INTERNATIONAL ATOMIC ENERGY AGENCY
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AUTUMN COLLEGE
ON
THE TROPOSPHERE, STRATOSPHERE AND MESOSPHERE
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ATMOSPHERIC SCIENCES 541

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ATMOSPHERIC SCIENCES 541
Course Outline

1. Fundamentals

   Equations of motions, continuity and energy. Rotation. Spherical coordiantes. Hydrostatic equilibrium. Scaling. Isobaric coordinates -- natural coordinates. Geostrophic cyclostrophic and gradient flows. Vertical motion.

   (Holton, Chapters 1-3)

2. Vorticity

   Vorticity equation. Kelvin & Bjerknes circulation theorems. Ertel potential vorticity. Applications. Scale analysis of vorticity equation.

   (Holton, Chapter 4)

3. Planetary boundary layer

   Ekman layers and spin-down

   (Holton, Chapter 5)

4. Linear wave theory

   Internal gravity waves and Rossby waves. Phase speed, group velocity. WKBJ theory. Wave, mean-flow interaction. A simple model of the quasi-biennial oscillation.

   (Holton, Chapter 7)

TEXT: Holton, An Introduction to Dynamic Meteorology, 2nd Edition.

Other (not necessary), Pedlosky, Geophysical Fluid Dynamics

1. Stokes' Theorem

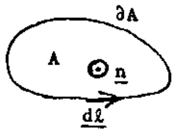
(a)

$$\int_{\partial V} \underline{B} \cdot \underline{dA} = \int_V \underline{\nabla \cdot B}\, dV \qquad \text{where } \underline{dA} = \underline{n}\, dA, \quad \underline{n} \text{ is the outward unit normal on } \partial V$$

(b)

$$\int_{\partial A} \underline{B} \cdot \underline{d\ell} = \int_A (\underline{\nabla} \times \underline{B}) \cdot \underline{dA}$$

where



2. $\underline{\nabla}$, $\underline{\nabla}\cdot$, $\underline{\nabla}\times$ in Cartesian and spherical coordinates (Holton, p. 369)

3. Substantial derivative

Consider some property ψ of the fluid. Then

$\psi = \psi(\underline{x}, t)$ Eulerian description

or $\psi = \psi(\underline{\xi}, t)$ Lagrangian description

where $\underline{\xi} = \underline{\xi}(\underline{x}, t)$ is the position vector of each particle of the fluid.

Using the chain rule for partial derivatives

$$\left.\frac{\partial\psi}{\partial t}\right|_{\xi=\text{const}} = \left.\frac{\partial\psi}{\partial t}\right|_{x=\text{const}} + \frac{\partial\psi}{\partial \underline{x}} \cdot \left.\frac{\partial x}{\partial t}\right|_{\xi=\text{const}}$$

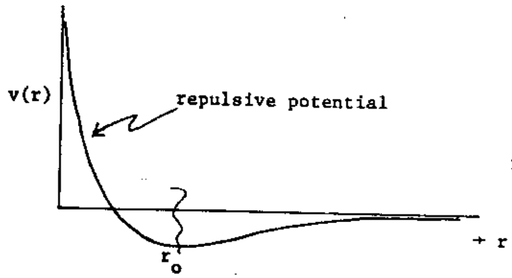
i.e., $\frac{d\psi}{dt}$ (or $\frac{D\psi}{Dt}$) $= \frac{\partial\psi}{\partial t} + \underline{v} \cdot \nabla\psi$

if $\psi = \underline{x}$ $\qquad \underline{v} = \frac{d\underline{x}}{dt}$

Assume partial derivatives hold the respective Eulerian variables constant.

1. Continuum

We treat geophysical fluids as continuous media: their properties are assumed to vary smoothly on the scale of interest. On a finer sacle, the media have molecular properties:



$r_o \sim 1\text{-}2$ Å

In air at STP there are 2.69×10^{25} molec/m^3(n), → average space between molecules is $\sim 1/(27 \times 10^{24})^{1/3} \sim 3 \times 10^{-9}$ m. $\sim 20\, r_o$.

But the mean free path, λ is

$$\lambda = \frac{\bar{c}}{n \cdot \pi(2r_o)^2 \bar{c}} = \frac{1}{n\pi(2r_o)^2} \sim 10^{-7} \text{ m} \sim 500\, r_o \qquad \text{(hence ideal gas behavior)}$$

For Venus $p \sim 100$ p (Earth), $T = 700$ K $\sim 7/3\, T_{(\text{earth})}$, $n \sim 40$ n (Earth).
Mean spacing $\sim$ Mean spacing(Earth)/$\sqrt[3]{40} \sim 6r_o$, λ(Venus) $\sim 2 \times 10^{-9}$ m $\sim 15\, r_o$.
Departures from ideal gas behavior may occur at the high surface pressures of Venus.

For water $n \sim 2 \times 10^3$ n(air), Mean spacing is $\sim 1.5\, r_o$, and the concept of mean free-path is no longer applicable. The molecules move as aggregates. Liquids are not compressible, but they respond to applied stress with continuously varying deformation. Rate of strain $\propto$ stress. This is in contrast to crystalline solids for which strain itself is proportional to stress.

The continuum hypothesis works because λ is much smaller than the scales of interest, and there are so many molecules in even the smallest measurement vo. that molecular statistical fluctuations are fully smoothed out for any measurement (e.g., in 1 mm^3 of air at STP, there are 3×10^{16} molec.). For this reason, the fluid equations are applicable at the laboratory scale. But we are concerned with geophysical macroscales. For example: radiosonde measures a volume $\sim km^3$. But it is assumed to represent a volume $\gtrsim 100 \times 100 \times 1$ km over time $\gtrsim 1/4$ day. Hence all fluctuations at smaller scales are smoothed out by the measurement analysis process.

However, there is an analog between the geophyiscal scale and the laboratory scale. On the lab scale, transports of properties (mass, momentum, energy) at scales below the measurement scale are assumed due to molecular transport processes -- i.e., molecular diffusion, viscosity, heat conduction. We can think of parcels of fluid in the lab which move and change shape with the mass motion of the fluid. Properties of these parcels then change with time as a result of processes other than the mass motion, e.g., radiation, chemical reaction, and molecular transport across parcel boundaries. This is the basic conservation principle of the fluid dynamics equations.

Now on the geophysical super macroscale, the same fundamental principle applies, except that in addition to the processes which affect parcel properties on the lab scale, we must add: transports across parcel boundaries due to fluid motions at scales smaller than the measurement scale, i.e., to geophysical turbulence. These transports are in fact much more important than molecular transports for geophysical flows. The difficulty is that the fluxes by

turbulence are not well understood. Thus, in applying the fluid equations to the macroscale, we are implicitly assuming either that the turbulent eddy transports are small, or that they can be in some way approximately characterized in terms of large-scale processes.

These considerations, in fact, help to specify the scale at which measurements must be made. For terrestrial mid-latitudes the radiosonde network works very well -- it measures at scales $\sim$ 100-1000 km above which most the energy lies. Examples of possible problem areas however are: the large scale ocean circulation, the tropical atmosphere, the atmosphere of Venus.

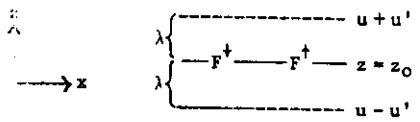
We shall develop and apply the fluid equations to atmospheric problems, but it is well to keep this basic limitation in mind.

1. Continuity: Fixed control volume, change in mass = mass flux

$$\frac{\partial}{\partial t}\int_V \rho dv = \int_V \frac{\partial\rho}{\partial t} dv = -\int_A \rho\underline{u}\cdot\hat{n}\, dA = -\int_V \nabla\cdot(\rho\underline{u})\, dv \Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$$

Corollary: $\rho\frac{d\psi}{dt} = \frac{\partial}{\partial t}(\rho\psi) + \frac{\partial}{\partial x_i}(\rho\psi u_i)$, $\psi(\underline{x},t)$ is any scalar, vector, or tensor field (substantial derivative form is equivalent to flux form).

2. Momentum: Consider a parcel of fluid. We must distinguish long range forces, such as gravity, which act uniformly over the parcel and short range forces, due to molecular diffusion, which act only within a mean free path length of the boundary of the fluid parcel. Let $P_i(\hat{n})$ = t stress (force/unit area) exerted by the fluid on the side of a surface to which the normal $\hat{n}$ points, on the fluid on the side of the surface from which $\hat{n}$ points. Note that by Newton's third law $P_i(\hat{n}) = -P_i(-\hat{n})$. We can write $P_i(\hat{n}) = \sigma_{ij}n_j$ in terms of components, n_j, of $\hat{n}$. σ_{ij} = stress tensor, ith component of stress across surface whose outward normal is in j^{th} coordinate direction. In general, $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$; $p = \frac{1}{3}\sigma_{kk}$ = pressure, τ_{ij} = viscous stress. Properties of τ_{ij}:
(i) Its trace $(=\tau_{kk}) \equiv 0$, (ii) Its components depend linearly on components of rate of strain tensor, $\partial u_i/\partial x_j$, (A molecular interpretation of this relationship can be given for a fluid which flows uniformly in the x-direction with a shear in the z-direction.



$F^{\downarrow}$ is the downward flux of x-component momentum on the surface $z = z_o$, carried by molecules which have arrived from the surface $z = z_o + \lambda$. Similarly $F^{\uparrow}$ is the upward flux of x-component momentum on $z = z_o$, carried by molecules which have arrived from $z = z_o - \lambda$.

$$F^{\downarrow} = \frac{1}{6}\rho c(u+u') = \frac{1}{6}\rho c(u+\lambda\frac{\partial u}{\partial z}); \quad F^{\uparrow} = \frac{1}{6}\rho c(u-u') = \frac{1}{6}\rho c(u-\lambda\frac{\partial u}{\partial z})$$

New upward momentum flux $= F^{\uparrow} - F^{\downarrow} = -\frac{1}{3}\rho\bar{c}\lambda\frac{\partial u}{\partial z}$. Hence the x-component of stress due to this momentum flux is proportional to $\partial u/\partial z$ and the constant of proportionality, μ, is equal to $\frac{1}{3}\rho\bar{c}\lambda$.)

(iii) It depends only on symmetric components of $\partial u_i/\partial x_j$:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \quad \text{(antisymmetric part, } \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right),$$

corresponds to solid rotation, and there is no viscous stress in solid rotation). Most general form of τ_{ij} satisfying (i)-(iii) is:

$$\boxed{\tau_{ij} = 2\mu(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij})} \quad ; \quad \mu = \frac{1}{3}\rho\bar{c}\lambda = \rho\nu; \quad \nu = \text{kinematic viscosity}$$

Momentum eqn:
$$\underbrace{\frac{\partial}{\partial t}\int_V \rho u_i dv}_{\text{change in momentum}} = -\underbrace{\int_A \rho u_i u_j n_j dA}_{\text{flux of momentum}} + \underbrace{\int_A \sigma_{ij}n_j dA}_{\text{surface forces}} + \underbrace{\int_V \rho F_i dv}_{\text{body forces}}$$

$$\boxed{\rho\frac{Du_i}{Dt} = \frac{\partial\rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\rho u_i u_j = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho F_i = -\frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ij}}{\partial x_j} + \rho F_i}$$

(a) (b)

$\underline{F}$ = body force $= -\frac{GM}{r^2}\hat{\underline{r}} \equiv -g^*\hat{\underline{r}}$, $g^* = g_o^*(r_o/r)^2$.

Optional exercise: Show that $\partial\tau_{ij}/\partial x_j = \mu\nabla^2 u_i + \frac{\mu}{3}\frac{\partial}{\partial x_i}(\nabla\cdot\underline{u})$.

Reynolds Number, Re = ratio of term (a) to (b), $= \rho U^2/L$: $\rho\bar{c}\lambda U/L^2 = UL/\bar{c}\lambda$
$= UL/\nu \ggg 1$.
(for geophysical fl

3. Energy: $\underline{F} = g^*\hat{\underline{r}} = -\nabla\phi$, $K = \frac{1}{2}u^2$.

KE + PE eqn: $$\rho\frac{D}{Dt}(K+\phi) = u_i\frac{\partial}{\partial x_j}\sigma_{ij} = \underset{(a)}{p\frac{\partial u_i}{\partial x_i}} - \underset{(b)}{\tau_{ij}\frac{\partial u_i}{\partial x_j}} + \underset{(c)}{\frac{\partial}{\partial x_j}[-pu_j + u_i\tau_{ij}]}$$

(a) = $p\nabla\cdot u$, pressure work (reversible), (b) = $-\rho\delta$, $\delta \geq 0$ = dissipation rate/unit mass (irreversible), (c) = boundary work done by stress.

($\because$ rate at which stress does work = $\int_A P_i u_i dA = \int_A \sigma_{ij}u_i n_j dA = \int \nabla_j(\sigma_{ij}u_i)dv$ $= \int\nabla_j\{u_i(-p\delta_{ij} + \tau_{ij})\}dv$

Conservation of the total energy of a system (or first law of thermodynamics states that the change of total thermodynamic energy of a system is equal to the net heating of the system + the rate at which work is done on the system by external forces.

Let $e \equiv$ internal energy/unit mass.

$$\rho\frac{D}{Dt}(\text{total energy}) = \rho\frac{D}{Dt}(K+\phi+e) = -\frac{\partial}{\partial x_i}(R_i + c_i) + \rho(\dot{Q}_L + \dot{Q}_c) + \frac{\partial}{\partial x_j}[-pu_j + u_i\tau_{ij}].$$

R_i and c_i are radiative and molecular conduction fluxes

$\dot{Q}_L$, $\dot{Q}_c$ latent and chemical heating. Last term is boundary work.

Subtract KE eqn. from total E eqn.

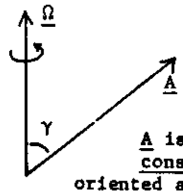
$$\rho\frac{De}{Dt} + p\nabla\cdot\underline{u} = -\nabla\cdot(\underline{R}+\underline{c}) + \rho(\dot{Q}_L + \dot{Q}_c) + \rho\delta, \quad \text{or with } \alpha = 1/\rho$$

$$\boxed{\frac{De}{Dt} + p\frac{D\alpha}{Dt}} = \dot{Q} + \delta; \quad \alpha \equiv 1/\rho, \quad \dot{Q} \equiv \dot{Q}_L + \dot{Q}_c - \frac{1}{\rho}\nabla\cdot(\underline{R}+\underline{c}).$$

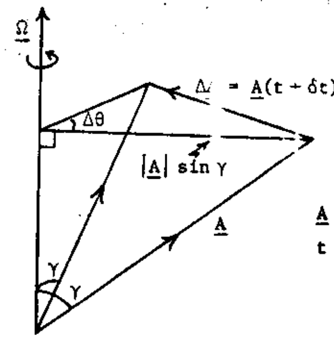
For an ideal gas, $\boxed{e = c_v T}$, $\boxed{p = R\rho T,\ \rho = nm,\ R = k/m}$.

where c_v is specific heat at constant volume, k is Boltzman's constant, m is the mean molecular mass.

Rotating Frames:



$\underline{A}$ is a vector of constant length oriented at an angle γ with respect to the axis of its rotation.



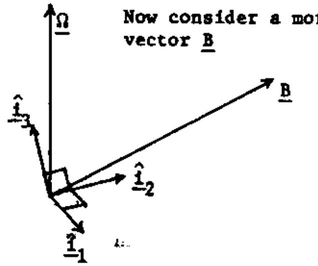
$\underline{A}$ at times t and t + δt.

$\Delta\underline{A}$ is perpendicular to $\underline{A}$ (since $|\underline{A}|$ = const)

$\Delta\underline{A}$ is perpendicular to $\underline{\Omega}$ (since $\underline{A}$ rotates about $\underline{\Omega}$)

$\Delta\underline{A}$ is parallel to $\underline{\Omega}\times\underline{A}$

$$|\Delta\underline{A}| = |\underline{A}|\sin\gamma\,\Delta\theta \Rightarrow \frac{|\Delta\underline{A}|}{dt} = |\underline{A}||\underline{\Omega}|\sin\gamma = |\underline{\Omega}\times\underline{A}| \qquad \therefore \frac{d\underline{A}}{dt} = \underline{\Omega}\times\underline{A}$$



Now consider a more general vector $\underline{B}$

basis $\hat{\underline{i}}_j$ rotates about axis with angular velocity $\underline{\Omega}$

$$\underline{B} = \hat{\underline{i}}_j B_j$$

$$\left(\frac{d\underline{B}}{dt}\right)_{\text{rotating}} = \hat{\underline{i}}_j\frac{dB_j}{dt}$$

$$\left(\frac{d\underline{B}}{dt}\right)_{\text{inertial}} = \hat{\underline{i}}_j\frac{dB_j}{dt} + \frac{d\hat{\underline{i}}_j}{dt}B_j$$

$$= \left(\frac{d\underline{B}}{dt}\right)_{\text{rot}} + \underline{\Omega}\times\hat{\underline{i}}_j B_j$$

$$= \left(\frac{d\underline{B}}{dt}\right)_{\text{rot}} + \underline{\Omega}\times\underline{B}$$

(n.b., Both observers agree on $d\underline{\Omega}/dt$) n.b. If ψ is a scalar, dψ/dt is coordinate independent.

<u>Basic equations</u>: $\underset{\sim}{\dfrac{D\underline{u}}{Dt}} = -\dfrac{1}{\rho}\nabla p + \dfrac{1}{\rho}\nabla\cdot\mathcal{T} + \underline{F} = \underset{\sim}{-\dfrac{1}{\rho}\nabla p} + \underset{\sim}{\mu\nabla^2\underline{u}} + \dfrac{\mu}{3}\nabla(\nabla\cdot\underline{u}) + \underset{\sim}{\underline{F}}$,

$$\frac{D\rho}{Dt} + \rho\,\underset{\sim}{\nabla\cdot\underline{u}} = \underset{\sim}{0}\,, \quad c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = \dot{Q}'\,, \quad \text{or} \quad c_p\frac{DT}{dt} - \alpha\frac{Dp}{Dt} = \dot{Q}'\ .$$

$\mathcal{T}$ = viscous stress tensor, $\dot{Q}'$ includes dissipation

Underlined terms comprise the <u>Navier-Stokes equations</u>. ($c_p = c_v + R$)

<u>Rotation</u>: Let $\underline{A}$ be any vector in a coordinate system rotating at rate $\underline{\Omega}$. Then the rate of change of $\underline{A}$ due to rotation alone is $d\underline{A}/dt = \underline{\Omega} \times \underline{A}$, and the total change rate of $\underline{A}$,

$$\frac{d\underline{A}}{dt_a} = \frac{d\underline{A}}{dt_r} + \underline{\Omega}\times\underline{A}$$

where d/dt_r is the rate of change measured by an observer in the rotating system. Let $A = \underline{r}$, then

$$\frac{d\underline{r}}{dt_a} = \frac{d\underline{r}}{dt_r} + \underline{\Omega}\times\underline{r} = \underline{u}_r + \underline{\Omega}\times\underline{r}$$

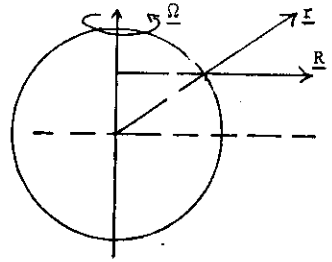
where $\underline{r}$ = distance from some origin. Then

$$\frac{d}{dt_a}\left(\frac{d\underline{r}}{dt_a}\right) = \frac{d\underline{u}_a}{dt_a} = \frac{d}{dt_r}\left(\frac{d\underline{r}}{dt_r} + \underline{\Omega}\times\underline{r}\right) + \underline{\Omega}\times\left(\frac{d\underline{r}}{dt_r} + \underline{\Omega}\times\underline{r}\right) = \frac{d\underline{u}_r}{dt_r} + 2\underline{\Omega}\times\underline{u}_r + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})\ .$$

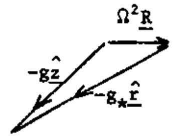
$d\underline{u}_a/dt_a$ can be identified with $D\underline{u}/Dt$ above. Hence

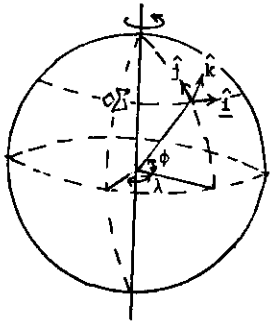
$$\boxed{\frac{D\underline{u}_r}{Dt_r} = -\frac{1}{\rho}(\nabla p + \nabla\cdot\mathcal{T}) - g_*\hat{\underline{r}} - \underbrace{2\underline{\Omega}\times\underline{u}_r}_{\substack{\text{Coriolis}\\ \text{Force}}} - \underbrace{\underline{\Omega}\times(\underline{\Omega}\times\underline{r})}_{\substack{\text{Centrifugal}\\ \text{Force}}}}$$

where $\underline{r}$ is taken to be radially outward from the center of gravitation, so that $\underline{F} = -g_*\hat{\underline{r}}$. Now $\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = -\Omega^2\underline{R}$ where $\underline{R}$ is radial distance out from axis of rotation



We can combine $g_*\hat{\underline{r}} + \Omega^2\underline{R}$ into $g\hat{\underline{z}}$, thus defining the local vertical and the local gravity, g. Note that $g_*\hat{\underline{r}} = -\nabla\phi_*$, $\Omega^2\underline{R} = -\nabla\phi_c$, $\phi_c = -\frac{1}{2}\Omega^2R^2$, $\phi_c + \phi_* = \tilde{\phi}$. Surfaces of constant $\tilde{\phi}$ are slightly flattened spheroids and are the <u>level surfaces</u>. By <u>defining</u> g and $\hat{\underline{z}}$ in this way, we remove the need for further specific consideration of <u>centrifugal force</u>.





Curvilinear Coordinates

$\underline{U} = u\hat{\underline{i}} + v\hat{\underline{j}} + w\hat{\underline{k}}$

$(\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}})$ right-handed set of orthogonal unit vectors. In terms of "spherical" coordinates (λ, ϕ, z)

$$\left.\begin{aligned} u &= \frac{dx}{dt} = r\cos\phi\,\frac{d\lambda}{dt} \\ v &= \frac{dy}{dt} = r\,\frac{d\phi}{dt} \\ w &= \frac{dz}{dt} \end{aligned}\right\}$$

n.b $r = a + z$ and $z \ll a \rightarrow r$ and a are interchangeable, where a is the radius of the earth, which is taken to be spherical.

Now

$$\frac{d\underline{U}}{dt} = \hat{\underline{i}}\,\frac{du}{dt} + \hat{\underline{j}}\,\frac{dv}{dt} + \hat{\underline{k}}\,\frac{dw}{dt} + u\,\frac{d\hat{\underline{i}}}{dt} + v\,\frac{d\hat{\underline{j}}}{dt} + w\,\frac{d\hat{\underline{k}}}{dt}$$

and

$$\left.\begin{aligned} \frac{d\hat{\underline{i}}}{dt} &= u\,\frac{\partial\hat{\underline{i}}}{\partial x} = \frac{u}{a\cos\phi}\,(\hat{\underline{j}}\sin\phi - \hat{\underline{k}}\cos\phi) \\ \frac{d\hat{\underline{j}}}{dt} &= u\,\frac{\partial\hat{\underline{j}}}{\partial x} + v\,\frac{\partial\hat{\underline{j}}}{\partial y} = -\frac{u\tan\phi}{a}\,\hat{\underline{i}} - \frac{v}{a}\,\hat{\underline{k}} \\ \frac{d\hat{\underline{k}}}{dt} &= u\,\frac{\partial\hat{\underline{k}}}{\partial x} + v\,\frac{\partial\hat{\underline{k}}}{\partial y} = \frac{u}{a}\,\hat{\underline{i}} + \frac{v}{a}\,\hat{\underline{j}} \end{aligned}\right\}$$

see Holton, p. 32-33

Also

$$\underline{\Omega} = \Omega\cos\phi\,\hat{\underline{j}} + \Omega\sin\phi\,\hat{\underline{k}} \qquad \Omega = |\underline{\Omega}|$$

$$\underline{g} = -g\,\hat{\underline{k}}$$

$$\nabla p = \frac{\partial p}{\partial x}\,\hat{\underline{i}} + \frac{\partial p}{\partial y}\,\hat{\underline{j}} + \frac{\partial p}{\partial z}\,\hat{\underline{k}}$$

n.b. $\hat{\underline{k}}$ defined to be parallel to the direction of "effective gravity" $\underline{g}$.

neglect any effects of the oblateness of the earth in calculating $\partial\hat{\underline{i}}/\partial x$ etc.

Also let frictional force $\underline{\mathcal{F}} = \mathcal{F}_x\hat{\underline{i}} + \mathcal{F}_y\hat{\underline{j}} + \mathcal{F}_z\hat{\underline{k}}$.

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + \mathcal{F}_x$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \mathcal{F}_y$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + \mathcal{F}_z$$

"Primitive" equations of motion

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{a\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(v\cos\phi) + \frac{\partial w}{\partial z} = 0$$

Continuity equation in spherical coordinates

$$c_p\,\frac{dT}{dt} - \alpha\,\frac{dp}{dt} = \dot{Q}'$$

Thermodynamic equation

n.b.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}$$

1. <u>Coriolis Force</u> can be though of either as the excess centrifugal force acting on the zonal component of flow, or as a manifestation of conservation of angular momentum for flow in the meridional plane.

(a) If a particle is given a zonal velocity, u, in the rotating frame, the centrifugal force acting on it is

$$\left(\Omega + \frac{u}{R}\right)^2 \underline{R} = \Omega^2\underline{R} + \frac{2\Omega u}{R}\underline{R} + \frac{u^2\underline{R}}{R^2}$$

For motions where $u << \Omega R$ the excess centrifugal force is

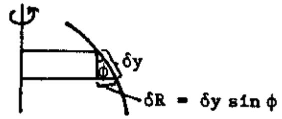
$$2\Omega u\hat{\underline{R}} = \underbrace{-\,2\Omega u \sin\phi\,\hat{\underline{j}}}_{\substack{\text{southward}\\ \text{acceleration}}} + \underbrace{2\Omega u \cos\phi\,\hat{\underline{k}}}_{\substack{\text{upward}\\ \text{acceleration}}}$$

(b) If a particle initially at rest in the rotating frame is displaced towards the equator with velocity v, conservation of angular momentum about axis c rotation gives

$$\Omega R^2 = \left(\Omega + \frac{\delta u}{R+\delta R}\right)(R + \delta R)^2 \simeq \left(\Omega + \frac{\delta u}{R}\right) R^2 \left(1 + \frac{2\delta R}{R}\right)$$

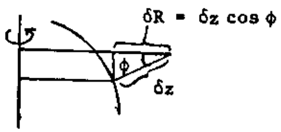
where δu is the required gain in zonal velocity to balance angular momentum.

$$\Rightarrow \frac{\delta u}{\delta t} = -\,2\Omega\,\frac{\delta R}{\delta t} = 2\Omega v \sin\phi$$



If the particle is launched vertically with velocity w

$$\frac{\delta u}{\delta t} = -\,2\Omega\,\frac{\delta R}{\delta t} = -\,2\Omega w \cos\phi$$



n.b. n the Northern Hemisphere the horizontal deflection is to the right of the particle's velocity (Holton, p. 13).

2. <u>Hydrostatic Equilibrium</u>

In the absence of any atmospheric motion

$$\frac{1}{\rho}\nabla p + g\hat{\underset{\sim}{z}} = 0 \Leftrightarrow dp/dz = -\,g\rho$$

Define the geopotential

$$\Phi(z) = \int_0^z g\,dz$$ (= work required to raise unit mass to height z from mean sea level, MSL)

then $dp = -\,\rho d\Phi$ or $d\Phi = -\,RT\,d\,\ell n\,p$. (This is the hypsometric equation.)

Define geopotential height $Z = \Phi/g_o$ where $g_o = 9.80665\ \mathrm{ms^{-2}}$ (average MSL value).

Then the thickness $\Delta Z(p_2,p_1)$ between pressure levels p_2 and p_1

$$= \frac{R}{g_o}\int_{p_2}^{p_1} T d\ell n p = H\ \ell n\,(p_1/p_2)$$

where $H = R\overline{T}/g_o$ and $\overline{T} = \int_{p_2}^{p_1} T\,d\,\ell n\,p \Big/ \int_{p_2}^{p_1} d\,\ell n\,p$

This thickness is proportional to the mean temperature for a given pressure layer. For an isothermal atmosphere H = const. and

$$p(Z) = p(0)e^{-Z/H}$$

$H \sim 7.8$ km in the troposphere and is called the scale height. For an isothermal atmosphere pressure decreases exponentially with Z by a factor e^{-1} per scale height.

Scaling

In this course we are concerned with finding a simplified set of equations which adequately describe mid-latitude synoptic scale weather systems in the atmosphere. To do this we use a preconception of the qualitative nature of the motions of such systems in terms of their scales and amplitudes. The equations of motion do not dictate what scales we must choose; rather they guide us in deciding whether our choice of scales is consistent.

We choose:

(1) A horizontal length scale $L = 10^6$ m

(2) A horizontal velocity $U = 10$ ms^{-1}

(3) A vertical depth $D = 10^4$ m (≈ height of tropopause and tropospheric scale hei

(4) A time scale U/L (equal to the advective time scale)

Also -- $f_o = 2\Omega \sin \phi_o = 10^{-4}$ s^{-1} ($\phi_o \sim 45°$)

$\nu = 10^{-5}$ m^{2}s^{-1}

$g = 10$ ms^{-2}

$a = 10^7$ m

$c^2 = (\gamma RT) \sim 10^5$ m^{2}s^{-2} (c = speed of sound; $\gamma = c_p/c_v \sim 1$)

So -- $Re = UL/\nu = 10^{12}$ (Molecular friction utterly negligible. In this section we shall neglect all friction but cf. lecture 1)

$Ro = U/fL = 10^{-1} = O(\varepsilon)$ (ratio of inertial to Coriolis acceleration -- Rossby number)

$\delta = D/L = 10^{-2} = O(\varepsilon^2)$ (aspect ratio)

$\tilde{\delta} = L/a = 10^{-1} = O(\varepsilon)$

$Ma^2 = (U/c)^2 = 10^{-3} = O(\varepsilon^3)$ (Mach number squared)

$F = f_o^2 L^2/gD = 10^{-1} = O(\varepsilon)$ (Froude number)

These dimensionless numbers suggest we scale the equations of motion according to:

(a) slow motions compared with the speed of sound

(b) thin atmosphere

(c) slow motions compared with planetary rotation and small scale compared with planetary scale.

Remember that the total pressure (or density or tempe ature) at a point can be thought of as the sum of the hydrostatic pressure $p_o(z)$ (the pressure field of an atmosphere with no motions) together with a hor zontally varying pressure which is associated with the atmospheric motions

$$p = p_o(z) + p'(x,y,z,t)$$

$$\rho = \rho_o(z) + \rho'(x,y,z,t)$$

$$T = T_o(z) + T'(x,y,z,t)$$

(5) Since the Rossby number is small, we shall assume that the horizontal pressure gradient is the same order as the Coriolis acceleration (rather than the same order as the acceleration $d\underline{u}_h/dt$)

i.e., $p'/L\rho \sim f_o U$

or $$\frac{p'}{p} = \frac{f_o UL}{RT} = Ro^{-1}\, Ma^2 = O(\varepsilon^2)$$

(6) We also assume that

$$\rho'/\rho_o = Ro^{-1}\, Ma^2$$

We shall show that this assumption is consistent with assuming that motions are approximately adiabatic.

We can now scale the continuity equaiton

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}\nabla_h \cdot (\rho \underline{u}_h) + \frac{1}{\rho}\frac{\partial}{\partial z}(\rho w) = 0$$

where $\underline{u}_h$ is the horizontal velocity and ∇_h is the horizontal derivative. To lowest order

$$\frac{1}{\rho_o}\frac{\partial \rho'}{\partial t} + \nabla_h \cdot \underline{u}_h + \frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o w) = 0$$

If the horizontal divergence scales as U/L, then the ratio of first to second terms equals

$$\frac{Ro^{-1} Ma^2\ U/L}{U/L} = O(\varepsilon^2)$$

and to $O(\varepsilon^2)$ the continuity equation can be written as

$$\nabla_h \cdot \underline{u}_h + \frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o w) = 0$$

i.e., $W/D \lesssim U/L$ where W is an estimate of w.

n.b. If the horizontal velocities tend to be non-divergent (which, as we shall see, they do) the U/L is an overestimate of $\nabla_h \cdot \underline{u}_h$ -- hence the < in the inequality for vertical velocity. We shall find that the horizont momentum equations demand that the horizontal convergence scales as Ro U/ so that the ratio of first to second terms in the full continuity equatio is $O(\varepsilon)$, rather than C

Now consider the vertical momentum equation

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = \frac{1}{(\rho_o+\rho')}\frac{\partial(p_o+p')}{\partial z} - g + 2\Omega u \cos\phi$$

Using the fact that the basic state is hydrostatic and expanding $1/(\rho_o+\rho')$ to $O(\varepsilon^2)$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = \frac{1}{\rho_o}\frac{\partial p'}{\partial z} - \frac{g\rho'}{\rho_o} + 2\Omega u \cos\phi$$

Scaling we have

$$\frac{WU}{L} \qquad \frac{U^2}{a} \qquad \frac{f_o UL}{D} \qquad gRo^{-1}Ma^2 \qquad f_o U$$

i.e.,

$$\frac{U^2}{L}\left[\ \le\delta \qquad \tilde{\delta} \qquad Ro^{-1}\delta^{-1} \qquad \underbrace{Ro^{-3}Ma^2F^{-1}\delta^{-1}}_{O(Ro^{-1})} \qquad Ro^{-1}\ \right]$$

$$O(\varepsilon^2) \qquad O(\varepsilon) \qquad O(\varepsilon^{-3}) \qquad O(\varepsilon^{-3}) \qquad O(\varepsilon^{-1})$$

Hence the perturbations are to $O(\delta)$, hydrostatic, i.e., with small aspect ratio, or in a "thin atmosphere" motions are hydrostatic.

Now let us study the horizontal momentum equations

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi$$

$$\frac{U^2}{L}\left[\ 1 \qquad \tilde{\delta} \qquad \lesssim \delta\tilde{\delta} \qquad Ro^{-1} \qquad Ro^{-1} \qquad \lesssim Ro^{-1}\delta\ \right]$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi$$

$$\frac{U^2}{L}\left[\ 1 \qquad \tilde{\delta} \qquad \lesssim \delta\tilde{\delta} \qquad Ro^{-1} \qquad Ro^{-1}\ \right]$$

$$1 \qquad O(\varepsilon) \qquad O(\varepsilon^3) \qquad O(\varepsilon^{-1}) \qquad O(\varepsilon^{-1})$$

To the lowest order we have the geostrophic approximation (i.e., neglecting $O(\varepsilon)$ corrections)

$$\left.\begin{aligned} 2\Omega v\sin\phi &= +\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (\equiv 2\Omega v_g \sin\phi) \\ 2\Omega u\sin\phi &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (\equiv 2\Omega u_g \sin\phi) \end{aligned}\right\} + O(\varepsilon)$$

$$\text{i.e., } u = u_g + O(\varepsilon), \quad v = v_g + O(\varepsilon)$$

Note from this that since $\tilde{\delta} = O(Ro)$,

$$\nabla_h \cdot \underline{u}_h = Ro \cdot U/L$$

so that our estimation of w can be tightened to

$$W/D = Ro\ U/L\ .$$

The geostrophic approximation is a useful diagnostic relationship but it cannot determine the evolution of the motion, i.e., it is not prognostic. To have a prognostic system we must return $O(\varepsilon)$ corrections. The next order of approximation can be referred to as the 'thin atmosphere - synoptic scale' approximation. Recognizing the improved scaling of vertical velocity demanded by the geostrophic approximation the thin atmosphere - synoptic scale approximation gives

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi \end{aligned}\right\} + O(\varepsilon^2)$$

For completeness we note that if $O(\varepsilon^2)$ corrections are retained then we have the pure "thin atmosphere" equations

$$\left.\begin{aligned} \frac{du}{dt} - \frac{uv\tan\phi}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi \\ \frac{dv}{dt} + \frac{u^2\tan\phi}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi \end{aligned}\right\} + O(\varepsilon^3)$$

In vector notation the thin atmosphere synoptic scale equations may be written in the form

$$\frac{d\underline{v}}{dt} + f\,\underline{k} \times \underline{v} = -\frac{1}{\rho}\nabla p$$

where $\underline{v} = u\hat{\underline{i}} + v\hat{\underline{j}}$ is the horizontal velocity $\qquad f = 2\Omega\sin\phi$

Entropy and Potential Temperature

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = \dot{Q} \quad \text{(drop the prime on } \dot{Q}' \text{ for convenience)}$$

$$\text{or} \quad c_p \frac{d\ell nT}{dt} - R \frac{d\ell np}{dt} = \frac{\dot{Q}}{T}$$

Let $S(T,p) = c_p \ell nT - R\ell np + \text{const} = \text{entropy/unit mass}$. Then $dS/dt = \dot{Q}/T$. Writing $S = c_p \ell n\theta + \text{const}$, then $\theta = T(p_s/p)^{R/c_p}$ is the potential temperature. So $d\ell n\theta/dt = \dot{Q}/Tc_p$. p_s is a (constant) reference pressure.

An atmosphere is statically stable, or stably stratified if $\partial\theta/\partial z > 0$. For a neutrally stable atmosphere $\partial\theta/\partial z = 0$, i.e., $dT/dz = -g/c_p \sim 10 \text{ K km}^{-1}$. [Consider a parcel of air and displace it a small distance δz without disturbing its hydrostatic environment. The vertical acceleration of the parcel is

$$\frac{dw}{dt} = \frac{d^2}{dt^2}(\delta z) = -g - \frac{1}{\rho_{par}} \frac{\partial p_{par}}{\partial z} = g \left(\frac{\rho_{env} - \rho_{par}}{\rho_{par}}\right) = g \left(\frac{\theta_{par} - \theta_{env}}{\theta_{env}}\right). \quad (p_{env} = p_{pa}$$

For an adiabatic displacement potential temperature is conserved, i.e., $\theta_{par} = \theta_s = \text{const}$. Putting $\theta_{env}(\delta z) = \theta_s + (d\theta_{env}/dz)\delta z$ we have $(d^2/dt^2)(\delta z) = -$ where $N = \sqrt{(g/\theta_{env})(d\theta_{env}/dz)}$ is the Brunt-Vaisalla frequency $\simeq 10^{-2} \text{ s}^{-1}$ in the troposphere.]

We can now scale the thermodynamic equation. Remember we scaled p'/p_o as $Ro^{-1}Ma^2$ to balance pressure gradients against Coriolis forces. We also scaled $p'/p_o \sim \rho'/\rho_o$. From the ideal gas law we must have that $T'/T_o \lesssim Ro^{-1}Ma^2 = O(\varepsilon^2)$. If we put $\theta'/\theta_o = O(\varepsilon^2)$ so that $\ell n\theta = \ell n\theta_o + (\theta'/\theta_o) + O(\varepsilon^4)$ then to $O(\varepsilon^2)$ the thermodynamic equation becomes

$$\frac{d_h}{dt}(\theta'/\theta_o) + w \frac{d}{dz}(\ell n\, \theta_o) = \dot{Q}/c_p T \qquad \text{(A)}$$

If we require that the equation should balance with purely adiabatic terms then $(U/L)[O(\varepsilon^2) + w(N^2/g)(L/U)] = 0$. Since $W = URo\delta = UO(\varepsilon^3)$, then a balance obtains if $N^2L/g = O(\varepsilon^{-1})$, ($\Rightarrow N^2 \sim 10^{-4} \text{ s}^{-2} \sim$ observed value) or, alternatively, if $d\theta_o/dz = O(\varepsilon)(\theta_o/D)$. (i.e., the basic state temperature has a vertical scale of RoD). In terms of the Burger number $B = N^2D^2/f_o^2 L^2$, $B = O(1)$.

Equation (A) can be used to estimate vertical velocity from estimates of diabatic heating, local horizontal velocity, the horizontal temperature field, and the local rate of change of temperature-- better than continuity see Holton r If $\dot{Q}$ is assumed to be small, then the estimation of w from (A) is called the adiabatic method.

Isobaric Coordinates

By the chain rule

$$\left.\frac{\partial F}{\partial x}\right|_s = \left.\frac{\partial F}{\partial x}\right|_z + \left.\frac{\partial F}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_s$$

for functions $F = F(x,y,z,t)$ $s = s(x,y,z,t)$ with s a monotonic function of z.

If $F = s = p$

$$\left.\frac{\partial p}{\partial x}\right|_z = -\left.\frac{\partial p}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_p$$

Divide by ρ and use the hydrostatic equation (ok for a thin atmosphere)

$$\frac{1}{\rho}\left.\frac{\partial p}{\partial x}\right|_z = g\left.\frac{\partial z}{\partial x}\right|_p = \left.\frac{\partial \Phi}{\partial x}\right|_p$$

or $\frac{1}{\rho}\nabla_z p = \nabla_p \Phi$

and the horizontal (i.e., thin atmosphere, synoptic scale) momentum equation may be written as

$$\frac{d\underline{v}}{dt} + f\,\hat{\underline{k}} \times \underline{v} = -\nabla_p \Phi$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

and $\underline{v} = (u,v) = \text{horizontal velocity} = \underline{u}_h$

The continuity equation can be derived by noting that the mass δM of a small element of fluid can be written as

$$\delta M = \rho\, dxdydz = -g\, dxdydp$$

i.e., $M = -g \int_V dxdydp$

For a comoving volume

$$\frac{dM}{dt} = -g\int_V \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}\right) dxdydp$$

Since mass is conserved

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \qquad \text{density does not appear}$$

n.b. $\frac{d}{dt}(\delta x) \simeq u(x+\delta x) - u(\delta x) \simeq \frac{u(x+\delta x) - u(\delta x)}{\delta x}\delta x \simeq \frac{\partial u}{\partial x}\delta x$ etc.

Thermodynamic equation

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$$

i.e., $$c_p \frac{dT}{dt} - \alpha\omega = \dot{Q}$$

or

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\omega = \dot{Q}/c_p$$

where

$$S_p = \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}$$

which is the static stability for the isobaric system. Unlike $\partial\theta/\partial z$, $\partial\theta/\partial p$ increases exponentially with height.

The relationship between ω and w is given by

$$\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \underline{v}\cdot\nabla p + w\frac{\partial p}{\partial z}$$

Now

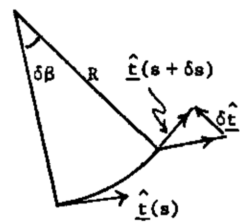
$$\underline{v} = \underline{v}_g + O(Ro) \quad \text{and} \quad \underline{v}_g\cdot\nabla p = \frac{1}{\rho f}(\hat{k}\times\nabla p)\cdot\nabla p = 0$$

We can write $\frac{\partial p}{\partial t} = \frac{p'U}{L}$ $\quad w\frac{\partial p}{\partial z} = -g\rho w \quad \therefore \quad \omega = -g\rho w\left(1 + \frac{p'U}{Lgw\rho_o}\right) = -g\rho w(1+F)$

(remember $F = f_o^2L^2/gD$ = Froude number $= O(\varepsilon)$). To lowest order $\omega = -g\rho w$

Wind in Natural Coordinates

Basis of unit vectors $(\hat{\underline{t}}, \hat{\underline{n}}, \hat{\underline{k}})$ such that $\underline{v} = v\hat{\underline{t}}$ where $\underline{v}$ is the horizontal velocity, s the path length along trajectories of $\underline{v}$, $\hat{\underline{n}}$ is positive to the left of the flow direction, and $\hat{\underline{k}}$ is as before.

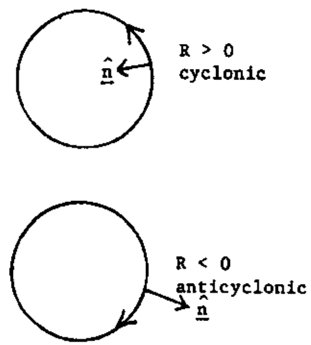


$$\delta\hat{\underline{t}} = \hat{\underline{n}}\,\delta\beta \Rightarrow \frac{d\hat{\underline{t}}}{ds} = \frac{\hat{\underline{n}}}{R}$$

R is the radius of curvature of the flow.

R is taken to be positive when the center of curvature is in the positive $\hat{\underline{n}}$ direction.

$$\frac{d\underline{v}}{dt} = \frac{dv}{dt}\hat{\underline{t}} + v\frac{d\hat{\underline{t}}}{ds}\cdot\frac{ds}{dt} = \frac{dv}{dt}\hat{\underline{t}} + \frac{v^2}{R}\hat{\underline{n}}$$



in N. Hemisphere

where $v = |\underline{v}| \geq 0$. Also, $f\hat{\underline{k}}\times\underline{v} = fv\hat{\underline{n}}$ so

$$\left.\begin{aligned} \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial s} \\ \frac{v^2}{R} + fv &= -\frac{1}{\rho}\frac{\partial p}{\partial n} \end{aligned}\right\} \begin{aligned}&\text{thin atmosphere}\\ &\text{equations in natural coordinates}\end{aligned}$$

Special cases where the instantaneous fluid velocity is parallel to the isobars
(so that $dv/dt = 0$)

(a) Cyclostrophic balance $v^2/R = -\frac{1}{\rho}\frac{\partial p}{\partial n}$ (fv is small here -- applies to tornadoes, i.e., $Ro = O(\varepsilon^{-1})$, $p'/p_o = Ma^2$

(b) Inertial $(\partial p/\partial n = 0)$ $v = -Rf$ (Can only be anticylonic. Inertial oscillations have been observed in the ocean. see Holton p. 60)

(c) Geostrophic $(|R| \to \infty)$ $fv = -\frac{1}{\rho}\frac{\partial p}{\partial n}$ (Here there is no curvature in the isobars.

(d) Gradient wind

$$v = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - \frac{R}{\rho}\frac{\partial p}{\partial n}\right)^{1/2}$$

For a regular low $R > 0$, $\partial p/\partial n < 0$ (+ root)

For a regular high $R < 0$, $\partial p/\partial n < 0$ and $v < -fR/2$ (− root)

(d) Gradient wind (continued)

There are two other 'anomalous' solutions

For an anomalous low $R < 0$, $\partial p/\partial n > 0$ (+ root)

For an anomalous high $R < 0$, $\partial p/\partial n < 0$ and $v > -fR/2$ (+ root)

These anomalous highs and lows are not observed. As the wind field accelerates in response to an imposed pressure distribution a regular high/low will result.

Note that for a high $f^2R^2/4 - R/\rho(\partial p/\partial n)$ is positive only if $|\partial p/\partial n| < \rho|R|f$

i.e., as R decreases $\partial p/\partial n$ decreases. It is for this reason that pressure gradients are weak and winds are slack near the center of a high.

The gradient wind can also be written as

$$\frac{v^2}{R} + fv - fv_g = 0 \qquad v\left(1 + \frac{v}{fR}\right) = v_g$$

To O(Ro) $\quad v/fR = v_g/fR \qquad v = v_g/(1 + v_g/fR) = v_g(1 - v_g/fR + O(\varepsilon^2))$

$\Rightarrow v \le v_g$ in cyclones $\quad v \ge v_g$ in anticyclones.

n.b. The curvature referred to is the curvature of the trajectories of the fluid i.e., the paths followed by parcels of fluid over a finite period of time. In practice R is often estimated by using the radius of curvature of the isobars. However, the isobars are actually streamlines of the gradient wind (i.e., lines which are everywhere parallel to the instantaneous wind velocity). If β is the angular direction of the wind and R_t and R_s are the radii of curvature of the trajectories and streamlines respectively then

$$\frac{d\beta}{ds} = \frac{1}{R_t} \qquad \frac{\partial\beta}{\partial s} = \frac{1}{R_s} \qquad \frac{d\beta}{dt} = \frac{d\beta}{ds}\cdot\frac{ds}{dt} = \frac{v}{R_t} \quad \text{and} \quad \frac{d\beta}{dt} = \frac{\partial\beta}{\partial t} + v\frac{\partial\beta}{\partial s} = \frac{\partial\beta}{\partial t} + \frac{v}{R_s}$$

$$\Rightarrow \frac{\partial\beta}{\partial t} = v\left(\frac{1}{R_t} - \frac{1}{R_s}\right) \qquad (= 0 \text{ if and only if } R_t = R_s)$$

If a circular pressure pattern is advected eastward with velocity $\underline{c}$, i.e., $\partial\beta/\partial t = -\underline{c}\cdot\nabla\beta$ and $R_s = R_t\,[1 - (|\underline{c}|\cos\gamma/v)]$ where γ is the angle between $\underline{c}$ and the streamlines. See Holton, p. 67 for diagrams of the difference this makes.

Thermal Wind

In isobaric coordinates

$$v_g = \frac{1}{f}\frac{\partial\Phi}{\partial x} \qquad u_g = -\frac{1}{f}\frac{\partial\Phi}{\partial y}$$

The hydrostatic equation is

$$\frac{\partial\Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p}$$

Hence

$$p\frac{\partial v_g}{\partial p} = \frac{p}{f}\frac{\partial}{\partial x}(\partial\Phi/\partial p) = -\frac{R}{f}(\partial T/\partial x)_p$$

$$p\frac{\partial u_g}{\partial p} = -\frac{p}{f}\frac{\partial}{\partial y}(\partial\Phi/\partial p) = \frac{R}{f}(\partial T/\partial y)_p$$

Vectorially we have the thermal wind equation

$$\frac{\partial \underline{v}_g}{\partial \ln p} = -\frac{R}{f}\,\hat{\underline{k}} \times (\nabla_p T)$$

For a barotropic atmosphere $\rho = \rho(p)$, i.e., $T = T(p)$ and $\partial\underline{v}_g/\partial\ln p = 0$.

For a baroclinic atmosphere $\rho = \rho(p,T)$, $\partial\underline{v}_g/\partial\ln p \ne 0$

The difference in geostrophc wind between the top and bottom of a layer of thickness $\Delta Z = g_o(\Phi_1 - \Phi_o)$ is defined to be the thermal wind, $\underline{v}_T$, for that layer. Hence

$$\underline{v}_T = g_o/f\,\hat{\underline{k}} \times \nabla(\Delta Z) = \frac{1}{f}\,\hat{k} \times \nabla(\Phi_1 - \Phi_o)$$

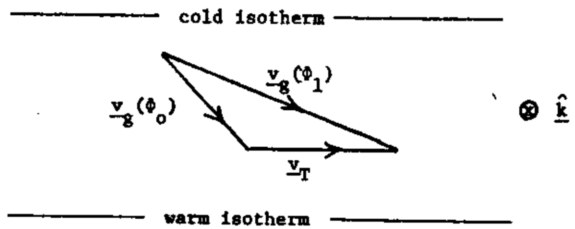
Hence the thermal wind blows parallel to the lines of constant thickness (lines of constant mean temperature).

Rule of thumb:

Wind backing with height = cold air is being advected by mean geostrophic wind

Wind veering with height = warm air is being advected by mean geostrophic wind

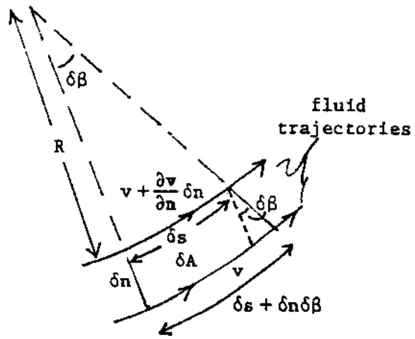
(abc: anticlockwise = backing = cold advection)



Vorticity

$\underline{\omega} = \nabla \times \underline{u}$

For solid body rotation $\underline{u} = \underline{\Omega}_o \times \underline{r} \Rightarrow \underline{\omega} = 2\underline{\Omega}_o$. Otherwise, consider the natural coordinate system $(\hat{\underline{t}}, \hat{\underline{n}}, \hat{\underline{k}})$.



$$\int_A \underline{\omega} \cdot \underline{dA} = \int_{\partial A} \underline{v} \cdot \underline{d\ell}$$

Put $\underline{dA} = \delta A \hat{\underline{k}}$, $\quad \zeta = \underline{\omega} \cdot \hat{\underline{k}}$

Then for the small area δA of the figure $\zeta \delta A = \delta c$ where

$$\delta c = v(\delta s + \delta n \delta\beta) - \left(v + \frac{\partial v}{\partial n}\delta n\right)\delta s$$

$$= v\, \delta n \delta\beta - \frac{\partial v}{\partial n}\delta n \delta s$$

$$\zeta = v\frac{\delta n \delta \beta}{\delta n \delta s} - \frac{\partial v}{\partial n}\frac{\delta n \delta s}{\delta n \delta s} = \Big(\underbrace{\frac{v}{R}}_{\text{curvature vorticity}} - \underbrace{\frac{\partial v}{\partial n}}_{\text{shear vorticit}}\Big)$$

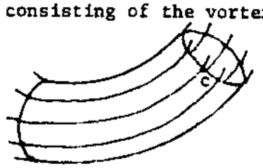
In an inertial frame

$$\underline{\omega}_a = \nabla \times (\underline{u} + \underline{\Omega} \times \underline{r}) = \underbrace{\underline{\omega}}_{\text{relative vorticity}} + \underbrace{2\underline{\Omega}}_{\text{planetary vorticity}}$$

Relative vorticity/planetary vorticity $\sim \frac{U/L}{f} = Ro$ $\therefore$ if $Ro = O(\varepsilon) \Rightarrow$ flow has vort. dominated by planetary rot.

Kinematic relationship $\nabla \cdot \underline{\omega} = \nabla \cdot \underline{\omega}_a = 0$ (immediate from definition)

Define a vortex filament as a line in the fluid which at each point is parallel to the vorticity vector. A vortex tube is formed by the surface consisting of the vortex filaments which pass through a closed curve c.



$\Rightarrow$ vorticity tubes end only on the surface which bounds the fluid or else close on themselves

Define $\Gamma_a = \int_A \underline{\omega}_a \cdot \underline{dA}$ = absolute circulation $= \int_c \underline{u}_a \cdot \underline{d\ell}$

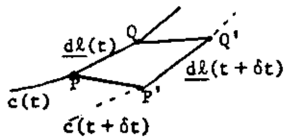
$\Gamma = \int_A \underline{\omega} \cdot \underline{dA}$ = relative circulation $= \int_c \underline{u} \cdot \underline{d\ell}$ $\qquad c = \partial A$

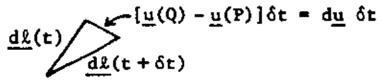
Γ, Γ_a do not depend on the choice of cross sections of the vortex tube.

$$\boxed{\Gamma_a = \Gamma + 2\Omega A_n \quad \text{where } A_n \text{ is the projected area of A normal to } \underline{\Omega} \quad (\Omega = |\underline{\Omega}|)}$$

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int_c \underline{u} \cdot \underline{d\ell} = \int_c \frac{d\underline{u}}{dt} \cdot \underline{d\ell} + \int_c \underline{u} \cdot \frac{d}{dt}(\underline{d\ell})$$

If c is a material curve





$\frac{d}{dt}(\underline{d\ell}) = \underline{du} \qquad \int_c \underline{u} \cdot \underline{du} = \int_c \frac{1}{2} d\, |\underline{u}|^2 = 0 \qquad$ (c is closed; i.e., $c = \partial A$)

$$\frac{d\Gamma}{dt} = \underbrace{-\int_c (2\underline{\Omega} \times \underline{u}) \cdot \underline{d\ell}}_{①} \underbrace{- \int \frac{\nabla p}{\rho} \cdot \underline{d\ell}}_{②} + \underbrace{\int \frac{\mathcal{F}}{\rho} \cdot \underline{d\ell}}_{③}$$

[n.b. $\int_c \nabla\phi \cdot \underline{d\ell} = \int_A (\nabla \times \nabla\phi) \cdot dA = 0$
i.e., conservative body forces cannot change Γ]

where $\mathcal{F} = \mu\nabla^2\underline{u} + \frac{\mu}{3}\nabla\ (\nabla \cdot \underline{u})$ $\qquad$ Bjerknes circulation theorem

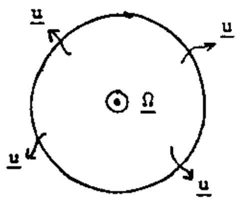
Similarly

$$\frac{d\Gamma_a}{dt} = -\int_c \frac{\nabla p}{\rho} \cdot d\ell + \int_c \frac{\mathcal{F}}{\rho} \cdot \underline{d\ell} \qquad \text{Kelvin's circulation theorem}$$

(n.b. $\mathcal{F}_a = \mathcal{F}$)

Let us study the terms in the Bjerknes form of the circulation theorem

① $\quad -\int_c (2\underline{\Omega} \times \underline{u}) \cdot \underline{d\ell}$



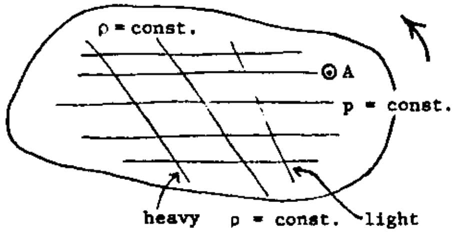
Coriolis force to the right, produces a relative circulatio in the anticyclonic sense.

Also, since $\Gamma_a = \Gamma + 2\Omega A_n$ we can write $-\oint_c (2\Omega \times \underline{u}) \cdot \underline{d\ell} = -2\Omega \frac{dA_n}{dt}$ (subtracting Kelvin's circulation theorem from Bjerknes' circulation theorem).

So if A_n increases, the relative circulation decreases, i.e., as A_n expands the flux of relative vorticity through A will decrease in direct proportion as the number of planetary vorticity filaments captured by c is increased.

② $$-\int \frac{\nabla p}{\rho} \cdot \underline{d\ell} = -\int \nabla \times \left(\frac{\nabla p}{\rho}\right) \cdot \underline{dA} = \int \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot \underline{dA}$$

If the surfaces of constant pressure and constant density do not coincide, i.e., the fluid is baroclinic, then the 'solenoidal' term will generate circulati

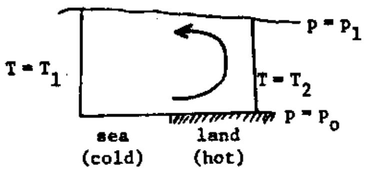


Lighter fluid will rise more rapidly than heavier fluid because they both experience the same upwar pressure gradient force. ∴ The acceleration varies as $1/\rho \Rightarrow$ cyclo circulation.

This term can also be written as

$$-\oint_c RT\, \nabla(\ell n\, p) \cdot \underline{d\ell} = -\oint_c RT\, d(\ell n\, p) \approx R(T_2 - T_1)\, \ell n(p_o/p_1)$$

for the diagram on the left.

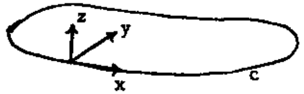


Sea breeze circulation

(n.b., does not, without inclusion of friction, give realistic estimates. cf. Holton, p. 83)

③ $$\int \frac{\mathcal{F}}{\rho} \cdot \underline{d\ell} = \nu \int (\nabla^2 \underline{u}) \cdot \underline{d\ell} = -\nu \int (\nabla \times \underline{\omega}) \cdot \underline{d\ell}$$

n.b. $\nabla \times (\nabla\psi) = 0$ and $\nabla \times (\nabla \times \underline{u}) = \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$

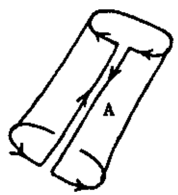


Consider a local Cartesian coordinate system and suppose $\underline{\omega} = \omega\hat{\underline{k}}$, then locally $-\nu(\nabla \times \underline{\omega}) \cdot \underline{d\ell} = -\nu \frac{\partial \omega}{\partial y} dx$, i.e., the effect of viscosity is to reduce or increase the vortex tube strength encompassed by c by a diffusion of vorticity down the vorticity gradient. e.g., consider a solid body impulsively accelerated from rest. Initially there is an infinite sheet of vorticity on the surface of the body since viscosity requires that the fluid in contact with the body has a velocity equal to that of the body. Hence the initially irrotational fluid acquires vorticity through viscous diffusion and advection of vorticity from the viscous boundary layer (cf, thermal conduction and thermal advection.

If the fluid is barotropic and frictionless, then

$$\frac{d\Gamma_a}{dt} = 0 \qquad \text{(Kelvin's theorem)}$$

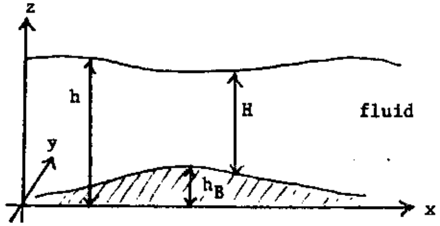
Consider a vortex tube. The circulation around any area on the surface of the tube is equal to zero. Consider the following 'blanket' A which initially lies on the surface of the vortex tube, and is defined to be a material surface.



Γ_a is initially zero. The blanket is a material surface. Γ_a is always zero by Kelvin's theorem. ∴ absolute vorticity filaments can never penetrate A. Therefore A is always a blanket, and absolute vorticity tubes move with the fluid.

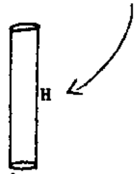
In the limit as the blanket shrinks down onto a vortex filament, we can infer that vortex filaments move with the fluid.

Consider an incompressible homogeneous inviscid fluid bounded above and below



and two dimensional fluid motion. For simplicity, suppose first of that the vortex filaments are vertically aligned. Then we know from Kelvin's theorem that a small column of fluid which is initially aligned vertically will remain aligned vertically for all time.

The material tube has conserved mass and is always vertically oriented.



i.e., $\frac{d}{dt}\,[\rho H \delta A] = 0$. But $\frac{d}{dt}\,\delta A = \nabla \cdot \underline{u}_h\, \delta A$

where $\underline{u}_h$ = horizontal velocity

$\therefore$ since ρ = const., $\frac{dH}{dt} + H\,\nabla \cdot \underline{u}_h = 0$

The above fluid satisfies the conditions for Kelvin's theorem, i.e.,

$\frac{d}{dt}\int_{\delta A} \underline{\omega}_a \cdot \underline{dA} = 0$ where we suppose that δA is a horizontal surface and

$$\underline{\omega}_a \cdot \underline{dA} = \eta dA = (\underline{\omega} + \underline{2\Omega}) \cdot \hat{\underline{k}}\, dA$$

But $\frac{d}{dt}\int_{\delta A} \eta\, dA = \int_{\delta A} \frac{d\eta}{dt}\, dA + \int_{\delta A} \eta \frac{d}{dt}\, dA = \int_{\delta A} \frac{d\eta}{dt}\, dA + \int_{\delta A} \eta\, \nabla \cdot \underline{u}_h\, dA$

$$= \int_{\delta A} \frac{d\eta}{dt}\, dA - \int_{\delta A} \frac{\eta}{H}\frac{dH}{dt}\, dA = 0$$

since δA is arbitrary $\frac{d}{dt}\left(\frac{\eta}{H}\right) = 0$

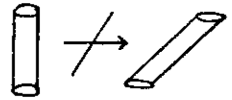
Hence the vertical component of absolute vorticity increases as the vortex tube is stretched in the vertical, i.e., as H increases. (cf. ice skater).

If H is constant then $\frac{d}{dt}\eta = 0$.

To show that in a barotropic atmosphere for quasi-geostrophic flow $\frac{d}{dt}\left(\frac{\zeta_g + f}{H}\right) = 0$

For a barotropic inviscid fluid vorticity filaments move with the fluid.

Earlier we considered inviscid barotropic flow of variable depth H. If the vorticity filaments of such a flow are vertically aligned then fluid contained within a vertical column will always be contained in a vertical column. The column cannot slant over as in the diagram.



because the vorticity filaments which were initially vertically aligned would subsequently become aligned at an angle to the vertical which, by hypothesis, does not happen. It is this vertical alignment of a column of fluid that is the crucial assumption. That this is so can be seen by relaxing the requirement that vorticity filaments are aligned vertically but demanding that the horizontal fluid velocity has no vertical shear. If the horizontal velocity has no vertical shear, then, once again a vertical column of fluid always remains vertically oriented; the column cannot be sheared out of the vertical. As the depth of fluid shrinks then the horizontal cross sectional area of the column must expand to conserve the mass of fluid within the column, i.e., for a small column of horizontal cross section δA and height H

$$\frac{d}{dt}\,(\delta A H) = 0 \Rightarrow \frac{d}{dt}\,\delta A + \frac{\delta A}{H}\frac{dH}{dt} = 0 \qquad \text{if } \rho = \text{constant.}$$

(Notice that if the column did not remain vertical, then its volume would no longer be δAH. This is why the assumption that vertical columns remain vertical columns is so crucial.) Now Kelvin's theorem applies for a barotropic inviscid fluid

$$\text{i.e.,}\quad \frac{d}{dt}\int_A \underline{\omega}_a \cdot \underline{dA} = 0$$

where A is a material surface. In particular we could choose for our material surface a horizontal cross section of our vertically oriented column. So Kelvin's theorem becomes

$$\frac{d}{dt}\,(\underline{\omega}_a \cdot \underline{\delta A}) = 0$$

But $\underline{\delta A} = \hat{\underline{k}}\,\delta A$, and $\underline{\omega}_a \cdot \hat{\underline{k}} = (\zeta + f)$, hence

$$\frac{d}{dt}\,[(\zeta + f)\delta A] = 0 \quad .$$

Notice that only the component of $\underline{\omega}_a$ normal to δA is important here.

Hence $$\frac{d}{dt}(\zeta + f) - \frac{(\zeta + f)}{H}\frac{dH}{dt} = 0 \quad \text{using the continuity equation}$$

or $$\frac{d}{dt}\left(\frac{\zeta + f}{H}\right) = 0$$

In summary, if the horizontal velocity has no shear, then the potential vorticity $(\zeta + f)/H$ is conserved. In particular, the geostrophic wind which is by definition horizontal, has no vertical shear in a barotropic atmosphere, so that

$$\frac{d}{dt}\left(\frac{\zeta_g + f}{H}\right) = 0 \quad \text{for quasi-geostrophic flow.} \qquad \text{(A)}$$

It is important to notice that if the flow is geostrophic upwind of some topographic ridge, then in order that the depth of the fluid decrease, or correspondingly the cross sectional area increase there must be some horizontal divergence of fluid. This cannot be accomplished by the geostrophic wind. Hence in order to flow over the ridge the horizontal flow must become at least partially ageostrophic, or quasi-geostrophic as it is usually called. This is why equation (A) is called a (barotropic) quasi-geostrophic potential vorticity equation: the flow <u>cannot</u> be exactly geostrophic if it is to flow over the ridge. On the other hand if the topographic feature was a somewhat localized bump, then the fluid could flow around the bump rather than over it. The fluid could then be geostrophic everywhere except in the Taylor column that would extend above the bump. If the density of the fluid varied with height then the mass of the vertically aligned column could be written as

$$\delta A \, H \, \overline{\rho}$$

where $$\overline{\rho} = \frac{1}{H}\int_0^H \rho \, dz$$

and conservation of mass gives

$$\frac{d}{dt}\delta A + \frac{\delta A}{H\overline{\rho}}\frac{dH\overline{\rho}}{dt} = 0$$

and conservation of potential vorticity gives

$$\frac{d}{dt}\left\{\frac{\zeta_g + f}{H\overline{\rho}}\right\} = 0 \quad .$$

We can summarize some of our findings in the three Helmholtz laws.

(1) Vortex lines never end in the fluid.

(2) A fluid line which at any instant of time coincides with a vortex line will coincide with a vortex line forever.

(3) On a vortex of fixed identity, the ratio of the vorticity to the product of the fluid density with the length of the line remains constant in time (i.e., if the vortex line is stretched, the vorticity increases).

<u>Vorticity Equation</u>

<u>Identity</u> $$\underline{\omega} \times \underline{u} = (\underline{u}\cdot\nabla)\underline{u} - \nabla\left(\frac{|\underline{u}|^2}{2}\right) \Rightarrow \frac{\partial \underline{u}}{\partial t} + (2\underline{\Omega} + \underline{\omega}) \times \underline{u} = -\frac{\nabla p}{\rho} + \nabla\left(\phi - \frac{|\underline{u}|^2}{2}\right) + \frac{\underline{\mathcal{F}}}{\rho}$$

take the curl of this

$$\frac{\partial \underline{\omega}}{\partial t} + \nabla \times \{\underline{\omega}_a \times \underline{u}\} = \frac{\nabla\rho \times \nabla p}{\rho^2} + \nabla \times (\underline{\mathcal{F}}/\rho)$$

<u>Identity</u>

$$\nabla \times (\underline{A} \times \underline{B}) = \underline{A}(\nabla\cdot\underline{B}) + (\underline{B}\cdot\nabla)\underline{A} - \underline{B}(\nabla\cdot\underline{A}) - (\underline{A}\cdot\nabla)\underline{B}$$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}_a\cdot\nabla\underline{u} - \underline{\omega}_a\,\nabla\cdot\underline{u} + \frac{\nabla\rho \times \nabla p}{\rho^2} + \nabla \times (\underline{\mathcal{F}}/\rho)$$

This is the vorticity equation.

n.b. for a steady inviscid homogeneous fluid

$$\underline{u} \times \underline{\omega}_a = \frac{1}{\rho}\nabla\underbrace{(\underbrace{p}_{\text{static pressure}} + \frac{1}{2}\rho\,|\underline{u}|^2 - \rho\phi)}_{\text{stagnation pressure}}$$

Crocco's Theorem: The stagnation pressure is constant along each streamline and varies between streamlines only if vorticity is present.

In order to obtain a vorticity equation for the 'thin atmosphere, synoptic scale' system we can either scale the above equation, or compute a vorticity equation for the already scaled equations of motion. The former course is carried out in Professor Leovy's notes. For simplicity, I shall follow the latter course.

Thin atmosphere - synoptic scale vertical vorticity equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (i)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (ii)$$

In fact we could scale this equation further by neglecting $w\frac{\partial u}{\partial z}$ and putting $\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho_o}\frac{\partial p'}{\partial x}$. It is, however, illuminating to retain these terms for the time being and scale them out later in the vorticity equation, after we have examined their physical significance. $\partial(ii)/\partial x - \partial(i)/\partial y$ and putting $\zeta = \partial v/\partial x - \partial u/\partial y$ we obtain

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

i.e., $$\frac{d_h}{dt}(\zeta + f) = -(\zeta + f)\nabla\cdot\underline{u}_h - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right]$$

(since $v\frac{df}{dy} = \frac{d_h f}{dt}$)

1st term on right = generation of vertical vorticity by vortex stretching

2nd term on right = generation of vertical vorticity by the tilting of horizontall oriented components of vorticity into the vertical by a nonuniform vertical motion field

3rd term on right microscopic equivalent of the solenoidal term in the circulation theorem.
$= \underline{\hat{k}}\cdot(\nabla\rho \times \nabla p)/\rho^2$.

Scale analysis of the thin atmosphere-synoptic scale vertical vorticity equation, for low Rossby number.

$$\underset{\text{I}}{\left(\frac{\partial}{\partial t} + \underline{u}_h\cdot\nabla_h\right)\zeta} + \underset{\text{II}}{w\frac{\partial \zeta}{\partial z}} + \underset{\text{III}}{v\frac{df}{dy}} = \underset{\text{IV}}{-(\zeta + f)\nabla\cdot\underline{u}_h} - \underset{\text{V}}{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)} + \underset{\text{VI}}{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}$$

$\zeta \sim U/L$ (I) $\sim U^2/L^2$

(II) $\sim w\frac{U}{LD} \sim \mathrm{Ro}\,\delta\frac{U^2}{L^2}\delta^{-1} \sim \mathrm{Ro}\frac{U^2}{L^2}$

Define a coordinate $y = a(\phi - \phi_o)$ then $f = f_o + \beta_o y + \ldots.$ where

$$\beta_o = \left.\frac{df}{dy}\right|_{\phi=\phi_o} = \frac{2\Omega}{a}\cos\phi_o \qquad \beta_o y/f_o \lesssim f_o L/af_o = L/a = \tilde{\delta} = O(\varepsilon)$$

$f = f_o + \beta_o y + O(\varepsilon^2)$ - β-plane approximation

(III) $\sim U\beta_o \sim Uf_o/a \sim \frac{U^2}{L^2}\frac{f_o L}{U}\cdot\frac{L}{a} \sim \frac{U^2}{L^2}\mathrm{Ro}^{-1}\tilde{\delta}$

(IV) $(\zeta + f)\nabla\cdot u_h = (1 + \mathrm{Ro}^{-1})\frac{U^2}{L^2}\mathrm{Ro}$

(V) $\frac{W}{L}\frac{U}{D} = \mathrm{Ro}\,\delta\frac{U}{L}\cdot\frac{U}{D} = \mathrm{Ro}\frac{U^2}{L^2}$

(VI) $\frac{1}{L^2}\frac{\delta\rho}{\rho}\cdot\frac{\delta p}{\rho} = \frac{RT}{L^2}\frac{\delta\rho}{\rho}\cdot\frac{\delta p}{p} = \frac{U^2}{L^2}\mathrm{Ma}^{-2}\cdot[\mathrm{Ro}^{-1}\mathrm{Ma}^2]^2 = \frac{U^2}{L^2}\mathrm{Ro}^{-2}\mathrm{Ma}^2$

$$\text{i.e.,}\quad \frac{U^2}{L^2}\begin{array}{cccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI} \\ \left[1\right. & \mathrm{Ro} & \mathrm{Ro}^{-1}\tilde{\delta} & 1 & \mathrm{Ro} & \left.\mathrm{Ro}^{-2}\mathrm{Ma}^2\right] \\ 1 & \varepsilon & 1 & 1 & \varepsilon & \varepsilon \end{array}$$

Balance for small Rossby number is

$$\left(\frac{\partial}{\partial t} + \underline{u}_g\right)\zeta_g + \beta_o v_g = -f_o\nabla\cdot\underline{u}_h = \frac{f_o}{\rho_o}\frac{\partial}{\partial z}(\rho_o w)$$

where $\underline{u}_g = (u_g, v_g)$; $\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$ since $\underline{u} = \underline{u}_g + O(\varepsilon)$ and $\zeta = \zeta_g + O(\varepsilon)$

This is the quasi-geostrophic vorticity equation. It contains time derivatives of geostrophic velocities, which are forced by the $\boxed{\text{ageostrophic}}$ horizontal divergence. This equation is central to much of dynamical meteorology.

Taylor-Proudman Theorem

This is a result for small Ro which can be derived from the vorticity equation without the prior restriction of the 'thin atmosphere' approximation (i.e., would apply to rapidly rotating annuli).

Suppose Ro << 1 so that $\omega_a \approx 2\Omega$ and consider a steady inviscid incompressible barotropic flow so the vorticity equation

$$\frac{d\underline{\omega}_a}{dt} = \underline{\omega}_a \cdot \nabla \underline{u} - \underline{\omega}_a \, \nabla \cdot \underline{u} + \frac{\nabla\rho \times \nabla p}{\rho^2} + \nabla \times (\underline{\mathcal{F}}/\rho)$$

becomes

$$(2\underline{\Omega} \cdot \nabla)\underline{u} = 0 \qquad \text{(A)}$$

i.e., $\underline{u}$ is independent of the coordinate direction which is parallel to $\underline{\Omega}$. For example, if a body, such as a sphere or cylinder is towed in a homogeneous fluid on a path perpendicular to the rotation axis, fluid must stream around the object as it passes through the fluid. If (A) is correct the motion is strictly two dimensional. Fluid above and below the body must imitate the fluid parted by the body and allow a 'phantom body' consisting of the fluid contained in the so-called "Taylor column' formed by the projection of the body along the rotation axis, to pass through the fluid as if it too were solid.

Ertel's Potential Vorticity

Let us return to the full vorticity equation and derive an important result: the (Ertel) potential vorticity equation. Consider an inviscid fluid

$$\frac{d\underline{\omega}_a}{dt} = \frac{d\underline{\omega}}{dt} = \underline{\omega}_a \cdot \nabla\underline{u} - \underline{\omega}_a \, \nabla \cdot \underline{u} + \frac{\nabla\rho \times \nabla p}{\rho^2} \qquad \text{i.e., since } \nabla \cdot \underline{u} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{d}{dt}\left(\frac{\underline{\omega}_a}{\rho}\right) = \left(\frac{\underline{\omega}_a}{\rho} \cdot \nabla\right)\underline{u} + \frac{\nabla\rho \times \nabla p}{\rho^3} \qquad (1)$$

Consider some scalar fluid property Λ which satisfies $d\Lambda/dt = \Psi$

Now a straightforward calculation gives

$$\frac{\underline{\omega}_a}{\rho} \cdot \frac{d}{dt} \nabla\Lambda = \left(\frac{\underline{\omega}_a}{\rho} \cdot \nabla\right)\frac{d\Lambda}{dt} - \left\{\left(\frac{\underline{\omega}_a}{\rho} \cdot \nabla\right)\underline{u}\right\} \cdot \nabla\Lambda \qquad (2)$$

(you may find it easiest to verify this using indices). The scalar product of (1) with $\nabla\Lambda$ gives

$$\nabla\Lambda \cdot \frac{d}{dt}\left(\frac{\underline{\omega}_a}{\rho}\right) = \left\{\left(\frac{\underline{\omega}_a}{\rho} \cdot \nabla\right)\underline{u}\right\} \cdot \nabla\Lambda + \nabla\Lambda \cdot \left\{\frac{\nabla\rho \times \nabla p}{\rho^3}\right\}$$

Using (2) for 1st term on the RHS

$$\nabla\Lambda \cdot \frac{d}{dt}\left(\frac{\underline{\omega}_a}{\rho}\right) = \left(\frac{\underline{\omega}_a}{\rho} \cdot \nabla\right)\frac{d\Lambda}{dt} - \frac{\underline{\omega}_a}{\rho} \cdot \frac{d}{dt}\nabla\Lambda + \nabla\Lambda \cdot \left\{\frac{\nabla\rho \times \nabla p}{\rho^3}\right\}$$

If i) Λ is conserved ($\Psi = 0$)

ii) No frictional forces

iii) Fluid barotropic <u>or</u> $\Lambda = \Lambda(\rho, p)$

then

$$\Pi = \frac{(\underline{\omega} + 2\underline{\Omega})}{\rho} \cdot \nabla\Lambda$$

is conserved, i.e., $d\Pi/dt = 0$

The quantity Π is called the Ertel potential vorticity. There is a close relationship between Ertel's potential vorticity theorem and Kelvin's theorem. Remember that

$$\frac{d\Gamma_a}{dt} = \frac{d}{dt}\int_A \underline{\omega}_a \cdot \underline{dA} = \int_A \left(\frac{\nabla\rho \times \nabla p}{\rho^2}\right) \cdot \underline{dA}$$

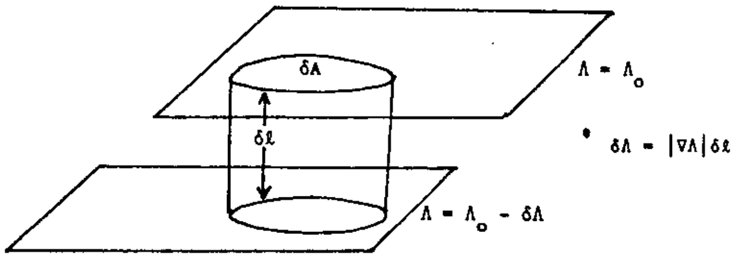
where A is a material surface. If Λ is conserved then Λ = const. is a material surface. Hence with A lying in $\Lambda = \Lambda_o$ say

$$(\nabla\rho \times \nabla p) \cdot \underline{dA} = (\nabla\rho \times \nabla p) \cdot \nabla\Lambda = 0$$

$$(\text{since } \nabla\Lambda = (\partial\lambda/\partial p)\nabla p + (\partial\lambda/\partial\rho)\nabla\rho \qquad \text{i.e., } \frac{d}{dt}\int_A \underline{\omega}_a \cdot \underline{dA} = 0$$

Now consider a small area δA then

$$\frac{d}{dt}(\underline{\omega}_a \cdot \underline{n} \, \delta A) = 0$$



The cylinder is an element of mass bounded by two Λ-surfaces, δA is an area defined by fluid particles $\therefore \delta m = \rho\delta A\delta\ell$ is conserved. i.e., $\rho\delta A\delta\Lambda/|\nabla\Lambda|$ is conserved. Hence

$$\frac{d}{dt}\left[\frac{\underline{\omega}_a \cdot \underline{n}}{\rho} |\nabla\Lambda| \frac{\delta m}{\delta\Lambda}\right] = 0$$

and since δm and $\delta\Lambda$ are constants following the motion, and since $\nabla\Lambda = \underline{n}\,|\nabla\Lambda|$ we have

$$\frac{d}{dt}\left(\underline{\omega}_a \cdot \frac{\nabla\Lambda}{\rho}\right) = 0$$

i.e., we can derive Ertel's potential vorticity theorem in terms of Kelvin's theorem for a special $(\Lambda = \Lambda_o)$ contour.

Examples of potential vorticity

(i) Two dimensional horizontal flow $w = dz/dt = 0$, i.e., put $\Lambda = z$ and Ertel becomes

$$\frac{d}{dt}(\zeta + f) = 0 \qquad \text{(c.f., Kelvin)}$$

(ii) Incompressible homogeneous two dimensional flow (cf., p. 5 of vorticity not

$$\frac{d}{dt}\left(\frac{z - h_B}{H}\right) = 0$$

This states that during the stretching or contraction of each column of fluid, the relative position of a fluid element in the column is unchanged. $\therefore$ put $\Lambda = (z - h_B)/H$ and Ertels pv becomes

$$\frac{d}{dt}\left(\frac{\zeta + f}{H}\right) = 0 \qquad \text{(c.f., Kelvin)}$$

(iii) Adiabatic flow $d\theta/dt = 0$, i.e., put $\Lambda = \theta$

$$\frac{d}{dt}\left(\frac{\underline{\omega}_a}{\rho} \cdot \nabla\theta\right) = 0$$

(iv) Quasi-geostrophic adiabatic flow. i.e., scale (iii) for lowest non-trivial order in Ro.

First of all remember that

$$\frac{d}{dt} = \underset{\sim \frac{U}{L}}{\frac{d_h}{dt}} + \underset{O(\epsilon)\frac{U}{L}}{w\frac{\partial}{\partial z}}$$

where

$$\frac{d_h}{dt} = \left(\frac{\partial}{\partial t} + \underline{v}_g \cdot \nabla\right) + O(\epsilon)$$

Second

$$\omega^z (= \zeta) = \partial v/\partial x - \partial u/\partial y \sim U/L$$
$$\omega^x = \partial w/\partial y - \partial v/\partial z \sim U/D$$
$$\omega^y = \partial u/\partial z - \partial w/\partial x \sim U/D$$

But to lowest order

$$\omega^z\, \partial\theta/\partial z \sim \omega^z\, d\theta_o/dz \sim O(\epsilon)\,\frac{U}{L}\frac{\theta_o}{D}$$

$$\omega^x\, \partial\theta/\partial x \sim \omega^x\, \partial\theta'/\partial x \sim \frac{U}{D}\frac{\theta'}{L} \sim \frac{U}{D}\frac{\theta_o}{L}\left(\frac{\theta'}{\theta_o}\right) \sim O(\epsilon^2)\,\frac{U}{L}\frac{\theta_o}{D}$$

Similarly

$$\omega^y\, \partial\theta/\partial y \sim O(\epsilon^2)\,\frac{U}{L}\frac{\theta_o}{D} \sim O(\epsilon^3)\, f_o\theta_o/D$$

Hence, to lowest order we can neglect the horizontal vorticity components in calculating potential vorticity: even though ω^x can be two order larger than ω^z (in a baroclinic atmosphere), $\partial\theta/\partial x$ is three orders smaller than $\partial\theta/\partial z$. It was for this reason we wrote

$$\Pi \sim \frac{(\zeta + f)\partial\theta/\partial z}{\rho_o}$$

when considering the qualitative effect of flow over a mountain.

so that

$$\Pi = \frac{1}{\rho_o} \left\{ (\omega^z + f_o + \beta y) \frac{d\theta_o}{dz} + f_o \frac{\partial \theta'}{\partial z} \right\} + O(\varepsilon^3)$$

$$\left[O(\varepsilon) f_o \quad f_o \quad O(\varepsilon) f_o \right] \quad O(\varepsilon) \frac{\theta_o}{D} \quad f_o O(\varepsilon^2) \frac{\theta_o}{D}$$

(since $\theta'/\theta_o \sim O(\varepsilon^2)$) and Ertel's pv equation is

$$\frac{d_h}{dt} \left[\frac{1}{\rho_o} (\zeta_g + \beta y) \frac{d\theta_o}{dz} + \frac{f_o}{\rho_o} \frac{\partial \theta'}{\partial z} \right] + w \frac{\partial}{\partial z} \left[\frac{f_o}{\rho_o} \frac{d\theta_o}{dz} \right] = 0 \qquad \text{(A)}$$

where ζ has been replaced by its geostrophic value ζ_g.

[n.b. (i) $\frac{d_h}{dt} \left(\frac{f_o}{\rho_o} \frac{d\theta_o}{dz} \right) = 0$, (ii) $\frac{d\theta_o}{dz} \sim O(\varepsilon) \frac{\theta_o}{D}$, (iii) the term

$$-\frac{f_o}{\rho_o} \frac{d\theta_o}{dz} \frac{\rho'}{\rho_o} \sim O(\varepsilon^3) \frac{f_o \theta_o}{\rho_o D}$$ and $\therefore$ does not appear to $O(\varepsilon^2)$.]

If we now use the scaled thermodynamic equation

$$\frac{d_h}{dt} \theta' + w \frac{d\theta_o}{dz} = 0$$

to substitute for θ' we have (using the commutivity of $\frac{d_h}{dt}$ and $\frac{\partial}{\partial z}$; see appendix)

$$\frac{d_h}{dt} \left[\frac{1}{\rho_o} (\zeta_g + \beta y) \frac{d\theta_o}{dz} \right] - \frac{f_o}{\rho_o} \frac{\partial}{\partial z} \left(w \frac{d\theta_o}{dz} \right) + w \frac{\partial}{\partial z} \left(\frac{f_o}{\rho_o} \frac{d\theta_o}{dz} \right) = 0$$

i.e., $$\frac{d_h}{dt} \zeta_g + \beta v_g - \frac{f_o}{\rho_o} \frac{\partial}{\partial z} (\rho_o w) = 0$$

which is identical to the quasi-geostrophic vorticity equation obtained by scaling the full vorticity equation. Alternatively, if we substitute for w in (A) from the thermodynamic equation we have

$$\frac{d_h}{dt} \left[\frac{1}{\rho_o} (\zeta_g + \beta y) \frac{d\theta_o}{dz} \right] + \frac{d_h}{dt} \left[\frac{f_o}{\rho_o} \frac{\partial \theta'}{\partial z} \right] - \frac{d_h}{dt} \left[\frac{f_o \theta'}{d\theta_o/dz} \frac{\partial}{\partial z} \left(\frac{d\theta_o/dz}{\rho_o} \right) \right] = 0$$

multiplying by ρ_o and rearranging

$$\frac{d_h}{dt} \left[(\zeta_g + \beta y) \frac{d\theta_o}{dz} \right] + \frac{d_h}{dt} \left[f_o \left(\frac{\partial \theta'}{\partial z} + \frac{d\theta_o/dz}{\rho_o} \frac{\partial}{\partial z} \left\{ \frac{\rho_o}{d\theta_o/dz} \right\} \theta' \right) \right] = 0$$

i.e., $$\frac{d_h}{dt} \left[(\zeta_g + \beta y) \frac{d\theta_o}{dz} + \frac{f_o}{\rho_o} \frac{d\theta_o}{dz} \frac{\partial}{\partial z} \left(\frac{\rho_o \theta'}{d\theta_o/dz} \right) \right] = 0$$

Finally, dividing by $d\theta_o/dz$ we have the conservation of quasi-geostrophic potential vorticity

$$\left(\frac{\partial}{\partial t} + \underline{v}_g \cdot \nabla \right) q = 0$$

where

$$q = (\zeta_g + \beta y) + \frac{f_o}{\rho_o} \frac{\partial}{\partial z} \left(\rho_o \frac{\theta'}{d\theta_o/dz} \right)$$

This equation ranks as one of the most important in the dynamical theory of mid-latitude synoptic scale motions.

Appendix

When we substituted for θ' in the Ertel potential vorticity equation from the scaled thermodyanmic equation (page 12) we had to use the fact that

$$\frac{\partial}{\partial z}\left(\frac{d_h\theta'}{dt}\right) = \frac{d_h}{dt}\left(\frac{\partial\theta'}{\partial z}\right)$$

Writing

$$\frac{d_h\theta'}{dt} = \frac{\partial\theta'}{\partial t} + \underline{v}_g\cdot\nabla_h\theta'$$

we see that this implies that

$$\frac{\partial v_g}{\partial z}\cdot\nabla_h\theta' = 0$$

If we had been working in isobaric coordinates then

$$\frac{\partial v_g}{\partial p}\cdot\nabla_p\theta' = 0$$

immediately since

$$\nabla_p\theta' = \left(\frac{p_s}{p}\right)^{R/c_p}\nabla_p T'$$

and

$$\frac{\partial v_g}{\partial p} = -\frac{R}{f_o p}\,\hat{\underline{k}}\times(\nabla_p T')$$

In height coordinates some analysis is required. Using the identity

$$\ln\theta = \frac{1}{\gamma}\ln p - \ln\rho \qquad (\gamma = c_p/c_v)$$

then

$$\ln\{\theta_o(1+\theta'/\theta_o)\} = \frac{1}{\gamma}\ln\{p_o(1+p'/p_o)\} - \ln\{\rho_o(1+\rho'/\rho_o)\}$$

To lowest order

$$\ln\theta_o = \frac{1}{\gamma}\ln p_o - \ln\rho_o \tag{1}$$

To the next order

$$\frac{\theta'}{\theta_o} = \frac{1}{\gamma}\frac{p'}{p_o} - \frac{\rho'}{\rho_o} \tag{2}$$

Differentiating (1) and using the hydrostatic relationship we have

$$\frac{1}{\theta_o}\frac{d\theta_o}{dz} = -\frac{g\rho_o}{\gamma p_o} - \frac{1}{\rho_o}\frac{d\rho_o}{dz}$$

which, rearranging, gives

$$\frac{1}{\gamma p_o} = -\frac{1}{g\rho_o}\frac{1}{\theta_o}\frac{d\theta_o}{dz} - \frac{1}{g\rho_o^2}\frac{d\rho_o}{dz}\ .$$

Substituting for $1/\gamma p_o$ in (2) gives

$$\frac{\theta'}{\theta_o} = -\frac{p'}{g\rho_o}\frac{1}{\theta_o}\frac{d\theta_o}{dz} - \frac{p'}{g\rho_o^2}\frac{d\rho_o}{dz} - \frac{\rho'}{\rho_o}$$

Using the fact that thin atmosphere motions are hydrostatic

$$\frac{\theta'}{\theta_o} = -\frac{p'}{g\rho_o}\frac{1}{\theta_o}\frac{d\theta_o}{dz} - \frac{p'}{g\rho_o^2}\frac{d\rho_o}{dz} + \frac{1}{g\rho_o}\frac{\partial p'}{\partial z} = -\frac{p'}{g\rho_o}\frac{1}{\theta_o}\frac{d\theta_o}{dz} + \frac{1}{g}\frac{\partial}{\partial z}\left(\frac{p'}{\rho_o}\right) \tag{3}$$

Now since

$$\frac{d\theta_o}{dz} \sim Ro\cdot\theta_o/D$$

the ratio of first to second terms on the right hand side of (3) is $O(\varepsilon)$, i.e., to lowest order

$$\frac{\theta'}{\theta_o} = \frac{\partial}{\partial z}(p'/\rho_o g) \tag{4}$$

Now since

$$\frac{\partial v_g}{\partial z} = \frac{1}{f_o}\,\underline{\hat{k}}\times\nabla_h\left\{\frac{\partial}{\partial z}\left(\frac{p'}{\rho_o}\right)\right\}$$

then we have the desired result, i.e.,

$$\frac{\partial v_g}{\partial z}\cdot\nabla_h\left(\frac{\theta'}{\theta_o}\right) = 0$$

Notice also that from equation (4)

$$\theta' = \frac{\theta_o f_o}{g}\frac{\partial\psi}{\partial z}$$

where ψ is the geostrophic streamfunction. Using this the quasi-geostrophic potential vorticity can be written as

$$q = \beta y + \nabla^2\psi + \frac{f_o^2}{\rho_o}\frac{\partial}{\partial z}\left\{\frac{\rho_o}{N^2}\frac{\partial\psi}{\partial z}\right\} \qquad \left(N^2 = \frac{g}{\theta_o}\frac{d\theta_o}{dz}\right)$$

so that the equation for conservation of potential vorticity can be written in terms of one dependent variable.

The equivalent β-effect

The quasi-geostrophic vorticity equation for a homogeneous fluid of depth H (e.g., fluid in a rotating annulus) is

$$\frac{d_h}{dt}\left(\frac{\zeta + f}{H}\right) = 0$$

If

$$H = D_o + h_B$$

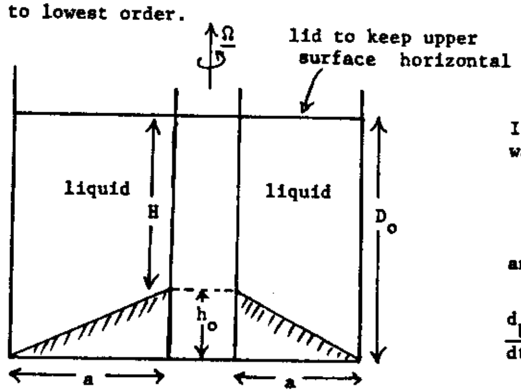
where D_o is a constant and $h_B << D_o$ then

$$\frac{d_h}{dt}\left[\frac{(\zeta + f)}{D_o}\left(1 - \frac{h_B}{D_o}\right)\right] = 0$$

In a rotating annulus $\underline{\Omega} = f_o\hat{k}$ i.e., $f = f_o$; there is no β-effect. However, since $\zeta << f$ for low Rossby number,

$$\frac{d_h}{dt}\left(\zeta + f_o - \frac{f_o h_B}{D_o}\right) = 0$$

to lowest order.



If y = distance from the outer wall, then

$$H = D_o - \frac{h_o}{a}y$$

and, with $h_o << D_o$, $h_B = - h_o y/a$ and

$$\frac{d_h}{dt}\left(\zeta + f_o + \frac{h_o f_o}{aD_o}y\right) = 0$$

Hence the sloping bottom provides an effective β (i.e., linearly varying Coriolis parameter) with

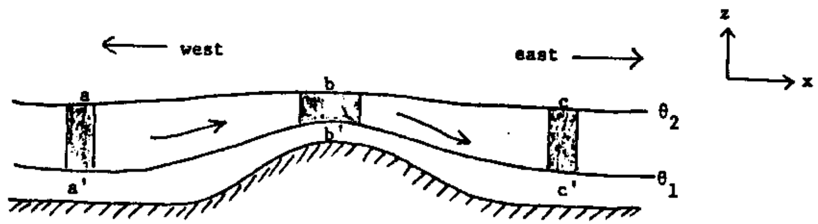
$$\beta_{eff} = f_o h_o/aD_o$$

and Rossby waves can be generated in the annulus.

VORTICITY AND ITS APPLICATIONS

(x) Applications of vorticity concepts to the atmosphere

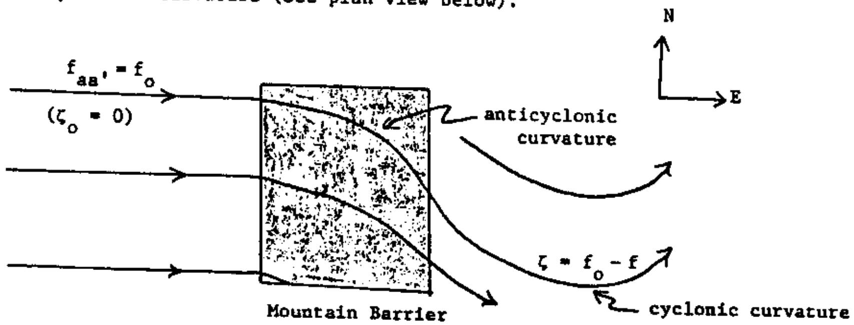
(a) Vortex stretching in flow over large-scale mountain barriers:



adiabatic flow with $(\underline{\omega} + 2\underline{\Omega})\cdot\nabla\theta \sim \eta\,\frac{\partial\theta}{\partial z}$

Suppose there is nearly uniform flow over the barrier in the layer between θ surfaces θ_1 and θ_2 $(\theta_2 > \theta_1)$. Then the column of air aa' shrinks vertically as it moves to bb', with the result that η must decrease. This can be accomplished by anticyclonic curvature developing at bb' so that η becomes negative there. At cc', the column has expanded again, so that the original

vorticity is restored. However, because of moving anticyclonically at bb', the column now is at a lower latitude, and hence has a smaller (positive) value of f. The result is that ζ must assume a significant positive value so that $(\zeta + f)_{cc'} \simeq (\zeta + f)_{aa'} \sim f_{aa'}$, and to do this the flow assumes a positive curvature (see plan view below).



These factors tend to produce ridges over and to windward of large ranges and a lee-side trough.

(b) Downstream waves - Rossby waves - In the example above $\eta = (\zeta + f)$ is conserved for 2-dimensional barotropic flow over flat terrain,

$$\frac{D\eta}{Dt} \simeq \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} + v\beta = 0 \qquad (3.26)$$

where $\beta \equiv df/dy$, and we neglect terms of order L/a. Consistent with this, we also assume β = const. $= 2\Omega \cos\phi_o/a$ at some mean latitude ϕ_o. (These assumptions comprise the mid-latitude β-plane approx.). For the assumptions,

$\nabla_h \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, and the flow can be represented by means of a stream function, ψ:

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}, \quad \vec{u} = \hat{k} \times \nabla_h\psi \ .$$

Then

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \nabla^2\psi \ , \qquad (3.27)$$

and (3.26) becomes

$$\frac{\partial}{\partial t}(\nabla^2\psi) + J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0 \qquad (3.28)$$

where

$$J = \frac{\partial\psi}{\partial x}\cdot\frac{\partial}{\partial y}\nabla^2\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi \ .$$

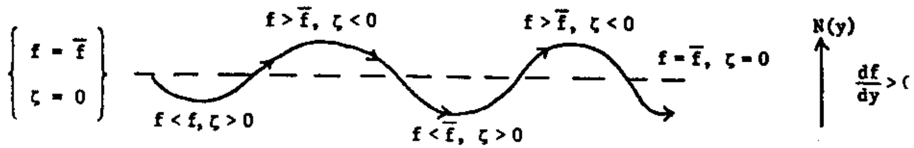
Suppose $\psi = -Uy + \psi' = -Uy + \{A\cos(kx - \omega t)\cdot\cos \ell y\}$ representing a zonal flow with superposed traveling cosine wave. Note that $J(\psi,\nabla^2\psi) = U[k(k^2 + \ell^2)\sin(kx - \omega t)\cos \ell y]$, the terms $J(\psi',\nabla^2\psi')$ vanishing identically because the lines of constant ψ' and $\nabla^2\psi'$ superpose. Hence solutions exist for

$$[\omega - kU](k^2 + \ell^2) + k\beta = 0, \quad \text{or} \quad c \equiv \omega/k = U - \beta/(k^2 + \ell^2) \ . \qquad (3.29)$$

Such wave solutions are Rossby waves. Notice that for the special assumptions (β-plane, simple waves) the linear solution is the solution to the non-linear problem. Rossby waves propagate only westward with respect to the flow.

Stationary waves having $c = 0$, $k^2 = \frac{4\pi^2}{L_x^2} = \beta/U$ may exist, and can be expected downwind of ranges producing a disturbance with wave-length component L_x.

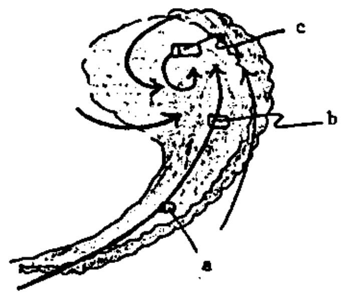
The propagation mechanism for Rossby waves is as follows: a parcel displaced poleward from its original latitude which conserves $\eta = \zeta + f$ will have to develop negative ζ, or anticyclonic curvature to compensate the increase of f with latitude. If the parcel is traveling eastward with respect to the disturbance pattern associated with displacement, it curves equatorward. In other words it returns to its original latitude, then overshoots. The parcel then finds itself too far equatorward and has to recurve poleward to conserve η. The process then repeats, the parcel oscillating about its equilibrium latitude. The process works only if the parcel is moving eastward with respect to the disturbance pattern (i.e., the wave pattern)



arrows indicate motion of parcel with respect to wave pattern. When the parcel moves toward the west with respect to the wave, it cannot balance f departures ($f \neq \bar{f}$) with ζ departures ($\zeta \neq 0$).

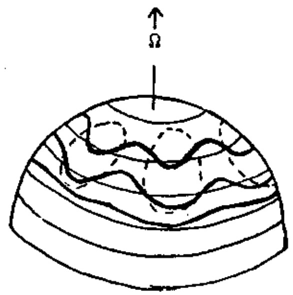
For west to east flow over large N-S aligned topography, stationary Rossby waves can produce lee Rossby wave trains. Such trains are not produced in E W flow over topography. The wave response of flow to steady zonal winds over tropography is much stronger in westerlies than in easterlies.

(c) Vorticity production by heating -- Heating in the lower troposphere tends to produce vertical motion in the mid-troposphere, and hence vertical stretching of vortex lines. This tends to increase vorticity despite the fact that potential vorticity is not conserved. This allows latent heat realease in incipient storms to contribute to storm development.



Active comma cloud system viewed from above and right. a,b,c represent stretching of vertical column as it moves toward cloud center. This is only one factor which may contribute to development.

(d) Use of the Barotropic vorticity eqn. Eqn. (3.26) is sometimes useful in forecasting mid-tropospheric flow where other terms in (3.21) are sometimes small. The vorticity distribution consists of a large component of planetary vorticity whose contours are latitude circles and a small component of relative vorticity.



On this N. Hemisphere view, the thin parallel lines are f-countours (latitudes). The dashed lines represent the ζ field of synoptic systems. The heavy solid lines are contours of $\eta = \zeta + f$.

The contours of η are advected by the flow and evolution of the ψ field is predicted by eqn. (3.28) since $\eta = f + \nabla^2\psi$. The evolution involves simple advection and a tendency for wave dispersion according to (3.29).

(e) Vertical velocity and vorticity - We show later that the dominant terms in (3.21) are $D\eta/Dt$ and $\eta\nabla_h \cdot \vec{u}_h \simeq f\nabla_h \cdot \vec{u}_h$ since $f >> \zeta$ normally. Consider a steady flow with a small amplitude wave superposed. Let the flow relative to the wave be U, and the downstream coordinate x. Then the vorticity eqn. is approximately

$$U \frac{\partial \eta}{\partial x} + f\nabla_h \cdot \vec{u}_h \simeq 0 \quad . \tag{3.30}$$

Then if $U \frac{\partial \eta}{\partial x}$ increases with height, as it generally does with cold troughs, divergence $(- \nabla_h \cdot \vec{u}_h)$ must also increase with height to produce compensating

decrease of vorticity with height needed to maintain the steady wave pattern. If divergence increases with height (large above and small below) the, by continuity, ρw must be positive through the region of increasing $(-\nabla_h \cdot \vec{u}_h)$. This follows from $\nabla_h \cdot \vec{u}_h \approx -\frac{1}{\rho}\frac{\partial \rho w}{\partial z}$. We have the rule of thumb: upward motion is associated with increase of positive vorticity advection upward, or, since vorticity advection tends to be small in the lower troposphere, simply with positive vorticity advection aloft. Note that for positive vertical wind shear (from the west), and a vertically coherent wave pattern $U\frac{\partial \eta}{\partial x}$ tends to be positive ahead of the trough at upper levels, negative ahead of the trough at low levels.

Vertical velocity is also related to horizontal temperature advection. For adiabatic motion, flow is along θ surfaces. Where the flow is along an upward sloping θ surface, $w > 0$. Since θ increases upward, this occurs where warm advection is occuring in coordinates moving with the local θ-pattern, i.e.: $U\frac{\partial \ell n\theta}{\partial x} + w\,(\nu_B^2/g) = 0$ in adiabatic flow with assumptions analogous to those in (3.30). Hence when $U\frac{\partial \ell n\theta}{\partial x} < 0$ (warm advection), $w > 0$, and conversely. The pattern of vertical velocity with mid-latitude storms can be interpreted in terms of these two factors.

The Planetary Boundary Layer (Holton, chapter 5)

We have neglected the effect of molecular viscosity in the synoptic-scale thin atmosphere equations of motion because the Reynolds number was so large. Near the ground however transport of heat and momentum by turbulent eddies may have an appreciable influence on motions throughout the so-called planetary boundary layer (≈ the lowest kilometer of the atmosphere). For a statically stable atmosphere the lowest few meters of the boundary layer are known as the surface layer, where the characteristic scale of the turbulent eddies is observed to grow linearly with the distance above the ground. The region from the top of the surface layer to the top of the boundary layer is called the Ekman layer in which the characteristic scale of the turbulent eddies is observed to be nearly constant with height. These eddies are generated primarily by dynamical instability due to vertical shears of the wind near the ground.

In the (incompressible) Navier-Stokes equations the effect of molecular transport of momentum was written as $\mu\nabla^2\underline{u}$. Hence for atmospheric boundary layer flow the simplest assumption we could make is that in the Ekman layer geophysical eddies can be parameterized in exactly the same way, and instead of a molecular viscosity ν we have a so-called eddy viscosity coefficient κ. However, this is perhaps an oversimplification. The basic anisotropy of large scale systems betwe the horizontal and vertical scales of the flow suggest that the mixing of large-scale momentum in the two directions cannot be expected to be the same. Hence we define two coefficients κ_H and κ_V called the horizontal and vertical eddy viscosity coefficients respectively. Using these coefficients we would express the forcing by the eddies as

$$\kappa_H\left(\frac{\partial^2\langle\underline{u}\rangle}{\partial x^2}+\frac{\partial^2\langle\underline{u}\rangle}{\partial y^2}\right)+\kappa_V\frac{\partial^2\langle\underline{u}\rangle}{\partial z^2}$$

The operator $\langle\ \rangle$ is a suitable time average which will be defined to have the property that if we write $x = \langle x\rangle + x'$ then $\langle x'\rangle = 0$. i.e., writing

$$\langle X\rangle = \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} X\,dt$$

then over periods of time short compared with the natural time scales of the synoptic scale flow, $\langle x\rangle$ is time independent. Whether it is really possible to find averaging times long enough for this condition to hold whilst being short enough compared with the natural time scales of the large scale is problematic. We assume here that it is possible.

Using this decomposition in the thin atmosphere synoptic scale x component momentum equation we have

$$\left[\frac{\partial}{\partial t}+(\langle u\rangle+u')\frac{\partial}{\partial x}+(\langle v\rangle+v')\frac{\partial}{\partial y}+(\langle w\rangle+w')\frac{\partial}{\partial z}\right](\langle u\rangle+u')$$

$$-f(\langle v\rangle+v') = -\frac{1}{\rho_o}\frac{\partial\langle p\rangle}{\partial x}-\frac{1}{\rho_o}\frac{\partial p'}{\partial x}$$

(Remember that here p' is a turbulent eddy pressure.)

Averaging this equation gives

$$\frac{\partial\langle u\rangle}{\partial t} + \langle u\rangle\frac{\partial\langle u\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle u\rangle}{\partial y} + \langle w\rangle\frac{\partial\langle u\rangle}{\partial z} - f\langle v\rangle = -\frac{1}{\rho_o}\frac{\partial\langle p\rangle}{\partial x} - \left[\frac{\partial}{\partial x}(\langle u'u'\rangle) + \frac{\partial}{\partial y}(\langle u'v'\rangle) + \frac{\partial}{\partial z}(\langle u'w'\rangle)\right]$$

with a similar equation for $\partial\langle v\rangle/\partial t$ (and we assume, for simplicity, that the average fluctuation velocities are non-divergent (the boundary layer depth is small compared with a scale-height). If we define the Reynolds stress tensor

$$\tau'_{ij} = \rho\langle u'_i u'_j\rangle$$

then our parameterization hypothesis is that

$$\tau_{xx} = 2\rho\,\kappa_H\,\partial\langle u\rangle/\partial x$$

$$\tau_{yy} = 2\rho\,\kappa_H\,\partial\langle v\rangle/\partial y$$

$$\tau_{zz} = 2\rho\,\kappa_V\,\partial\langle w\rangle/\partial z$$

$$\tau_{xy} = \tau_{yx} = \rho\,\kappa_H\left(\frac{\partial\langle v\rangle}{\partial x} + \frac{\partial\langle u\rangle}{\partial y}\right)$$

$$\tau_{xz} = \tau_{zx} = \rho\left[\kappa_V\frac{\partial\langle u\rangle}{\partial z} + \kappa_H\frac{\partial\langle w\rangle}{\partial x}\right]$$

$$\tau_{yz} = \tau_{zy} = \rho\left[\kappa_V\frac{\partial\langle v\rangle}{\partial z} + \kappa_H\frac{\partial\langle w\rangle}{\partial y}\right]$$

which, apart from the anisotropic eddy viscosity coefficients, is completely equivalent to the parameterization of molecular transport (stress α to rate of strain). In the following, however, we shall suppose that the flow is horizontally homogeneous and put $\kappa_V = \kappa$. Because the eddies have a constant size in the Ekman layer we assume κ = constant. For convenience we shall also drop the bracket $\langle\ \rangle$.

Including the effect of eddy viscosity in the thin atmosphere synoptic scale equations we have

$$du/dt - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \kappa\,\partial^2 u/\partial z^2$$

$$dv/dt + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \kappa\,\partial^2 v/\partial z^2$$

For small Rossby number the Coriolis acceleration is much larger than the inertial acceleration so that

$$-fv = -fv_g\ \ \kappa\,\partial^2 u/\partial z^2 \qquad + fu = fu_g + \kappa\,\partial^2 v/\partial z^2$$

or, combining these equations

$$\kappa\frac{\partial^2(u+iv)}{\partial z^2} - if(u+iv) = -if(u_g + iv_g) \qquad (\text{where } i = \sqrt{-1})$$

For a barotropic atmosphere a particular solution of this equation is

$$u + iv = u_g + iv_g$$

Let us suppose that the flow is oriented such that $v_g = 0$, then for a barotropic atmosphere

$$(u+iv) = A\exp[(if/\kappa)^{1/2}z] + B\exp[-(if/\kappa)^{1/2}z] + u_g$$

where A and B are constants of integration. As boundary conditions we require that far from the ground the flow becomes geostrophic, which can only be accomplished if $A = 0$; and that, for simplicity, $u = v = 0$ at $z = 0$, i.e., $B = -u_g$. Hence

$$(u + iv) = -u_g\exp[-(if/\kappa)^{1/2}z] + u_g$$

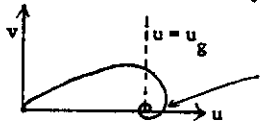
Since

$$(if/\kappa) = \{(1 + i)\gamma\}^2$$

where $\gamma = (f/2\kappa)^{1/2}$, then the solution can be written as

$$u = u_g(1 - e^{-\gamma z}\cos\gamma z) \qquad v = u_g(e^{-\gamma z}\sin\gamma z)$$

which is known as the Ekman spiral.



(see Holton p. 109)

When $\gamma z = \pi$, $v = 0$ and u is slightly greater than u_g. This is designated the top of the boundary layer solution, i.e., the depth, De, of Ekman layer is taken to be

$$De = \pi/\gamma\ .$$

Putting $De = 10^3$ m, $\kappa \sim 5$ m^2 s^{-1}.

For molecular viscosity $\nu \sim c\lambda$ where c was a typical (rms) molecular speed, and λ was a mean free path. For a geophysical eddy let us define an analogous quantity ℓ', the mixing length, which is the vertical distance a parcel of fluid will carry the mean horizontal velocity of its level of origin. Furthermore we shall suppose that a horizontal eddy velocity $u' = -\ell'(\partial\langle u\rangle/\partial z)$. For a neutrally stable atmosphere (i.e., no buoyancy forces) we would expect $w \sim u'$.

Hence if we put

$$\kappa \sim w'\ell' \sim (\ell')^2 \frac{\partial \langle u \rangle}{\partial z}$$

then with $\partial\langle u\rangle/\partial z = 5$ m s^{-1} km^{-1}, $\kappa \approx 5$ m^2 s^{-1}, $\ell' \approx 30$ m. This length is indeed small compared with the depth of the boundary layer as it should be if the mixing length, and eddy viscosity are to be useful concepts.

If, instead of the simple boundary condition $(u + iv) = 0$ at $z = 0$ (top of surface layer) we put

$$u + iv = C_o e^{i\alpha} \quad \text{at} \quad z = 0$$

i.e., at the top of the surface layer the wind has magnitude C_o and is oriented at an angle α relative to the isobars, then the general solution becomes

$$u + iv = (C_o e^{i\alpha} - u_g)\, e^{-(1+i)\gamma z} + u_g \qquad \text{(A)}$$

We can determine the constant C_o by requiring that the top of the surface layer match smoothly to the bottom of the Ekman layer. In the surface layer the wind increases in strength with height, but does not change direction, i.e., in the surface layer

$$\frac{\partial}{\partial z}\left(\frac{u}{v}\right) = 0 \Rightarrow \frac{1}{v}\frac{\partial u}{\partial z} - \frac{u}{v^2}\frac{\partial v}{\partial z} = 0 \Rightarrow \frac{1}{u}\frac{\partial u}{\partial z} = \frac{1}{v}\frac{\partial v}{\partial z} \quad \text{i.e.,} \quad (u+iv) = C\frac{\partial}{\partial z}(u+iv)$$

where C depends only on height. Hence, to match the Ekman layer to the surface layer we require that at $z = 0$ (bottom of Ekman layer)

$$(u + iv) = \hat{C}\frac{\partial}{\partial z}(u + iv)$$

where $\hat{C}$ is the value of C at the top of the surface layer, i.e., $\hat{C}$ is a real con[illegible]
Substituting (A) into the above boundary condition and equating real and imaginary parts we have

$$C_o \cos\alpha = \gamma\hat{C}[C_o(\sin\alpha - \cos\alpha) + u_g]$$

$$C_o \sin\alpha = \gamma\hat{C}[-(C_o(\sin\alpha + \cos\alpha) + u_g]$$

which gives (divide one equation by the other)

$$C_o = u_g(\cos\alpha - \sin\alpha)$$

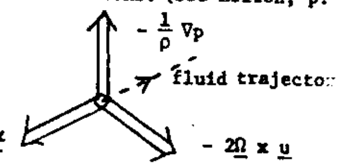
Substituting for this in (A) and taking real and imaginary parts

$$u = u_g[1 - \sqrt{2}\sin\alpha\, e^{-\gamma z}\cos(\gamma z - \alpha + \tfrac{\pi}{4})]$$

$$v = u_g \sqrt{2}\sin\alpha\, e^{-\gamma z}\sin(\gamma z - \alpha + \tfrac{\pi}{4})$$

which gives the modified Ekman spiral. The parameter α (surface wind angle) along with κ, can be chosen to give the best fit to observations. (see Holton, p.

Note in the Ekman layer the relative direction of the three forces, i.e., the flow is cross-isobaric towards low pressure.
It turns out that the Ekman layer wind profile is generally unstable for a neutrally buoyant atmosphere, though the vertically integrated horizontal mass transport in the boundary layer is still directed toward lower pressure.



Secondary circulation and spin-down

We have assumed $v_g = 0$ so that the v component of wind is cross isobaric. Hence mass flow towards lower pressure integrated over the depth of the Ekman layer is for a column of unit width

$$M = \int_0^{De} \rho v\, dz$$

From the continuity equation (with $w = 0$ at $z = 0$)

$$(w\rho)_{z=De} = -\int_0^{De}\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right] dz$$

Substituting for the Ekman spiral solution we obtained in the previous lecture

$$(w\rho)_{z=De} = -\frac{\partial}{\partial y}\int_0^{De} \rho u_g e^{-\pi z/De}\sin(\pi z/De)\, dz \qquad (\partial u_g/\partial x = 0 \text{ when } v_g = 0)$$

Hence the vertical flux at the top of the boundary layer is equal to the horizontal convergence $-\partial M/\partial y$ of mass in the boundary layer.

Since

$$\zeta_g = -\partial u_g/\partial y$$

then neglecting any variation of ρ in the boundary layer, and putting $1 + e^{-\pi} \approx 1$

$$w_{De} = \zeta_g(\kappa/2f)^{1/2}$$

(e.g., if $\zeta_g \sim 10^{-5}$ s^{-1}, $w \sim 10^{-1}$ cm s^{-1}). In this way the effect of friction in the boundary layer is communicated directly to the free atmosphere through a forced secondary circulation rather than indirectly by the slow process of viscous diffusion (cf., the decay of circulation created when a cup of tea is stirred). See Holton p. 114.

For a homogeneous fluid of depth H we showed that the potential vorticity $(\zeta + f)/H$ was conserved for two dimensional flow, i.e.,

$$\frac{d}{dt}(\zeta + f) = (\zeta + f)\frac{\partial w}{\partial z} \qquad \left(\frac{1}{H}\frac{dH}{dt} = \nabla_h \cdot \underline{u}_h\right)$$

For $\zeta \ll f$ and $df/dy \approx 0$ (i.e., ignore the β-effect), we can integrate this equation from $z = H$ to $z = De$ to give

$$\frac{d\zeta}{dt} \times H = -f\,(w)_{z=De}$$

where we have set $w = 0$ at $z = H$. Using the Ekman layer value $(w)_{z=De}$

$$\frac{d\zeta}{dt} = -\frac{f}{H}(\kappa/2f)^{1/2}\,\zeta = -(f\kappa/2H^2)^{1/2}\,\zeta$$

i.e.,

$$\zeta = \zeta(0)\exp\{-(f\kappa/2H^2)^{1/2}\,t\}$$

The time $\tau_e = (2H^2/f\kappa)^{1/2}$ is the time it takes a barotropic vortex of height H to <u>spin-down</u> to e^{-1} of its original value. For a midlatitude synoptic scale disturbance this time is on the order of a few days, which is much shorter than the time scale for diffusion (H^2/κ) which is $\sim$ 100 days.

n.b. For a statically stable atmosphere, the buoyancy force will suppress vert motion since air lifted vertically in a stable environment will be denser than the environmental air. Hence the secondary flow will quickly spin-down the vorticity at the top of the Ekman layer without appreciably affecting the higher levels. Ultimately there will result a baroclinic vortex with a vertical shear of azimuthal velocity just strong enough to bring ζ to zero at the top of the boundary layer, implying a radial temperature gradient by the thermal wind relationship; this is brought about by the adiabatic cooling of the air forced out of the Ekman layer. (Holton, p. 115)

For the quasi-geostrophic barotropic vorticity equation

$$\frac{d_h\zeta_g}{dt} + v_g\beta_o = \frac{f}{\rho_o}\frac{\partial}{\partial z}(\rho_o w)$$

multiplying by ρ_o, integrating from z = top to z = De, and using the fact that ζ_g and v_g are independent of z, then

$$\int_{De}^{top}\rho_o\,dz\left(\frac{d_h\zeta_g}{dt} + v_g\beta_o\right) = -f\rho_o w|_{z=De} \qquad (w|_{z=top} = 0)$$

But

$$\int_{De}^{top}\rho_o dz = \frac{1}{g}\int_{p_{top}}^{p_{De}} dp_o = \frac{p_{De}}{g} \qquad \text{since } p_{De} >> p_{top}$$

$$\Rightarrow \quad \frac{d_h\zeta_g}{dt} + v_g\beta_o = -\frac{fp_{De}}{RT_{De}}\cdot\frac{g}{p_{De}}\,w|_{z=De} = -\frac{f}{H}\,w|_{z=De}$$

where $H = RT_{De}/g$. Hence $d_h\zeta_g/dt + v_g\beta_o = -\zeta_g(\kappa f/2H^2)^{1/2}$.

Putting $\psi = -Uy + \varepsilon\psi'$, where ψ is the geostrophic streamfunction, U is a constant and ψ' depends only on x, then to $O(\varepsilon)$

$$U\frac{d^3\psi'}{dx^3} + \beta\frac{d\psi}{dx} = -\left(\frac{\kappa f}{2H^2}\right)^{1/2}\frac{d^2\psi}{dx^2} \quad \text{or} \quad \frac{d^2\psi'}{dx^2} + \frac{1}{U}\left(\frac{\kappa f}{2H^2}\right)^{1/2}\frac{d\psi}{dx} + \frac{\beta}{U}\,\psi = 0$$

which is an equation for a damped harmonic oscillator with damping length UH (2.

Waves

1. Mean flows, perturbations, and wave, mean-flow interaction

Let ψ be any fluid variable and define $\overline{\psi}$ to be either the average of ψ around a circle of constant latitude, or the average of ψ along a horizontal Cartesian coordinate direction (according the problem in question). We can define ψ' to be the difference between ψ and $\overline{\psi}$, i.e.,

$$\psi = \overline{\psi} + \psi' \tag{1}$$

Suppose ψ satisfies the equation

$$d\psi/dt = F \quad .$$

Using (1) this can be written as

$$\frac{\partial}{\partial t}(\overline{\psi} + \psi') + (\overline{\underline{u}} + \underline{u}')\cdot\nabla(\overline{\psi} + \psi') = \overline{F} + F' \tag{2}$$

If we apply the overbar operator on this equation and note that for any quantity X

$$\overline{\overline{X}} = \overline{X} \quad (\text{i.e., } \overline{X'} = 0\,) \qquad \overline{\frac{\partial X}{\partial t}} = \frac{\partial}{\partial t}\overline{X}\,, \qquad \overline{\nabla X} = \nabla\overline{X}$$

then we have the mean equation

$$\frac{\partial\overline{\psi}}{\partial t} + \overline{\underline{u}}\cdot\nabla\overline{\psi} + \overline{\underline{u}'\cdot\nabla\psi'} = \overline{F} \tag{3}$$

Subtracting (3) from (2) we have the deviation equation

$$\frac{\partial\psi'}{\partial t} + \overline{\underline{u}}\cdot\nabla\psi' + \underline{u}'\cdot\nabla\overline{\psi} + \underline{u}'\cdot\nabla\psi' - \overline{\underline{u}'\cdot\nabla\psi'} = F' \tag{4}$$

Now suppose that all deviation variables are small compared with their meaned counterparts (e.g., $\psi' << \overline{\psi}$), then we can introduce a small parameter $\varepsilon << 1$ and, for example, expand $\overline{\psi}$ and ψ' in powers of ε i.e.,

$$\overline{\psi} = \overline{\psi}_o + \varepsilon\overline{\psi}_1 + \varepsilon^2\overline{\psi}_2 + \ldots \tag{5}$$

$$\psi' = \quad \varepsilon\psi_1' + \varepsilon^2\psi_2' \tag{6}$$

with similar expansions for $\underline{u}$ and F.

Retaining terms only of O(1), the mean equation, (3), is

$$\frac{\partial\overline{\psi}_o}{\partial t} + \overline{\underline{u}}_o\cdot\nabla\overline{\psi}_o = \overline{F}_o \tag{7}$$

and both sides of the deviation equation are equal to zero.

Retaining terms up to $O(\varepsilon)$ and subtracting (7), the mean equation is

$$\frac{\partial\overline{\psi}_1}{\partial t} + \overline{\underline{u}}_o \cdot \nabla\overline{\psi}_1 + \overline{\underline{u}}_1 \cdot \nabla\overline{\psi}_o = \overline{F}_1 \tag{8}$$

and the deviation equation (4) is

$$\frac{\partial\psi_1'}{\partial t} + \overline{\underline{u}}_o \cdot \nabla\psi_1' + \underline{u}_1' \nabla\overline{\psi}_o = F_1' \tag{9}$$

Notice that this equation is linear in the deviation variables. It is often called a linearized perturbation equation. When this equation admits wavelike solutions it is called a linearized wave equation. Solutions of this equation will depend on the $O(1)$ mean state.

Retaining terms up to $O(\varepsilon^2)$, and subtracting (7) and (8), the mean equation is

$$\frac{\partial\overline{\psi}_2}{\partial t} + [\overline{\underline{u}}_2 \cdot \nabla\overline{\psi}_o + \overline{\underline{u}}_1 \cdot \nabla\overline{\psi}_1 + \overline{\underline{u}}_o \cdot \nabla\overline{\psi}_2] + \overline{\underline{u}_1' \cdot \nabla\psi_1'} = \overline{F}_2$$

The most important feature of this equation for our purposes is that to $O(\varepsilon^2)$ the linearized perturbations can, through the term $\overline{\underline{u}_1' \cdot \nabla\psi_1'}$, react back and alter the mean flow. If the linearized perturbation equation yields wavelike solutions, then the process of these waves both being influenced by the mean state, and influencing the mean state is often referred to as wave, mean-flow interaction. We shall not be concerned with the $O(\varepsilon^2)$ deviation equations.

2. Representation of the perturbation variables

In a bounded domain of length L (in the x-direction, say) the perturbation variables can be written as a Fourier series of sinusoidal components.

$$\psi' = \sum_{n=1}^{\infty} \left(\Psi_c(n) \cos\frac{2\pi nx}{L} + \Psi_s(n) \sin\frac{2\pi nx}{L}\right) .$$

Since

$$\frac{1}{L}\int_0^L \sin\frac{2\pi mx}{L} \sin\frac{2\pi nx}{L}\, dx = \frac{1}{2}\,\delta(n,m)$$

$$\frac{1}{L}\int_0^L \cos\frac{2\pi mx}{L} \cos\frac{2\pi nx}{L}\, dx = \frac{1}{2}\,\delta(n,m)$$

$$\frac{1}{L}\int_0^L \sin\frac{2\pi mx}{L} \cos\frac{2\pi nx}{L}\, dx = 0 \quad \text{for all } n,m$$

(where $\delta(n,m) = 0$ if $n \neq m$; $=1$ if $n = m$). Then

$$\Psi_s(n) = \frac{2}{L}\int_0^L \psi' \sin\frac{2\pi nx}{L}\, dx$$

$$\Psi_c(n) = \frac{2}{L}\int_0^L \psi' \cos\frac{2\pi nx}{L}\, dx.$$

Another useful way of writing a Fourier series is in the complex form

$$\psi' = \sum_{n=-\infty}^{\infty} \Psi_n\, e^{2\pi inx/L}$$

Since ψ' is real we must have that $\Psi_{-n} = \Psi_n^*$, where * denotes the complex conjugate. It is easily shown that

$$\Psi_c(n) = 2\mathrm{Re}(\Psi_n) \qquad \Psi_s(n) = -2\mathrm{Im}(\Psi_n)$$

Note that for two perturbation variables ψ' and ϕ', then if the overbar is defined as the average from L to 0,

$$\overline{\Psi'\phi'} = \frac{1}{L}\int_0^L \sum_{n=0}^{\infty} \left(\Psi_c(n) \cos\frac{2\pi nx}{L} + \Psi_s(n) \sin\frac{2\pi nx}{L}\right)$$

$$\times \sum_{m=0}^{\infty} \left(\Phi_c(n) \cos\frac{2\pi mx}{L} + \Phi_s(n) \sin\frac{2\pi mx}{L}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (\Psi_c(n)\Phi_c(n) + \Psi_s(n)\Phi_s(n)) = \sum_{n=-\infty}^{\infty} \psi_n\phi_n$$

using the orthonormal relations for sin and cos. When the bounded domain is the latitude circle $0 \leq \lambda < 2\pi$, then $L = 2\pi a \cos\phi$ and $x = a\lambda \cos\phi$, so that

$$\psi' = \sum_{k=1}^{\infty} (\Psi_c(k) \cos(k\lambda) + \Psi_s(k) \sin k\lambda) = \sum_{k=-\infty}^{\infty} \Psi_k e^{ik\lambda}$$

and k is called the zonal wavenumber.

(In an unbounded domain, the corresponding Fourier integral representation applies

$$\psi' = \int_{-\infty}^{\infty} \Psi(k)\, e^{ikx}\, dk; \quad \Psi(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi'(x) e^{-ikx}\, dx \quad .)$$

If the overbar represents an average over x, so that all $O(1)$ variables are independent of x, then the 'elementary' form $\psi_k e^{ikx}$ will be a solution of the linearized perturbation equation if the $O(1)$ mean flow is independent of time there will be solutions taking the form $\Psi_k e^{i(kx-\omega t)}$. If the linearized perturbation equation can be satisfied with real k and ω then $\Psi_k e^{i(kx-\omega t)}$ is a wave solution which oscillates at a frequency $\omega(k)$, determined by a linearized differential equation such as (9).

For example, consider a barotropic fluid and put $u = \bar{u}_o + u_1'$, $v = v_1'$, where $\overline{(\;)}$ is the zonal average. If $\bar{u}_o$ is independent of y so that $\bar{\zeta}_o = 0$ then on the β-plane the linearized barotropic quasi-geostrophic vorticity equation is

$$\left(\frac{\partial}{\partial t} + \bar{u}_o \frac{\partial}{\partial x}\right) \nabla^2 \psi_1' + \beta \frac{\partial \psi_1'}{\partial x} = 0$$

where ψ is the geostrophic streamfunction. Putting $\psi_1' = \mathrm{Re}\{\Psi_k e^{ikx}\}$

$$\left(\frac{\partial}{\partial t} + ik\bar{u}_o\right) (-k^2\Psi_k) + ik\beta\Psi_k = 0$$

or

$$\frac{\partial \Psi_k}{\partial t} + ik(\bar{u}_o - \beta/k^2)\Psi_k = 0$$

If $\bar{u}_o$ is also independent of time, then $\Psi_k = A_k e^{-i\omega t}$ where A_k is independent of y,x,t. $\omega(k) = k\bar{u}_o - k\beta/k^2$ for waves $A_k e^{i(kx-\omega t)}$. The equation $\omega = \omega(k)$ is the dispersion equation (sometimes called the frequency equation).

3. Phase speed and group speed -- The quantity $\phi = kx - \omega t$ is the wave phase. It is constant at points moving with phase speed $c = \omega/k$, since $\phi = \phi(x,t)$, $\delta\phi = \phi_x\delta x + \phi_t\delta t$, $\phi_x = \partial\phi/\partial x$, $\phi_t = \partial\phi/\partial t$, $(dx/dt)_{\phi=\mathrm{const}} = -\phi_t/\phi_x$ and $\phi_t = -\omega$, $\phi_x = k$. A more important property of waves is the group velocity (or in the one-dimensional case, the group speed: $c_g = \partial\omega/\partial k$. Consider a disturbance consisting of 2 waves of nearly the same wave-number $k + \delta k$ and $k - \delta k$, and equal amplitudes. Then

$$\psi(x,t) = \mathrm{Re}\,\{e^{i[(k+\delta k)x - (\omega+\delta\omega)t]} + e^{i[(k-\delta k)x-(\omega-\delta\omega)t]}\}$$

$$= 2\mathrm{Re}\,\cos(\delta kx - \delta\omega t) e^{i(kx-\omega t)}$$

representing a wave form $e^{i(kx-\omega t)}$ modulated by amplitude $2\cos(x\delta k - t\delta\omega)$. The modulation envelope moves at the groupspeed $\delta\omega/\delta k$. That is, the modulation is constant for points for which $x/t = \delta\omega/\delta k$. In the limit $\delta k \to 0$, this gives $x/t = c_g(k)$.

This group speed is therefore an appropriate measure of the speed at which the 'energy' carried by the wave is propagating. This idea may be familiar from the study of radio wave transmission where information (a low frequency signal) is mixed with a high frequency 'carrier' wave resulting in the transmission of an amplitude modulated radio signal. The information or 'energy' will therefore travel at the speed of the modulations. It is something of an act of faith to accept that these ideas are applicable to studies of Rossby wave propagation but (see section 4) they have proved to be useful in explaining some of the large scale features of the general circulation of the atmosphere.

4. Generalization to two dimensions.

$\underline{k} = \hat{\imath}k + \hat{\jmath}\ell$, $\phi = \underline{k}\cdot\underline{x} - \omega t$. $\underline{k}$ is $\perp$ to surfaces of constant ϕ; $c^x = \omega/k$, $c^y = \omega/\ell$, $c_g^x = \partial\omega/\partial k$, $c_g^y = \partial\omega/\partial\ell$. For example, for a barotropic fluid again with $\bar{u}_o$ independent of y and t, the elementary wave

$$A_{k\ell}\, e^{i(kx+\ell y-\omega t)}$$

will satisfy the linearized barotorpic vorticity equation with the dispersion relationship

$$\hat{\omega} = -\beta k/(k^2 + \ell^2)$$

where

$$\hat{\omega} = \omega - k\bar{u}_o$$

is the wave frequency relative to the mean flow. The dispersion relation confirms that the phase of these two dimensional Rossby waves always propagates westward relative to the mean zonal flow. The group velocity velocity $\underline{c}_g = (c_g^x, c_g^y)$ is given by

$$c_g^x = \frac{\partial\omega}{\partial k} = \bar{u}_o + \beta(k^2 - \ell^2)/(k^2 + \ell^2)^2$$

$$c_g^y = \frac{\partial\omega}{\partial\ell} = \frac{\partial\hat{\omega}}{\partial\ell} = 2\beta\, k\ell/(k^2 + \ell^2)^2$$

For stationary ($\omega = 0$) Rossby waves, substituting for $\bar{u}_o$ from the dispersion relation into the expression for c_g^x

$$c_g^x = 2\beta k^2/(k^2 + \ell^2)^2 > 0$$

The direction of the group velocity vector is given by $\tan\chi = c_g^y/c_g^x = \ell/k$ for a stationary Rossby wave, where χ is the angle to the x-direction.

Figure 1 is a schematic illustration of the hypothesized global pattern of middle and upper tropospheric geopotential height anomalies during a Northern Hemisphere winter which falls within an episode of warm sea-surface temperatures in the equatorial Pacific. The shading indicates regions of enhanced rainfall. The centres of the height anomalies follow a great circle path. Great circles are the ray paths for Rossby wave trains on the sphere. (A line of shortest distance between two points of the surface of the sphere is a great circle and therefore the natural generalization of a straight line for the β-plane.)

Figure 2 is taken from 500 mb height data over a number of Northern Hemisphere winters. The contours are isopleths of correlation coefficient between height fields at one fixed point (marked x in the diagram) and the rest of the hemisphere. The correlation of point x with other points is not a simultaneous correlation but is one with point x correlated with every other gridpoint in the hemisphere two days earlier. The figure is characterized by strong upstream centers of action over Scandinavia and the North Atlantic indicative of wave-activity propagating in the great circle route. The figure i an example of a so-called teleconnection map.

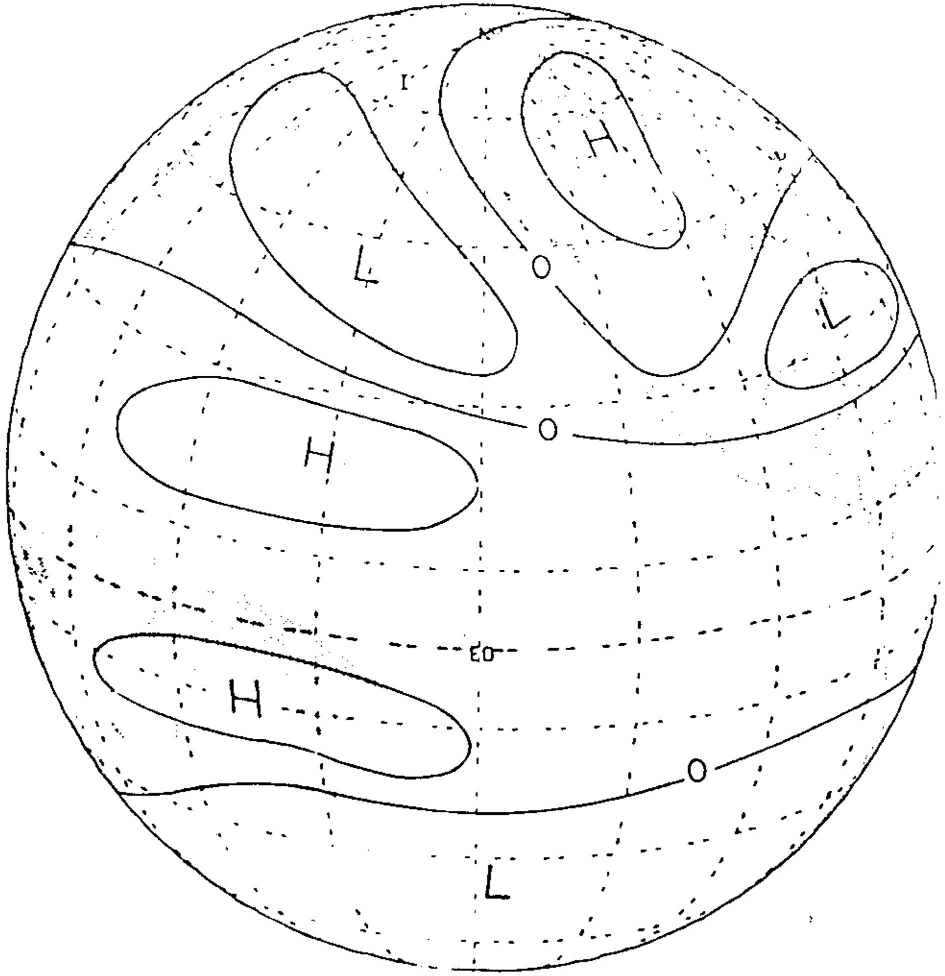


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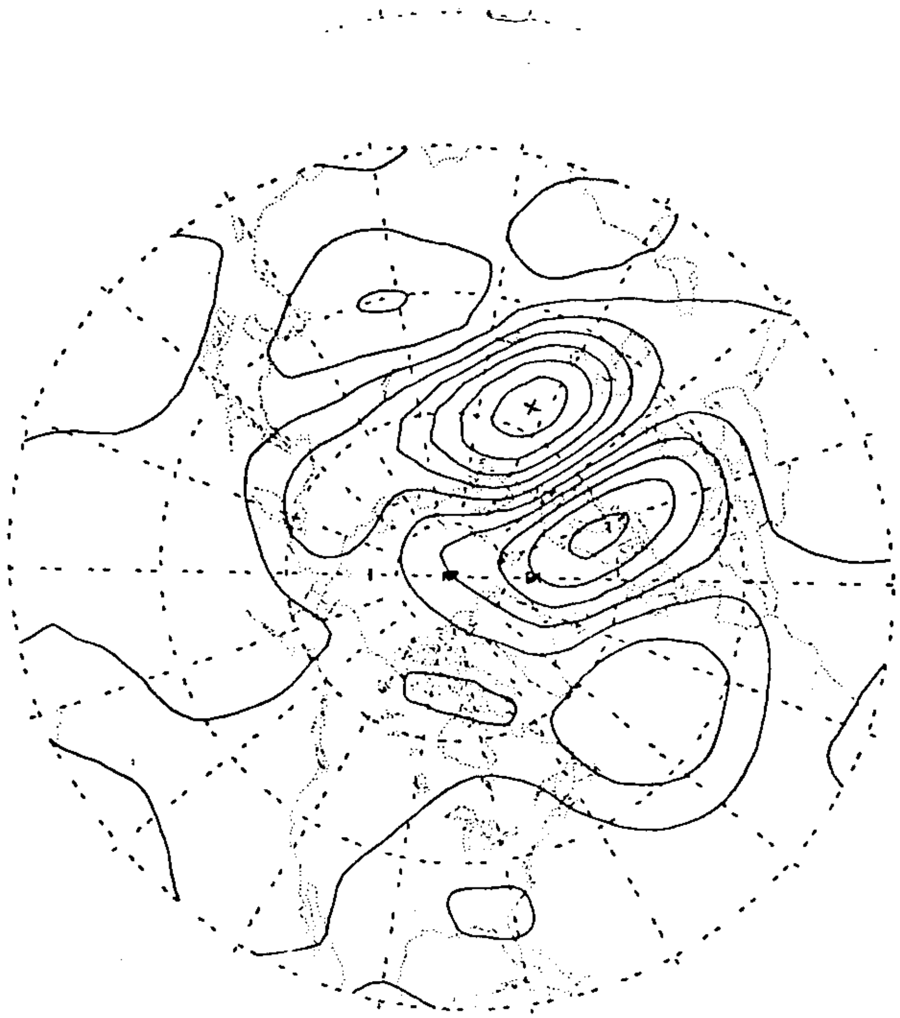
FIG 1

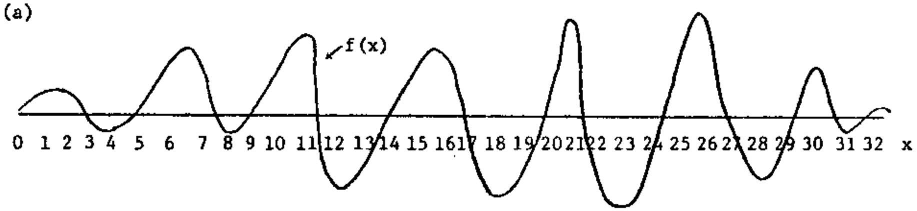


CONTOUR FROM -1.0000 TO 1.0000 CONTOUR INTERVAL OF .10000 PT(3,3)= 3.7975E-02

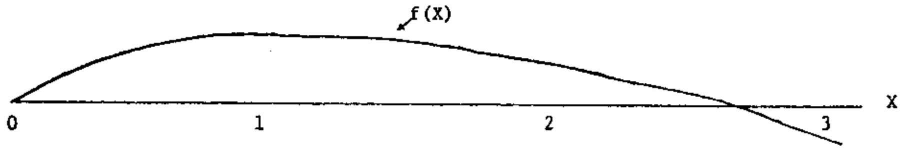
FIG 2

<u>The transformation $X = \tau x \quad \tau \ll 1$</u>

(a)



(b)



$X = \tau x$ where $\tau = 1/10$ (so when $x = 30$, $X = 3$).

In this example (b) is a magnification of (a) in the interval $0 < x \leq 3$.

It has been suggested that Rossby waves can be generated in low latitudes by thermal forcing due to anomalously warm tropical sea surface temperatures, and that Rossby wave activity will propagate into mid-latitudes along so-called 'ray paths' (lines parallel to the group velocity vector and have a significant effect on the evolution of regular mid-latitude synoptic systems. For such stationary forcing the direction of such ray paths must have an eastward component (see diagram).

5. <u>WKBJ (or 'slowly varying') approximation</u> (after Wentzel, Kramers, Brillouin and Jeffreys)

The equation

$$d^2y/dx^2 + k^2y = 0 \qquad k = \text{constant}$$

has wavelike solutions $y = Ae^{ikx}$. What about an equation like

$$d^2y/dx^2 + f^2(x)y = 0 \quad ; \tag{1}$$

under what conditions will this have wavelike solutions? If we substitute $y = e^{i\phi(x)}$ then the above differential equation becomes

$$-(d\phi/dx)^2 + i(d^2\phi/dx^2) + f^2 = 0 \tag{2}$$

If $d^2\phi/dx^2$ is small compared with the other terms in (2) then as a first guess we can completely neglect $d^2\phi/dx^2$ and (2) becomes

$$d\phi/dx = \pm|f| \tag{3}$$

Hence we require

$$\left|\frac{d^2\phi/dx^2}{f^2}\right| \ll 1$$

for (3) to be a good approximation, or, differentiating (3)

$$|df/dx| \ll f^2 \quad \text{(i.e., f is slowly varying)} \tag{4}$$

(From (1), $2\pi/f$ is roughly one wavelength in y, hence we require that the change in f over one wavelength in y should be small compared with $|f|$).

From (3) we have

$$\frac{d^2\phi}{dx^2} = \pm\frac{d|f|}{dx}$$

Substituting this into (2) we have a second order (and hence better) solution given by $(d\phi/dx)^2 = f^2 \pm i\, d|f|/dx$

i.e., using (4), $d\phi/dx = \pm|f| + \frac{i}{2}\frac{1}{|f|}\frac{d|f|}{dx}$ or $\phi = \pm\int|f|\,dx + \frac{i}{2}\ln|f|$

and the second order solution becomes the WKBJ solution

$$y(x) = \frac{A}{|f|^{1/2}}e^{\pm i\int|f|dx}$$

The solution fails if f changes too rapidly or if f passes through zero. A convenient form to express the WKBJ solution is in terms of a 'long' space variable $X = \tau x$, $\tau \ll 1$, over which f is presumed to have O(1) variation. (i.e., $|df/dX| \sim f^2$). In terms of this

$$y(x) = y_o(X)e^{i\Phi(X)/\tau}$$

where $\Phi(X) = \pm\int|f|dX$, $y_o(X) = A/|f|^{1/2}$

<u>Internal Gravity (or Buoyancy) Waves</u>

These waves exist in an atmosphere that is stably stratified so that a fluid parcel displaced vertically will undergo buoyancy oscillations with frequency N (see earlier discussion in the notes). We suppose that the horizontal scale L of these waves is sufficiently small that Ro >> 1 and the equations of motion can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \delta_x \tag{i}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \delta_z \tag{ii}$$

where, for simplicity we limit our discussion to two-dimensional internal gravity waves propagating in the x-z plane. The terms δ_x, δ_z stand for any additional terms (e.g., acceleration due to viscous forces) that may be present in the momentum equation. We shall also suppose that the vertical scale of the motions is much smaller than a typical scale-height, i.e., D << H, so that the continuity equation can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{iii}$$

Finally, the thermodynamic equation is

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \mu \tag{iv}$$

where μ is a diabatic term.

First of all write

$$p = \tilde{p}(z) + \hat{p} \qquad \rho = \tilde{\rho}(z) + \hat{\rho} \qquad \theta = \tilde{\theta}(z) + \hat{\theta}$$

where $\tilde{p}$, $\tilde{\rho}$, $\tilde{\theta}$ are hydrostatic values for a resting atmosphere (note the new not

Now in the horizontal momentum equation (1), the pressure gradient must be balanced against the inertial acceleration (Ro >> 1), i.e.,

$$U^2/L \sim \hat{p}/\tilde{\rho}L$$

where U is a measure of <u>u</u>. Hence using the gas law

$$\hat{p}/\tilde{p} \sim Ma^2 \sim 10^{-5} \quad \text{if} \quad U \sim 1\ m\,s^{-1}$$

On the other hand, the adiabatic thermodynamic equation will balance if

$$\frac{\hat{\theta}}{\tilde{\theta}}\frac{U}{L} \sim \hat{w}\frac{1}{\tilde{\theta}}\frac{d\tilde{\theta}}{dz}$$

Now from (iii), $\hat{w} \sim U(D/L)$ $\quad (\partial u/\partial x \sim U/L)$

hence the thermodynamic equation will balance if

$$\frac{\hat{\theta}}{\tilde{\theta}} \sim \frac{D}{\tilde{\theta}}\frac{d\tilde{\theta}}{dz} \sim \frac{DN^2}{g}$$

If D << H then $D \leq 10^3$ m. With $N^2 = 10^{-4}\ s^{-2}$, $D \sim 10^3$ m

$$\hat{\theta}/\tilde{\theta} \sim 10^{-2}$$

Hence

$$\hat{\theta}/\tilde{\theta} >> \hat{p}/\tilde{p}$$

However, from the definition of potential temperature and the ideal gas law

$$\ln\rho = \frac{1}{\gamma}\ln p - \ln\theta + \text{const.} \qquad \gamma = c_p/c_v$$

Putting $\rho = \tilde{\rho}\,(1 + \hat{\rho}/\tilde{\rho})$ etc., then $\frac{\hat{\rho}}{\tilde{\rho}} \simeq \frac{1}{\gamma}\frac{\hat{p}}{\tilde{p}} - \frac{\hat{\theta}}{\tilde{\theta}}$

Hence for these buoyancy wave motions density fluctuations due to pressure changes are small compared with those due to temperature changes.

Therefore we can put

$$\hat{\rho}/\tilde{\rho} = -\hat{\theta}/\tilde{\theta}$$

and treat density
as a constant except where it is coupled to gravity in the buoyancy term in the vertical momentum equation. This is known as the Boussinesq approximation.

We now define the overbar operator as an average along the x-axis, and assume that the deviations of fluid variables from their average value are small,

i.e., we write

$$u = \bar{u}_o + \varepsilon(\bar{u}_1 + u_1') + \dots$$
$$w = \varepsilon(\bar{w}_1 + w_1') + \dots$$
$$\rho = \bar{\rho}_o + \varepsilon(\bar{\rho}_1 + \rho_1') + \dots$$
$$p = \bar{p}_o + \varepsilon(\bar{p}_1 + p_1') + \dots$$
$$\theta = \bar{\theta}_o + \varepsilon(\bar{\theta}_1 + \theta_1') + \dots$$
$$\delta_x = (\bar{\delta}_x)_o + \varepsilon((\bar{\delta}_x)_1 + (\delta_x)_1') + \dots \qquad (\text{etc. for } \delta_z \ \&\ \mu)$$

Notice we have assumed that the flow is horizontal to O(1). We shall also assume that the O(1) flow is hydrostatic, i.e.,

$$\partial\bar{p}_o/\partial z = -g\bar{\rho}_o\ ,$$

and that the Boussinesq approximation holds.

To O(1) in the expansion, equations (i), (ii) and (iv) become, respectively

$$\frac{\partial\bar{u}_o}{\partial t} = (\bar{\delta}_x)_o \tag{0.1}$$

$$0 = (\bar{\delta}_z)_o \quad \text{(because the flow is hydrostatic)} \tag{0.2}$$

$$\frac{\partial\bar{\theta}_o}{\partial t} = \bar{\mu}_o \tag{0.3}$$

The $O(\varepsilon)$ mean flow equations are of no interest to us here since the waves do not couple to the mean flow to $O(\varepsilon)$. The $O(\varepsilon)$ deviation equations are the linearized perturbation equations and take the form

$$\frac{\partial u_1'}{\partial t} + \bar{u}_o\frac{\partial u_1'}{\partial x} + w_1'\frac{\partial\bar{u}_o}{\partial z} + \frac{1}{\bar{\rho}_o}\frac{\partial p_1'}{\partial x} = -\frac{1}{2}\,\delta u_1' \tag{1.1}$$

$$\frac{\partial w_1'}{\partial t} + \bar{u}_o\frac{\partial w_1'}{\partial x} + \frac{1}{\bar{\rho}_o}\frac{\partial p_1'}{\partial z} - \frac{\theta_1'}{\bar{\theta}_o}\,g = -\frac{1}{2}\,\delta w_1' \tag{1.2}$$

$$\frac{\partial u_1'}{\partial x} + \frac{\partial w_1'}{\partial z} = 0 \tag{1.3}$$

$$\frac{\partial\theta_1'}{\partial t} + \bar{u}_o\frac{\partial\theta_1'}{\partial x} + w'\frac{d\bar{\theta}_o}{dz} = -\frac{1}{2}\,\mu\theta_1' \tag{1.4}$$

In equations (1.1) and (1.2) we have written

$$(\delta_x')_1 = -\frac{1}{2}\,\delta u_1' \qquad (\delta_z')_1 = -\frac{1}{2}\,\delta w_1'$$

which is a common form of parameterizing the effect of eddy friction. The constant δ is known as the Rayleigh friction coefficient. Similarly in (1.4), the effect of diabatic cooling μ_1' has been parameterized using the Newtonian cooling coefficient μ. In (1.2) we have used the Boussinesq approximation to put $\rho_1'/\rho_o = \theta_1'/\theta_o$.

If we average (1), then, to $O(\varepsilon^2)$ the mean flow equation is

$$\frac{\partial \bar{u}_2}{\partial t} + \overline{w_1' \frac{\partial u_1'}{\partial z}} = (\bar{\delta}_x)_2$$

To get this we have used the Boussinesq approximation to put $\rho = \bar{\rho}_o$ in the pressure gradient term. Remember also that

$$\partial \bar{u}_o/\partial x = \partial \bar{u}_2/\partial x = 0$$

and

$$\overline{u_1' \frac{\partial u_1'}{\partial x}} = \overline{\frac{1}{2}\frac{\partial}{\partial x}(u_1')^2} = 0$$

Furthermore, using the continuity equation (1.3) we can write the $O(\varepsilon^2)$ mean flow equation as

$$\frac{\partial \bar{u}_2}{\partial t} + \frac{\partial}{\partial z}(\overline{u_1'w_1'}) = (\bar{\delta}_x)_2 \tag{2.1}$$

Hence, if the Reynolds stress $\overline{u'w'}$ varies in the vertical, then the perturbation induce an acceleration in the mean flow, to $O(\varepsilon^2)$. The right hand side of (2.1) represents the effect of mean flow viscosity.

A wave energy equation can be formed by multiplying (1.1) by u' and (1.2) by w', adding these together, and averaging, i.e.,

$$\frac{1}{2}\frac{\partial}{\partial t}(\overline{u_1'^2}+\overline{w_1'^2}) + \overline{u_1'w_1'}\frac{\partial \bar{u}_o}{\partial z} + \frac{1}{\bar{\rho}_o}\overline{u_1'\frac{\partial p_1'}{\partial x}} + \frac{1}{\bar{\rho}_o}\overline{w_1'\frac{\partial p_1'}{\partial z}} - \frac{\overline{w_1'\theta_1'}}{\bar{\theta}_o}g = -\frac{1}{2}\delta(\overline{u_1'^2}+\overline{w_1'^2})$$

Using (1.3)

$$\frac{1}{2}\frac{\partial}{\partial t}(\overline{u_1'^2}+\overline{w_1'^2}) + \overline{u_1'w_1'}\frac{\partial \bar{u}_o}{\partial z} + \frac{1}{\bar{\rho}_o}\frac{\partial}{\partial z}(\overline{w_1'p_1'}) - \frac{\overline{w_1'\theta_1'}}{\bar{\theta}_o}g = -\frac{1}{2}\delta(\overline{u_1'^2}+\overline{w_1'^2}) \tag{2.2}$$

Multiplying 1.4 x θ'

$$\frac{1}{2}\frac{\partial}{\partial t}(\overline{\theta_1'^2}) + \overline{w_1'\theta_1'}\frac{d\bar{\theta}_o}{dz} = -\frac{1}{2}\mu\overline{\theta_1'^2} \Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\overline{B_1'^2}}{N^2}\right) + \frac{\overline{w_1'\theta_1'}}{\bar{\theta}_o}g = -\frac{1}{2}\mu\frac{\overline{B_1'^2}}{N^2} \tag{2.3}$$

where

$$B_1' = \theta_1' g/\bar{\theta}, \quad N^2 = \frac{g}{\bar{\theta}_o}\frac{d\bar{\theta}_o}{dz}$$

Adding 2.2 and 2.3 we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\overline{u_1'^2} + \overline{w_1'^2} + \frac{\overline{B_1'^2}}{N^2}\right) + \frac{1}{\bar{\rho}_o}\frac{\partial}{\partial z}(\overline{w_1'p_1'}) + (\overline{u_1'w_1'})\frac{\partial u_o}{\partial z} = -\frac{1}{2}\left\{\delta(\overline{u_1'^2}+\overline{w_1'^2}) + \mu\frac{\overline{B_1'^2}}{N^2}\right\} \quad (2.$$

The number of variables in (1.1) to (1.4) can be reduced by introducing the y component of perturbation vorticity

$$\eta_1' = \partial w_1'/\partial x - \partial u_1'/\partial z$$

and y component of mean flow vorticity

$$\bar{\eta}_o = -\partial \bar{u}_o/\partial z$$

Then $\partial/\partial x(1.2) - \partial/\partial z(1.1)$ gives

$$\left(\frac{\partial}{\partial t} + \bar{u}_o\frac{\partial}{\partial x}\right)\eta_1' - \left(\frac{\partial \bar{u}_o}{\partial z}\frac{\partial u_1'}{\partial x} + \frac{\partial w_1'}{\partial z}\frac{\partial \bar{u}_o}{\partial z}\right) - w_1'\frac{\partial^2 \bar{u}_o}{\partial z^2}\left(+\frac{1}{\bar{\rho}_o}\frac{\partial^2 p_1'}{\partial z \partial x} - \frac{1}{\bar{\rho}_o}\frac{\partial^2 p_1'}{\partial z \partial x}\right) - \frac{\partial B_1'}{\partial x} = -\frac{\delta}{2}\eta$$

which, using the continuity equation, gives

$$\left(\frac{\partial}{\partial t} + \bar{u}_o\frac{\partial}{\partial x}\right)\eta_1' + w_1'\frac{\partial \bar{\eta}_o}{\partial z} - \frac{\partial B_1'}{\partial x} = -\frac{\delta}{2}\eta_1' \tag{1.5}$$

The thermodynamic equation, (1.4), can be written

$$\left(\frac{\partial}{\partial t} + \bar{u}_o\frac{\partial}{\partial x}\right)B_1' + N^2 w_1' = -\frac{1}{2}\mu B_1' \tag{1.6}$$

Finally, from (1.3) we can introduce a streamfunction ψ, such that

$$\left.\begin{aligned} u_1' &= -\partial\psi_1'/\partial z \\ w_1' &= \partial\psi_1'/\partial x \end{aligned}\right\} \qquad \eta_1' = \nabla^2\psi_1' \tag{1.7}$$

and (1.5) and (1.6) are two coupled equations for the two perturbation variables, ψ_1' and B_1'.

To start with consider a fluid with constant (positive) static stability and constant O(1) zonal velocity $\bar{u}_o$. Suppose the fluid is bounded below by a regularly corrugated surface with sinusoidal variation in the x-direction with fixed wavelength $2\pi/k$, no variation in the y-direction, and moving in the +x-direction with fixed speed $\omega/k > 0$ (see earlier notes) Suppose also that δ and μ are equal to zero.

Using (1.7) we can substitute the expressions

$$\psi_1' = \mathrm{Re}\,\{\psi\, e^{i(kx-\omega t+mz)}\}$$

$$B_1' = \mathrm{Re}\,\{B\, e^{i(kx-\omega t+mz)}\}$$

into (1.5) and (1.6) which gives

$$-(\omega - k\bar{u}_o)(k^2+m^2)\psi + kB = 0$$

$$(\omega - k\bar{u}_o)B = kN^2\psi$$

which can be combined to give the real dispersion relation

$$\hat{\omega} = \pm\, kN/\sqrt{k^2+m^2} \qquad (1.8)$$

so that $\hat{\omega} = \omega - k\bar{u}_o$ is the wave frequency relative to the zonal mean wind.

The vertical phase speed is

$$\omega/m = k\bar{u}_o/m \pm kN/m\sqrt{k^2+m^2}$$

i.e., relative to the mean wind the vertical phase speed is

$$c^z = \hat{\omega}/m = \pm\, kN/m\sqrt{k^2+m^2}$$

The vertical group velocity is

$$c_g^z = \partial\hat{\omega}/\partial m = \partial\omega/\partial m = \mp\, mkN/(k^2+m^2)^{3/2} = -\,c^z\,\frac{m^2}{k^2+m^2} \qquad (1.9)$$

Notice that downward phase propagation in a frame moving with the fluid
$\Rightarrow$ upward group velocity, (and vice versa)

Other properties of note are

(1) From the dispersion relation wavelike solutions only obtain if $N^2 \geq \hat{\omega}^2 > 0$.

(2) When $\hat{\omega} \to 0$, then, according to the dispersion relation, $m \to \infty$ (i.e., the vertical wavelength $\to 0$). When $\hat{\omega} \to N$ then $m \to 0$, i.e., phase lines become vertically oriented. In general if $\hat{\omega}^2$ decreases, then m^2 must increase, and the lines of constant phase become more horizontally aligned.

(3) A straightforward calculation gives $\partial(c_g^z)^2/\partial m^2 < 0$ if $\hat{\omega}^2 < 2/3\,N^2$. In other words providing $\hat{\omega}^2 < 2/3\,N^2$ then if m^2 increases, $(c_g^z)^2$ decreases. Combining this with the result in (2), then $(c_g^z)^2$ varies directly with $\hat{\omega}^2$ if $\hat{\omega}^2 < 2/3\,N^2$.

Now suppose that instead of the basic state mean flow having no vertical shear and the waves are inviscid, we allow, in the spirit of the WKBJ approximation, a small amount of vertical shear $\partial\bar{u}_o/\partial z$ (i.e., a slowly varying mean wind $\bar{u}_o$) and a small amount of Rayleigh friction and Newtonian cooling. Formally, this can be achieved by introducing a small parameter $\tau \ll 1$ and introducing a long space variable

$$Z = \tau z$$

so that

$$\bar{u} = \bar{u}(Z) \qquad \delta = \tau\delta_o \qquad \mu = \tau\mu_o \qquad (\delta_o, \mu_o = O(1))$$

and seek WKBJ solutions

$$\psi_1' = \mathrm{Re}\left\{\psi(Z)e^{i(kx-\omega t+\Phi/\tau)}\right\} \qquad B_1' = \mathrm{Re}\left\{B(Z)e^{i(kx-\omega t+\Phi/\tau)}\right\}$$

where

$$m(Z) = \partial\Phi/\partial Z \left(= \frac{1}{\tau}\frac{\partial\Phi}{\partial z}\right)$$

Now if we substitute these solutions into 1.5, 1.6 and 1.7, then, to lowest order in τ

$$\frac{\partial}{\partial z}\left\{\psi(Z)e^{i(kx-\omega t+\Phi/\tau)}\right\} = i\,\frac{1}{\tau}\frac{\partial\Phi}{\partial z}\,\psi(Z)e^{i(kx-\omega t+\Phi/\tau)} + \tau\,\frac{\partial\psi}{\partial Z}\,e^{i(kx-\omega t+\Phi/\tau)}$$

$$= i\,m(Z)\;\psi(Z)e^{i(kx-\omega t+\Phi/\tau)} + O(\tau)$$

i.e., to lowest order in τ there is no difference between the WKBJ solution and the elementary solution with m = constant. In particular, the dispersion relation 1.9 and conclusions (1)-(3) will all hold to lowest order in τ.

A difference arises; however, when we consider the energy equation, which, since we assume a steady state (i.e., no time variation in wave amplitudes) becomes

$$\frac{1}{\rho_o}\frac{\partial}{\partial z}\,(\overline{w_1'p_1'}) + (\overline{u_1'w_1'})\,\frac{\partial\bar{u}_o}{\partial z} = -\frac{1}{2}\left\{\delta(\overline{u_1'^2} + \overline{w_1'^2}) + \mu\,\frac{\overline{B_1'^2}}{N^2}\right\} \qquad (2.5)$$

Now since

$$u_1' = -\,\partial\psi_1'/\partial z$$

$$w_1' = \partial\psi_1'/\partial x$$

then

$$(\overline{u_1'^2}) + (\overline{w_1'^2}) = \overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2} + \overline{\left(\frac{\partial\psi_1'}{\partial z}\right)^2} = (k^2+m^2)|\psi|^2/2 \qquad (2.6)$$

Since $\mu = O(\tau)\mu_o$, then, from 1.6, to lowest order in τ

$$(\omega - k\bar{u}_o)B - kN^2\psi = 0$$

i.e.,

$$\frac{\overline{B_1'^2}}{N^2} = \frac{|B|^2}{2N^2} = \frac{|\psi|^2 k^2 N^2}{2\hat{\omega}^2} = (k^2 + m^2)|\psi|^2/2 \qquad (2.7)$$

using the dispersion relation (1.8).

Again since $\delta = O(\tau)\delta_o$, then to lowest order in τ 1.1 is

$$\left(\frac{\partial}{\partial t} + \overline{u}_o\frac{\partial}{\partial x}\right) u_1' = -\frac{1}{\overline{\rho}_o}\frac{\partial p_1'}{\partial x}$$

i.e.,

$$\hat{\omega}u_1' = \frac{k}{\overline{\rho}_o} p_1'$$

or

$$P = -\frac{\overline{\rho}_o\hat{\omega}}{k}(-im\psi) \qquad \text{where } p_1' = \mathrm{Re}[P(z)e^{i(kx - \omega t + \phi/\tau)}]$$

so that

$$\overline{w_1'p_1'} = -\overline{\rho}_o m\hat{\omega}|\psi|^2/2 \qquad (2.8)$$

Finally

$$\overline{u_1'w_1'} = -mk|\psi|^2/2 \qquad (2.9)$$

Substituting 2,6, 2.7, 2.8 and 2.9 into 2.5 and defining

$$\alpha = \frac{1}{2}(\delta + \mu)$$

we get

$$\frac{\partial}{\partial z}(m\hat{\omega}|\psi|^2) + mk|\psi|^2\frac{\partial\overline{u}_o}{\partial z} = \alpha(k^2 + m^2)|\psi|^2 \qquad (2.10)$$

Now for the simple case with no shear and no dissipation then both sides of this equation are trivially zero since m and $|\psi^2|$ would be constants. With weak shear and weak dissipation this equation gives an expression for how wave amplitudes must vary with height. Notice that 2.10 can be written purely in terms of the long space coordinate (since m, ω, u_o and ψ are all functions of Z only) i.e.

$$\frac{d}{dZ}(m\hat{\omega}|\psi|^2) + mk|\psi|^2\frac{d\overline{u}_o}{dZ} = \alpha_o(k^2 + m^2)|\psi|^2 \qquad (2.11)$$

where $\alpha = \tau\alpha_o$.

Now if we define a quantity

$$A = (k^2 + m^2)|\psi|^2/\hat{\omega}$$

called wave-action, then from the definition of group velocity c_g^z we can write (from (1.8) and (1.9))

$$c_g^z A = -m|\psi|^2$$

and 2.11 can be written as

$$\frac{d}{dZ}(c_g^z A\hat{\omega}) + kc_g^z A\frac{d\overline{U}_o}{dZ} = -\alpha_o A\hat{\omega}$$

But

$$\frac{d\hat{\omega}}{dZ} = \frac{d\omega}{dZ} - k\frac{d\overline{U}_o}{dZ} = -k\frac{d\overline{U}_o}{dZ}$$

Hence

$$\frac{d}{dZ}(c_g^z A) = -\alpha_o A \qquad (2.12)$$

which is a differential equation for the depletion of wave-action due to the combined effects of dissipation and diabatic cooling.

Writing 2.12 as

$$\frac{d}{dZ}(c_g^z A) = -(c_g^z A)/D$$

where $D = c_g/\alpha_o$, then the solution

$$Ac_g^z\Big|_z = Ac_g^z\Big|_{z=0} e^{-\int_0^z D^{-1}dz}$$

of 2.12 is easily obtained. Furthermore, since the wave momentum flux

$$\overline{u_1'w_1'} = -\frac{1}{2}km|\psi|^2 = \frac{1}{2}kAc_g^z$$

then 2.6 can also be written as

$$\overline{u_1'w_1'}\Big|_z = \overline{u_1'w_1'}\Big|_{z=0} e^{-\int_0^z D^{-1}dz}$$

and according to 2.1, the waves induce an $O(\varepsilon^2)$ acceleration

$$-\frac{\partial}{\partial z}\,(\overline{u_1'w_1'}) = D^{-1}\,\overline{u_1'w_1'}\Big|_{z=0}\, e^{-\int_0^z D^{-1}dz}$$

to the mean flow.

The quantity D is a height scale for wave dissipation. If α is small <u>or</u> c_g is large (so that D is large) the waves propagate a large depth before the momentum flux is substantially depleted. Consequently the mean flow is accelerated weakly over a large depth of the fluid. Conversely if α is large <u>or</u> c_g is small then the waves are dissipated over a shallow depth of the fluid and the mean flow acceleration is concentrated in that shallow depth.

We now recall the following pieces of information:

(1) For fixed α, the dissipation height scale D varies directly with c_g.

(2) For $\hat{\omega}^2 < 2/3\, N^2$, c_g^z varies directly with $\hat{\omega}$.

(3) For waves with positive (Doppler shifted) frequency $\hat{\omega}$, wave, mean-flow interaction through wave-dissipation will decrease $\hat{\omega}$ (i.e., the wave momentum flux will tend to accelerate the mean flow to the speed of the corrugated boundary which forces the waves).

This gives us a positive feedback loop. If $\hat{\omega}$ decreases then from (2) c_g^z decreases, so from (1) D decreases. The waves therefore dissipate over a smaller depth above the lower boundary. Hence the mean flow accelerates more strongly in this region -- hence $\hat{\omega}$ decreases more strongly in this region: so c_g^z decreases more strongly -- hence D decreases still further.....until finally $\hat{\omega}$ is reduced to zero immediately above the lower boundary and no further wave propagation can occur.

A surface where $\hat{\omega} = 0$ is called a critical surface, and according to the wave dispersion relation, $m^2 \to \infty \Rightarrow c_g^z \to 0$, i.e., $D \to 0$. Hence for any non-zero value of α, critical layers are (for linear theory) strong absorbers of wave-activity.

If the boundary executes a standing wave oscillation

$$z = A \sin k(x + ct) + A \sin k(x - ct)$$

where $c = \omega/k$, then even when the component moving with phase speed $+c$ is trapped, the component moving with phase speed $-c$ is not trapped -- relative to this phase speed the Doppler shifted frequency is 2ω. This wave can induce wave mean flow deceleration, resulting in the vacillating mean flow described in the earlier notes, and demonstrated by the Plumb and McEwan experiment.

WKBJ analysis is, in fact, rather more powerful than we have so far been able to show. It turns out that it is possible to carry out a WKBJ analysis to the full nonlinear internal gravity wave equations. Under such circumstances wave, mean-flow interaction would be accounted for to all orders of τ, and due to such interaction the waves and mean-flow would be slowly varying in time. Under these circumstances 2.12 generalizes to

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}\,(c_g^z A) = -\,\alpha_o A \tag{2.13}$$

where $A = A(T,Z)$

and $T = \mu t$

Equation 2.13 is the full equation for conservation of wave action. In the absence of any dissipation or diabatic cooling then

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}\,(c_g^z A) = 0$$

so that the rate of change of wave-action in any depth $\Delta z = z_2 - z_1$ of the fluid

$$= \frac{\partial}{\partial t}\int_{z_1}^{z_2} A dz = \tau \int_{z_1}^{z_2} \frac{\partial A}{\partial T}\, dz = -\tau \int_{z_1}^{z_2} \frac{\partial}{\partial Z}\,(c_g^z A) dz$$

$$= -\int_{z_1}^{z_2} \frac{\partial}{\partial z}\,(c_g^z A) dz = c_g^z A\Big|_{z=z_1} - c_g^z A\Big|_{z=z_2}$$

i.e., the wave-action in any depth Δz will change solely due to the advection of wave-action through the boundary of Δz by the group velocity c_g^z.

Appendix

1. Non-acceleration theorem for internal gravity waves.

Multiply (1.5) by B_1' and average

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{B_1'^2} + \overline{w_1'B_1'}\,N^2 = -\frac{1}{2}\mu\,\overline{B_1'^2} \tag{A.1}$$

Multiply (1.6) by η_1' and average

$$\overline{\eta_1'\frac{\partial B_1'}{\partial t}} + \bar{u}_o\,\overline{\eta_1'\frac{\partial B_1'}{\partial x}} + N^2\,\overline{w_1'\eta_1'} = -\frac{1}{2}\mu\,\overline{B_1'\eta_1'} \tag{A.2}$$

Multiply (1.5) by B_1' and average

$$\overline{B_1'\frac{\partial \eta_1'}{\partial t}} + \bar{u}_o\,\overline{B_1'\frac{\partial \eta_1'}{\partial x}} + \overline{w_1'B_1'}\,\frac{\partial\bar{\eta}_o}{\partial z} = -\frac{1}{2}\delta\,\overline{B_1'\eta_1'} \tag{A.3}$$

Add A2 and A3

$$\frac{\partial}{\partial t}(\overline{\eta_1'B_1'}) + \overline{w_1'B_1'}\,\frac{\partial\bar{\eta}_o}{\partial z} + \overline{w_1'\eta_1'}\,N^2 = -\alpha\,\overline{B_1'\eta_1'} \tag{A.4}$$

From (A.1), $\overline{w_1'B_1'}$ is zero if the waves are steady and non-dissipative. Hence, from (A.4) $\overline{w_1'\eta_1'}$ is zero if the waves are steady and non-dissipative.

But

$$\overline{w_1'\eta_1'} = -\overline{w_1'\partial u_1'}/\partial z = -\frac{\partial}{\partial z}(\overline{u_1'w_1'})$$

Hence $(\partial/\partial z)(\overline{u_1'w_1'})$ is non-zero if the waves are steady and non-dissipative.
From (2.1)

$$\frac{\partial\bar{u}_2}{\partial t} + \frac{\partial}{\partial z}(\overline{u_1'w_1'}) = (\bar{\delta}_x)_2$$

Hence to $O(\varepsilon^2)$ the waves cannot change the mean zonal flow if they are steady and non-dissipative.

This is called the non-acceleration theorem for internal gravity waves.
There is a similar theorem for quasigeostrophic waves.

2. Conservation of wave-action for a basic state which is slowly varying in time.

Put $T = \tau t$

$Z = \tau z \qquad \tau \ll 1$

$\bar{u}_o = \bar{u}_o(Z,T) \qquad \alpha = \tau\alpha_o$

and seek WKBJ solutions

$$\psi_1' = \mathrm{Re}\,\{\psi(Z,T)e^{i(kx+\Phi(Z,T)/\tau)}\}$$

$$B_1' = \mathrm{Re}\,\{B(Z,T)e^{i(kx+\Phi(Z,T)/\tau)}\}$$

and

$$\omega(Z,T) = -\partial\Phi/\partial T \qquad m(Z,T) = \partial\Phi/\partial Z$$

n.b.

$$\partial\omega/\partial Z + \partial m/\partial T = 0 .$$

With $\hat{\omega} = \omega - k\bar{u}_o$, then since, from the dispersion relation $\hat{\omega} = \hat{\omega}(m)$

$$\frac{\partial\hat{\omega}}{\partial T} = \frac{\partial\hat{\omega}}{\partial m}\frac{\partial m}{\partial T} = c_g^z\frac{\partial m}{\partial T} = -c_g^z\frac{\partial\omega}{\partial Z} \tag{A.5}$$

from the above relation.

The energy equation (2.4) is

$$\frac{\partial}{\partial t}\{(k^2+m^2)|\psi|^2\} - \frac{\partial}{\partial z}\{m\hat{\omega}|\psi|^2\} - mk|\psi|^2\frac{\partial\bar{u}_o}{\partial Z} = -\alpha(k^2+m^2)|\psi|^2 \tag{A.6}$$

As before define the wave-action

$$A = (k^2+m^2)|\psi|^2/\hat{\omega} \Rightarrow c_g^zA = -m|\psi|^2$$

Substituting this into (2.6) and noting that each term is $O(\tau)$ then

$$\frac{\partial}{\partial T}\{A\hat{\omega}\} + \frac{\partial}{\partial Z}\{c_g^zA\hat{\omega}\} + c_g^zAk\frac{\partial\bar{u}_o}{\partial Z} = -\alpha_oA\hat{\omega} \tag{A.7}$$

But using (A.5) and the identity

$$k\frac{\partial\bar{u}_o}{\partial Z} = \frac{\partial\omega}{\partial Z} - \frac{\partial\hat{\omega}}{\partial Z} \Rightarrow c_g^zk\frac{\partial\bar{u}_o}{\partial Z} = -\frac{\partial\hat{\omega}}{\partial T} - c_g^z\frac{\partial\hat{\omega}}{\partial Z}$$

(A.7) becomes, dividing by

$$\frac{1}{\hat{\omega}}\frac{\partial}{\partial T}(A\hat{\omega}) - A\frac{1}{\hat{\omega}}\frac{\partial\hat{\omega}}{\partial T} + \frac{1}{\hat{\omega}}\frac{\partial}{\partial Z}\{c_g^z A\hat{\omega}\} - c_g^z A\frac{1}{\hat{\omega}}\frac{\partial\hat{\omega}}{\partial Z} = -\alpha_o A$$

which can be rearranged to give

$$\boxed{\frac{\partial A}{\partial T} + \frac{\partial}{\partial Z}(c_g^z A) = -\alpha_o A}$$

which is an equation for conservation of wave-action in the absence of dissipation.

1. Show from the vector momentum equation in a rotating frame of reference that for an atmosphere which is motionless in the rotating frame, the pressure and density fields p_o and ρ_o must satisfy the equation for hydrostatic equilibrium

$$\nabla p_o = \rho_o \underline{g} \qquad (1)$$

where $\underline{g}$ is effective gravity.

By taking the curl of (1), show that a state of exact hydrostatic equilibrium is impossible unless there are no variations of density on surfaces of constant geopotential.

Assuming that the continuity equation can be written in the form

$$\nabla_h \cdot \underline{u}_h + \frac{1}{\rho_o}\partial(\rho_o w)/\partial z = 0$$

show that the ratio of the terms dw/dt and $\rho_o^{-1}\partial p'/\partial z$ in the vertical momentum equation is of order

(a) $\delta^2 Ro^2$ if $Ro \ll 1$

(b) δ^2 or less if $Ro \gg 1$

Here p' is the pressure deviation from a motionless atmosphere, δ is the ratio of vertical to horizontal length scales of the fluid motions, and Ro is the Rossby number.

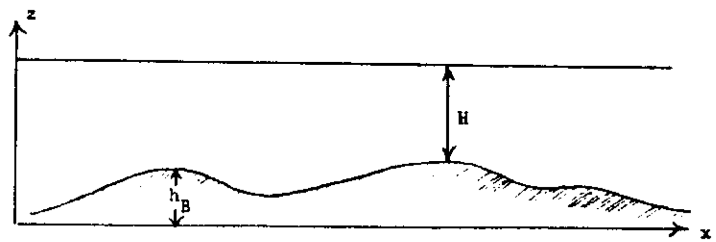
Explain what is meant by the statement that fluid motions are hydrostatic in a thin atmosphere.

2. State the conditions under which the Ertel potential vorticity

$$\Pi = \frac{\underline{\omega}_a \cdot \nabla \Lambda}{\rho}$$

is conserved, where $\underline{\omega}_a$ is the absolute vorticity and Λ is some scalar property of the fluid.

Consider homogeneous incompressible fluid flow of variable depth H(x) over a lower boundary given by $z = h_B(x)$



2. (continued)

If there is no fluid motion perpendicular to the (x-z) plane, and the x-component of fluid velocity is independent of height, give a physical explanation of why you would expect

$$\frac{d}{dt}\left(\frac{z - h_B(x)}{H(x)}\right) = 0$$

Hence show that with $\Lambda = (z - h_B)/H$ the Ertel potential vorticity is

$$\Pi = (\omega_a)^z/H \qquad (2)$$

and satisfies $d\Pi/dt = 0$.

A uniform and steady west to east flow of magnitude U_o with vertically and horizontally uniform flow impinges on a large-scale midlatitude N-S extending mountain range. The height of the mountain range is 5 km above mean sea level, and upstream of the mountain range the flow extends from mean sea level to a height of 10 km. Putting $(\omega_a)^z = \zeta + f$ in (2) show that an estimate of the radius of flow curvature near the summit is $2U_o/f$, assuming that the meridional deflection of the streamlines from their upstream latitudes to the summit is small.

3. Give a physical explanation of the mechanism for Rossby wave oscillations and why their phase speed in the x-direction must be westward relative to the basic state flow.

Writing the geostrophic streamfunction

$$\psi = -Uy + \varepsilon\psi' \qquad \varepsilon \ll 1, \quad U = \text{const.}$$

the barotropic vorticity equation for stationary perturbations $\psi' = \psi'(x)$ in the presence of Ekman friction can be written, to $O(\varepsilon)$, as

$$\frac{d^2\psi'}{dx^2} + \frac{\beta}{U}\psi' + \left(\frac{fde}{2HU}\right)\frac{d\psi'}{dx} = 0 \qquad (3)$$

where de is the depth of the Ekman layer and H is an atmospheric scale height.

Show that for synoptic scales with de = 1 km

$$\left(\frac{fde}{2HU}\right)^2 \Big/ \frac{4\beta}{U} \sim 10^{-1}$$

3. (continued)

Hence demonstrate that equation (3) has solutions

$$\psi' = \text{Re}\,\{Ae^{i(\pm(\beta/U)^{1/2} + ifde/4HU)x}\}$$

Using this, find an expression for the number of observable wavelengths produced in a steady and uniform west-to-east flow downstream of a N-S extending mountain range of unbounded latitudinal extent. Assume that waves can only be observed within one decay length 2HU/fde of the mountain range.

4. For steady non-dissipative linear Boussinesq internal gravity wave oscillations propagating in the (x-z) plane on a basic state u_o = constant, the streamfunction ψ_1' can be written

$$\psi_1' = \text{Re}\,\{\psi_o\, e^{i(kx + mz - \omega t)}\}$$

where k, m, ω and ψ_o are cosntants.

Show that the group velocity relative to the basic state flow is parallel to lines of constant phase.

Show that

$$\overline{u_1'w_1'} = \frac{1}{2}kAc_g^z$$

where

$$A = (k^2 + m^2)|\psi_o|^2/(\omega - ku_o)$$

is the wave-action.

If the phase speed of the wave in the x-direction is positive, what sign must m have to ensure upward propagation of wave-action?

If the basic state flow has a small vertical shear and the waves are weakly dissipative, the equation for depletion of wave-action by the (small) dissipation coefficient α is

$$\frac{d}{dz}(c_g^z A) = -\alpha A \quad .$$

Show from this that the eddies induce a change in the basic state mean flow proportional to

$$X = D^{-1} e^{-\int_0^z D^{-1} dz}$$

where $D = c_g^z/\alpha$. Why is the restriction of weak vertical shear and small dissipation necessary.

Treating c_g^z and α as constants, sketch the function X as a function of z (a) for large c_g^z (b) for small c_g^z.

Explain why critical lines are good absorbers of wave activity.

WAVE, MEAN-FLOW INTERACTION FROM EULERIAN AND LAGRANGIAN VIEWPOINTS

INTRODUCTION

A study of the mechanics of wave-like motions in the atmosphere can be divided in two parts, kinematics and dynamics. Kinematics can be thought of as the format in which you input and output information about the system under investigation, to and from the dynamics, the latter being a kind of 'central processor'.

A kinematical specification is important. It can make the difference between

1) mathematical analysis being easy or intractable
2) observational diagnostics being straightforward or impossible to calculate
3) physical interpretation being obvious or obscure.

Unfortunately 1), 2) and 3) are not always compatible.

Since the earth's rotation plays an important role in the dynamics of large scale atmospheric waves, one kinematical specification of an atmospheric fluid variable ψ is to write it as the sum of a zonal mean and a deviation from the zonal mean

$$\psi(z,\phi,\lambda) = \overline{\psi}(z,\phi) + \psi'(z,\phi,\lambda)$$

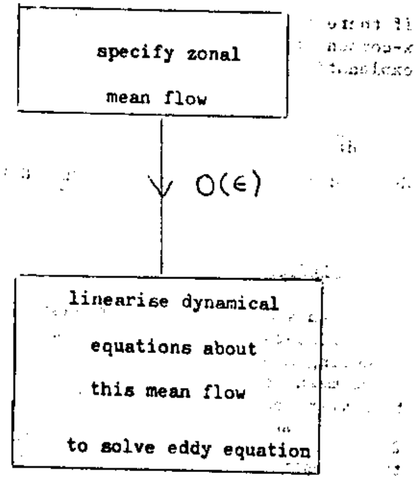
where $\overline{\psi}$ is the Eulerian mean of ψ round the latitude circle ϕ = constant, λ being longitude, z height above the earth's surface.

In analytical models, it is often assumed that ψ' is a small perturbation, ie

$$\psi' = O(\epsilon)\,\overline{\psi} \quad , \epsilon \ll 1$$

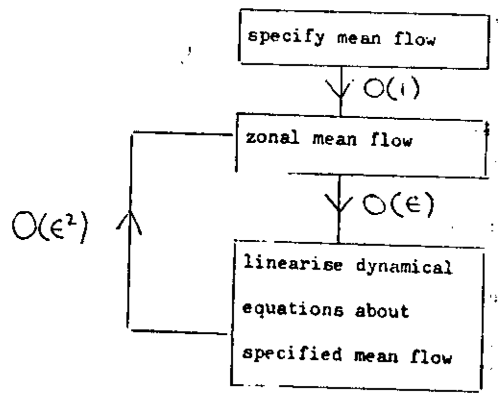
We call the primed variables, eddies.

To $O(\epsilon)$ the dynamics are linearised, and the Eulerian mean kinematical description takes the form



Waves are created (eg by instability) or modified (eg from wave-like forcing at a boundary) by the specified mean flow.

Since terms like $\overline{\psi'^2} \neq 0$, then to $O(\epsilon^2)$, the waves react back on the mean flow, ie



For finite amplitude waves (all wave amplitudes are finite in practice) we have a continuous feedback system, comprising the problem of

> wave, mean-flow interaction
>
> (i) How the waves change the mean flow
>
> (ii) How mean-flow profiles react back on the waves.

Whether (i) comes before (ii) or vice-versa is really a chicken-and-egg argument.

In these lectures I hope to show that

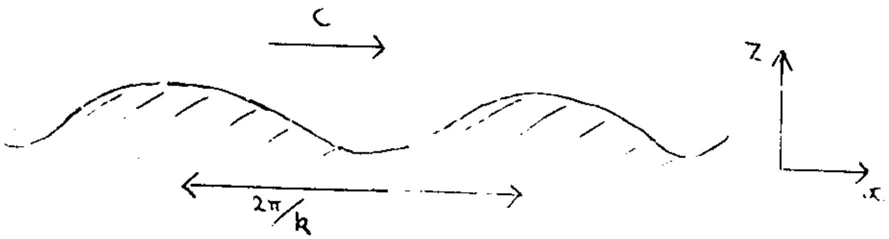
a) Wave mean-flow interaction is important in the atmosphere, and can result in some rather surprising fluid motions.

b) Blind use of the Eulerian mean kinematic description in rotating fluids can lead to results which are difficult to interpret physically, and only until recently have caused some confusion in the literature.

In order to convince you of points a) and b), I shall consider rather simple examples - the internal gravity wave, and the inertio-gravity wave. These waves illustrate the essentials of wave, mean-flow interaction occurring in the more complex atmospheric wave motions.

Internal gravity waves

We consider 2-dimensional steady, adiabatic buoyancy waves generated by a slippery boundary moving parallel to itself with constant velocity c (= phase speed of waves). Assume initial 'zonal' velocity (spacial average along z = constant) is zero.



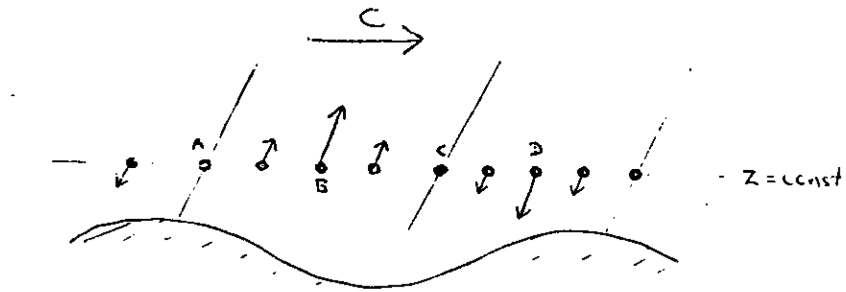
In order that buoyancy waves are generated and propagate upwards, the forcing frequency $\omega = ck$ must satisfy

$$0 < \omega \leqslant N$$

where N is the Brunt-Vaisalla frequency. These inequalities can be derived from the dispersion relation of the eddy wave equation, but can be seen by considering the analogy of a pendulum with natural frequency N, being forced by your hand (say) oscillating with frequency ω. If $\omega = 0$ nothing happens; if $\omega = N$ the bob oscillates at the natural frequency of the pendulum; if $\omega > N$ you destroy the oscillatory motion of the bob. So with the particles of the fluid. Note that if the initial zonal velocity of the fluid, $\bar{u}$, $\neq 0$ then the condition for no wave forcing is

$$\omega_{\text{DOPPLER}} \equiv (c - \bar{u})k = 0$$

With a little thought, we can draw the eddy velocities of the fluid particles as they respond to wave forcing and buoyancy forces



Particle A is at the top of its oscillation

Particle B is at the mid-point of its oscillation travelling up

Particle C is at the bottom of its oscillation

Particle D is at the mid-point of its oscillation travelling down.

hence

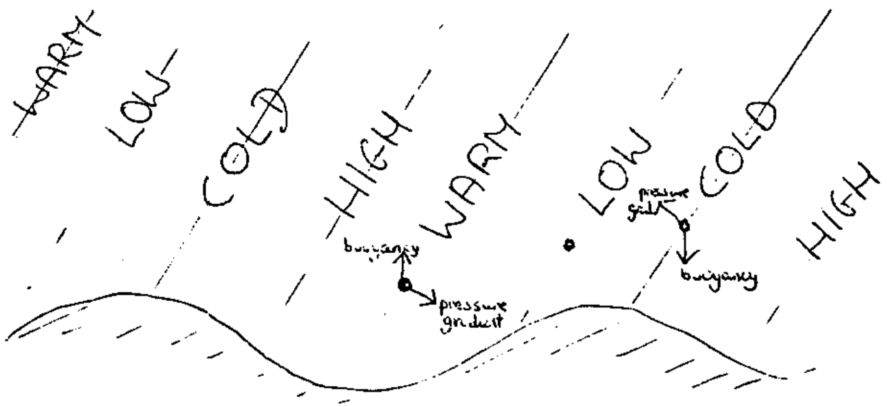
The temperature perturbation at A is negative (adiabatic ascent)

" " " " B is zero (neutral point)

" " " " C is positive (adiabatic subsidence)

" " " " D is zero (neutral point)

Note from the diagram that the zonal mean temperature perturbation is zero. The general temperature and pressure perturbation structure for the steady wave can easily be deduced



The smaller the (Doppler) frequency of wave forcing, the flatter the particle oscillations and the temperature and pressure phase lines.

Another result we shall need concerns the eddy velocity $\underline{u}' = (u', w')$. Note that u' is positively correlated with w' ie when $u' > 0$, $w' > 0$, and when $u' < 0$, $w' < 0$, so that $\overline{u'w'} > 0$. In a region of steady wave forcing with no wave dissipation, the eddy velocities increase as the fluid gets more rarified, keeping the upward flux of eddy momentum $\rho \overline{u'w'}$ constant with height. (Here, of course,

ρ is the zonal mean density of the fluid). Put another way; in the absence of wave transience and dissipation

$$\frac{\partial}{\partial z}\left(\rho\overline{u'w'}\right) = 0$$

If the waves are dissipating, then the eddy momentum flux will decrease with height. Where does this momentum go? It goes into accelerating the zonal mean flow. This information is contained in the zonal mean momentum equation

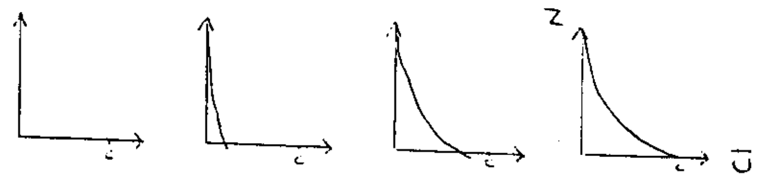
$$\frac{\partial \overline{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\overline{u'w'}\right) + \overline{X}$$

where $\overline{X}$ is a mean dissipative term. Note also that at the leading wave front, eddy momentum is injected into the mean flow, causing mean flow acceleration. We ascribe such a situation to wave transience. We could of course have obtained all the preceding information from the wave equation and resulting dispersion relation. Here I wish to emphasis what the particles are doing. The reason for this will become apparent in the next lecture.

Let us suppose some dissipation is present in the steady waves, and represent a 'dissipative height scale' by D, i.e., eddy momentum fluxes decrease as exp (-z/D). It turns out that the more horizontal particle oscillations are, the smaller will be the dissipative height scale. (To show this rigorously we would need to discuss the so-called conservation of wave-action equation, and the associated group velocity of the waves.)

Let us examine the complete wave, mean-flow interaction for this system. The dissipating wave will induce a mean-flow acceleration throughout a depth D. But now the Doppler-shifted frequency of the wave becomes less than it was before the mean-flow acceleration. Hence the particles oscillate in slightly more horizontally tilted lines.

Therefore the scale height D decreases, and further mean-flow acceleration occurs throughout a smaller depth. This feedback process continues, and a sizeable mean flow will develop near the bottom of the fluid. Eventually the mean flow velocity becomes equal to the velocity of the slippery corrugated boundary, ie the Doppler shifted frequency of the waves becomes equal to zero, and as we have seen, when this happens, the waves can no longer propagate up - we have reached impasse! Pictorially the mean-flow evolves as



We have assumed that mean-flow dissipation is unimportant.

If we now add to the input of waves at $Z = 0$, a component travelling with equal and opposite phase speed $-C$, something very interesting happens. If the two waves, with phase speeds $\pm C$, have equal amplitudes, then the boundary executes a standing wave

$$Z = a \sin k(x-ct) + a \sin k(x+ct)$$
$$= 2a \sin kx \cos kct$$

Now, as we have seen, the wave travelling in the positive x direction cannot propagate up, but the one travelling in the negative x direction can (providing its Doppler shifted frequency 2kc is less than N). Hence, this wave starts to accelerate the mean flow in the negative x direction. Because its Doppler shifted frequency is comparatively large, a negative mean acceleration is induced throughout a comparatively deep layer,

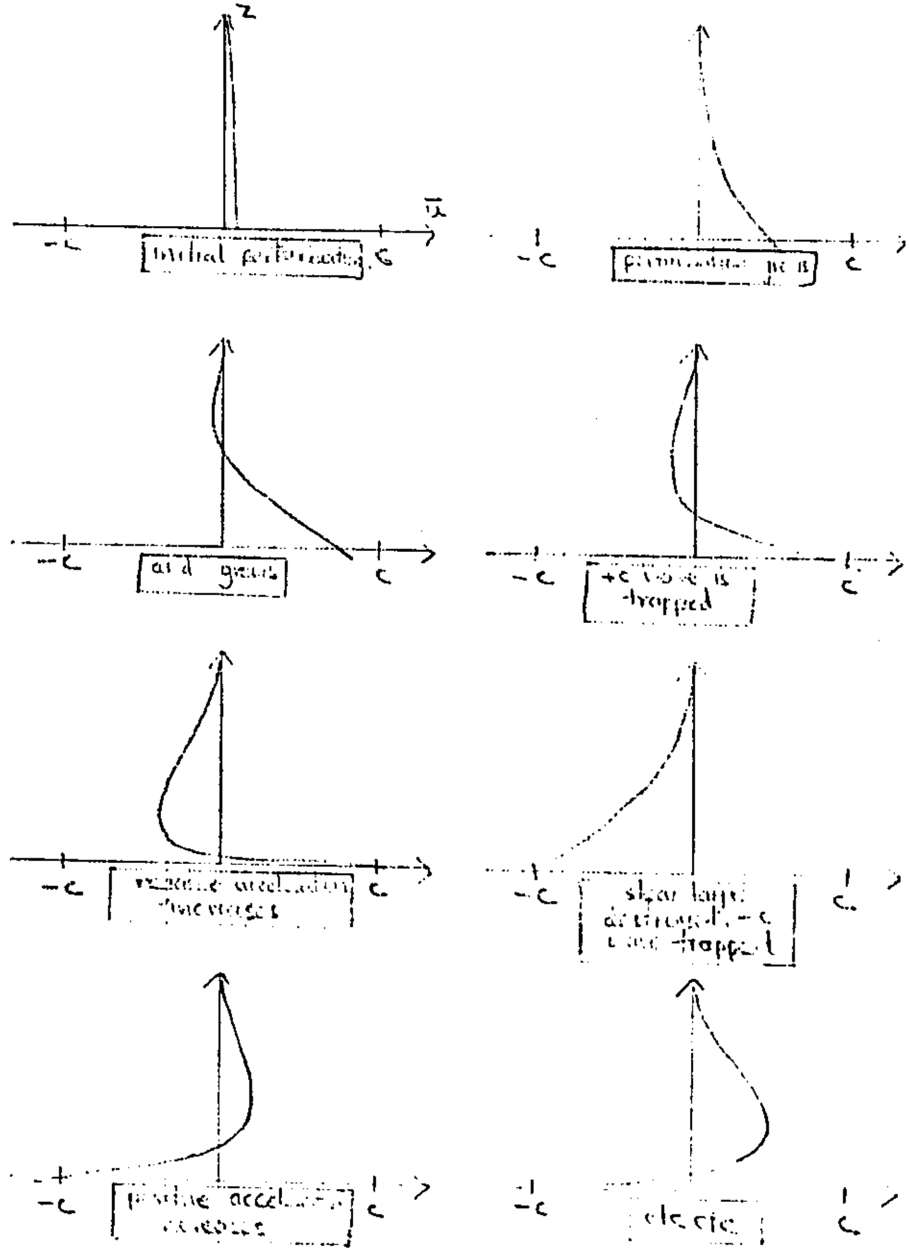


FIG 1

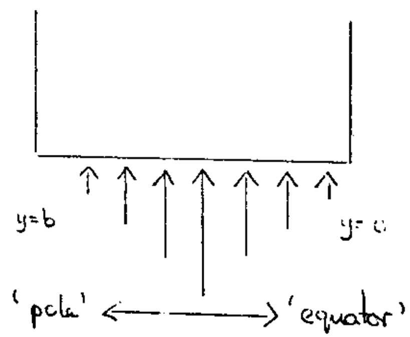
it demonstrates in the laboratory, an analogue of the stratospheric quasi-biennial oscillation (for more details of this, see the course on the stratosphere). Plumb and McEwan's film of the simulated QBO beautifully demonstrates wave, mean-flow interaction! Who would have thought that downward propagating easterlies, then westerlies, then easterlies was due to the input of a steady standing wave from below?

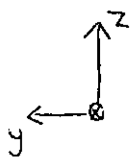
Inertia-gravity waves

Let us now consider steady buoyancy waves with the added complication of rotation. Again I shall be rather descriptive, and not bother too much with mathematical analysis. We shall find that the Eulerian-mean description of these waves is somewhat misleading, and this has led to the recent formulation of a "Lagrangian-mean" description of such motions. Such a description of fluid motions has proved useful in the stratosphere, both in understanding its dynamical behaviour, and in understanding how passive chemical tracers are transported.

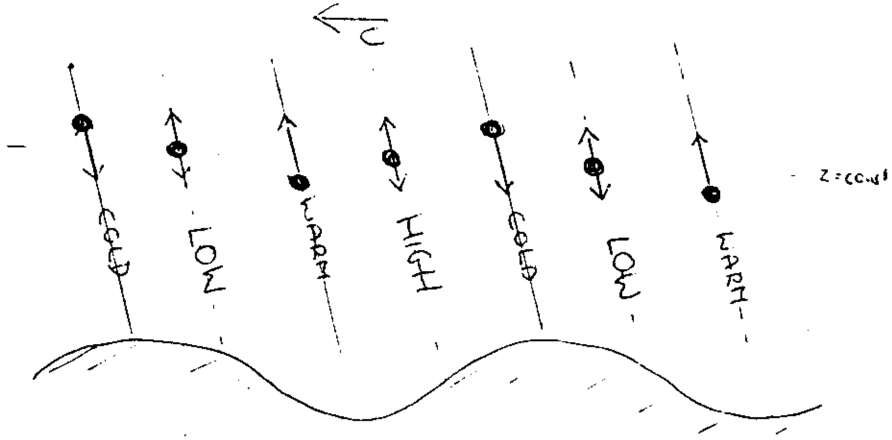
We consider the idealisation of an inviscid diffusionless fluid of constant buoyancy frequency N, rotating about the z-axis with constant angular velocity $\Omega = \frac{1}{2}f$. The fluid is contained in a channel, between rigid vertical walls at $y = 0, b$ and a moving lower boundary $z = h(x,y,t)$ with, as before, $\overline{h} = 0$.

Let's also suppose the forcing from below has maximum amplitude at mid-channel





If the wave forcing has phase speed -C (in the x direction), we can draw the particle oscillations in the x-z plane from our knowlege of buoyancy waves



The particles however also oscillate in the 'meridional' direction because of Coriolis forces. Let us determine the meridional velocity (qualitatively at least), from the geostrophic relationship

$$v' = \frac{1}{f\rho}\frac{\partial p'}{\partial x}$$

From this relationship, we can see that

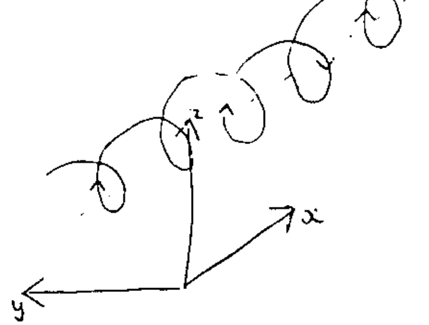
when $u' = 0$ (at top of oscillation), $\partial p'/\partial x < 0$, $v' < 0$

when $u' < 0$ (at mid-point, going down), $\partial p'/\partial x = 0$, $v' = 0$

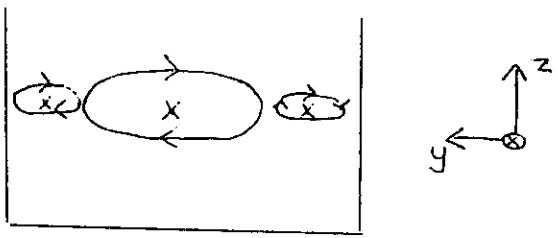
when $u' = 0$ (at bottom of oscillation), $\partial p'/\partial x > 0$, $v' > 0$

when $u' > 0$ (at mid-point, going up), $\partial p'/\partial x = 0$, $v' = 0$

Putting all this information together we see that the particles paths are spirals (if viewed from a frame in which the forcing is stationary)

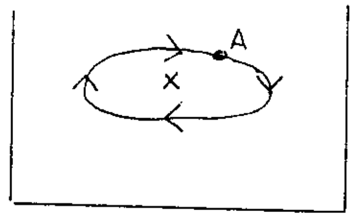


Projecting onto the y, z plane (the 'meridional plane'), the paths are ellipses



whose sizes are related to the amplitude of wave forcing. When the waves are steady, then clearly the average position of the particles, marked by Xs, does not change with time. (Note that since the particles have meridional components of velocity, Coriolis forces are also felt in the x-direction. This means that particle oscillations in the x-z plane are more horizontal than they would be if there was no rotation. In other words, **rotation** tends to supress the upward propagation of buoyancy waves. It turns out that when $\omega = f$, the waves are completely suppressed).

Assuming that $f < \omega < N$, what picture does the Eulerian mean formalism present for these steady waves? Consider first an ellipse whose centre is at the 'latitude' of maximum forcing



At the top of the ellipse $V' < 0$, but so also is T', $\therefore V'T' > 0$. At the bottom of the ellipse $V' > 0$, so also is T', $\therefore V'T' > 0$, ie V' is positively correlated with T' throughout the particle oscillation. If we take an arbitrary point A on the ellipse, then the zonal mean of $V'T'$ at A must also be positive, ie $\overline{V'T'} > 0$. Since A is arbitrary, $\overline{V'T'} > 0$ everywhere.

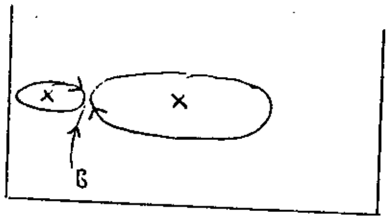
Hence we have a somewhat paradoxical picture of an eddy heat flux which is directed towards $y = 0$ ('poleward') everywhere, even though there is no net heat transport towards the pole (because, on average, particles are not going anywhere). In other words, even though the eddy heat flux is poleward, the pole is not being heated! Note the fact that $\overline{V'T'} > 0$ is independent of whether there exists a positive or negative zonal mean temperature gradient between y = b and y = 0. Hence, any attempt to parametrize the eddies with a diffusive type parameter

$$\overline{V'T'} = \kappa \frac{\partial \overline{T}}{\partial y}$$

is doomed to failure. The heat flux is related to the wave forcing not to the mean temperature gradient.

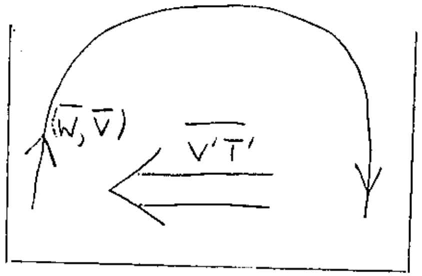
If the wave forcing had positive phase speed then it can easily be shown that $\overline{V'T'}$ would be negative everywhere, implying an 'equatorward' heat flux. Since particle trajectories are parallel to lines of constant phase ($\underline{u}' = R_e\{\underline{A}e^{i\Phi}\} \Rightarrow \mathrm{div}\,\underline{u}' = \underline{u}'.\nabla\Phi = 0$ (incompressibility) $\Rightarrow \underline{u}' \,//\,$ to $\Phi = \mathrm{const}$), we have derived the well-known relation that the phase lines must tilt west with height to get poleward heat fluxes.

What is the resolution of this paradox? Consider the two ellipses



At the point B, the particle of the large ellipse is moving upward, the particle of the small ellipse is moving downward. The net result is an upward velocity (the larger the amplitude, the larger the velocity). Hence the Eulerian mean vertical velocity at B is positive. In fact at any point 'north' of the centre of the ellipse at maximum forcing, the Eulerian mean vertical velocity is positive. Conversely, at any point 'south' of the centre of the ellipse, the Eulerian mean vertical velocity is negative.

Hence, the full Eulerian mean picture can be summarised by the following diagram

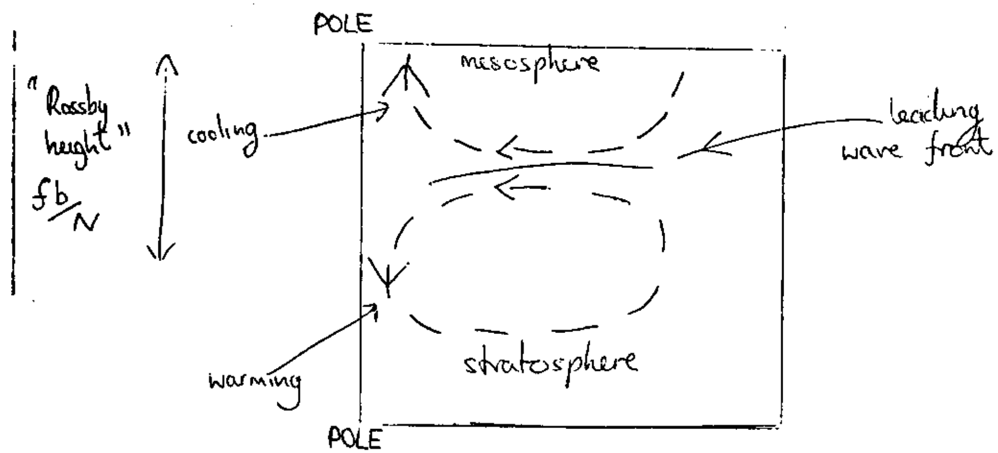


The eddy heat flux induces a mean meridional circulation. Adiabatic ascent over the pole balances the eddy heat flux input, and adiabatic subsidence over 'low-latitudes' balances the eddy heat flux output. The whole system organises itself so that no net heat or momentum is transferred to the mean-flow.

This result, that when waves are steady the zonal flow is not acclerated or heated, has the status of a theorem in the literature, the so-called Charney-Drazin theorem (Charney, J.G., and P.G. Drazin 1961, J. Geophys. Res. 66 83-109) and indeed it is a difficult result to prove in an Eulerian-mean kinematic description. Nevertheless we have seen that for steady, non-dissipative waves, the fluid particles do not on average get anywhere - a very straightforward physical result. Surely there must be some kinematic description in which the Charney-Drazin theorem is immediate? Indeed there is - it is the 'Lagrangian mean' description; Lagrangian because the averaging is related to particle trajectories, as opposed to Eulerian where the averaging is related to Eulerian where the averaging is related to fixed points in space.

Before describing a little of the kinematic formulation of Lagrangian mean theory, it is worth spending a little time discussing the meteorological relevance of the above considerations.

Dynamic processes in the stratosphere result not from in situ instabilities, but rather from tropospheric forcing. Of particular interest are the mid-latitude tropospheric planetary waves. Occasionally, the amplitudes of these waves reach rather high values - we refer to the meteorological condition when this happens by the adjective blocking. If conditions are right (which only happens in the winter), these high amplitude waves propagate up into the stratosphere. As the leading wave front moves up, the manifestly transient conditions decelerate the westerly zonal mean flow. The effects of this deceleration are felt most keenly at high altitudes and high latitudes, firstly because air is rather tenuous at high altitudes, and secondly the mass of air between two neighbouring latitude circles decreases with latitude. Because the zonal flow is decelerated at the leading wave front, it is no longer in geostrophic balance, and air flows into the low pressure region over the stratospheric pole. The result is a Lagrangian meridional circulation with descending air over the stratospheric pole. By adiabatic subsidence, the stratospheric pole heats up very dramatically - sometimes by 40° in just a few days. The Lagrangian meridional circulation is illustrated below

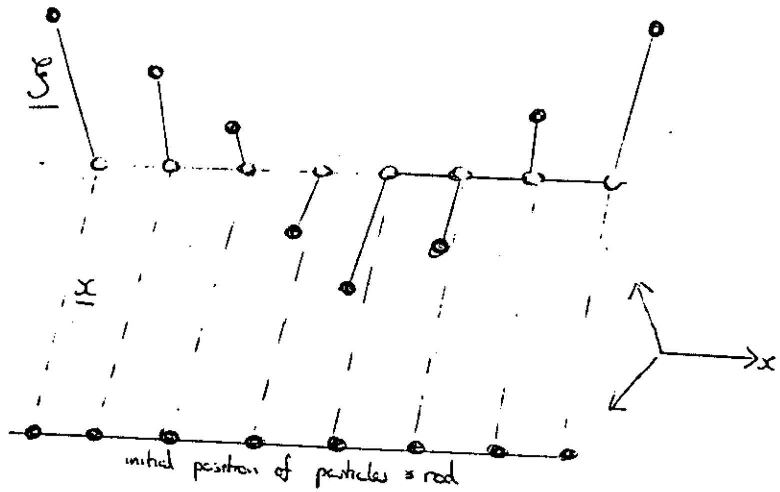


The magnitude of this Lagrangian circulation has been quantified by Matsuno and Nakumara (Matsuno T. and K. Nakaraira K. 1979, J. Atmos. Sci. 36, 640-654). The Lagrangian picture gives a rather straightforward explanation of this phenomenon, named, rather appropriately, the stratospheric sudden warming. By contrast, the Eulerian mean description replete with its ficticious eddy heat fluxes and mean meridional circulations, gives a rather complicated explanation of the sudden warming - certainly one would have little idea that the warming was due to mass subsidence. Consequently much of the literature on the subject is rather obscure as far as providing an understanding of the physical mechanisms of a warming.

Lagrangian-mean kinematics

The Lagrangian-mean theory, recently developed (Andrews D.G. and M.E. McIntyre 1978 J. Fluid. Mech. 89, 609-646) is a hybrid kinematic description, lying somewhere between Stokes classical idea of taking the time mean following a single air parcel, and the Eulerian mean description of taking means with respect to fixed latitude circles.

Consider a number of fluid particles lying along a latitude circle at an initial time t_0, before the wave forcing has begun. When the wave propagates up the particles are displaced. Imagine the particles are attached by elastic bands to a 'magic' rod of zero mass which is constrained, for all time, to be parallel to latitude circles. Initially the particles sit on the rod. When the particles are displaced, the rod is moved by the tension in the elastic bands, from its initial latitude circle, to another one. This is illustrated below



The position vector of the particles, $\underline{\xi}$, relative to the magic rod satisfies $\overline{\underline{\xi}} = 0$ for all time. The position vector of the points of attachment of the elastic bands to the rod, relative to their initial positions, is written as $\underline{x}$. The Lagrangian-mean of some property ψ of the fluid is defined by

$$\overline{\psi(\underline{x},t)}^{L} = \overline{\psi(\underline{x}+\underline{\xi}(\underline{x},t),t)}$$

i.e. each point of attachment assumes the value ψ of its associated particle, and the Lagrangian-mean is the zonal mean along the 'magic' rod. The Lagrangian-mean velocity of the fluid is defined to be the velocity of the points of attachment.

I have deliberately stripped this description of its mathematics; if you are concerned that the formalism lacks rigour, I recommend you look up the Andrews and McIntyre reference. (1)

As an example of the conceptual simplicity of the Lagrangian mean kinematical description, the Lagrangian mean potential temperature, Θ, in an adiabatic atmosphere, satisfies

$$\left(\frac{\partial}{\partial t} + \overline{V}^{L}\frac{\partial}{\partial y} + \overline{W}^{L}\frac{\partial}{\partial z}\right)\overline{\theta}^{L} = 0$$

Compare this with the Eulerian mean equation

$$\left(\frac{\partial}{\partial t} + \overline{V}\frac{\partial}{\partial y} + \overline{W}\frac{\partial}{\partial z}\right)\overline{\theta} = -\frac{1}{\cos\phi}\frac{\partial}{\partial y}\left(\overline{v'\theta'}\cos\phi\right) - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\overline{w'\theta'}\right)$$

where ϕ is latitude.

The formalism has successfully been applied to such problems as stratospheric sudden warmings (see above), baroclinic instability* and chemical tracer transport+.

An important and immediate consequence of Lagrangian mean theory is that for steady non-dissipative waves in a zonal mean flow

$$\overline{V}^{L} = \overline{W}^{L} = 0$$

This result, as we have seen, is definitely not true of the Eulerian velocities $\overline{V}$, $\overline{W}$. If we write

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* Uryu, M., 1979, J. Met. Soc. Japan 57, 1-20.

+ Dunkerton, T., 1978, J. Atmos. Sci., 35, 2325-2333.

$$\bar{V} = \bar{V}^L + \bar{V}^S$$
$$\bar{W} = \bar{W}^L + \bar{W}^S$$

then $(\bar{V}^S, \bar{W}^S)$ is called the "Stokes drift" in the meridional plane. For steady non-dissipative linear waves, the Stokes drift may be written

$$\bar{V}^S = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\, \overline{v'\theta'} \Big/ \frac{\partial\bar{\theta}}{\partial z}\right)$$

$$\bar{W}^S = \frac{1}{\cos\phi}\frac{\partial}{\partial y}\left(\cos\phi\, \overline{v'\theta'} \Big/ \frac{\partial\bar{\theta}}{\partial z}\right)$$

Hence, if we define the Eulerian-mean "residual" velocities

$$\bar{V}^* = \bar{V} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\, \overline{v'\theta'} / \partial\bar{\theta}/\partial z\right)$$

$$\bar{W}^* = \bar{W} + \frac{1}{\cos\phi}\frac{\partial}{\partial y}\left(\cos\phi\, \overline{v'\theta'} / \partial\bar{\theta}/\partial z\right)$$

then although $(\bar{V}^*, \bar{W}^*)$ are not Lagrangian mean velocities in general they should be closer to them than the Eulerian mean pair $(\bar{V}, \bar{W})$. Since the Lagrangian mean displacement vector is not directly observable from radiosonde or satellite radiance measurements, a kinematic formalism based on these 'residual' velocities should be a nice compromise between the theoretically important Lagrangian mean theory, and the Eulerian mean formalism (which has the advantage of fitting rather straightforwardly into an observational diagnostic scheme).

Indeed wave, mean-flow interactions in the stratosphere have been diagnosed using measurements from the Meteorological Office's Stratospheric Sounding Units, flown on the Tiros-N and Noaa-A satellites, and they have given interesting new insights into the dynamics of the stratosphere during a sudden warming.