united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR.1071/6

THIRD SCHOOL ON NONLINEAR FUNCTIONAL ANALYSIS
AND APPLICATIONS TO DIFFERENTIAL EQUATIONS

(12 - 30 October 1998)

Spike-layers in diffusion and cross-diffusion systems

W.-M. Ni

School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S.E.
Minneapolis, MN 55455
U.S.A.

These are preliminary lecture notes, intended only for distribution to participants

strada costiera, 11 - 34014 trieste italy - tel. +39 40 2240111 fax +39 40 224163 - sci_info@ictp.trieste.it - www.ictp.trieste.it

Outline

$\boxed{\text{Systems}}$

Spike-layer steady states
(pattern formation:
stability, dynamics ⋯)

Kinetic (ODE) ⟶ Diffusion ⟶ Cross-Diffusion

- Diffusion-driven instability
- Diffusion-induced extinction
- Diffusion-induced blow-up

- Positive taxis
- Negative taxis

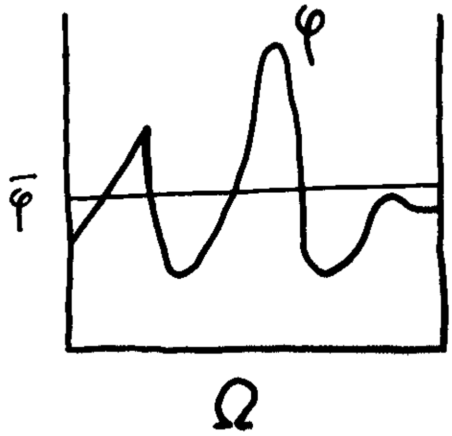
Diffusion

(I) Single Equations

Diffusion: Smoothing, trivializing

(i) Heat equation

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = \varphi(x) \geq 0 & \text{in } \Omega \end{cases}$$



$\Delta = \sum_1^n \dfrac{\partial^2}{\partial x_i^2}$: Laplacian

Ω : bounded smooth domain in $\mathbb{R}^n$

ν : unit outward normal to $\partial\Omega$

- $u(x,t)$ becomes smooth as soon as $t > 0$ and

$$u(\cdot, t) \longrightarrow \frac{1}{|\Omega|}\int_\Omega \varphi(x)\,dx \quad (\equiv \text{Constant})$$

as $t \to \infty$.

(ii) Reaction-Diffusion

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = \varphi(x) & \text{in } \Omega \end{cases}$$

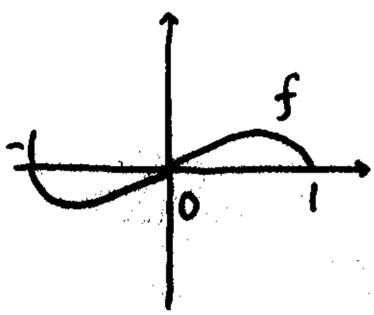
• (Matano, Casten-Holland, 1978-79)

Ω convex $\Rightarrow$ no stable nonconstant s.s.

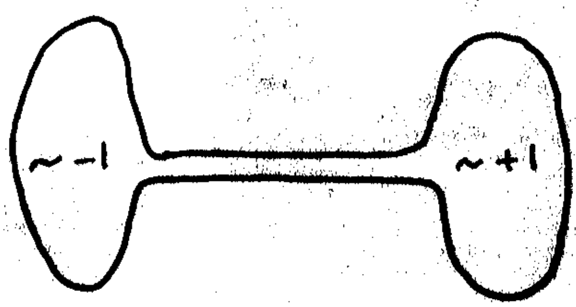
(s.s. : steady states)

• (Matano)

$\exists\, f, \Omega$ s.t. $\exists$ stable nonconstant s.s.



"bistable"



- Hale + Vegas (1984), Jimbo + Morita (1988-89)

(II) 2 × 2 Systems

$$\begin{cases} u_t = d_1 \Delta u + f(u,v) & \text{in } \Omega \times (0,T) \\ v_t = d_2 \Delta v + g(u,v) & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases}$$

d_1, d_2 : diffusion rates for u, v, resp.

- Such a system is often compared to its "kinetic" system (ODE)

$$\begin{cases} \bar{u}_t = f(\bar{u}, \bar{v}) \\ \bar{v}_t = g(\bar{u}, \bar{v}) \end{cases}$$

- Diffusion-driven instability (A. Turing, 1952)
- Diffusion-induced extinction
  (Iida, Muramatsu, Ninomiya & Yanagida, 1995)
- Diffusion-induced blow-up
  (Mizoguchi, Ninomiya & Yanagida, 1997)
  (Weinberger 1998 : $d_1 = d_2$)

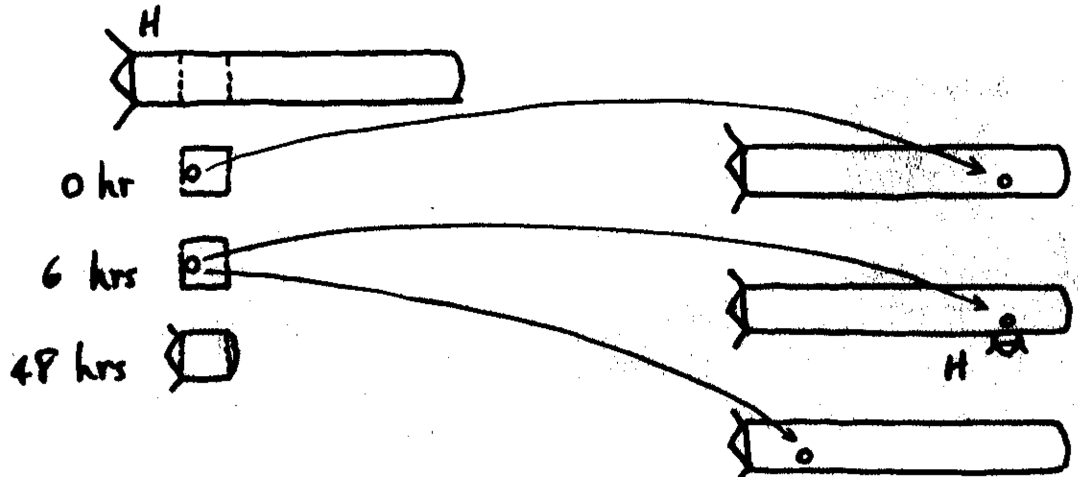
An Activator-Inhibitor System

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{v^q} & \text{in } \Omega \times \mathbb{R}^+ \\ \tau v_t = d_2 \Delta v - v + \dfrac{u^r}{v^s} & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

$$d_1 << d_2 , \quad 0 < \frac{p-1}{q} < \frac{r}{s+1}$$

- Gierer & Meinhardt (1972) $(2,1,2,0)$
- Meinhardt 1982

- Regeneration of hydra : A. Trembley (1744)
- Hydra : Biological experiments



$\Rightarrow$ Not due to cell orientation

but to a "graded" property :

$$\begin{cases} u : \text{activator (slowly diffusing)} \\ v : \text{inhibitor (rapidly diffusing)} \end{cases}$$

[A. Gierer: Prog. Biophys. molec. Biol. 37 (1981)]

An Activator-Inhibitor System

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{v^q} & \text{in } \Omega \times R^+ \\ \tau v_t = d_2 \Delta v - v + \dfrac{u^r}{v^s} & \text{in } \Omega \times R^+ \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times R^+ \end{cases}$$

$$d_1 << d_2 , \quad 0 < \frac{p-1}{q} < \frac{r}{s+1}$$

- Gierer & Meinhardt (1972) $(2, 1, 2, 0)$
- Meinhardt, 1982

• Diffusion-driven instability: (Turing)

* The constant s.s. $u \equiv v \equiv 1$ is asymp. globally stable if $\tau < \frac{s+1}{p-1}$ without diffusion, i.e. for its kinetic system

$$\begin{cases} u_t = -u + \dfrac{u^p}{v^q} \\ \tau v_t = -v + \dfrac{u^r}{v^s} . \end{cases}$$

However, with diffusion introduced, say, d_1 small & d_2 large, then the solution $u \equiv v \equiv 1$ becomes unsta

Heuristically: Replacing Δ by one of its eigenvalues λ_k in the linearized operator

$$\begin{pmatrix} p-1 & -q \\ \frac{r}{\tau} & -\frac{s+1}{\tau} \end{pmatrix} \xrightarrow[\text{introduced}]{\text{with diffusion}} \begin{pmatrix} \underline{-d_1\lambda_k}+(p-1) & -q \\ \frac{r}{\tau} & \underline{-d_2\lambda_k} - \frac{s+1}{\tau} \end{pmatrix}$$

An Activator-Inhibitor System

$$
\begin{cases}
u_t = d_1 \Delta u - u + \dfrac{u^p}{v^q} & \text{in } \Omega \times R^+ \\
\tau v_t = d_2 \Delta v - v + \dfrac{u^r}{v^s} & \text{in } \Omega \times R^+ \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times R^+
\end{cases}
$$

$$d_1 << d_2 , \quad 0 < \frac{p-1}{q} < \frac{r}{s+1}$$

- Gierer & Meinhardt (1972) $(2,1,2,0)$
- Meinhardt, 1982

• The Shadow System

$d_2 \to \infty \Rightarrow$ (heuristically) $\Delta v(\cdot,t) \to 0$ for each t

$\Rightarrow v(\cdot,t) \to \xi(t)$

$$
\begin{cases}
u_t = d_1 \Delta u - u + \dfrac{u^p}{\xi^q} & \text{in } \Omega \times R^+ \\
\text{[illegible]} - \xi + \text{[illegible]} \\
\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times R^+
\end{cases}
$$

• Suitably scaled, we have

$$
(*) \begin{cases}
\varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega \\
u > 0 \text{ in } \Omega , \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega
\end{cases}
$$

§. What do we know about (*) ? $1 < p < \frac{n+2}{n-2}$, ε small

1986 ~ 1993 in a series of papers, [Lin, Ni, Takagi
variational functional ("energy") on $H^1(\Omega)$

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) - \frac{1}{p+1}\int_\Omega u_+^{p+1}$$

Th'm. $\exists$ a least-energy solution u_ε which has exactly one (local, thus global) maximum point P_ε in $\overline{\Omega}$ (i.e. the single peak). Moreover

- $P_\varepsilon \in \partial\Omega$ and $H(P_\varepsilon) \to \max_{\partial\Omega} H$, H denotes the mean curvature of $\partial\Omega$; (i.e. the "most-curved" part,
- $u_\varepsilon(\varphi_\varepsilon(\varepsilon y)) \to w(y)$ as $\varepsilon \to 0$, where φ_ε^{-1} is a diffeomorphism straightening $\partial\Omega$ near P_ε & w is the unique positive solution of

$$\begin{cases} \Delta w - w + w^p = 0 \text{ in } \mathbb{R}^n, \\ w(0) = \max_{\mathbb{R}^n} w \quad \& \quad w \to 0 \text{ at } \infty. \end{cases}$$

Idea: $J_\varepsilon(u_\varepsilon) = \varepsilon^n\left(\frac{1}{2} I(w) - \varepsilon \cdot C \cdot H(P_\varepsilon) + o(\varepsilon)\right)$

(Singular perturbation, non-traditional)

Ideas of the Proof

- Existence of $u_\varepsilon \not\equiv$ Const. (for small ε):
  (Mountain-Pass & Nehari's constrained minimization)
- A spike-layer has its energy concentrated in a small nbhd of its peaks
  $\Rightarrow$ u_ε has only one peak $P_\varepsilon \in \bar{\Omega}$
- Energy of a "boundary-peak"
  $\approx \frac{1}{2} \cdot$ energy of an "interior-peak"
  $\Rightarrow P_\varepsilon \in \partial\Omega$
- More delicate: Where on $\partial\Omega$ should P_ε be?

$$(\dagger) \qquad J_\varepsilon(u_\varepsilon) = \varepsilon^n\left(\tfrac{1}{2} I(w) - C\varepsilon H(P_\varepsilon) + o(\varepsilon)\right)$$

$\Rightarrow$ To minimize $J_\varepsilon(u_\varepsilon)$, need to maximize $H(P_\varepsilon)$.

- To obtain $(\dagger)$, need to first obtain the/an accurate "profile" of u_ε.

Structure of the Solution Set of (*).

- For ε large, $u \equiv 1$ is the only solution of (*).
- Multi-peak Spike-Layers: Pushing the "energy" idea/method further, Gui & Wei (1997) proved, for any integer $k > 0$, $\exists$ a solution of (*) with exactly k peaks, if ε is sufficiently small.
  - interior peaks: all alike, ~ "sphere packing"
  - boundary peaks: all alike also.
- Related works: Alikakos, Bates, Fusco, Kowalczyk, Chen, del Pino, Felmer, Li, Alama, Rabinowitz, Nirenberg, ⋯
- If we view spike-layers as 0-dim'l ($\because$ the set where a spike-layer does not tend to 0 as $\varepsilon \to 0$ consists of only isolated points), then we have

Conjecture: For any integer k between 0 & $n-1$, (*) has "k-dim'l layer" solutions.

Rk: Energy levels of "k-dim'l layer" solutions are

$$\sim C\varepsilon^{n-k}$$

- Critical exponent: X. Wang, Pan, Z. Wang, [NT] Adimurthi, Yadava, Pacella, ⋯ (NOT "Robust")

Returning to :

- the shadow system ($d_1 = \varepsilon^2$)

Let u_ε be a least-energy solution of (*). Set

$$\xi_\infty^{-\alpha} = \int_\Omega u_\varepsilon^r \quad \& \quad u_{\varepsilon,\infty} = \xi_\infty^{q/(p-1)} u_\varepsilon ,$$

where $\alpha = \frac{qr}{p-1} - (s+1) > 0$. Then $(u_{\varepsilon,\infty}, \xi_\infty)$ is a s.s. of the shadow system &

$$u_{\varepsilon,\infty} = \varepsilon^{-nq/[qr-(p-1)(s+1)]} u_\varepsilon$$

$$\xi_\infty = \varepsilon^{-n(p-1)/[qr-(p-1)(s+1)]} \frac{1}{2|\Omega|} \int_{\mathbb{R}^n} w^r + \cdots$$

- the Gierer-Meinhardt system

(Only for $n=1$ or Ω : axially symmetric)

Th'm. ~~For every small ε,~~ $\exists \varepsilon_0 > 0$ small & $\exists D$ large s.t. $\begin{cases} \forall \varepsilon < \varepsilon_0 \ \& \\ \forall d_2 > D, \end{cases}$

$\exists$ s.s. $(u_{\varepsilon,d_2}, v_{\varepsilon,d_2})$ of (GM) with

$$u_{\varepsilon,d_2} = u_{\varepsilon,\infty} + \varphi(x;\varepsilon,d_2)$$

$$v_{\varepsilon,d_2} = \xi_\infty + \psi(\cdot;\varepsilon,d_2)$$

where $|\varphi|_{L^\infty} + |\psi|_{L^\infty} \longrightarrow 0$ as $d_2 \longrightarrow \infty$.

► <u>Stability</u> : $n=1$ & Ω = the unit interval $(0,1)$
(Joint work : Ni, Takagi & Yanagida)

- <u>Shadow System</u> : Set $\alpha = \frac{qr}{p-1} - (s+1) > 0$ & $\underline{r=2}$.

- <u>Th'm</u>. If $\underline{1<p<5}$, then for α and $d_1 = \varepsilon^2$ small, $\exists\ \tau_c = \tau_c(p,q,r,s,\varepsilon) > 0$ s.t.
  (i) $0<\tau<\tau_c \Rightarrow (u_{\varepsilon,\infty}, \xi_\infty)$ is <u>stable</u>.
  (ii) $\tau = \tau_c \Rightarrow \exists$ a family of <u>periodic</u> solutions bifurcating from $(u_{\varepsilon,\infty}, \xi_\infty)$.
  (iii) $\tau > \tau_c \Rightarrow (u_{\varepsilon,\infty}, \xi_\infty)$ is <u>unstable</u>.

- <u>Th'm</u>. If $\underline{p \geq 5}$, then $(u_{\varepsilon,\infty}, \xi_\infty)$ is always <u>unstable</u> for α & ε suff. small.

- <u>Gierer-Meinhardt system</u>
  - Similar conclusions hold for d_2 suff. large

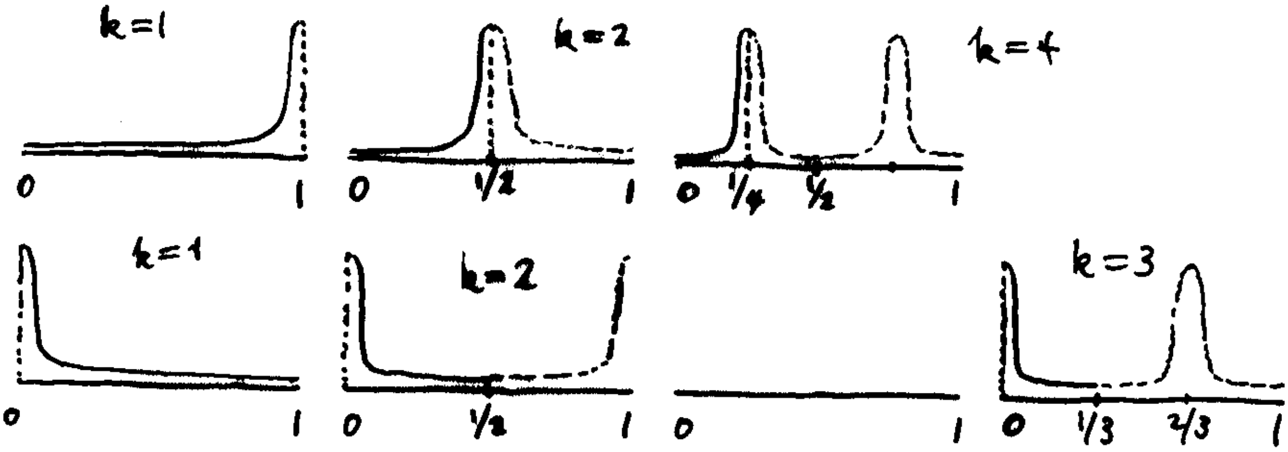
- <u>OPEN</u>: The rich structure of steady states makes the <u>dynamics</u> extremely interesting and challenging.

- **Multi-peak Spike-Layer Steady States**

  Again, assume $n=1$, $\Omega=(0,1)$ & $\underline{r=2}$.

Shadow System (S.S.)

- Existence: Construction of solutions of "mode k" by reflection



- **Instability Theorem**: Steady state solutions of mode k, $k \geq 2$, are always unstable.
  [NTY]

Gierer-Meinhardt System

The above result holds if d_2 is suff. large.

$\boxed{\text{Conjecture:}}$ As d_2 decreases, more & more multi-peak spike-layer steady states become stable.

The key points here are

(1) the gap between the diffusion coeffs. d_1, d_2 (i.e. the activator u must diffuse <u>slowly</u>, & the inhibitor v must diffuse <u>rapidly</u>)

(2) the combination of (1) with the "correct" reaction terms.

To illustrate (2), we shall consider the classical Lotka-Volterra competition-diffusion system.

EXAMPLE: Lotka-Volterra Competition system

$$\begin{cases} u_t = d_1 \Delta u + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,T) \\ v_t = d_2 \Delta v + v(a_2 - b_2 u - c_2 v) & \text{———} \\ u(x,0) = u_0(x), \; v(x,0) = v_0(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases}$$

All constants a_i, b_i, c_i, d_i are positive and set $A = \frac{a_1}{a_2}$ $B = \frac{b_1}{b_2}$ $C = \frac{c_1}{c_2}$

Cases:

(i) $A > \max\{B, C\}$

(ii) $A < \min\{B, C\}$

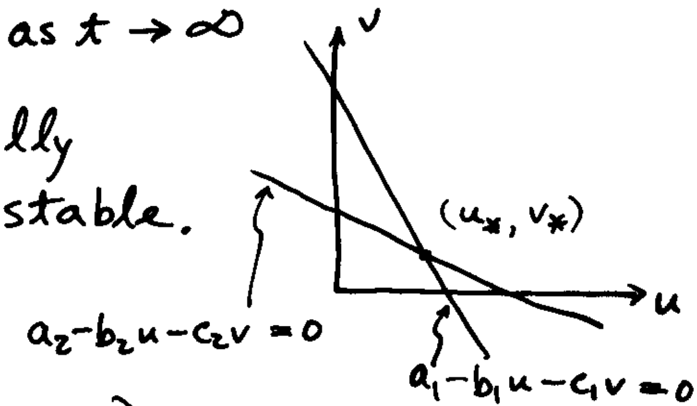
(iii) $B > A > C$ (weak competition):

$(u, v) \to (u_*, v_*)$ as $t \to \infty$

i.e. (u_*, v_*) is globally asymptotically stable.

(In particular, this implies that there is no nonconstant s.s.)

[no matter what d_1, d_2 are!]



A Cross-Diffusion System in Population Dynamics

(Shigesada, Kawasaki & Teramoto: J. Theo. Biol. 1979)

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{11}u + \rho_{12}v)u] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,T) \\ v_t = \Delta[(d_2 + \rho_{21}u + \rho_{22}v)v] + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases}$$

$\rho_{ij} \geq 0$ (ρ_{11}, ρ_{22} : self-diffusion, ρ_{12}, ρ_{21} : cross-diffusion)

- To model "segregation" phenomena in population dynamics
- Parabolic : Kim (1984), Deuring ('87), Yagi ('93), Amann ('90~'93) : Local existence, Wu, Lou, Ni + Wu ('97)
- Elliptic (i.e. s.s.): Matano + Mimura ('83), Mimura, Nishiura, Tesei & Tsujikawa ('84), ($n=1$, $\rho_{11} = \rho_{12} = \rho_{22} = 0$: transition layers) Kan-on ('93): "Stability"

Biodiffusions: Derivation of the model

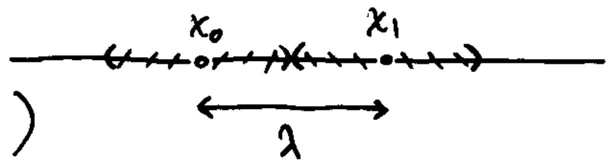
(Okubo) u: density

J: flux vector

f: sink / source

Conservation of mass $\quad u_t + \operatorname{div} J = f$

$n=1$ (for simplicity)



$P(x_0 \to x_1 ; t) =$ the probability of an individual moving from x_0 to x_1 in the time span $(t, t+\tau)$

$\boxed{\text{Assumption}}$: $\boxed{\dfrac{\lambda^2}{\tau} P(x_0 \to x_1, t) \longrightarrow D(x,t)}$ as $\lambda, \tau \to 0$

flux
$$J = \frac{1}{\tau}\Big[P(x_0 \to x_1, t)\lambda u(x_0,t) - P(x_1 \to x_0; t)\lambda u(x_1,t) \Big]$$
$$= \frac{\lambda^2}{\tau}\left[\frac{P(x_0 \to x_1, t) u(x_0,t) - P(x_1 \to x_0, t) u(x_1,t)}{x_1 - x_0} \right]$$

3 types: (heuristic)

(i) Random: $P(x_0 \to x_1, t) = P(x_1 \to x_0, t) = P(x_0, x_1; t)$

$$\Rightarrow J = \frac{\lambda^2}{\tau} P(x_0, x_1; t)\left[\frac{u(x_0,t) - u(x_1,t)}{x_1 - x_0}\right]$$

$$\rightsquigarrow -D(x,t)\nabla u$$

and the equation becomes

$$\boxed{u_t = \nabla\cdot[D(x,t)\nabla u] + f}$$

(ii) Repulsive: $P(x_0 \to x_1, t) = P(x_0, t)$
$P(x_1 \to x_0, t) = P(x_1, t)$

$$\Rightarrow J = \frac{\lambda^2}{\tau}\left[\frac{P(x_0,t)\,u(x_0,t) - P(x_1,t)\,u(x_1,t)}{x_1 - x_0}\right]$$

$$\rightsquigarrow \frac{D(x_0,t)\,u(x_0,t) - D(x_1,t)\,u(x_1,t)}{x_1 - x_0}$$

$$\rightsquigarrow -\nabla[D(x,t)\,u(x,t)]$$

$\therefore$ Equation: $\boxed{u_t = \Delta[D(x,t)u] + f}$

(iii) Attractive: $P(x_0 \to x_1, t) = P(x_1, t)$
$P(x_1 \to x_0, t) = P(x_0, t)$

$$\Rightarrow J = \cdots \sim D^2(x,t)\,\nabla\cdot\left[\frac{u}{D(x,t)}\right]$$

$$\therefore \boxed{u_t = \nabla\cdot\left[D^2(x,t)\nabla\left(\frac{u}{D(x,t)}\right)\right] + f}$$

A Cross-Diffusion System in Population Dynamics

(Shigesada, Kawasaki & Teramoto, J. Theo. Biol. 1979)

$$
(C)\begin{cases}
u_t = \Delta\left[(d_1 + \rho_{11} u + \rho_{12} v)u\right] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,T) \\
v_t = \Delta\left[(d_2 + \rho_{21} u + \rho_{22} v)v\right] + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0,T) \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T)
\end{cases}
$$

$\rho_{ij} \geq 0$ (ρ_{11}, ρ_{22} : <u>self-diffusion</u>, ρ_{12}, ρ_{21} : <u>cross-diffusion</u>)

- <u>Goal</u> : To study the possibility of nonconstant s.s. created by the <u>cross-diffusion</u> pressures & their [illegible] properties.

(<u>Joint work with [illegible]</u>)

- <u>Self-Diffusion</u>

<u>Th'm</u>. $\exists$ Constant $K(a_i, b_i, c_i, d_i, \rho_{12}, \rho_{21})$ s.t. if either $\rho_{11} > K$ or $\rho_{22} > K$, then (C) has no nonconstant s.s.

<u>Pf</u> : a priori estimates.

▸ Diffusion vs. Cross–Diffusion : Weak Competition

(Assume $\rho_{11} = \rho_{22} = \rho_{21} = 0$ for simplicity.)

$$(C)' \begin{cases} \Delta\left[(d_1 + \rho_{12} v)u\right] + u(a_1 - b_1 u - c_1 v) = 0 & \text{in } \Omega \\ \Delta(d_2 v) + v(a_2 - b_2 u - c_2 v) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of $-\Delta$ with multiplicity m_k under 0 Neumann boundary condition.

▸ Existence. Suppose $B > A > (B+C)/2$ & m_k is odd for some $k \geq 1$. Then $\exists$ positive constants $K_1(a_i, b_i, c_i) < K_2(a_i, b_i, c_i)$ s.t. $\forall\, d_1 > 0$, $d_2 \in (K_1, K_2)$, the system $(C)'$ has at least a nonconstant solution if $\rho_{12} \geq K_3$ for some positive constant $K_3(a_i, b_i, c_i, d_1)$.

▸ Nonexistence. Suppose $B > A > C$. Then $\exists$ positive constant $K_4(a_i, b_i, c_i)$ s.t. (u_*, v_*) is the only positive solution of $(C)'$ if any one of the three quantities ρ_{12}/d_1, ρ_{12}/d_2 or $\rho_{12}/\sqrt{d_1 d_2}$ is small.

Rk. In both "weak" and "strong" cases, d_2 is required to stay in some appropriate range, for existence. This turns out to be crucial.

→ **Theorem**. Suppose that $C \neq A \neq B$ & $n \leq 3$. Then, for fixed a_i, b_i, c_i, $\exists\, K = K(a_i, b_i, c_i)$, indep. of d_1 & ρ_{12} s.t. if $d_2 \geq K \Rightarrow (C)'$ has no nonconstant positive solution.

→ That is, without the "cooperation" of diffusion, just increasing the cross diffusion pressure ρ_{12} may NOT help in creating nontrivial solutions.

- Q: What happens if $\rho_{12} \to \infty$?

- Classification of Limiting Behavior.

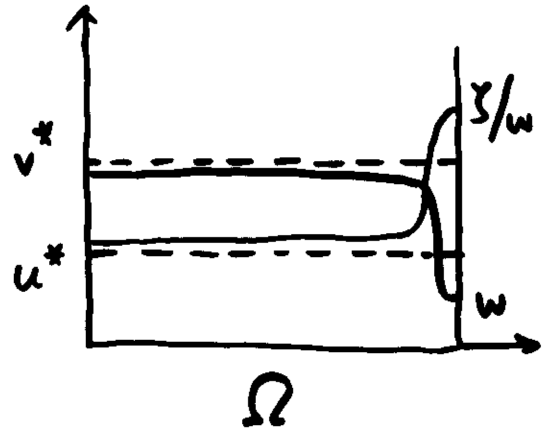
Suppose $n \le 3$, $C \ne A \ne B$ & $a_2/d_2 \ne \lambda_k \ \forall k$. Let (u_j, v_j) be a nonconstant solution of (C') with $\rho_{12} = \rho_{12,j}$. Then (by passing to a subseq.) either (i) or (ii) holds as $\rho_{12,j} \to \infty$ where

(i) $(u_j, v_j) \to (\frac{\zeta}{w}, w)$ uniformly, $\zeta > 0$ is a constant, $w > 0$ satisfies
$$\begin{cases} d_2 \Delta w + (a_2 - c_2 w) w - b_2 \zeta = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \int_\Omega \frac{1}{w}\left(a_1 - b_1 \frac{\zeta}{w} - c_1 w\right) = 0 \end{cases}$$

(ii) $(u_j, \frac{\rho_{12,j}}{d_1} v_j) \to (u, v)$ uniformly, $u > 0$, $v > 0$ &
$$\begin{cases} d_1 \Delta[(1+v)u] + u(a_1 - b_1 u) = 0 & \text{in } \Omega \\ d_2 \Delta v + v(a_2 - b_2 u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Rk: Both cases can happen.

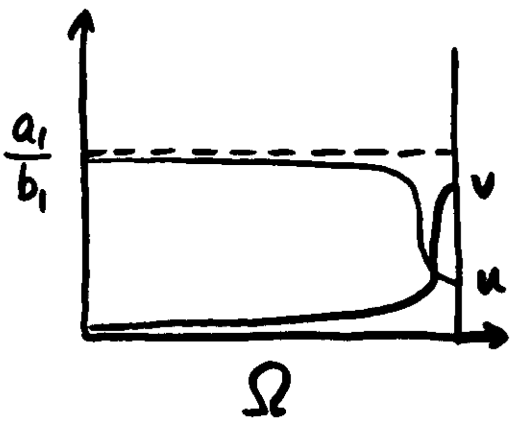
(i)



$b_2 u^*/c_2 v^* \in (\frac{1}{3}, 1)$

d_2 small

(ii)



d_2 small, d_1 large (fixed)

$A > B$

($n = 1$, $d_2 > 0$ small)

Rk: There may be many solutions to (i) & (ii)

Q: Stability? Thus, it is incomplete.

- What really matters is <u>the ratio</u> ρ_{12}/d_1.

- <u>Theorem</u>. Same hypothesis as before. Let (u_j, v_j) be a nonconstant solution of $(C)'$ with $\rho_{12} = \rho_{12,j}$ & $d_1 = d_{1,j}$. Then as $\rho_{12,j} \to \infty$ (by passing to a subseq. if necessary) we have:

(I) If $\rho_{12,j}/d_{1,j} \to \infty$ & $d_{1,j} \to d_1 \in (0,\infty) \Rightarrow$ (i) or (ii) holds

(II) $\rho_{12,j}/d_{1,j} \to \infty$ & $d_{1,j} \to \infty \Rightarrow$ (i) or (ii)$'$ holds,

(III) $\rho_{12,j}/d_{1,j} \to r \in (0,\infty) \Rightarrow$ (iii) holds, where

(ii)$'$ $\left(u_j, \dfrac{\rho_{12,j}}{d_{1,j}} v_j\right) \to \left(\dfrac{\zeta}{1+v}, v\right)$ unif., $\zeta > 0$ is a constant, $v >$

satisfies
$$\begin{cases} d_2 \Delta v + v\left(a_2 - \dfrac{b_2 \zeta}{1+v}\right) = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega \frac{a_1}{1+v} = \int_\Omega \frac{b_1 \zeta}{(1+v)^2} & \end{cases}$$

(iii) $(u_j, v_j) \to \left(\dfrac{\zeta}{1+rv}, v\right)$ unif., $\zeta > 0$ is a constant, $v >$

satisfies
$$\begin{cases} d_2 \Delta v + v\left(a_2 - \dfrac{b_2 \zeta}{1+rv} - c_2 v\right) = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega \frac{a_1 - c_1 v}{1+rv} = \int_\Omega \frac{b_1 \zeta}{(1+rv)^2} & \end{cases}$$
.