B_\rho, \ u = 0 & \text{on} \quad \partial B_\rho, \end{cases} \tag{2.2}$$

and its associated functional $J_\rho : H_0^1(B_\rho) \to I\!R$ given by

$$J_\rho(u) = \frac{1}{2}\int_{B_\rho} |\nabla u|^2 + u^2 - \int_{B_\rho} F(u). \tag{2.3}$$

This functional has a least positive critical value, denoted by c_ρ, which can be characterized similarly to (2.1). Using Schwarz's symmetrization, we find at least one radially symmetric least energy solution of (2.2).

The following lemma is a crucial step in the proof.

Lemma 2.1

$$c_\rho = c_* + \exp[-2\rho(1 + o(1))], \tag{2.4}$$

where c_* is given by (0.5).

Here where $o(1)$ approaches 0 as $\rho \to \infty$.

We postpone the proof of this result to the end of this section. Next prove Theorem 0.2.

Proof of Theorem 0.2 Let u_ε be a least energy solution of (0.7) and x_ε a maximum point of it. Standard compactness arguments yield that the scaled function $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$ converges locally in the C^1-sense to a least energy solution w of the limiting equation. Moreover, this convergence is also uniform and in the $H^1 - sense$ globally. Furthermore v_ε and its derivatives has a uniform exponential decay for large $|y|$.

To prove statements (1) and (2) we first observe that the previous lemma provides an upper estimate right away. In fact, we consider a ball of maximal radius contained in Ω, centered at a point $\bar{x}$. Then since the least energy values for I_ε in Ω and that in the ball are ordered, namely $c_\varepsilon \le \varepsilon^n c_\rho$ with $\rho = \text{dist}(\bar{x}, \partial\Omega)/\varepsilon$, we obtain from Lemma 2.1,

$$c_\varepsilon \le \varepsilon^N \{c_* + \exp[-\frac{2}{\varepsilon}\text{dist}(\bar{x}, \partial\Omega)(1 + o(1))]\}. \tag{2.5}$$

Next we will estimate c_ε from below. We may assume, passing to a subsequence, that x_ε converges to a point $x_0 \in \bar{\Omega}$. Thus

$$d_\varepsilon \equiv \text{dist}(x_\varepsilon, \partial\Omega) \longrightarrow d_0 \equiv \text{dist}(x_0, \partial\Omega) \quad \text{as } \varepsilon \longrightarrow 0.$$

Given $\delta > 0$, let us choose a number $d_0' > 0$ so that

$$\text{vol}(B(x_0, d_0')) = \text{vol}(\Omega \cap B(x_0, d_0 + \delta)).$$

Next we take a $\delta' > 0$ slightly smaller than δ with $d_0' < d_0 + \delta'$.

Let us consider a C^∞ cut-off function η_ε such that
$\eta_\varepsilon(s) = 1$ for $0 \le s \le d_\varepsilon + \delta'$ and $\eta_\varepsilon(s) = 0$ if $s \ge d_\varepsilon + \delta$,
with $0 \le \eta_\varepsilon \le 1$ and with uniformly bounded derivative. Let us set $\tilde{u}_\varepsilon = u_\varepsilon \eta_\varepsilon(|x_\varepsilon - x|)$. We find that

$$c_\varepsilon \ge I_\varepsilon(tu_\varepsilon) \ge I_\varepsilon(t\tilde{u}_\varepsilon) - \exp[-\frac{2}{\varepsilon}(d_\varepsilon + \delta')], \tag{2.6}$$

for all $t \in [0, 2]$, for ε sufficiently small. Here we have used the fact, obtained from the maximum principle by comparison with a suitable test function, that $u_\varepsilon(x) \le e^{(|x - x_\varepsilon| + o(1))/\varepsilon}$. A similar estimate also holds true for $|\nabla u_\varepsilon(x)|$.

Let us consider the number $R_\varepsilon = d_\varepsilon'/\varepsilon$, where d_ε' is chosen such that

$$\text{vol}(B(x_\varepsilon, d_\varepsilon')) = \text{vol}(\Omega \cap B(x_\varepsilon, d_\varepsilon + \delta)).$$

Using Schwarz's symmetrization we then obtain

$$I_\varepsilon(t\tilde{u}_\varepsilon) \ge \varepsilon^N J_{R_\varepsilon}(tu_\varepsilon^*) \tag{2.7}$$

for all $t \in [0, 2]$, where u_ε^* is the standard radially decreasing rearrangement of $\tilde{u}_\varepsilon$ and J_{R_ε} is given by (2.3) for $\rho = R_\varepsilon$. Next, let us take a number $t^* > 0$ so that $J_{R_\varepsilon}(tu_\varepsilon^*) \le J_{R_\varepsilon}(t^* u_\varepsilon^*)$ for all $t > 0$. Then combining the lower estimate in Lemma 2.1, (2.6) and (2.7) we obtain

$$c_\varepsilon \ge \varepsilon^N\{c_* + \exp[-2R_\varepsilon(1 + o(1))] - \exp[-\frac{2}{\varepsilon}(d_\varepsilon + \delta')]\} \ge \varepsilon^N\{c_* + \exp[-\frac{2}{\varepsilon}(d_\varepsilon + \delta)]\}.$$

Thus we have proven that given $\delta > 0$ this inequality holds if ε is small enough. Therefore,

$$c_\varepsilon \ge \varepsilon^N\{c_* + \exp[-\frac{2}{\varepsilon}(d_\varepsilon + o(1))]\}. \tag{2.8}$$

From (2.5) and (2.8) Statements 1. and 2. follow, thus finishing the proof.□

It only remains to prove Lemma 2.1. To do this we will make use of the elementary estimates contained in the following two lemmas.

Lemma 2.2 *If w and w_ρ are solutions of equations (0.11) and (2.2), respectively, then the following estimates hold*

$$w_\rho(\rho-1) = \exp[-\rho(1+o(1))] \text{ and } w(\rho-1) = \exp[-\rho(1+o(1))],$$

where $o(1) \to 0$ as $\rho \to 0$.

Proof. The proofs of both estimates are similar, so that we only perform that for w_ρ.

Given numbers $R > 0$ and $\varepsilon > 0$, we consider the solution of the equation

$$u'' - (1-\varepsilon/2)u = 0 \quad \text{in } (R,\rho)$$

with boundary conditions $u(R) = 1$ and $u(\rho) = 0$. Clearly $u(\rho-1) \leq \exp[-(1-\varepsilon)\rho]$ is ρ is large enough. Let us choose a fixed number R, independent of ρ, so that $w_\rho(R) \leq 1$,. This is always possible since w_ρ converges up to subsequences uniformly over compacts to a radial least energy solution of the equation in entire $\mathbb{R}^N$ and the set of such solutions is compact in $H^1(\mathbb{R}^N)$.

Making R larger if necessary, we also obtain that u is a supersolution of (2.2) in $[R,\rho)$. Therefore $w_\rho \leq u$ in that range, and the estimate from above readily follows.

As for the lower estimate, given $R > 0$ and $\varepsilon > 0$, we consider the solution of the equation

$$u'' + \frac{N-1}{R}u' - u = 0 \quad \text{in } (R,\rho)$$

such that $u(R) = w_\rho(R)$ and $u(\rho) = 0$. We can easily see that for a large enough R we have $u(\rho-1) \geq \exp[-(1-\varepsilon)\rho]$, for all large ρ. Let us observe that this u is a subsolution of (2.2). From here the desired lower estimate immediately follows.□

Lemma 2.3 *Let u and v solutions of the equations*

$$u'' + \frac{N-1}{r}u' - u = 0 \qquad \text{in } (\rho-1,\infty) \tag{2.9}$$

$$v'' + \frac{N-1}{r}v' - e(\rho)v = 0 \qquad \text{in } (\rho-1,\rho) \tag{2.10}$$

with boundary conditions $u(\rho-1) = v(\rho-1) = 1$ *and* $u(+\infty) = 0$, $v(\rho) = 0$, *where* $e(\rho) \to 1$ *as* $\rho \to \infty$ *uniformly. Then for some* $\lambda_0 > 0$

$$u'(\rho-1) - v'(\rho-1) \geq \lambda_0,$$

for large ρ.

Proof. The proof consists of finding upper and lower solutions similar to those in the previous proof. We omit the details.□

Proof of Lemma 2.1. First we find an upper estimate for c_ρ with the required form. Let v_ρ be the solution of the equation

$$\Delta u - u = 0 \tag{2.11}$$

in $B_\rho \setminus \bar{B}_{\rho-1}$, with boundary conditions $v_\rho(\rho-1) = w(\rho-1)$, and $v_\rho(\rho) = 0$. We define $\bar{w}_\rho(r) = w(r)$ if $0 \leq r \leq \rho - 1$, and $\bar{w}_\rho(r) = v_\rho(r)$ if $\rho - 1 \leq r \leq \rho$. Then we have

$$c_\rho = J_\rho(w_\rho) = \max_{t\geq 0} J_\rho(tw_\rho) \leq \max_{t \geq 0} J_\rho(t\bar{w}_\rho) = J_\rho(t_\rho \bar{w}_\rho). \tag{2.12}$$

Since $\bar{w}_\rho$ converges in the H^1 sense to w, it is easy to see that $t_\rho \to 1$ as $\rho \to \infty$. Next we see that

$$\begin{aligned} J_\rho(t_\rho \bar{w}_\rho) &\leq \frac{t_\rho^2}{2}\int_{B_{\rho-1}} |\nabla w|^2 + w^2 - \int_{B_{\rho-1}} F(t_\rho w) + \frac{t_\rho^2}{2}\int_{B_\rho \setminus \bar{B}_{\rho-1}} |\nabla v_\rho|^2 + v_\rho^2 \\ &\leq I(t_\rho w) + \frac{t_\rho^2}{2} \int_{\partial B_{\rho-1}} \frac{\partial v_\rho}{\partial \nu} v_\rho \\ &\leq c_* - a_N r^{N-1} t_\rho^2 v_\rho'(r) v_\rho(r)|_{r=\rho-1}. \end{aligned}$$

On the other hand, Lemma 2.2 plus the use of an appropriate supersolution of the equation yields that

$$v_\rho(\rho-1), -v_\rho'(\rho-1) \leq \exp[-\rho(1+o(1))].$$

The upper estimate then follows directly from the above two inequalities.

As for the lower estimate, let $\bar{v}_\rho$ be the solution of equation (2.11) in $\mathbb{R}^N \setminus \bar{B}_{\rho-1}$, with boundary conditions $\bar{v}_\rho(\rho-1) = w_\rho(\rho-1)$ and $\bar{v}_\rho(+\infty) = 0$.

Let us define $\bar{w}_\rho(r) = w_\rho(r)$ if $0 \le r \le \rho - 1$, and $\bar{w}_\rho(r) = \bar{v}_\rho(r)$ if $\rho - 1 \le r < +\infty$. Then we have, for all $t > 0$ that

$$\begin{aligned} c_\rho = J_\rho(w_\rho) \;&\ge\; I(t\bar{w}_\rho) + \frac{t^2}{2}\int_{B_\rho \setminus B_{\rho-1}} |\nabla w_\rho|^2 + e(\rho,t)w_\rho^2 \\ &-\; \frac{t^2}{2}\int_{\mathbb{R}^N \setminus B_{\rho-1}} |\nabla \bar{v}_\rho|^2 + \bar{v}_\rho^2, \end{aligned} \tag{2.13}$$

where $e(\rho,t) = \max\{(1 - F(tw_\rho(r))/(tw_\rho(r))^2) \;/\; \rho - 1 \le r \le \rho\}$. Now we choose t_ρ so that $I(t_\rho \bar{w}_\rho) \ge c$. Since $w_\rho \to w$ in the H^1 sense, we see that $t_\rho \to 1$ as $\rho \to \infty$. We then also have $e(\rho, t_\rho) \to 1$ as $\rho \to \infty$. We consider next the comparison function z_ρ given as the solution of the equation

$$\Delta u - e(\rho, t_\rho)u = 0 \qquad \text{in} \quad B_\rho \setminus \bar{B}_{\rho-1}$$

with boundary conditions $z_\rho(\rho - 1) = w_\rho(\rho - 1)$ and $z_\rho(\rho) = 0$. Then, it follows from (2.13) that

$$c_\rho \ge c - \frac{t_\rho^2}{2}\int_{\partial B_{\rho-1}} \left(\frac{\partial z_\rho}{\partial \nu} z_\rho - \frac{\partial \bar{v}_\rho}{\partial \nu}\bar{v}_\rho\right) = c_* - \frac{t_\rho^2}{2} a_N r^{N-1} w_\rho(r)(z_\rho'(r) - \bar{v}_\rho'(r))|_{r=\rho-1}.$$

But one has

$$\bar{v}_\rho'(\rho - 1) - z_\rho'(\rho - 1) \ge w_\rho(\rho - 1)\lambda_0 \ge \exp[-\rho(1 + o(1))]$$

as it follows directly from Lemmas 2.2 and 2.3. The desired lower estimate is then a consequence from the above two relations, and Lemma 2.2. □

PART II.- LOCALIZING SPIKE-LAYER PATTERNS IN SINGULARLY PERTURBED ELLIPTIC PROBLEMS

Let Ω be a domain in $I\!R^N$, not necessarily bounded, with smooth or empty boundary. In this part we will review some results concerning the problem of finding nontrivial, finite energy solutions to an equation of the form

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0, \quad u \in H_0^1(\Omega), \tag{0.1}$$

where $1 < p < (N+2)/(N-2)$ and $V(x) \geq \alpha > 0$. Equations of this form arise in different models where the presence of a small diffusion parameter ε becomes natural. For instance, in the study of *standing waves* of the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma \mid \psi \mid^{p-1} \psi. \tag{0.2}$$

Namely solutions of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$, reduces to an equation like (0.1). See [15], [24].

Again, we are concerned with the study of solutions exhibiting "spike-layer patterns" as $\varepsilon \longrightarrow 0$. Let us observe, that similarly to the type of phenomena described in the first part of these lectures, if u_ε solves equation (0.1) in Ω and a_ε is a point in Ω where u_ε maximizes, then the function $v_\varepsilon(y) = u_\varepsilon(a_\varepsilon + \varepsilon y)$ maximizes at the origin and satisfies

$$\Delta v_\varepsilon - V(a_\varepsilon + \varepsilon y)v_\varepsilon + v_\varepsilon^p = 0. \tag{0.3}$$

in the expanding domain $\varepsilon^{-1}\{\Omega - a_\varepsilon\}$ which as $\varepsilon \longrightarrow 0$ becomes entire $I\!R^n$ or a half-space. Let us assume for instance that $a_\varepsilon \longrightarrow \bar{a}$ as $\varepsilon \longrightarrow 0$. Equation (0.3) thus becomes in the limit

$$\Delta v - \beta v + v^p = 0. \tag{0.4}$$

in entire $I\!R^N$, where $\beta = V(\bar{a}) > 0$. Let us recall that equation (0.4) for subcritical p posseses a unique solution w_β in entire $I\!R^N$ maximizing at zero and vanishing at infinity which turns out to be radially symmetric and radially exponentially decreasing. If indeed v_ε converged to w_β as $\varepsilon \longrightarrow 0$, in some appropiate sense, then it would be natural to expect $u_\varepsilon(x)$ to look approximately like $w_\beta((x - a_\varepsilon)/\varepsilon)$, a function exhibiting a sharp spike shape near $\bar{a}$ while vanishing at an exponential rate in $1/\varepsilon$ elsewhere in Ω.

The first result in this line for the Schrödinger equation when $N = 1$ and $p = 3$ seems due to Floer and Weinstein [15], who found such a concentrating family via a Lyapunov-Schmidt reduction, around any nondegenerate critical point of the potential $V(x)$. Later Oh [24], [25] extended this result to higher dimensions when $1 < p < \frac{N+2}{N-2}$, with potentials V which exhibit "mild behavior at infinity", also constructing multiple-peaked solutions. Ambrosetti, Badiale and Cingolani [1] partially lifted the nondegeneracy assumption, obtaining existence of a single peak solution when the potential has a local minimum or maximum with nondegenerate *mth*-derivative. The first result for equation (0.1) in $I\!R^N$ in the possibly degenerate setting seems due to Rabinowitz [26], see also Ding and Ni [14] for an independent related result. In [26] it was shown that if $\inf_{I\!R^N} V < \liminf_{|x|\to\infty} V(x)$, then the mountain-pass value for the associated energy functional provides a solution for all small ε. This solution indeed concentrates around a global minimum of V as $\varepsilon \to 0$, as shown later by X. Wang in [29]. Moreover, Wang observed that concentration of any family of solutions with uniformly bounded energy may occur only at critical points of V.

The work [4] seems to be one of the first attempts to attack the degenerate case in (0.1) in a local setting. Here a penalization approach was devised, which permitted to find *local mountain passes* around a local minimum of V with arbitrary degeneracy. More precisely, given a bounded open set Λ such that

$$\inf_{\Lambda} V < \inf_{\partial\Lambda} V, \tag{0.5}$$

a family u_ε exhibiting a single spike in Λ, at a point x_ε such that $V(x_\varepsilon) \to \inf_\Lambda V$, is constructed.

In [5] this approach was extended to the construction of a family of solutions with several spikes located around any prescribed finite set of local minima of V in the sense of (0.5). More precisely, the following result holds.

Theorem 0.1 *Assume that there are bounded domains Λ_i, mutually disjoint, compactly contained in Ω, $i = 1, \dots, K$, such that*

$$\inf_{\Lambda_i} V < \inf_{\partial\Lambda_i} V. \tag{0.6}$$

Then there is an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ a positive solution $u_\varepsilon \in H_0^1(\Omega)$ to problem (0.1) exists. Moreover, u_ε possesses exactly K local maxima $x_{\varepsilon,i}$, with $x_{\varepsilon,i}$ in Λ_i. We also have that $V(x_{\varepsilon,i}) \to \inf_{\Lambda_i} V$, and

$$u_\varepsilon(x) \le \alpha \exp(-\frac{\gamma}{\varepsilon}|x - x_{\varepsilon,i}|), \tag{0.7}$$

for all $x \in \Omega \setminus \cup_{j\neq i}\Lambda_j$, where α and γ are certain positive constants.

We shall next describe the main points in the proof of Theorem 1. We consider first the case $K = 1$, so that we are searching for a single peak solution concentrating in Λ such that (0.5) holds.

Associated to equation (0.1) is the "energy" functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + V(x)u^2 - \frac{1}{p+1}\int_\Omega u_+^{p+1}. \tag{0.8}$$

It is standard to check that the nontrivial critical points of E_ε correspond exactly to the positive classical solutions in $H_0^1(\Omega)$ of equation (0.1). On the other hand, E_ε has 0 as a strict local minimizer and it is unbounded below, so that it satisfies the assumptions of the mountain pass theorem except possibly for the P.S. condition due to the unboundedness of the domain.

We define a modification of this functional, introduced in [4], which satisfies the P.S. condition and for which the mountain pass theorem applies.

Let $a > 0$ be the value at which $a^{p-1} = \frac{\alpha}{2}$, where $V \ge \alpha$. Let us set

$$\tilde{f}(s) = \begin{cases} s^p & \text{if } s \le a \\ \frac{\alpha}{2}s & \text{if } s > a, \end{cases}$$

and define

$$g(x,s) = \chi_\Lambda(x)s^p + (1 - \chi_\Lambda(x))\tilde{f}(s),$$

where Λ is a bounded domain as in (0.5) and χ_Λ denotes its characteristic function. Let us denote $G(x,\xi) = \int_0^\xi g(x,\tau)d\tau$, and consider the modified functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + V(x)u^2 - \int_\Omega G(x,u) \quad . \tag{0.9}$$

whose critical points correspond to solutions of the equation

$$\varepsilon^2\Delta u - V(x)u + g(x,u) = 0 \qquad \text{in} \quad \Omega. \tag{0.10}$$

It is shown in [4] that J_ε satisfies the Palais-Smale condition, no matter whether Ω is bounded or not. The reason for this is the boundedness of Λ plus the choice of $\tilde{f}$ which roughly speaking, makes J_ε "coercive outside Λ". The mountain pass lemma yields a nontrivial solution u_ε to (0.10). Let us observe that a solution to (0.10) which satisfies that $u \le a$ on $\Omega \setminus \Lambda$ will also be a solution of (0.1). The main point of the proof is to establish that u_ε indeed satisfies this for sufficiently small ε. Note that u_ε satisfies outside Λ an equation of the form

$$\varepsilon^2 \Delta u - W_\varepsilon(x) u = 0,$$

where $W_\varepsilon(x) \ge \alpha/2$. This and the maximum principle imply that it suffices to check that $u_\varepsilon \le a$ on $\partial\Lambda$.

The latter fact is an immediate consequence of assumption (0.5) and the following statement: If $\varepsilon_n \downarrow 0$ and $z_n \in \overline{\Lambda}$ are such that $u_{\varepsilon_n}(z_n) \ge b > 0$, then

$$\lim_{n\to\infty} V(z_n) = V_0 = \inf_{\Lambda} V. \tag{0.11}$$

This also shows that the maximum point of u tends to minimize V in Λ as desired. Assume, passing to a subsequence, that $z_n \to \bar{z} \in \overline{\Lambda}$ and by contradiction that $V(\bar{z}) > V_0$. We consider the sequence $v_n(z) = u_n(z_n + \varepsilon_n z)$ which satisfies the equation

$$\left\{ \begin{array}{rcl} \Delta v_n - V(x_n + \varepsilon_n z) v_n + g(x_n + \varepsilon_n z, v_n) & = & 0 \text{ in } \Omega_n, \\ v_n & = & 0 \text{ on } \partial\Omega_n \end{array} \right. \tag{0.12}$$

where $\Omega_n = \varepsilon_n^{-1}\{\Omega - x_n\}$. It can be checked that v_n is uniformly bounded in $H^1(\mathbb{R}^N)$, and from elliptic estimates it can be assumed to converge uniformly on compacts subsets of $\mathbb{R}^N$ to a function $v \in H^1(\mathbb{R}^N)$. v will satisfy an equation of the form

$$\Delta v - V(\bar{z}) v + \bar{g}(z, s) = 0 \qquad \text{in } \mathbb{R}^N. \tag{0.13}$$

where $\bar{g}(z,s) = \chi(z) s^p + (1 - \chi(z)) \tilde{f}(s)$. Here $0 \le \chi \le 1$ a.e. Thus v is a nontrivial critical point of the functional

$$\bar{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(\bar{z}) w^2 - \int_{\mathbb{R}^N} \bar{G}(z, u) \quad , \quad u \in H^1(\mathbb{R}^N) \tag{0.14}$$

where $\bar{G}(z,s) = \int_0^s \bar{g}(z,\tau)d\tau$. Via concentration-compactness type arguments one can show that

$$\liminf_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_n) \geq \bar{J}(v).$$

On the other hand, an explicit upper estimate for the mountain pass value of J_ε using a suitable test path of functions supported near the minimum set of V in Λ yields that necessarily $\limsup_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_n) < \bar{J}(v)$, a contradiction which finally gives the result in the case $K = 1$. In the general case one considers a similar modification of the energy which penalizes concentration outside the $\Lambda_i's$ but with an extra term added which also avoids to have two local maxima of the solution in one of these sets More precisely, one takes

$$\tilde{J}_\varepsilon(u) = J_\varepsilon(u) + MP_\varepsilon(u)$$

where J_ε is defined as in (0.9) but with Λ replaced by the union of the $\Lambda_i's$. M is a large constant and the further penalization term P_ε is built as follows.

$$P_\varepsilon(u) = \sum_{i=1}^{K} \left\{ \left(J_\varepsilon^i(u)\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i)^{\frac{1}{2}} \right\}_+^2 . \qquad (0.15)$$

Here J_ε^i is defined as J_ε except that the integrals are taken just in small neighborhood of Λ_i. The numbers c_i may be chosen so that a critical point of $\tilde{J}_\varepsilon$ with more than one concentration point in one Λ_i makes the corresponding term in the sum "big" independently of M.

Then a min-max quantity for $\tilde{J}_\varepsilon$ taken over a suitable class of K-dimensional maps is defined which provides a critical value at the right energy order. Double concentration in one Λ_i is discarded thanks to the term P_ε which eventually makes the total energy larger than an explicit a priori upper estimate of the minmax quantity if M was chosen large enough. □

It should be remarked that in this result u^p can still be replaced by a nonlinearity $f(u)$ which satisfy appropriate growth and convexity assumptions. The homogeneity of the power nonlinearity is not used.

The results described above are certainly connected with the those described in the first part of these lectures.

We recall that Ni and Wei in [23] have considered the Dirichlet problem in a bounded domain when $V \equiv 1$ and found that the least energy solution concentrates around a global maximum of the distance to the boundary.

Reciprocally, a strict local maximum of this function yields a concentrating family, see [30]. Multiple spike solutions around a finite number of local maxima of this function have been constructed in [9] using again a penalization approach like the one already described plus estimates originated in the work [23]. More precisely, let us assume Ω is a smooth domain in R^N, not necessarily bounded, and there are K smooth bounded subdomains of Ω, $\Lambda_1, \cdots, \Lambda_K$, compactly contained in Ω, satisfying

(H1) $\max_{x\in\Lambda_i} d(x,\partial\Omega) > \max_{x\in\partial\Lambda_i} d(x,\partial\Omega)$, $i = 1, 2, \cdots, K$,

(H2) $\min_{i\neq k} d(\Lambda_i, \Lambda_k) > 2 \max_i \max_{x\in\Lambda_i} d(x,\partial\Omega)$

where $d(\Lambda_i, \Lambda_k)$ is the distance between Λ_i, Λ_k.

Under $(H1)$ and $(H2)$, there exists a solution u_ε which possesses exactly K local maximum points $x_{\varepsilon,1}, \cdots, x_{\varepsilon,K}$ with $x_{\varepsilon,i} \in \Lambda_i$. Moreover $d(x_{\varepsilon,i}, \partial\,\Omega) \to \max_{x\in\Lambda_i} d(x, \partial\,\Omega)$, as $\varepsilon \to 0$, for all $i = 1, \ldots, K$.

Another related example where the penalization approach has shown useful is the Ginzburg-Landau equation in a bounded domain in $\mathbb{R}^2$,

$$\varepsilon^2 \Delta u + (1 - |u|^2)u = 0 \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial\Omega$$

where $g : \partial\Omega \to S^1$ has degree $d > 0$. It was proven by Bethuel, Brézis and Hélein in [3], that if Ω is star-shaped, then the global minimizer of the associated energy converges smoothly to a harmonic map from Ω into S^1, away from d points, its singularities, all of them with degree 1. These d points happen to minimize globally a certain finite-dimensional functional called the *renormalized energy.* The star-shapeness assumption was lifted by Struwe in [27].

By studying the associated heat flow, F.H. Lin in [20] proved that a non-degenerate local minimizer of this functional determines a family of solutions exhibiting asymptotic singularities at the corresponding points. The variational penalization method in [4], [5], was extended in [7], to show that actually at a possibly degenerate local minimizer of the renormalized energy, in the same sense as in (0.5), the same answer is true, with the additional information that the associated solutions are local minimizers of the energy.

The penalization method is also useful to capture single point concentration at points that are not, loosely speaking, local minimizers of the underlying finite dimensional object determining the concentration phenomena. Coming back to Schrödinger equation, it has been established in [8] that a single-spike concentrating family of solutions for equation (0.1) exists around a region where a nontrivial change of topology of the level sets occurs, including as a special case a degenerate local maximum or saddle point situation.

Locally we consider the following setting. We assume that there is an open and bounded set Λ with smooth boundary such that $\overline{\Lambda} \subset \Omega$, and closed subsets of Λ, B, B_0 such that B is connected and $B_0 \subset B$. Let Γ be the class of all continuous functions $\phi : B \to \Lambda$ with the property that $\phi(y) = y$ for all $y \in B_0$. Define the min-max value c as

$$c = \inf_{\phi \in \Gamma} \sup_{y \in B} V(\phi(y)), \tag{0.16}$$

and assume additionally:
(H1)

$$\sup_{y \in B_0} V(y) < c.$$

(H2) For all $\phi \in \Gamma$, $\quad \phi(B) \cap \{y \in \Lambda \mid V(y) \geq c\} \neq \emptyset$.

We observe that in the standard language of calculus of variations, the sets B_0, B, $\{V \geq c\}$ *link* in Λ.

(H3) For all $y \in \partial\Lambda$ such that $V(y) = c$, one has $\partial_\tau V(y) \neq 0$, where ∂_τ denotes tangential derivative.

Standard deformation arguments show that these assumptions ensure that the min-max value c is a critical value for V in Λ, which is *topologically nontrivial.* In fact, assumption (H3) "seals" Λ so that the local linking structure described indeed provides critical points at the level c in Λ, possibly admitting full degeneracy.

It is not hard to check that all these assumptions are satisfied in a general local maximum, local minimum or saddle point situation. Our main resultin [8] asserts that there is a family of solutions to problem (0.1) concentrating around a critical point at the level c in Λ.

In fact, with the aid of the penalization method developed in [4], we find that the above min-max quantity for V inherits a min-max value for the energy associated to (0.1) which provides the desired solutions.

This method is also applicable in the case $V \equiv 1$, for Dirichlet and Neumann boundary conditions to predict existence of spike-layer solutions around a topologically nontrivial critical point situation of, respectively, the distance function to the boundary [10] (say, a domain exhibiting a "neck") and of the mean curvature [11]

Finally, we remark that in the works by Y.Y. Li [17], [18] and by Li & Nirenberg [19], a similar program has been carried out for respectively the Schrodinger and the autonomous Neumann and Dirichlet problems via an alternative notion of topological nontriviality, and a finite dimensional reduction method which lead to similar concentration results.

Also to be mentioned is that we *do not know* whether these results hold only under conditions (f1)-(f4) stated the introduction. The question of constructing spike-layers which are not of least energy nature, without any extra assumptions on the limiting equation is basically open.

On the other hand, there are recent results concerning existence of solutions in the Schrödinger equation having *infinitely many spikes.* We refer the reader to the recent paper [2] and the references therein. See also [28].

References

[1] A. Ambrosetti, M. Badiale , S. Cingolani. *Semiclassical states of nonlinear Schrödinger equations* Arch. Rat. Mech. Anal. 140, pp. 285-300 (1997).

[2] A. Ambrosetti, M. Berti, *Homoclinics and complex dynamics in slowly oscillating systems* Dicrete and Continuous Dynamical Systems 4, No 3, pp. 393-403 (1998).

[3] F. Bethuel , H. Brezis, F. Hélein. Ginzburg-Landau Vortices, PNLDE, 13, Birkhäuser, 1994. Boston.

[4] M. del Pino, P. Felmer *Local mountain passes for semilinear elliptic problems in unbounded domains.* Calc. Var. and PDE , 4(1996), 121-137.

[5] M. del Pino, P. Felmer *Multi-peak bound states of nonlinear Schrödinger equations.* Ann. Inst. H. Poincaré, Anal. Nonlineaire 15, pp. 127-149 (1998).

[6] M. del Pino, P. Felmer *Locally Energy-Minimizing Solutions of the Ginzburg Landau Energy.* C.R. Acad. Sci. Paris 321, p. 1207-1211 (1995).

[7] M. del Pino, P. Felmer *Local minimizers of the Ginzburg-Landau Energy.* Math. Z. 225, pp. 671-684 (1997)

[8] M. del Pino, P. Felmer *Semi-classical states for nonlinear Schrödinger equations.* Journal of Functional Analysis 149, pp. 245-265.

[9] M. del Pino, P. Felmer, J. Wei. *Multi-peak Solutions For Some Singular Perturbation Problems.* To appear, Calc. Var. and PDE's.

[10] M. del Pino, P. Felmer, J. Wei. *On the role of distance function in singularly perturbed elliptic problems,* To appear in Comm. in PDE's.

[11] M. del Pino, P. Felmer, J. Wei. *On the role of mean curvature in certain elliptic Neumann problems,* To appear in SIAM J. Math Anal.

[12] M. del Pino, P. Felmer *Spike-layered solutions of singularly perturbed elliptic problems in a dgenerate setting.* Preprint.

[13] M. Del Pino, C. Flores *Asymptotic behavior of extremals in trace embeddings in expanding domains.* In preparation.

[14] W.Y Ding, W.M. Ni, *On the existence of Positive Entire Solutions of a Semilinear Elliptic Equation.* Arch. Rat. Mech. Anal., 91, 283-308 (1986).

[15] A. Floer, A. Weinstein, *Nonspreading Wave Packets for the Cubic Schrödinger Equation with a Bounded Potential,* Journal of Functional Analysis 69, 397-408 (1986).

[16] B. Gidas, W-M. Ni, L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Adv. Math. Suppl. Stud. 7A, Math. Anal. Appl. Part A (1981) 369-402.

[17] Y. Y. Li *On a Singularly Perturbed Elliptic Equation.* To appear in Advances in Differential Equations.

[18] Y. Y. Li *On a Singularly Perturbed Elliptic Equation with Neumann boundary conditions.* To appear in Comm. in PDE's.

[19] Y. Y. Li, L. Nirenberg *The Dirichlet problem for singularly perturbed elliptic equations.* To appear in Comm. Pure Appl. Math.

[20] F.-H. Lin, *Solutions of Ginzburg-Landau Equations and Critical points of the Renormalized Energy,* Ann. Inst. H. Poincaré, 12, no 5, 599-622 (1995).

[21] W.M. Ni, I. Takagi *On the shape on least-energy solutions to a semilinear Neumann problem.* Comm. on Pure and App. Math. Vol. XLIV, 819-851, (1991).

[22] W.-M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem,* Duke Math. J. 70 (1993), 247-281.

[23] W.-M. Ni, J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems,* Comm. Pure Appl. Math. 48 (1995), 731-7 68.

[24] Y.J. Oh, *Existence of semi-classical bound states of nonlinear Schrödinger equations with potential on the class* $(V)_a$. Comm. Partial Diff. Eq. 13, 1499-1519 (1988).

[25] Y.J. Oh, *On positive multi-lump bound states nonlinear Schrödinger equations under multiple well potential.* Comm. Math. Phys., 131, 223-253 (1990).

[26] P. Rabinowitz, *On a class of nonlinear Schrödinger equations,* Z. Angew Math Phys 43, 270-291(1992).

[27] M. Struwe, *On the Asymptotic Behavior of Minimizers of the Ginzburg-Landau Model in 2 Dimensions* , Differential and Integral Equations, Vol.7, No 4, 1613-1624 (1994).

[28] N. Thandi, Ph. D. Thesis, University of Wisconsin, 1995.

[29] X. Wang *On concentration of positive bound states of nonlinear Schrödinger equations.* Comm. Math. Phys. 153, No 2, 229-244 (1993).

[30] J. Wei, *On the construction of single-peaked solutions to a singularly perturbed elliptic Dirichlet problem*, to appear in J. Diff. Eqns .