united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR/1108 - 25

COURSE ON "MEDITERRANEAN SEA(S) CIRCULATION & ECOSYSTEM FUNCTIONING"

2 - 20 November 1998

Trieste, Italy

∞∞∞

"Economy & Ecology"

R. MOSETTI
Osservatorio Geofisico Sperimentale
Trieste, Italy

Economy and Ecology

Ecology=*oìkos-logos*

Economy=*oìkos-nomos*

Ecosystem and Economic Systems:

Production

Consumption

Torna alla prima pagina

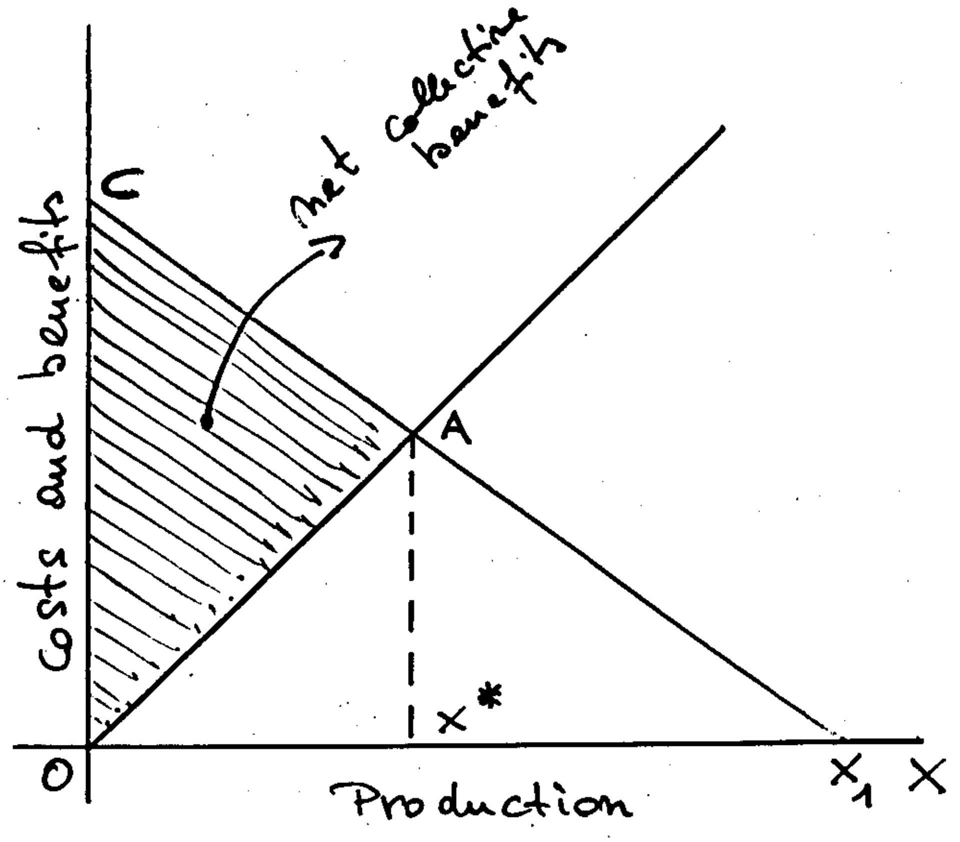
The problem of optimal solutions

- Conflicts between private needs and collective ones.

↓

the economical activity be such as

Social costs ≤ private benefits



$\overline{CX_1}$ ⇒ private marginal benefits
$\overline{OA}$ ⇒ social marginal costs

- For the private sector the optimal level of production is X_1 (area OCX_1)

- The social optimum is at level X^*

Three implications

- The level of production which maximize the collective benefit is less than the one which maximise the private benefit : $OX^* < OX_1$

- The social optimum imply a certain level of pollution

- The net collective benefit is:

$$OCA = OCAX^* - OAX^*$$

↓

A "Pareto optimum" i.e. a level in which no-one can grow his benefits without reducing the benefits of the others.

What does it mean in practical terms?

- We need a system in order that the firms that pollute be able to mantain the level $\underline{\underline{X^*}}$
  1) to pay for mantaining X^* (incentives)
  2) to do taxation on the firm in order to reach the net social benefit

However the point is:

> to insert in economical computation the social costs. [externalities]

↳ difficulties in monetary evaluation of the social/environmental costs!! |||

X Ecological Economy (Tiezzi ed

Socio-economic Analysis and Strategies for the Evolution of the Coastal Zones along North-Western Italy

S. Marzetti[1] and R. Mosetti[2]

1) Università di Bologna. Dipartimento di Scienze economiche

2) OGS Trieste

2.0 The case-study of Lido di Dante

The specific case of Lido di Dante permits some considerations to be made about the methodology of dealing with real environmental problems from an economic point of view.

2.1 Erosion and artificial preservation in Lido di Dante

Lido di Dante is a seaside resort near Ravenna (Italy) on the northern Adriatic coast (table 1 shows its geographical position) with wide beaches of white sand. In the last 50 years the beaches of Lido di Dante were subjected to a remarkable erosion, mainly generated by a reduction in sediment, yielded from the neighbouring rivers (Fiumi uniti and Bevano), and the strong subsidence present in the area due to groundwater exploitation. In order to protect seaside bathing facilities, the beach was artificially preserved in the following ways. In 1978 a groin was built to limit the northward sand drift. In 1983 two more groins were built together with beach nourishment protected by a sand bag barrier. However the erosion trend has not been stopped, as shown by table 1 and figure 2. Therefore in 1995-96 a new work of protection was constructed. This consisted of a sand beach nourishment protected by a submerged barrier of masses of stone, parallel to the beach, connected to the pre-existing northern groin by a submerged extension. Figure 3 shows the current state of the protection of the beach in Lido di Dante. We should note that the technical characteristics of the barriers have changed with the passing in time, from barriers above water level without nourishment to groins with nourishment protected by sand bag barriers and to nourishment protected by submerged barriers of masses of stone.

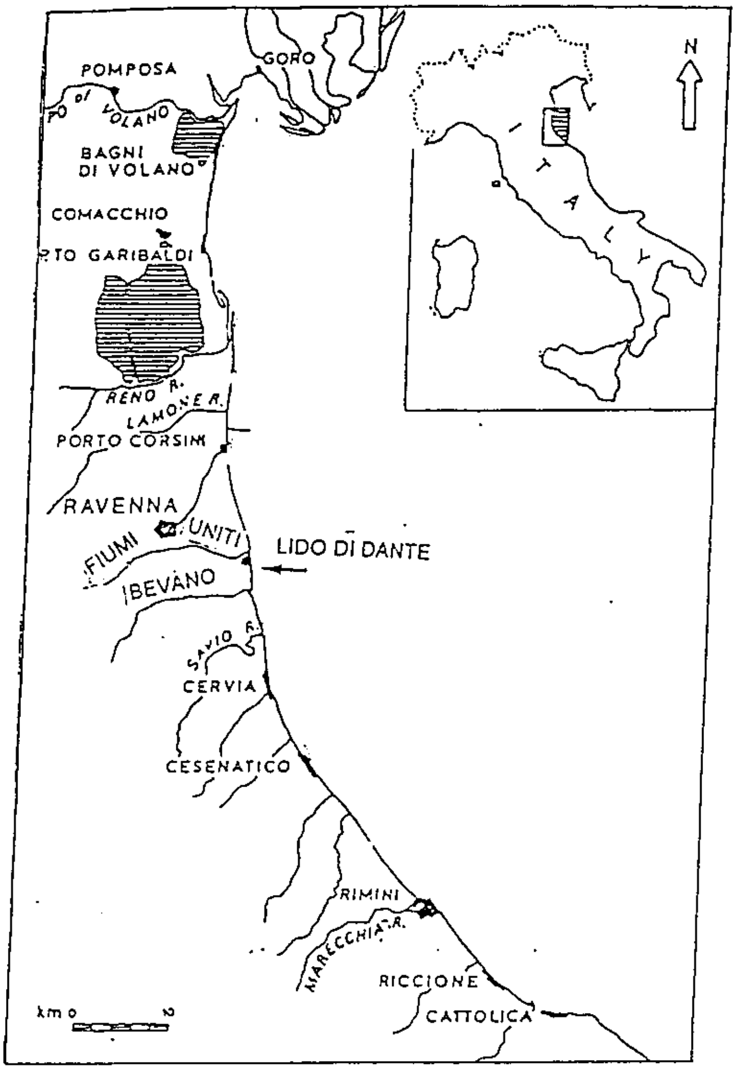

POMPOSA
GORO
PO DI VOLANO
BAGNI DI VOLANO
COMACCHIO
PORTO GARIBALDI
RENO R.
LAMONE R.
PORTO CORSINI
RAVENNA
FIUMI UNITI
LIDO DI DANTE
BEVANO
SAVIO R.
CERVIA
CESENATICO
RIMINI
MARECCHIA R.
RICCIONE
CATTOLICA
km 0
N
ITALY

Figure 1

X

AVERAGE WIDTH OF THE BEACH
(Northern and Southern cell)

Year	Northern cell Average width (m)	Soutern cell Average width (m)	Remarks
1957	88	97	
1968	74	80	
1971	49	65	
1978	54	70	Just after North groin construction
1980	29	50	
1983	43	56	After beach rehabilitation works
1991	28.5	46	
1993	24	48	
1995	22.5	30.5	Before beach rehabilitation works
1996	39	39	After beach rehabilitation works
1997	45.5	40.5	

Table 1

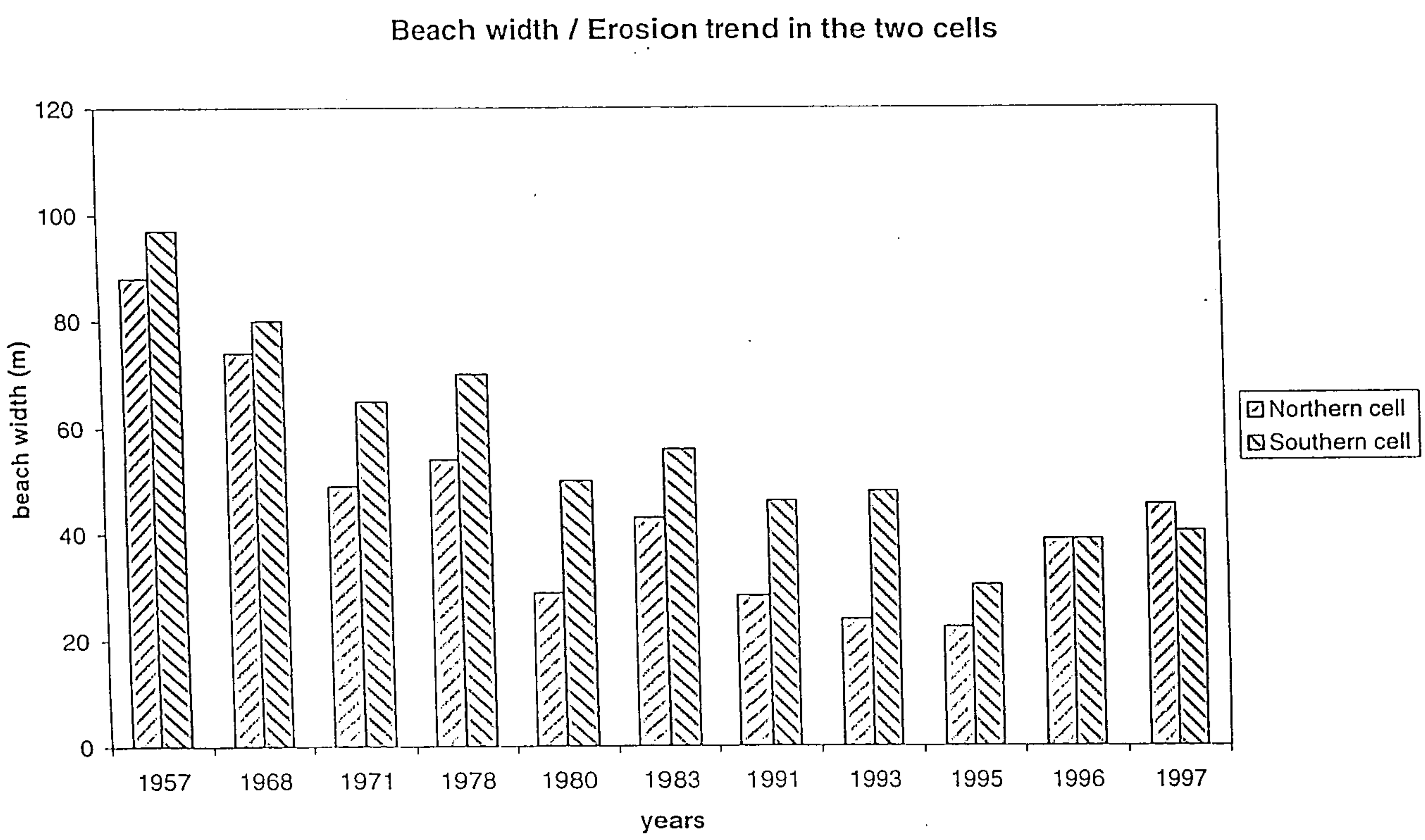
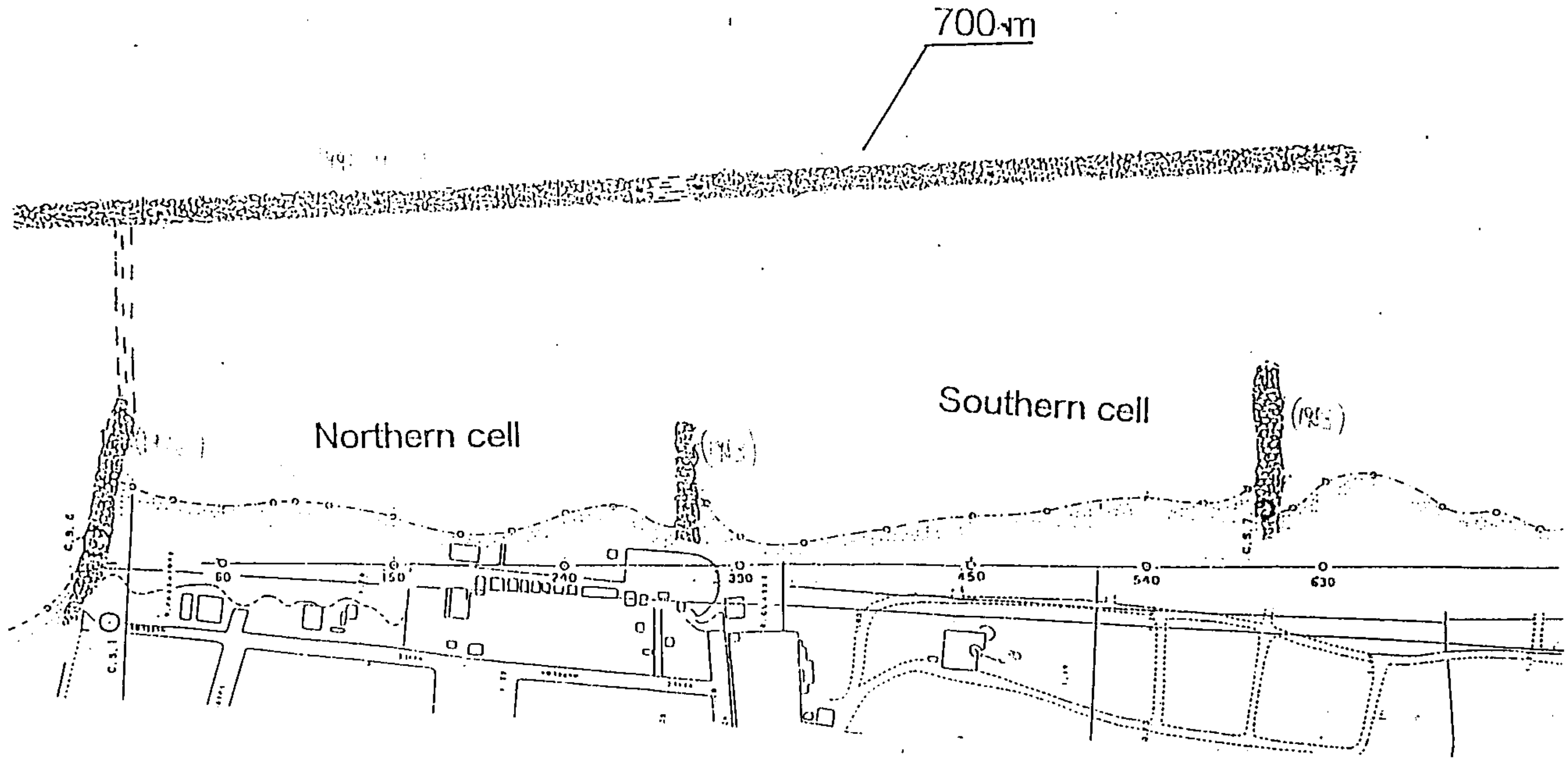

Beach width / Erosion trend in the two cells
120
100
80
60
40
20
0
beach width (m)
1957
1968
1971
1978
1980
1983
1991
1993
1995
1996
1997
years
Northern cell
Southern cell

figure 2



Protection of the beach in Lido di Dante

Figure 3

3.1 Tourist industry and environmental factors

A study about some economic aspects regarding the quality of the seawater (algae phenomenon) and the artificial protection of the beach in Lido di Dante, particularly about the submerged barrier, was made when construction of this barrier was *already* under way (Drei, 1977). The aims of the study were as follows: first of all, to see whether changes in the quality of seawater have some effect on the local tourist industry and real estate market; secondly whether beach erosion in Lido di Dante led to a reduction in tourist influxes; and finally whether beach protection work would affect the prices on the local estate real market.

In order to achieve these aims, data (updated to 1997 for the purpose of this article) were used about tourist influxes in 1978-95 together with a market research survey in 1996. More specifically, the tourist influxes to Lido di Dante were studied in relation to two environmental factors:

- water quality (the algae phenomenon);
- coastal erosion (beach width),

and two economic variables:

- inflation trend in Italy;
- trend of D.M. exchange rate.

The market survey, instead, involved 10 travel agencies. The managers of these agencies were interviewed, and the questions mainly concerned their opinions about the effects of the algae phenomenon and coastal erosion on the tourist industry, specifically on the real estate market.

However, the study focussed deliberately more on the negative effects of the algae phenomenon than on the best way to improve the quality of the seawater, and more on the possible benefits than on the possible costs of beach protection; thus it is not an application of the cost-benefit analysis. However, considering the sustainability principle, it will enable us to make some methodological considerations about the practical solution of environmental-economic problems such as in the case of Lido di Dante.

3.2. Algae phenomenon, tourist influxes and real estate market

Data about influxes in 1978-97 concern 'arrivals' (the number of tourists arriving in Lido di Dante in a year) and 'total night stays' (arrivals in a year multiplied by the average number of night stays of the representative tourist in the same year)[1] in hotels, flats and campsites. A plan of the tourist accommodation in Lido di Dante is shown in figure 4. Tourist influxes were divided in national and foreign arrivals, and national and foreign total night stays. Figures 5 and 6 show, respectively, their trend, which requires some explanation. In 1983-84 new buildings were available for rent, thus arrivals and total night stays (particularly foreign) increased remarkably. In 1988-90, instead, these figures decreased rapidly because of the algae phenomenon and the consequent deterioration in the water quality.

The effect of the algae phenomenon on the local tourist industry was remarkable also because of the emphasis placed on it by the mass media, both in Italy and abroad. Total arrivals decreased from about 7,900 to 4,000 units in 1989, and total night stays from about 106.900 to 51,500 units. This very negative effect is also evident when we considered arrivals and total night stays according to

---

[1] Data do not consider day trippers.
To be more precise (Marzetti, 1991, pp. 79-80), if N is the number of tourists arriving in a specific place in a year, and q the average night stays, then qN is the total night stays. Let p be the price of 24 hours of stay and w a vector of exogenous variables (for example environmental variables) which affect the demand of tourism in the area. The demand of a single tourist is given by $q = f(p, w)$ and $qN = N f(p, w)$ is the aggregate demand in the area.

the division between hotels, flats and campsites as shown in figures 7 and 8 respectively. Arrivals, for example, in flats fell from about 3,500

ACCOMODATION PLAN

1 hotel

2 campsites

about 200 flats for rent

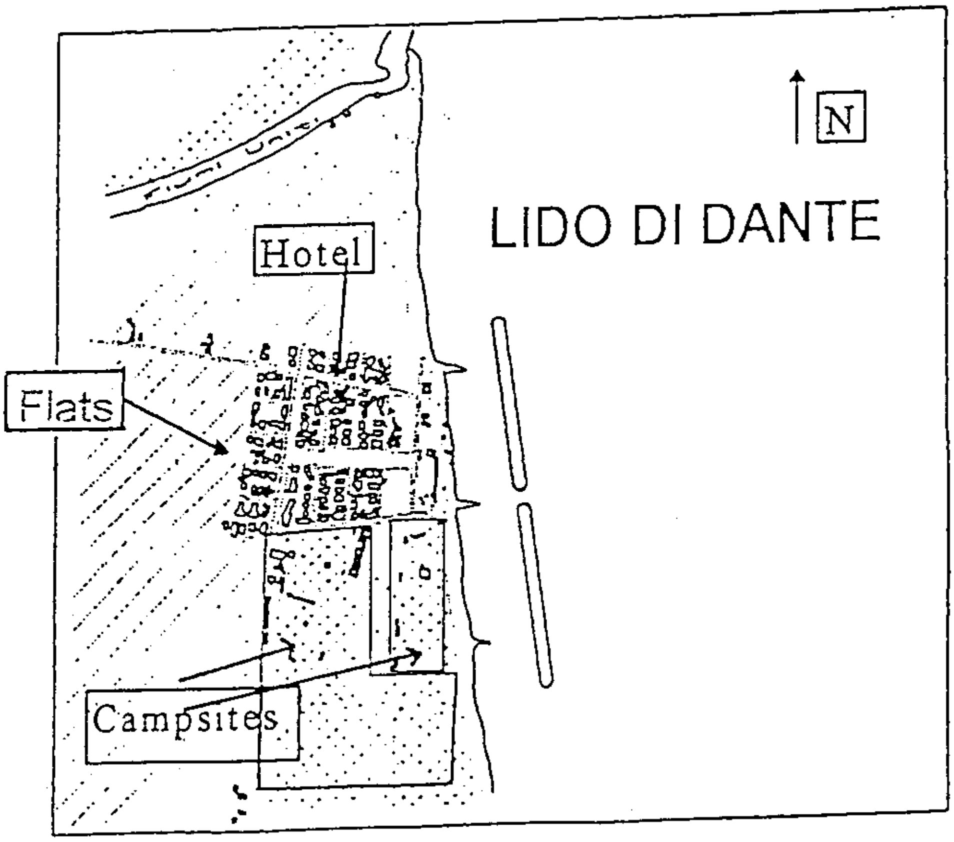


figure 4

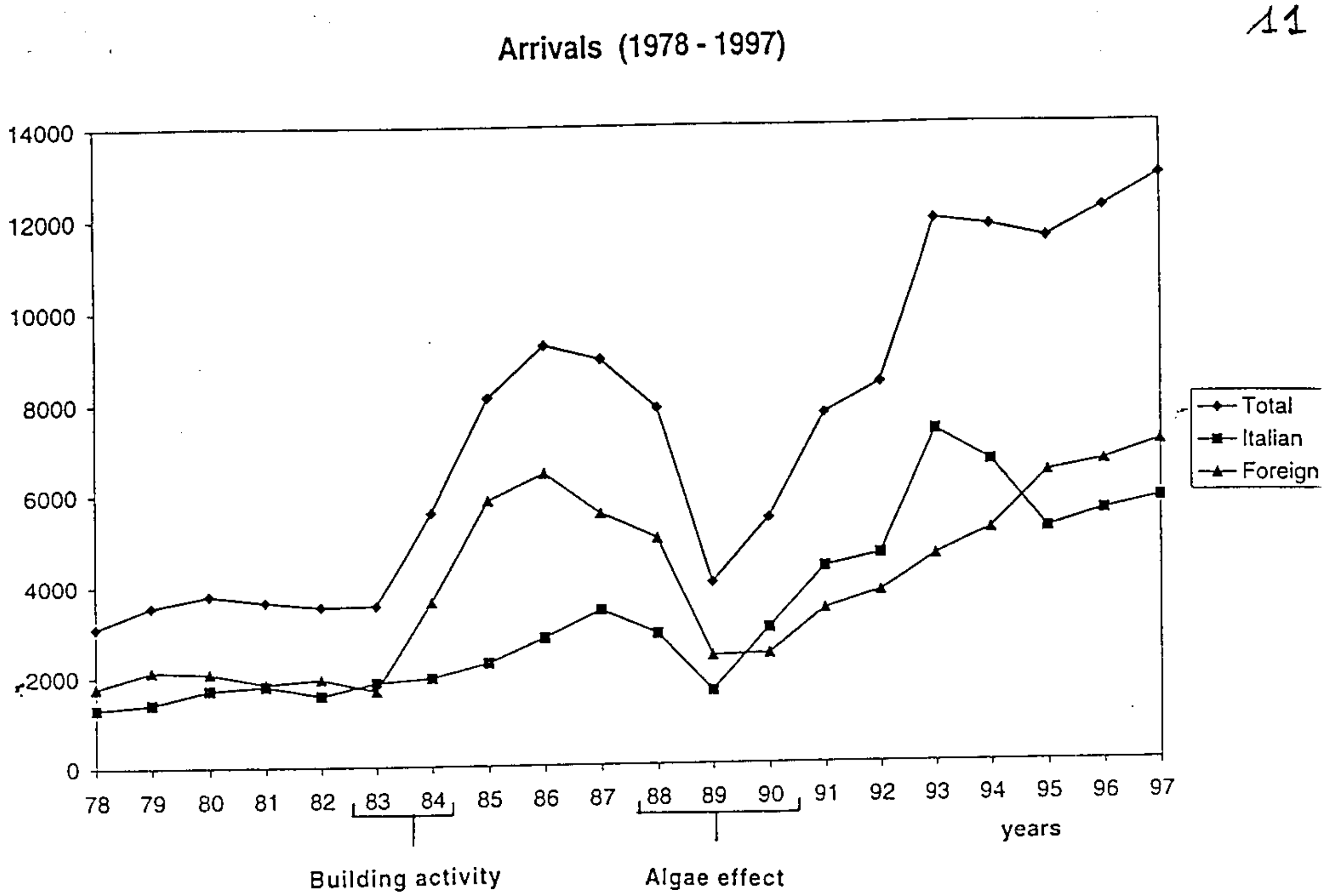
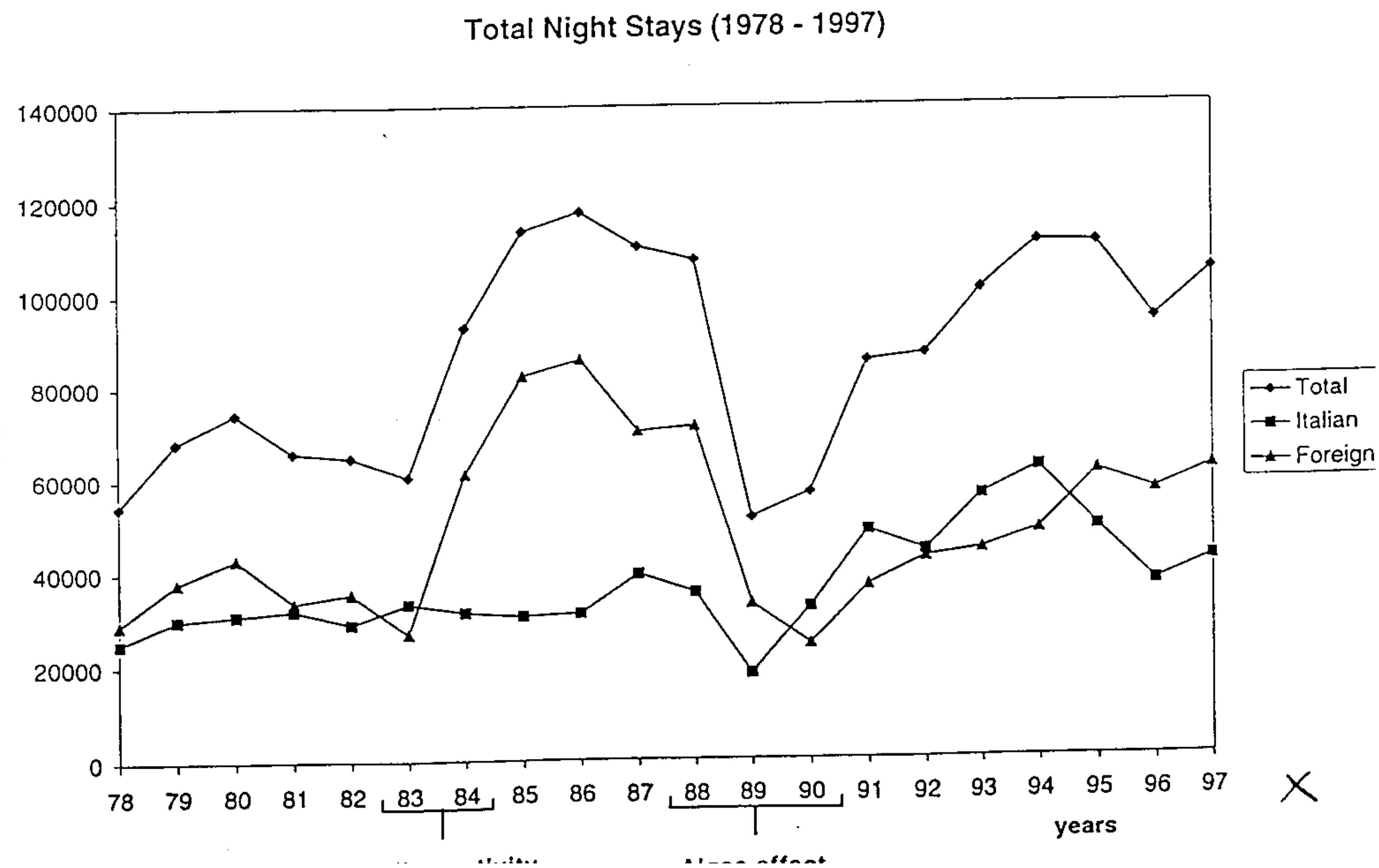

Arrivals (1978 - 1997)
14000
12000
10000
8000
6000
4000
2000
0
78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97
years
Building activity
Algae effect
Total
Italian
Foreign

figure 5


Total Night Stays (1978 - 1997)
140000
120000
100000
80000
60000
40000
20000
0
78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97
years
Total
Italian
Foreign

Arrivals - Accomodation (1978 - 1997)

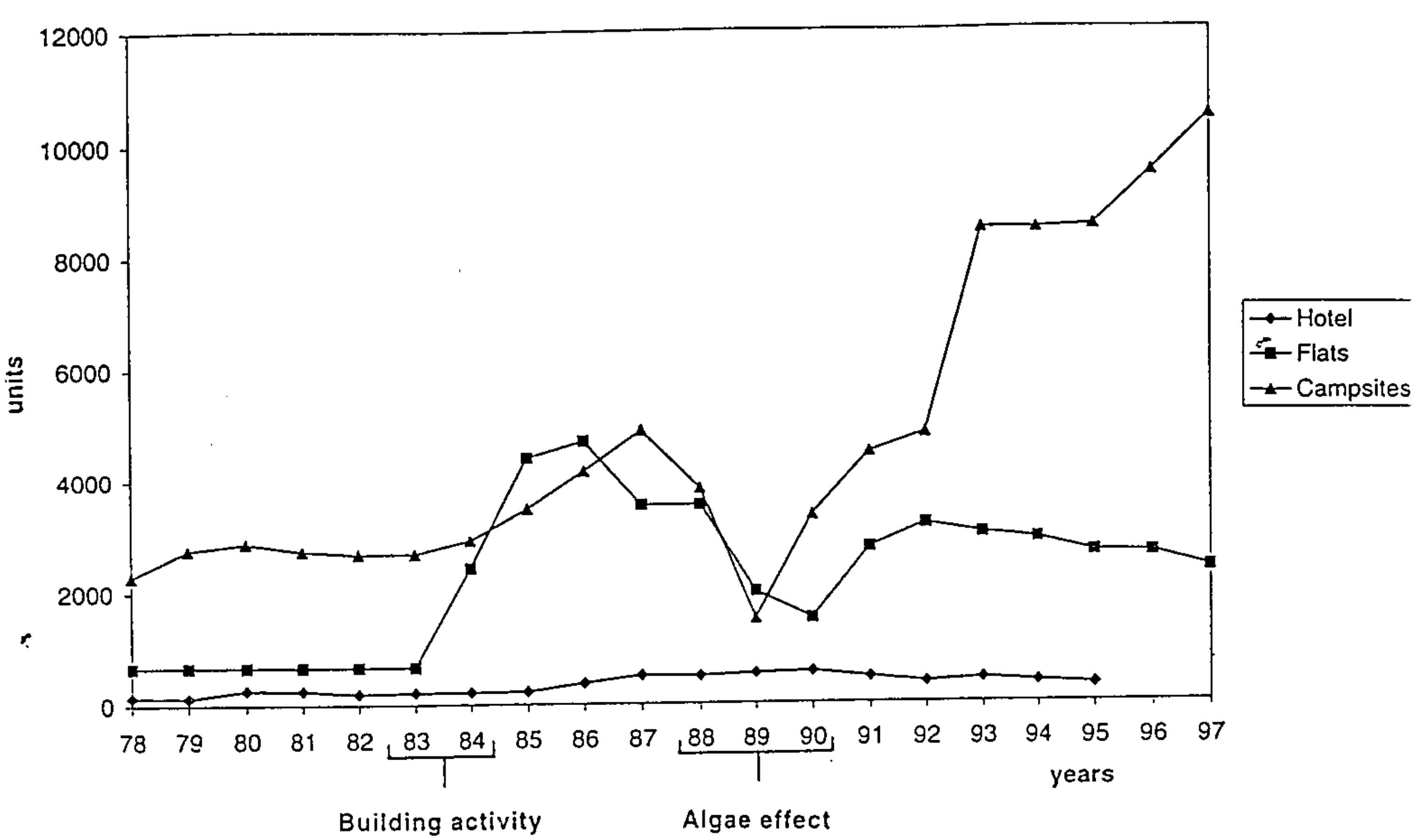


figure 7

Total Night Stays - Accomodation (1978 - 1997)

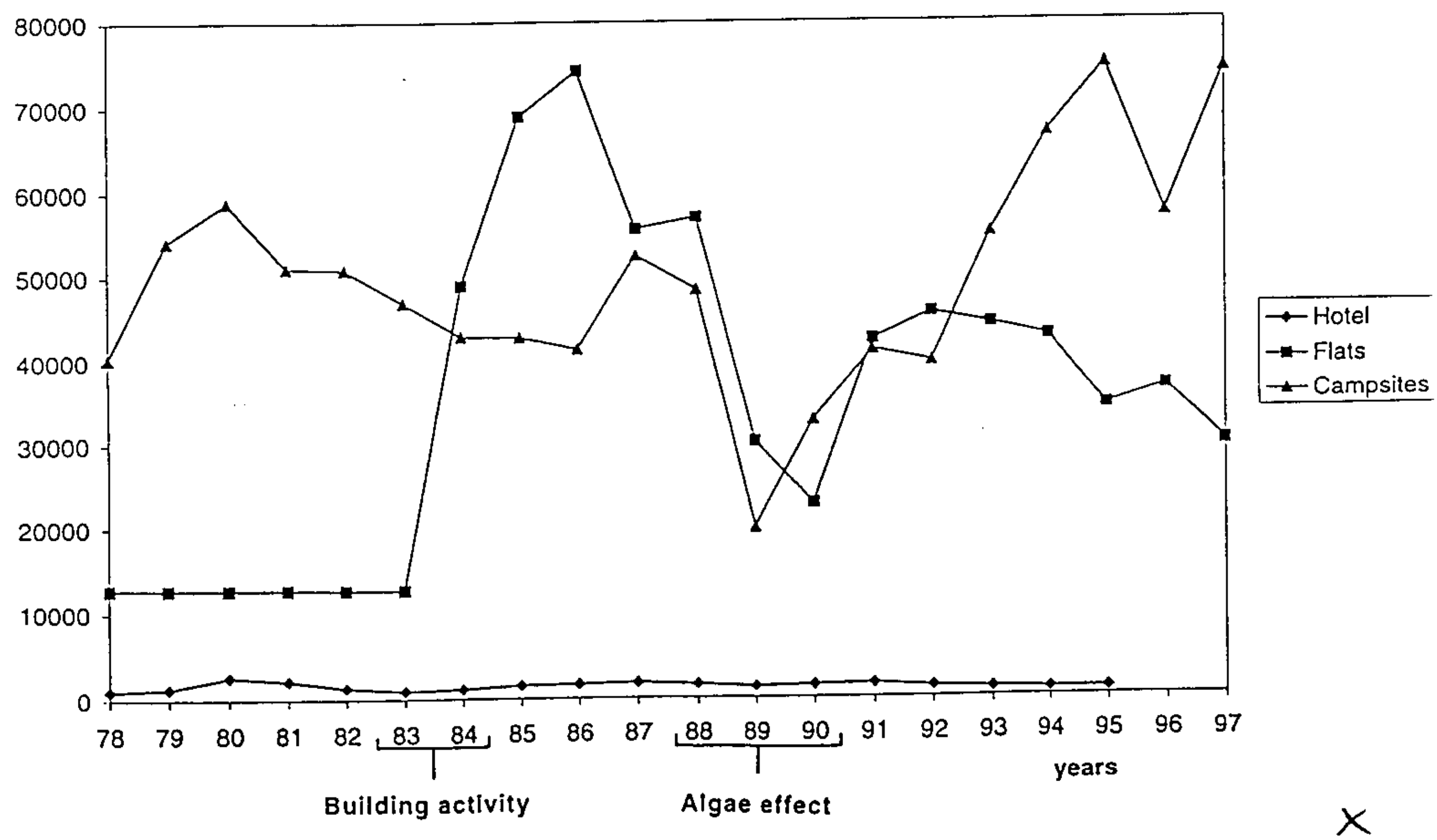


figure 8

X

units to about 1,500 in 1988-90, and on campsites fell from about 3,800 to 1,400 in 1989: more than a 50% reduction in arrivals in flats and a reduction of more than 3/5 in arrivals on campsites. The consequence was a drastic reduction in the income of the tourist industry, and a temporary decrease in rents (about 7-8%) and flat prices (about 30-40%), as shown by the market research survey (mentioned above) of 10 travel agencies of the area.

When the algae disappeared in 1991 the arrivals and total night stays began to pick up (figure 5 and 6), so the rents and the price on the real estate market reached their normal levels. Particularly, given the state of the natural environment, tourist arrivals and total night stays depend on economic variables. The study about Lido di Dante specifically considered the inflation trend in Italy (for Italian tourists), and the trend of exchange rate (for foreign tourists). Since the majority of foreign tourists in the area are German, the D.M. exchange rate was mainly considered. Figure 9 and 10 show respectively the inflation trend in Italy and the trend of the D.M. exchange rate in 1979-97. The first trend in 1988-90 (algae period) rose steadily and the second was stable; thus only inflation in Italy might have had a negative effect on tourism, if there had not been the algae effect, although not in the same drastic way.

In conclusion, data about the algae effect clearly show that water quality should be defended. A programme of exploitation of coasts for the tourist industry must be accompanied by a programme to identify the causes which may reduce the quality of seawater in order to eliminate them.

3.3 Beach erosion and tourist influxes

Let us consider table 2 about the trend of beach erosion in Lido di Dante. The first groin was built in 1978 and beach erosion reduced, Inflation trend in Italy - (1979 - 1997)

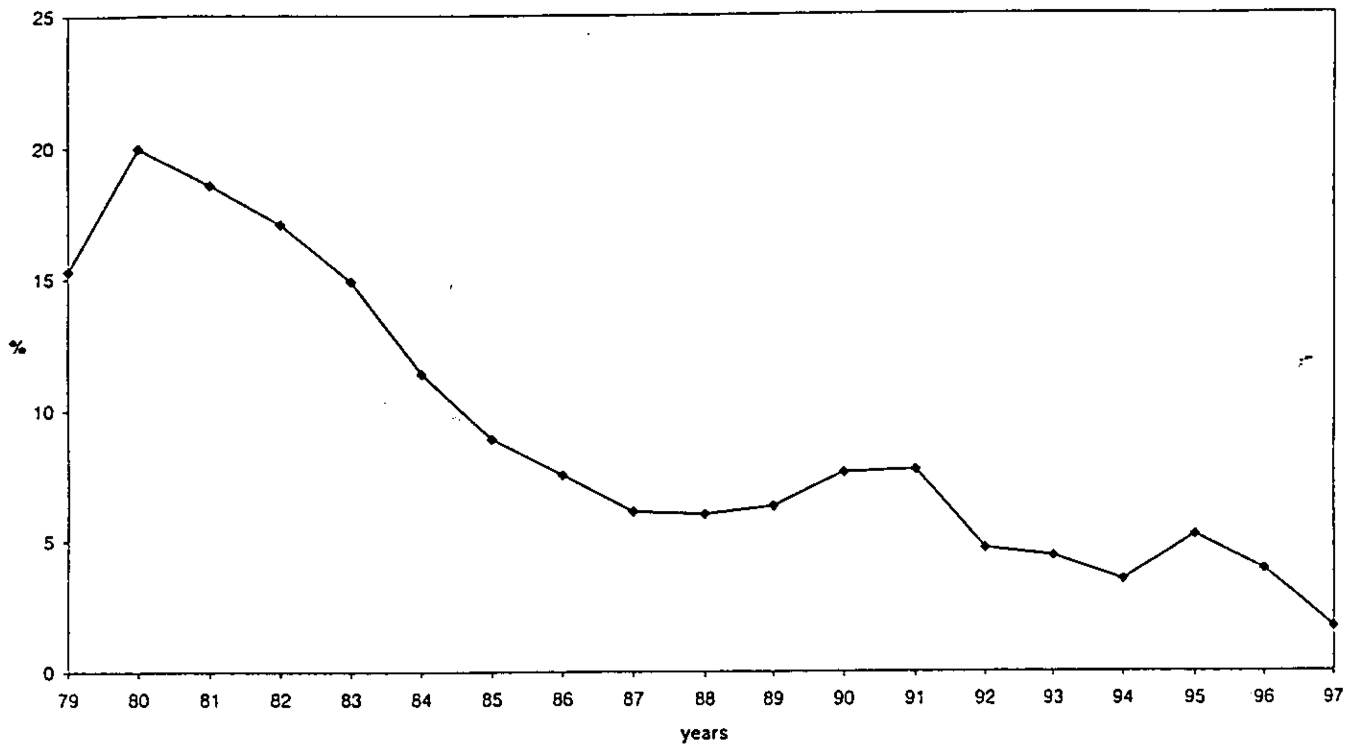


figure 9

Trend of DM exchange rate (1979 - 1997)

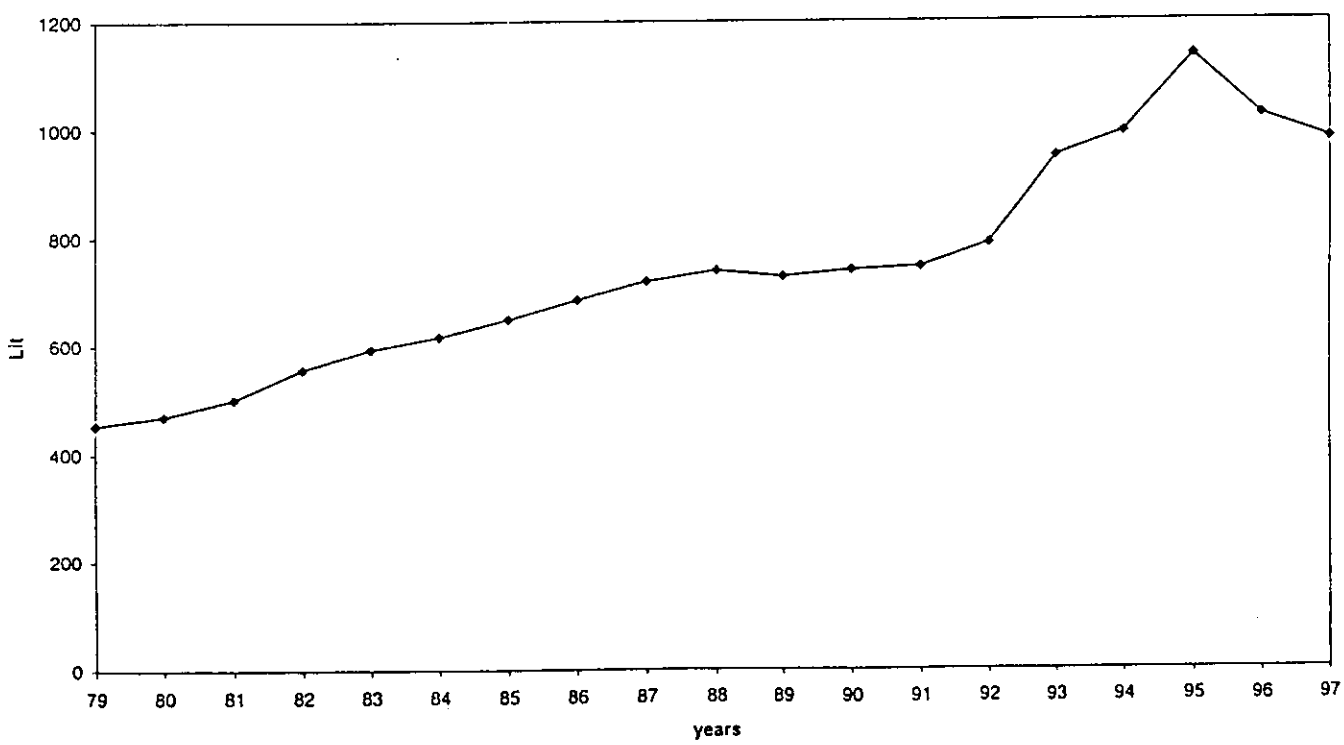


figure 10

but in 1980 the beach width decreased further. The same thing happened in 1983 when two groins were built and a beach nourishment was made: the beach width grew immediately due to human action, but before long the erosion continued. The last work in 1996, based on the new technology of submerged barrier, had the same immediate positive effect, but this is still under study.

In spite of human action, it is clearly evident that barriers and groins, whether or not accompanied by nourishment, did not solve the problem of beach erosion. In fact the true causes of erosion in Lido di Dante are:

- reduction of fluvial sediment load to the sea, caused by sand and gravel excavation;
- subsidence caused by groundwater exploitation.

For example, considering the river Fiumi Uniti near Lido di Dante, the vegetation which trapped a great quantity of solid materials was never cut for 25-30 years; only in 1996-97 was the vegetation cut, and now an increase in the size of beaches near the mouth of the river is expected.

However, while the trend of erosion increased between 1978-97, without considering the algae effect in 1988-90, the trend of arrivals and total night stays in the same period also increased and did not decrease, as figures 5 and 6, 7 and 8 show. Specifically, figures 7 and 9 seem to show in the last years only a certain change in the demand for tourist accommodation from flats to campsites. Thus there is no evident relation between coast erosion and tourist influxes, at least not so far: beach width has decreased but tourist influxes have tendentially increased. This fact leads us to think that, given the tourist accommodation structure, the carrying capacity of the beaches in Lido di Dante has so far been adequate for tourist influxes in spite of erosion. In other words, since the beaches were very large, erosion does not seem to have been a limiting factor to the growth of local tourism.

3.4 Beach protection and the real estate market

A different problem was to analyse the possible effect of beach protection on the prices of the real estate market. In the study mentioned above, an attempt was made to answer the following question: can beach nourishment and the submerged barrier lead to an increase in real estate prices at Lido di Dante? In other words: if nourishment and barriers protect against beach erosion, and thus defend the tourist industry, then estate prices could increase as a result.

Many variables, other than the cost of building, affect the real estate market price, some of these are: the characteristics of the place (distance from the city, from shopping centres, from the sea; facilities, ...). of the district (exclusive neighbourhood, different classes of residential neighbourhoods, ...), of the house (villa, flat, ...); the environmental quality (pollution, landscape, seascape, ...) etc. Specifically, in Lido di Dante there are mainly flats available for rent in summer, most of them near or very near to the sea, as figure 4 shows. Landscape and seascape are beautiful and wild, but facilities are not very satisfying, especially local transport. Finally, near Lido di Dante there are other seaside resorts with similar characteristics; and all of them are competitors as regard the tourist industry.

To find out how much influence beach erosion in Lido di Dante has had on building prices, the above mentioned market research survey was carried out among 10 agencies in the area. The result was that beach erosion has had no effect as yet on real estate prices. The managers of the travel agencies interviewed believe that in Lido di Dante this influence could occur only in the case of extreme beach erosion, should the carrying capacity of the beach become inadequate for bathing activity. In this case, if arrivals and total night stays fell drastically, followed by a decrease in the demand for rented accommodation, then flat prices would greatly decrease. As we have seen in par. 3.3, the beaches of Lido di Dante have a carrying capacity which is still adequate for the tourist influxes of the site. Therefore, from an economic point of view, man-made barriers and groins in this resort at the moment may be justified by the need to maintain the current carrying capacity of the beach, and thus to support the tourist industry, specifically the real estate market. Another economic justification may be the protection of the buildings closest to the sea, looking onto the beach; if erosion grows, their foundations may be damaged and buildings will lose their economic value. Considering a building cost of about 1.2 million (Lit.) m^2 in 1996, this value may be estimated at about 2,200 million (Lit.) against a cost of construction of about 230 million (Lit.) of the submerged barrier and the beach nourishment in 1996.

3.5 Sustainability and artificial conservation of the beach

The results described above are the conclusions of the study about some economic aspects of a specific project of coast conservation in Lido di Dante. Taking into account the principle of sustainability, they suggest the need to consider other costs of artificial conservation, other than the costs of construction, and particularly the social costs. In fact, respect of the principles of Sustainable Development, specifically of Sustainable Tourism Development, requires in general a management of all resources in order to fulfil economic, social and aesthetic needs, and to maintain ecological processes, biological diversity and life support systems. However, because in this paper our methodological aims are only general, we will not consider all the possible social costs, but only some of them as examples, nor will we take into account the question of their monetary evaluation.

As Preti, Carboni and Albertazzi (1997) underlined, the protection of the beach from natural erosion with barriers, reefs and groins causes the quality of the sea water to deteriorate; and tourists are sensitive to the quality of water, as shown by the algae phenomenon in 1988-90. In fact in 1997, because of reduced wave energy, in that part of the sea between the beach and the submerged barrier (constructed in 1996), algae have begun to appear. As we have seen in par. 3.2, this may lead to a reduction in arrivals and total night stays of tourists, and so in the income of the tourist industry.

Besides, around this submerged barrier the currents have created deep hollows in the seabed, and this phenomenon has not yet been studied due to lack of funds. From the technological point of view this type of barrier is new; therefore all its consequences require further study over a certain lapse of time before proceeding to the construction of the same structure in other sites. Experience has shown that short-term positive effects do not always justify long-term positive effects and possible negative consequences, today unknown, may reveal themselves to be a future cost for the local community.

Furthermore, if this artificial structure protects the beach in the limited area considered, it may lead to erosion in neighbouring areas according to the direction of the currents. From an economic point of view this damage would be a negative externality, i.e. an unpaid cost and this damage could suggest that a new barrier is necessary in these neighbouring areas. A cost-benefit analysis about the project should also take all the cost of the new erosion into account. In fact any damage from erosion - a form of degradation of the natural capital - would be deducted from EDP (environmental adjusted net domestic product), if a complete System for Environmentally adjusted Economic Accounts (SEEA) existed.

4.0 Final reflections

X

We have shown that economic rationality require a cost-benefit analysis about every single project. If we have to choose between different alternative projects, we have to select the one which gives the maximum net benefits. Specifically the classical economic approach to sustainability requires the maximization of the flow of net benefits 'possible from a given set of assets, without compromising the flow of future benefits. This requires the preservation or increase of the base of assets over time' (Munasinghe M. and McNeely J., 1995, p. 25).

This case study permits us to underline that, in practice, a full or even partial cost-benefit analysis is not always made, and the final project is often chosen according to other criteria. In fact not always is it possible to make a complete cost-benefit analysis, because not all the future consequences of a project are known before it is put into practice. Moreover, it is not always easy to establish the numerical probability of the consequences, and thus the situation is uncertain. Therefore, rationality is often bounded and agents proceeds by trial and error, and adjust their behaviour as they acquire experience.

We think that when the situation is of bounded rationality, as may be in the case of man-made works of beach protection, the *case-study approach* should prevail; in other words a sort of 'prototype' should be studied during the time necessary to reveal all its consequences. Only when all the consequences are revealed, can we perform a complete dynamic cost-benefit analysis, and only if the benefits are superior to the costs, can we proceed to setting up the specific project in other similar situations. This procedure may be considered a way of appling the 'precautionary approach', the essence of which is that no action should be taken to avoid any depreciation of natural capital in uncertain situations.

As regards the case study here considered and the building of barriers and groins on the coast of the Emilia Romagna Region in general, we feel we should make a methodological observation. If the procedure mentioned above had been followed in the past, 'the reaction chain induced by beach erosion, barrier building, erosion in the adjoining beach' (Preti, Carboni and Albertazzi, 1997, p. 1251) might have been prevented. In fact with this chain, the whole 110 km of shore line of the Emilia Romagna Region would have been protected with barriers since the Fifties without providing a final solution to the problem of beach erosion. What is more, there would no longer be the problem of recovering the area in which barriers and groins have damaged the environment: the economic cost of which would unavoidably be borne by the future community. With hindsight, this seems to be an example of choices made in the past without taking into account their negative effect on future generations; in other words, without taking into account their long-term consequences. Rationality requires instead consideration not only of the immediate consequences but also the future consequences of economic decisions. We ought to point out that not even at present, as far as we know, is the case study approach followed by the competent authorities.

Moreover rationality necessarily requires the elimination of the true causes of erosion, i.e. subsidence and the decrease in the fluvial sand flowing into the sea. If the 'prototype' approach had been followed, the need to know the true causes would soon have emerged. In fact, from a regional point of view, barriers, reefs and groins are revealed to be fallacious remedies of the local damage created by the erosion due to these other causes. The coastal plans of the Emilia Romagna Region have been moving in this direction only since the Eighties and some results have been obtained: in some areas shore subsidence is reducing and fluvial sand-flowing into the sea is increasing. Therefore more than 10 km of shore which in 1970 was under erosion or unstable is now advancing (Preti, Carboni and Albertazzi, 1997).

To conclude, as Munasinghe and Shearer (1995, p. XX) clearly affirm: 'Although sustainability is largely a natural state of affairs, unsustainability is the result of social actions, and, therefore, reducing these impacts involves a social process', which is often slow to occur.

Riferimenti bibliografici

Banca d'Italia (1997), *Relazione Annuale.*

Drei E. (1977), thesis of degree, *Analisi dell'intervento di ripascimento nella zona di Lido di Dante e di alcuni dei suoi aspetti economici*, University of Bologna, Italy, 1995-96 (coordinating professors: Lamberti A. for technical part, Marzetti S. for economic part).

Lutz E. and Munasinghe M. (1991), 'Accounting for the Environment', *Finance & Development*, march, pp. 19-21.

Marzetti Dall'Aste Brandolini S. (1991), *Microeconomia del consumo e della produzione. Un'applicazione ai problemi del turismo*, CLUEB, Bologna.

Marzetti Dall'Aste Brandolini S. (1994), *Guida alla controllabilità di modelli di politica economica, con applicazioni alla economia ambientale*, CLUEB, Bologna.

Munasinghe M. and McNeely (1995), 'Key Concepts and Terminology of Sustainable Development', in *Defining and Measuring Sustainability*, Munasinghe M. and Shearer W. (eds.), The World Bank, Washington D.C.

Munasinghe M. and Shearer W. (1995), 'An Introduction to the Definition and Measurement of Biogeophysical Sustainability', in *Defining and Measuring*

Sustainability, Munasinghe M. and Shearer W. (eds.), The World Bank, Washington D.C.

Preti M., Carboni E. and Albertazzi C. (1997), *Shore Protection Planning in Emilia-Romagna Region: A New Trend is in Act*, Proceedings of the Third International Conference on the Mediterranean Coastal Environment, MEDCOAST 97, November 11-14, Qawra, Malta, E. Özhan (Editor).

Simon H.A. (1957), *Models of Man, Social and Rational*, J. Wiley and Sons, New York.

- (1987), 'Bounded Rationality' in Eatwell J., Milgate M. and Newman P. (eds.), *The New Palgrave: Utility and Probability*, Macmillan, London, pp. 15-8.

Sinclair T.M. and Stabler M. (1997), *The Economics of Tourism*, Routlege, London.

Turner R.K., Pearce D.W. and Bateman I. (1994), *Environmental Economics. An Elementary Introduction*, Hemel Hempstead, Harvester Wheatsheaf.

Optimal Policies in a Bioeconomic Model of Eutrophication

Renzo Mosetti

Osservatorio Geofisico Sperimentale
Trieste, Italy

Transmitted by John Casti

ABSTRACT

We obtain optimal policies for the control of nutrients in a basin subjected to eutrophication. The aim is to minimize the costs of nutrient's removal plus the costs of the environmental damage due to poor water quality. The optimal solutions, in terms of phytoplankton biomass, are shown to be most rapid approach paths to a constant value.

I. INTRODUCTION

The problem of eutrophication of water basins can be very serious in places subjected to a continuous discharge of nutrients due to personal and industrial activities. This phenomenon not only is damaging to the water ecosystem, but also causes economic problems where tourism and fishery are among the principal sources of wealth. A typical example of this kind is the Adriatic coast of the Emilia-Romagna Region in Italy, where the growth of some species of phytoplankton (red algae) causes heavy damage to the local economy [1]. In this context it is of great importance to make a plan for the control of the input of nutrients, having in mind not only the protection of nature, but also the related economic aspects. In fact, it is evident that the removal of nutrients by technology has a cost that has to be sustained by the community in addition to the environmental damage.

Unfortunately, the usual cost-benefit analysis is not so easy to perform in the field of the pollution problems, as it is always impossible to evaluate the environmental damages, which are mainly of qualitative nature, in terms of monetary units. However, for the case here treated, we can suppose that it is possible to measure the monetary losses due to the effect of poor water quality on tourism as well as on fishery. So the aim of the paper is to study from a theoretical point of view, the policies minimizing the sum of the nutrient's removal costs and the costs of the environmental damage in a eutrophic system. To do this we shall use a simple dynamical model for phytoplankton growth.

2. THE MODEL

Consider a basin (lake, pond, or coastal sea) on which there is a discharge of nutrients due to personal and industrial activities. Assume that in a standard situation the level of nutrients in the basin has a constant value N and that growth of some species of phytoplankton, say red algae, is limited by these nutrients.

A simple growth model for red algae is the following:

$$\dot{x}(t) = ANx(t) - kx(t), \tag{1}$$

where:

$x(t)$ is the biomass of the red algae at time t;
A is a constant;
k is the mortality rate.

We assume $AN > k$, so that in the standard situation there is an increasing in the biomass.

We postulate that to a growth of red algae corresponds an amount of economic damage for the community living near the basin. We describe these costs by a function $P(x)$ that is increasing in x and such that $P(0) = 0$.

Suppose now that it is possible to remove nutrients in the basin by acting for instance at the source. In this way we can define the following control variable:

$$0 \leqslant u(t) \leqslant N =: u_M.$$

Substituting this into (1), we obtain the following bilinear control system:

$$\dot{x}(t) = Au(t)x(t) - kx(t).$$

Clearly, nutrient removal has a cost proportional to $1 - u/u_M$ that reaches a maximum value C when $u = 0$.

In view of these hypotheses, optimal management of this environmental system is to minimize over a long time period the sum of the cost due to the

environmental damage and the cost due to nutrient removal, i.e.

$$\int_0^\infty e^{-rt}\left[C\left(1 - \frac{u}{u_M}\right) + P(x)\right]dt,$$

where r is the discount rate.

3. SOLUTION OF THE OPTIMIZATION PROBLEM

The problem previously described can be formalized as follows: minimize the cost

$$J(u) = \int_0^\infty e^{-rt}\left[C\left(1 - \frac{u(t)}{u_M}\right) + P(x(t))\right]dt \tag{2}$$

on the set of admissible control functions

$$\mathscr{U} := \{u(\cdot):[0, +\infty) \to [0, u_M];\ u(\cdot) \text{ measurable}\} \tag{3}$$

and subjected to

$$\dot{x}(t) = Au(t)x(t) - kx(t), \tag{4}$$
$$x(0) = x_0 \in \mathbf{R}^+$$

where $r,\ k,\ A \in \mathbf{R}^+$, and where $P(x)$ is a strictly convex smooth function such that $P(0) = 0$ and $P_x(x) > 0\ \forall x > 0$. In the following we shall denote by

$$x_u(t) := x(x_0, u, t)$$

a solution of (4) with $u(\cdot) \in \mathscr{U}$, by Ω the set

$$\Omega := \{x_u(\cdot): \dot{x}_u(t) = Au(t)x_u(t) - kx_u(t);\ x_u(0) = x_0;\ u \in \mathscr{U}\},$$

and by T the set

$$T := \{t: t \in [0, +\infty)\}.$$

To solve the problem (2)–(4) we need two preliminary lemmas.

Lemma 3.1. *For every* $x_u(\cdot) \in \Omega$,

$$x_0 e^{-kt} \leqslant x_u(t) \leqslant x_0 e^{(Au_M - k)t} \qquad \forall t \in T.$$

Proof. From (4) we can write

$$\frac{\dot{x}_u(t)}{x_u(t)} = \frac{d}{dt} \ln x_u(t) = Au(t) - k,$$

so

$$\ln x_u(t) = \ln x_0 + \int_0^t [Au(t') - k]\, dt'.$$

Since $0 \leqslant u(t) \leqslant u_M$, the following inequality holds:

$$-\int_0^t k\, dt' \leqslant \int_0^t [Au(t') - k]\, dt' \leqslant \int_0^t [Au_M - k]\, dt'.$$

Hence

$$\ln x_0 - kt \leqslant \ln x_u(t) \leqslant \ln x_0 + (Au_M - k)t \qquad \forall t \in T,$$

from which we get the conclusion.

Lemma 3.2. *The optimization problem* (2)–(4) *is equivalent to minimizing*

$$J'(x_u) = \int_0^\infty e^{-rt} [M(x_u(t)) + rZ(x_u(t))]\, dt$$

on Ω, *where*

$$M(x_u(t)) = P(x_u(t)) + C\left(1 - \frac{k}{Au_M}\right),$$

$$Z(x_u(t)) = \frac{C}{Au_M} [\ln x_0 - \ln x_u(t)].$$

Proof. From (4) we have

$$u(t) = \frac{1}{A}\left(\frac{\dot{x}_u(t)}{x_u(t)} + k\right),$$

$$J(u) = \int_0^\infty e^{-rt} \left[P(x_u(t)) + C - \frac{Ck}{Au_M} - \frac{C}{Au_M} \frac{\dot{x}_u(t)}{x_u(t)} \right] dt. \tag{5}$$

Now

$$M(x_u(t)) := P(x_u(t)) + C\left(1 - \frac{k}{Au_M}\right).$$

Define

$$Z(x_u(t)) := -\frac{C}{Au_M} \int_{x_0}^{x_u(t)} \frac{1}{x'}\, dx' = \frac{C}{Au_M} [\ln x_0 - \ln x_u(t)],$$

$$\dot{Z}(x_u(t)) = -\frac{C}{Au_M} \frac{\dot{x}_u(t)}{x_u(t)}.$$

By inserting the last expression into (5) and by integrating by parts, we obtain

$$J(u) = \int_0^\infty e^{-rt} [M(x_u(t)) + rZ(x_u(t))]\, dt + \lim_{t \to \infty} [e^{-rt} Z(x_u(t))].$$

Now, to get the result we have to show that the limit is always zero. In fact, by Lemma 3.1 we easily note that

$$\lim_{t \to \infty} [e^{-rt} \ln x_u(t)] = 0 \qquad \forall x_u(\cdot) \in \Omega.$$

Now, from the hypotheses made on $P(\cdot)$ we can argue that the function $G: \mathbb{R} \to \mathbb{R}$ given by

$$G: x_u(t) \mapsto M(x_u(t)) + rZ(x_u(t))$$

is strictly convex and has a minimum (unique for $x > 0$) at $\hat{x}$ where

$$G_x(\hat{x}) = P_x(\hat{x}) - \frac{rC}{Au_M \hat{x}} = 0. \tag{6}$$

This solution $\hat{x}$ is the singular solution of the problem and it is a constant.

At this point we shall prove that, starting from every $x_0 \neq 0$, the optimal solution of our problem it is to reach the singular level $\hat{x}$ as fast as possible and then by applying the feasible control $\hat{u} = k/A$ to stay at the singular level for all the time.

We start with the following definition:

DEFINITION 3.3. A path $x^*_{u^*}(\cdot) \in \Omega$ is said to be a *most rapid approach path* (MRAP) to $\hat{x}$ if

$$|x^*_{u^*}(t) - \hat{x}| \leqslant |x_u(t) - \hat{x}| \qquad \forall t \in T$$

for every $x_u(\cdot) \in \Omega$.

We are now able to state the following theorem:

THEOREM 3.4. *The following policies define a MRAP to $\hat{x}$ and yield the optimal solution $x^*_{u^*}(\cdot)$ of* (2)–(4):

$$u^*(t) = \begin{cases} u_M, & t < \hat{t} \\ k/A, & t \geqslant \hat{t} \end{cases} \qquad if \quad x_0 < \hat{x}, \tag{7}$$

$$u^*(t) = \begin{cases} 0, & t < \hat{t}' \\ k/A, & t \geqslant \hat{t}' \end{cases} \qquad if \quad x_0 > \hat{x} \tag{8}$$

$$\hat{t} = \frac{1}{Au_M - k} \ln \frac{\hat{x}}{x_0},$$

$$\hat{t}' = \frac{1}{k} \ln \frac{x_0}{\hat{x}}.$$

PROOF. First we have to show that (7) or (8) defines a MRAP to $\hat{x}$. Suppose

$$x_0 < \hat{x}.$$

Then, consider the solution with $u = u_M$,

$$x_{u=u_M}(t) = x_0 e^{(Au_M - k)t}.$$

Clearly this solution reaches $\hat{x}$ for $t = \hat{t}$. By Lemma 3.1 we have

$$x_0 e^{(Au_M - k)t} \geqslant x_u(t) \qquad \forall t \in [0, \hat{t}], \quad \forall x_u(\cdot) \in \Omega.$$

Therefore (7) defines a MRAP to $\hat{x}$.

If

$$x_0 > \hat{x},$$

we consider the solution with $u = 0$,

$$x_{u=0}(t) = x_0 e^{-kt}.$$

This reaches $\hat{x}$ for $t = \hat{t}'$. Again by Lemma 3.1 we have that

$$x_0 e^{-kt} \leqslant x_u(t) \qquad \forall t \in [0, \hat{t}'], \quad \forall x_u(\cdot) \in \Omega,$$

so (8) defines a MRAP to $\hat{x}$.

Now, consider

$$J'(x_u) = \int_0^\infty e^{-rt} G(x_u(t))\, dt.$$

(a) If $x_0 < \hat{x}$, we write

$$J'(x_u) = \int_0^{\hat{t}} e^{-rt} G(x_u(t))\, dt + \int_{\hat{t}}^\infty e^{-rt} G(x_u(t))\, dt.$$

By Lemma 3.1 and by the convexity of $G(\cdot)$ it follows that

$$x_0 e^{(Au_M - k)t} \geqslant x_u(t) \quad \Rightarrow \quad G(x_u(t)) \geqslant G(x_0 e^{(Au_M - k)t})$$
$$\forall x_u(\cdot) \in \Omega, \quad \forall t \in [0, \hat{t}]$$

so that

$$\int_0^{\hat{t}} e^{-rt} G(x_0 e^{(Au_M - k)t})\, dt \leqslant \int_0^{\hat{t}} e^{-rt} G(x_u(t))\, dt.$$

Furthermore, since $G(\hat{x})$ is the minimum of $G(\cdot)$, we have

$$\int_{\hat{t}}^{\infty} e^{-rt} G(\hat{x})\, dt \leqslant \int_{\hat{t}}^{\infty} e^{-rt} G(x_u(t))\, dt.$$

Hence

$$J'(x^*_{u^*}) \leqslant J'(x_u) \qquad \forall x_u(\cdot) \in \Omega.$$

(b) If $x_0 > \hat{x}$, we write

$$J'(x_u) = \int_0^{\hat{t}'} e^{-rt} G(x_u(t))\, dt + \int_{\hat{t}'}^{\infty} e^{-rt} G(x_u(t))\, dt.$$

Again by Lemma 3.1 and by the convexity of $G(\cdot)$ it follows that

$$\int_0^{\hat{t}'} e^{-rt} G(x_0 e^{-kt})\, dt \leqslant \int_0^{\hat{t}'} e^{-rt} G(x_u(t))\, dt,$$

so as before we obtain

$$J'(x^*_{u^*}) \leqslant J'(x_u) \qquad \forall x_u(\cdot) \in \Omega.$$

4. FINAL CONSIDERATIONS

The solution obtained for the problem here described is a so-called "bang-bang singular solution," and it is related to the linear structure of the cost and the state equation with respect to the control variable. Examples of this kind can be found in models of economic growth [2, 3] as well as in the field of the optimal exploitation of renewable resources [4–6]. Heuristically, the philosophy of MRAP is that, due to the discount rate, we have to reach the singular level $\hat{x}$ (optimal equilibrium biomass) as fast as possible. If we start with an initial biomass $x_0 < \hat{x}$, we have to "pollute" at top level to reach in minimum time $\hat{x}$. If we start with $x_0 > \hat{x}$, we have on the contrary to remove all the nutrients until $\hat{x}$ is reached as rapidly as possible.

It is interesting to rewrite Equation (6) for the optimal equilibrium biomass as

$$r \frac{C/u_M}{P_x(\hat{x})} = A\hat{x}. \qquad (9)$$

Now, if we define Ax as a "marginal growth rate" (i.e., the variation of the growth rate of phytoplankton with respect to a variation of the input of nutrients), we observe that the optimal equilibrium level $\hat{x}$ is reached when the ratio between the marginal cost of nutrient removal and the marginal cost of the environmental damage multiplied by the discount rate equals the "marginal growth rate." This relation resembles, *mutatis mutandis*, some well-known rules of economic equilibrium.

By considering Equation (9) again, we can easily note that the larger r is, the larger is the optimal equilibrium biomass. This shows, also in this case, the negative effect of the discount rate on the conservation of the environment.

Finally, we have to discuss the choice of a linear model for phytoplankton growth, which might seem a little unrealistic. This can be justified by the following arguments. We suppose that a plan for the pollution control of an environment makes sense only if the environment is not yet deteriorated in an irreversible manner. This means that in our case the initial level of phytoplankton must not be very high. Thus, if we consider for example a logistic growth model, we are concerned with operation far from the saturation level of the biomass in a zone still dominated by exponential growth.

I would like to thank my colleague Dr. Fabio Cavallini for some stimulating discussions and suggestions.

REFERENCES

1 R. Marchetti (Ed.), Indagini sul problema dell'eutrofizzazione delle acque costiere dell'Emilia-Romagna, Report 35, Emilia-Romagna Region, 1984.
2 S. P. Sethi and G. L. Thompson, *Optimal Control Theory: Applications to Management Science*, Martinus Nijhoff, Boston, 1981.
3 G. Feichtinger and R. F. Hartl, *Optimale Kontrolle Ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*, de Gruyter, Berlin, 1986.
4 C. W. Clark, Profit maximization and the extinction of animal species, *J. Polit. Economy* 81(4):950–961 (1973).
5 C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, Wiley-Interscience, 1976.
6 G. Leitman, A minimum principle for a population equation, *J. Optim. Theory Appl.* 9:155–156 (1972).