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Lecture 1: SQUEEZED STATES OF THE RADIATION FIELD

Lecture 2: PHOTON STATISTICS OF A FREE ELECTRON LASER

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In this talk we shall try to answer the following questions concerning the Squeezed States of the radiation field.[1]

(i) WHAT are they ?

(ii) WHY are they important ?

(iii) HOW to generate them ?

WHAT ARE SQUEEZED STATES ?

In order to answer the first question, consider two dimensionless operators A and B which satisfy the following commutation relation:

$$[A,B] = C \quad . \tag{1}$$

Then, according to the Heisenberg's uncertainty relation

$$\Delta A \Delta B \geqslant \frac{1}{2} |\langle C \rangle| . \tag{2}$$

A state is called "squeezed" when uncertainty in one observable (say A) is less than that for the minimum uncertainty state, i.e.,

$$(\Delta A)^2 < \frac{1}{2} |\langle C \rangle| \tag{3}$$

Specifically let a^+ and a be the creation and destruction operators of a single - mode electromagnetic field with

$$[a, a^+] = 1 \tag{4}$$

Then the Hermitian amplitude operators a_1 and a_2 which are defined as

$$a = a_1 + ia_2, \tag{5a}$$

$$a^+ = a_1 - i\, a_2 \tag{5b}$$

satisfy the commutation relation

$$[a_1 \, , \, a_2] = \frac{i}{2} \tag{6}$$

The corresponding uncertainty relation is

$$\Delta a_1 \Delta a_2 \geqslant \frac{1}{4} \tag{7}$$

A state of the radiation field is squeezed if one of the amplitudes a_i (i= 1,2) satisfies

$$(\Delta a_i)^2 < \frac{1}{4} \tag{8}$$

We call a squeezed state an "ideal squeezed state" or "squeezed coherent state", if in addition to (8), we obtain

$$\Delta a_1 \Delta a_2 = \frac{1}{4} \tag{9}$$

Consider first a well-known state of the field-coherent state . This state, which is the state of an ideal laser, is an eigenstate of the destruction operator a, i.e.,

$$a|\alpha\rangle = \alpha |\alpha\rangle \tag{10}$$

A representation of this state is given as follows:

$$|\alpha\rangle = D(\alpha)\, |0\rangle , \tag{11}$$

where

$$D(\alpha) = \exp[\alpha a^+ - \alpha^* a] \tag{12}$$

is the displacement operator. It follows simply, on using Eq.(10), that in a coherent state

$$\begin{aligned} \Delta a_1^2 &= \langle\alpha|(a_1 - \langle a_1\rangle)^2|\alpha\rangle \\ &= \langle\alpha|a_1^2|\alpha\rangle - (\langle\alpha|a_1|\alpha\rangle)^2 \\ &= \frac{1}{4}[\langle\alpha|(a+a^+)^2|\alpha\rangle - (\langle\alpha|(a+a^+)|\alpha\rangle)^2] \\ &= \frac{1}{4}. \end{aligned} \tag{13}$$

Similarly

$$\Delta a_2^2 = \frac{1}{4} \, . \tag{14}$$

so that

$$\Delta a_1 \Delta a_2 = \frac{1}{4} \tag{15}$$

Coherent state is therefore not a squeezed state.

Next we consider the so-called two-photon state

$$|\alpha\,,\xi\rangle = D\,(\alpha)\;S\,(\xi)\;|o\rangle \tag{16}$$

where

$$S\,(\xi) = \exp\,(\tfrac{1}{2}\,\xi\;a^2 - \tfrac{1}{2}\,\xi^* a^{+2}) \tag{17}$$

is the "squeezing operator" and D (α), as before, is the displacement operator. The parameter

$$\xi = r\;e^{i\theta} \tag{18}$$

is a complex number. If we consider a particular choice of θ, namely $\theta = o$, then it can be shown that

$$\Delta a_1^2 = \frac{1}{4}\;e^{-2r} \tag{19}$$

$$\Delta a_2^2 = \frac{1}{4}\,e^{2r} \tag{20}$$

in a two-photon state. It follows from Eqs.(19) and (20) that

$$\Delta a_1 \Delta a_2 = \frac{1}{4} \tag{21}$$

It is therefore evident that for $r \neq o$, two-photon state is an ideal squeezed state.

WHY ARE SQUEEZED STATES IMPORTANT ?

The major interest in the squeezed states stems for two important reasons:

Firstly, the manifestation of the squeezed states of the radiation field is a purely quantum mechanical effect. In order to show this explicitly, we look for the condition on the coherent state representation when $\Delta a_i^2 < 1/4$ (i=1 or 2). The coherent state representation is defined by the following relation:

$$\rho = \int P(\alpha)\;|\alpha\rangle\langle\alpha|\;\;d^2\alpha \tag{22}$$

The normally ordered correlation functions of a and a^+ can be obtained from the coherent-state representation by using the methods of classical statistical mechanics. For example

$$\langle a^+a\rangle = \int \alpha^*\alpha\;\;P(\alpha)\;\;d^2\alpha \tag{23}$$

It can therefore be shown that, in terms of P (α),

$$\Delta a_1^2 = \frac{1}{4}\,[1 + \int P(\alpha)\;\;(\alpha + \alpha^*) - (\langle\alpha\rangle + \langle\alpha^*\rangle)\;^2\;d^2\alpha\;] \tag{24a}$$

$$\Delta a_2^2 = \frac{1}{4}\;\;[1 + \int P(\alpha)\;\;(\frac{\alpha - \alpha^*}{i}) - (\frac{\langle\alpha\rangle - \langle\alpha^*\rangle}{i})^2 d^2\alpha\;] \tag{24b}$$

It should be noted that the quantities inside the parenthesis ..... are real whose squares are positive. Therefore the squeezing condition

$$\Delta a_i^2 < \frac{1}{4}\;\;(i=1 \text{ or } 2) \tag{25}$$

is satisfied for those fields whose coherent state representation P(α) is NOT non-negative. Such fields have no classical analog.

The second motivation to study squeezed states comes from a different area of research-gravitational wave detection. One scheme to detect gravitation waves is based on Michelson interferometry. The sensitivity of this device is limited by quantum fluctuations. It has been proposed recently that a technique which uses the squeezed states of the radiation field could be employed to reduce the photon-counting fluctuation in the interferometer, thereby increasing the sensitivity of the device.

HOW TO GENERATE SQUEEZED STATES ?

A number of nonlinear optical systems have been considered that generate squeezed states. These include the degenerate parametric amplifier, four-wave mixing, resonance fluorescence, free-electron lasers, optical bistability, Jaynes-Cummings model, and the multiphoton absorption process.

In this talk we will analyse a degenerate parametric amplifier in some details. This device is a particularly important example of the systems that are predicted to exhibit squeezed states. Unlike many other systems, ideal squeezed states are generated in this nonlinear optical device, when the quantum fluctuations of the pump field are neglected. We shall look at the effect of these fluctuations on the squeezing using a Path-integral method which was developed recently by Mark Hillery and myself.

DEGENERATE PARAMETRIC AMPLIFIER

In a degenerate parametric amplifier, a pumping field of fequency 2ω interacts with a nonlinear medium and gives rise to a field of frequency ω. This process is described at exact resonance by the Hamiltonian

$$H = 2\omega b^+ b + \omega a^+ a + \frac{ik}{2}\,(a^2 b^+ - b a^{+2}), \qquad (26)$$

where b (b^+) and $a(a^+)$ are the destruction (creation) operators for the pump and signal modes respectively and κ is an appropriate coupling constant.

In the parametric approximation, pump field is treated classically and the pump deplection is neglected. We can then replace the operator b by $\beta \exp(-2i\omega t)$ in Eq.(26). For simplicity we assume β to be real. The

quantity β^2 then represents the number of photons in the pump mode. The resulting Hamiltonian is then given by

$$H = \omega a^+ a + \frac{ik}{2}\,(a^2 e^{2i\omega t} - a^{+2} e^{-2i\omega t}). \qquad (27)$$

In the interaction picture, it simplifies further and we obtain

$$H_1 = \frac{ik}{2}\,(a^2 - a^{+2}). \qquad (28)$$

The equations of motion for the signal mode operators

$$\dot{a} = i\,[a,H] = -K\beta a^+, \qquad (29a)$$

$$\dot{a}^+ = i\,[a^+, H] = -\kappa\beta\, a, \qquad (29b)$$

can be solved exactly

$$a(t) = a_o\, ch(\kappa\beta t) - a_o^+\, sh(\kappa\beta t), \qquad (30a)$$

$$a^+(t) = a_o^+ ch\,(\kappa\beta t) - a_o\, sh\,(\kappa\beta t). \qquad (30b)$$

where

$$a_o = a\,(o) \text{ and } a_o^+ = a^+\,(o)\ .$$

We assume the initial state of the signal mode to be vacuum, o . The various correlation functions necessary for determining a_1^2 and a_2^2 can then be obtained in a straightforward manner. The result is

$$\Delta a_1^2 = \frac{1}{4}\, e^{-2\kappa\beta t}, \qquad (31a)$$

$$\Delta a_2^2 = \frac{1}{4}\, e^{2\kappa\beta t}, \qquad (31b)$$

$$\Delta a_1 \Delta a_2 = \frac{1}{4} \qquad (31c)$$

i.e., we obtain ideal squeezed states.

So for we have neglected the quantum fluctuations. It is of considerable interest to investigate the effect of quantizing the pump mode on the statistical properties of the signal field. This problem is however very difficult due to the trilinear form of the Hamiltonian.

Only small time solutions can be obtained with relative ease. We present here a path-integral approach to solve this problem and obtain quantum corrections to the results obtained in the parametric approximation

PATH-INTEGRAL APPROACH:

We summarize here the basic elements of the path-integral approach to solve the problem at hand, namely degenerate parametric amplification process.

First we define a coherent state propagator. If we denote the eigenstate of the destruction operators a(t) and b(t) with eigenvalues and respectively by $|\alpha,\beta,t\rangle$, then the coherent state propagator is defined as the following inner product

$$K(\alpha_1,\beta_1,t_1;\alpha_0,\beta_0,o)=\langle\alpha_1,\beta_1,t_1|\alpha_0,\beta_0,o\rangle \quad (32)$$

For an initial coherent state $|\alpha_i,\beta_i,o\rangle$, the propagater is related to the Q-representation of the field, namely

$$Q(\alpha_f,\beta_f,t)=\frac{1}{\pi^2}|\langle\alpha_f,\beta_f,t|\rangle|\alpha_f,\beta_f,t\rangle \quad (33)$$

by a rather simple relation

$$Q(\alpha_f,\beta_f,t)=\frac{1}{\pi^2}\left|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,o)\right|^2 \quad (34)$$

Any antinormal ordered correlation function of a, a^+, b, b^+ can then obtained from Q (and, therefore, K using the method of statistical mechanics. For example

$$\langle a(t)a^+(t)\rangle=\frac{1}{\pi^2}\int d^2\alpha\int d^2\beta\,|\alpha|^2\,|K(\alpha,\beta,t;\alpha_i,\beta_i,o)|^2 \quad (35)$$

A path-integral representation of the coherent-state propagator can be obtained:

$$K(\alpha_1,\beta_1,t;\alpha_0,\beta_0,o)=\int D[\alpha(\tau)]\;D[\beta(\tau)]\;e^{iS} \quad (36)$$

where $\int D[\alpha(\tau)]$ and $\int D[\beta(\tau)]$ represent the integration over all paths $\alpha(\tau)$ and $\beta(\tau)$ such that

$$\alpha(t)=\alpha_1,\qquad \alpha(o)=\alpha_0 \quad (37a)$$

$$\beta(t)=\beta_1,\qquad \beta(o)=\beta_0 \quad (37b)$$

and

$$iS=\int_0^t d\tau\,[\tfrac{1}{2}(\dot\alpha^*\alpha-\alpha^*\dot\alpha)+\tfrac{1}{2}(\dot\beta^*\beta-\beta^*\dot\beta)-iH(\alpha^*,\alpha^*,\beta,\beta^*t)] \quad (38)$$

In Eq.(38), H is the Hamiltonian (26) such that the operators are replaced by their coherent state eigenvalues. The problem therefore is to solve Eq.(36) for the coherent state propagator and then obtain the necessary correlation functions by using Eq.(34).

Due to the trilinear form of the Hamiltonian, it is not possible to obtain an exact expression for the propagator in the present case. We therefore resort to a perturbation method such that the zeroeth order term in the expansion gives the contribution to the propagator corresponding to a classical description of the pump field, i.e., in the parametric approximation. The First order term in the perturbation expansion is the lowest order quantum correction.

The details of the calculations are rather complicated and we do not repeat them here. The interested reader is referred to Ref.2 and 3. The final expressions for Δa_1^2 and Δa_2^2 are

$$\begin{aligned}\Delta a_1^2=&\frac{1}{4}e^{-2\eta_0t}+\frac{1}{4\beta^2}[(\eta_0t)^2e^{-2\eta_0t}-(\eta_0t)\\&\times(e^{-2\eta_0t}+1)+(3\,\mathrm{sh}^2\eta_0t+2)\,\mathrm{Sh}\,\eta_0t\,e^{-\eta_0t}\\&-\mathrm{Sh}^2\eta_0t\,]\end{aligned} \quad (39)$$

$$\begin{aligned}\Delta a_2^2=&\frac{1}{4}e^{2\eta_0t}+\frac{1}{4\beta^2}[(\eta_0t)^2e^{2\eta_0t}+(\eta_0t)\times\\&\times(e^{2\eta_0t}+1)-(3\,\mathrm{Sh}^2\eta_0t+2)\,\mathrm{Sh}\,\eta_0t\,e^{\eta_0t}\\&-\mathrm{Sh}^2\eta_0t\,]\end{aligned} \quad (40)$$

where

$$\gamma_0 = 2\kappa\beta$$

The first terms in Eq. (39) and (40) correspond to the parametric approximation whereas the second terms give us the lowest order quantum corrections to the parametric approximation. The behaviour of the second term in Eq. (39) is such that Δa_1^2 first descrease and reaches a minimum and then increases. For large values of $\gamma_0 t$, the squeezing disappears.

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Path-integral approach to problems in quantum optics

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A formalism for applying path integrals to certain problems in nonlinear optics is considered. The properties of a coherent-state propagator are discussed and a path-integral representation for the propagator is presented. This representation is then employed in evaluating the propagator for general single-mode and multimode Hamiltonians which are at most quadratic in the creation and destruction operators of the field. Some examples involving parametric processes are given.

I. INTRODUCTION

Path integrals and the approximations to which they have led have been used very much in quantum field theory in recent years. The path-integral representation of the propagator allows one to see more clearly than the standard operator approach, the connection between the classical and quantum dynamics of a system. Semiclassical approximations can then be derived in a natural way.[1] So far, however, these techniques have not found much use in quantum optics.[2] In this paper we will develop some of the formalism which will be of use in applying path-integral techniques to certain problems in nonlinear optics.

The types of problems to which we would like to apply these techniques are those in which the medium with which the light interacts can be described by a nonlinear susceptibility tensor.[3] These include such processes as parametric amplification and harmonic generation. The interaction between the different modes is then described by products of various powers (depending upon the specific process) of the creation and destruction operators of the modes involved.

The type of path integral which we will consider is not the one usually used in quantum field theory in which one makes use of a coordinate representation of the field. We will be interested in problems in which only a few of the modes of the field are important and we will use a path integral which makes use of a representation of these modes in terms of coherent states. Because the Hamiltonians which we will consider will be expressed in terms of creation and destruction operators, and not the corresponding position and momentum operators, coherent states, which are eigenstates of the destruction operator, are natural objects to use. The coherent-state path integral can be used to calculate the matrix element of the time development transformation between two coherent states. This matrix element can be regarded as a type of propagator. This form of the path integral was first discussed by Klauder[4] and was subsequently examined by Schweber[5] in the context of Bargmann spaces. Klauder[6] in later work showed that the coherent-state path integral is but one example of a more general class of objects known as continuous representation path integrals.

In Sec. II, we discuss some properties of the propagator and show how it can be used to calculate various quantities of interest in quantum optics. In Sec. III, we derive formulas which can be used to calculate the propagator for single-mode systems with Hamiltonians at most quadratic in the creation and destruction operators. These are then used to calculate the propagator for the case of second subharmonic generation when the pump field is classical. In Sec. IV, we generalize our results and calculate the propagator for an N-mode system whose Hamiltonian is quadratic. This result is then used to calculate the propagator for a parametric amplifier with a classical pump field.

II. COHERENT-STATE PROPAGATOR

We consider a system which consists of one mode of the radiation field. Let the corresponding time-evolution operator be $U(t_2,t_1)$, i.e., if $|\psi(t_1)\rangle$ is the state of the system at time t_1 then the state at time t_2 is

$$|\psi(t_2)\rangle = U(t_2,t_1)|\psi(t_1)\rangle \ . \tag{1}$$

If the Hamiltonian governing the system is given by $H(t)$ then the time-evolution operator is (where we have chosen units such that $\hbar=1$)

$$U(t_2,t_1)=T\exp\left[-i\int_{t_1}^{t_2}H(t')dt'\right] , \tag{2}$$

where T is the Dyson time-ordering operator.

We will consider the propagator

$$K(\alpha_2,t_2;\alpha_1,t_1)=\langle\alpha_2|U(t_2,t_1)|\alpha_1\rangle , \tag{3}$$

where the coherent states $|\alpha_i\rangle$ are the eigenstates of the destruction operator a with eigenvalue α_i, at time $t=0$. Another expression for the propagator $K(\alpha_2,t_2;\alpha_1,t_1)$ can be derived by noting that the coherent state, at time t [i.e., the eigenstate of $a(t)$] is given by

$$|\alpha,t\rangle = U(t,0)^{-1}|\alpha\rangle \ . \tag{4}$$

We then obtain

$$K(\alpha_2,t_2;\alpha_1,t_1)=\langle\alpha_2,t_2|\alpha_1,t_1\rangle$$
$$=\langle\alpha_2|U(t_2,0)U(t_1,0)^{-1}|\alpha_1\rangle \ . \tag{5}$$

In quantum optics, one is usually interested in evaluating certain correlation functions of the field. For a one-mode field these are proportional to the expectation values of products of the creation and destruction operators. These correlation functions can be expressed in terms of the propagator $K(\alpha_2,t_2;\alpha_1,t_1)$. We assume that, at $t=0$, the density matrix has a P representation, i.e.,

$$\rho=\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \ , \tag{6}$$

so that the expectation value of any operator, $O(t)$, in the Heisenberg picture is given by

$$\langle O(t)\rangle = \mathrm{Tr}[\rho O(t)]$$
$$=\int d^2\alpha P(\alpha)\langle\alpha|O(t)|\alpha\rangle \ . \tag{7}$$

On using the completeness property of the coherent states, namely,

$$\frac{1}{\pi}\int d^2\alpha|\alpha,t\rangle\langle\alpha,t|=1 \ , \tag{8}$$

it can be easily shown that

$$\langle a(t)\rangle = \frac{1}{\pi}\int\int d^2\alpha_1 d^2\alpha_2 P(\alpha_2)|K(\alpha_1,t_1;\alpha_2,0)|^2\alpha_1 \ , \tag{9}$$

$$\langle a^\dagger(t_1)a(t_2)\rangle = \frac{1}{\pi^2}\int\int\int d^2\alpha_1 d^2\alpha_2 d^2\alpha_3 P(\alpha_3)K(\alpha_1,t_1;\alpha_2,t_2)K(\alpha_2,t_2;\alpha_3,0)K(\alpha_3,0;\alpha_1,t_1)\alpha_1^*\alpha_2 \ , \tag{10}$$

$$\langle a^\dagger(0)a^\dagger(t)a(t)a(0)\rangle = \frac{1}{\pi^2}\int\int\int d^2\alpha_1 d^2\alpha_2 d^2\alpha_3 P(\alpha_3)K^*(\alpha_2,t;\alpha_3,0)K(\alpha_1,t;\alpha_3,0)|\alpha_3|^2\alpha_2^*\alpha_1 \ . \tag{11}$$

The determination of the propagator thus enables us to calculate any correlation function of the field operators.

The propagator $K(\alpha_2,t_2;\alpha_1,t_1)$ is related to the Q representation of the radiation field, i.e.,

$$Q(\alpha,t)=\frac{1}{\pi}\langle\alpha,t|\rho|\alpha,t\rangle \ , \tag{12}$$

in a natural way. On substituting for ρ from Eq. (6), we obtain

$$Q(\alpha,t)=\frac{1}{\pi}\int d^2\alpha_1 P(\alpha_1)|K(\alpha,t;\alpha_1,0)|^2 \ . \tag{13}$$

In particular, for an initial coherent state, $P(\alpha_1)=\delta^2(\alpha_1-\alpha_0)$, and it follows from Eq. (13) that

$$Q(\alpha,t)=\frac{1}{\pi}|K(\alpha,t;\alpha_0,0)|^2 \ . \tag{14}$$

The Q representation has the property that the expectation value, at time t, of any antinormally ordered function $O_A(a,a^\dagger)$ of a and $a^\dagger$ may be determined via the relation

$$\langle O_A(a,a^\dagger)\rangle = \int d^2\alpha\, O_A(\alpha,\alpha^*)Q(\alpha,t) \ . \tag{15}$$

The close relation of propagator to the Q representation makes it easier to evaluate the expectation values of antinormally ordered products than the normally ordered products. For example, the mean number of photons at time t is most easily evaluated by using the commutation relation $[a,a^\dagger]=1$, as follows:

$$\langle a^\dagger(t)a(t)\rangle = \langle a(t)a^\dagger(t)\rangle - 1 = \frac{1}{\pi}\int d^2\alpha_1\int d^2\alpha_2 P(\alpha_2)|K(\alpha_1,t;\alpha_2,0)|^2|\alpha_1|^2-1 \ . \tag{16}$$

Finally, we note that the Q and P representations are related to each other via the following relationship[7]:

$$Q(\alpha,t)=\int d^2\alpha_1 P(\alpha_1,t)|K(\alpha,0;\alpha_1,0)|^2 \ . \tag{17}$$

We now turn to the calculation of the propagator itself for a particular set of systems.

III. REPRESENTATION OF THE PROPAGATOR

A. Path integral for the propagator

It is possible to express the coherent-state propagator in terms of a path integral. Here we outline the derivation of the path-integral representation which was first obtained by Klauder.[4]

We consider a system which is described by a Hamiltonian, $H(a^\dagger,a;t)$, which is expressed in terms of the creation and destruction operators $a^\dagger$ and a. We suppose further that $H(a^\dagger,a;t)$ is normally ordered. By inserting n resolutions of the identity into Eq. (5) we find that

$$K(\alpha_f,t_f;\alpha_i,t_i)=\left[\frac{1}{\pi}\right]^n\int d^2\alpha_1\cdots\int d^2\alpha_n\langle\alpha_f,t_f|\alpha_n,t_n\rangle\langle\alpha_n,t_n|\alpha_{n-1},t_{n-1}\rangle\cdots\langle\alpha_1,t_1|\alpha_i,t_i\rangle \ . \tag{18}$$

We also have that

$$\langle\alpha_j,t_j|\alpha_{j-1},t_{j-1}\rangle=\left\langle\alpha_j\left|T\exp\left[-i\int_{t_{j-1}}^{t_j}d\tau H(\tau)\right]\right|\alpha_{j-1}\right\rangle$$
$$\cong\left\langle\alpha_j\left|\left[1-i\int_{t_{j-1}}^{t_j}d\tau H(a^\dagger,a;\tau)\right]\right|\alpha_{j-1}\right\rangle$$
$$\cong\langle\alpha_j|\alpha_{j-1}\rangle[1-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]$$
$$\cong\exp[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})] \ , \tag{19}$$

where $\epsilon=(t_f-t_i)/n+1$, $t_j=t_i+j\epsilon$, and the function $H(\alpha''^*,\alpha';t)$ is defined as

$$H(\alpha''^*,\alpha';t)=\frac{\langle\alpha''|H(a^\dagger,a;t)|\alpha'\rangle}{\langle\alpha''|\alpha'\rangle} \ . \tag{20}$$

Inserting Eq. (19) into Eq. (18) immediately yields

$$K(\alpha_f,t_f;\alpha_i,t_i)=\lim_{n\to\infty}\left[\frac{1}{\pi}\right]^n\int d^2\alpha_1\cdots\int d^2\alpha_n\exp\left[\sum_{j=1}^{n+1}[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]\right] \ . \tag{21}$$

We note that

$$\sum_{j=1}^{n+1}[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})]$$
$$=\sum_{j=1}^{n+1}\left[-\tfrac{1}{2}\alpha_j^*\left[\frac{\alpha_j-\alpha_{j-1}}{\epsilon}\right]\epsilon+\tfrac{1}{2}\alpha_{j-1}\left[\frac{\alpha_j^*-\alpha_{j-1}^*}{\epsilon}\right]\epsilon-i\epsilon H(\alpha_j^*,\alpha_{j-1};t_{j-1})\right]$$
$$\to\int_{t_i}^{t_f}d\tau[\tfrac{1}{2}(\alpha\dot{\alpha}^*-\alpha^*\dot{\alpha})-iH(\alpha^*,\alpha;\tau)] \ , \tag{22}$$

as $\epsilon\to 0$. It then follows that

$$K(\alpha_f,t_f;\alpha_i,t_i)=\int \mathscr{D}[\alpha(\tau)]\exp\left\{\int_{t_i}^{t_f} d\tau[\tfrac{1}{2}(\alpha\dot{\alpha}^*-\alpha^*\dot{\alpha})-iH(\alpha^*,\alpha;\tau)]\right\}, \tag{23}$$

where $\int \mathscr{D}[\alpha(\tau)]$ designates the integration over all paths $\alpha(\tau)$, such that $\alpha(t_i)=\alpha_i$ and $\alpha(t_f)=\alpha_f$.

B. Quadratic Hamiltonian

If the Hamiltonian is at most quadratic in a and $a^\dagger$, it is possible to evaluate the path integral explicitly (Yuen[8] has calculated this propagator using a different method). The most general quadratic Hamiltonian is given by

$$H(a^\dagger,a;t)=\omega(t)a^\dagger a+f(t)a^2+f^*(t)a^{\dagger 2}+g(t)a+g^*(t)a^\dagger, \tag{24}$$

where $f(t)$ and $g(t)$ are arbitrary time-dependent functions. The evaluation of the path integral (21) corresponding to this Hamiltonian is outlined in Appendix A. The resulting expression for the propagator is

$$\begin{aligned}K(\alpha_f,t_f;\alpha_i,t_i)=\exp\Big\{&-i\int_{t_i}^{t_f} d\tau[2f(\tau)X(\tau)+f(\tau)Z^2(\tau)+g(\tau)Z(\tau)]\\&-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)+Y(t_f)\alpha_f^*\alpha_i+X(t_f)(\alpha_f^*)^2-i\alpha_i^2\int_{t_i}^{t_f} d\tau f(\tau)Y^2(\tau)+Z(t_f)\alpha_f^*\\&-i\alpha_i\int_{t_i}^{t_f} d\tau[g(\tau)+2f(\tau)Z(\tau)]Y(\tau)\Big\},\end{aligned} \tag{25}$$

where $X(t)$ satisfies the differential equation

$$\frac{dX}{dt}=-2i\omega(t)X-4if(t)X^2-if^*(t), \tag{26}$$

with $X(t_i)=0$ and

$$Y(t)=\exp\left\{-i\int_{t_i}^{t} d\tau[\omega(\tau)+4f(\tau)X(\tau)]\right\}, \tag{27}$$

$$Z(t)=-i\int_{t_i}^{t} d\tau[g^*(\tau)+2g(\tau)X(\tau)]\exp\left\{-i\int_{\tau}^{t} d\tau'[\omega(\tau')+4f(\tau')X(\tau')]\right\}. \tag{28}$$

The nonlinear differential Eq. (26) for $X(t)$ can be solved if we can express $f(t)$ as

$$f(t)=\tilde{f}(t)\exp\left\{2i\int_{t_i}^{t} d\tau\,\omega(\tau)\right\}, \tag{29}$$

where $\tilde{f}(t)$ is real or imaginary. We now consider a simple example where this condition is satisfied.

C. Degenerate parametric amplifier

The quantum statistical properties of the degenerate parametric amplifier have received considerable attention in recent years.[9] This nonlinear device is predicted to exhibit photon antibunching[10] which is a strictly quantum-mechanical effect. Squeezed states, which could prove to be useful in the efforts to detect gravitational waves, are also predicted to be generated in a degenerate parametric amplifier.[8,11]

The Hamiltonian that governs this nonlinear optical device is given by

$$H(t)=\omega a^\dagger a+\kappa(e^{2i\omega t}a^2+e^{-2i\omega t}a^{\dagger 2}), \tag{30}$$

where κ is a coupling constant and ω is the mode frequency. The Hamiltonian (30) is the same as that given by Eq. (24) if we make the following identifications:

$$\omega(t)=\omega,\quad f(t)=\kappa e^{2i\omega t},\quad g(t)=0. \tag{31}$$

Under these conditions Eq. (26) can be solved and we obtain

$$X(t)=\frac{1}{2i}e^{-2i\omega t}\tanh[2\kappa(t-t_i)], \tag{32a}$$

$$Y(t)=e^{-i\omega(t-t_i)}\operatorname{sech}[2\kappa(t-t_i)], \tag{32b}$$

$$Z(t)=0. \tag{32c}$$

On substituting from Eqs. (32a)–(32c) into Eq. (25) we obtain

$$\begin{aligned}K(\alpha_f,t_f;\alpha_i,t_i)=&\{\operatorname{sech}[2\kappa(t_f-t_i)]\}^{1/2}\\&\times\exp\{-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)+\alpha_f^*\alpha_i e^{-i\omega(t_f-t_i)}\operatorname{sech}[2\kappa(t_f-t_i)]\\&-\tfrac{1}{2}i(\alpha_f^*)^2e^{-2i\omega t_f}\tanh[2\kappa(t_f-t_i)]-\tfrac{1}{2}i\alpha_i^2e^{2i\omega t_i}\tanh[2\kappa(t_f-t_i)]\}.\end{aligned} \tag{33}$$

This expression for the propagator which we have derived using a path-integral approach can also be derived using a more conventional approach.[10] Correlation functions can be computed from this propagator in ways similar to those used in the one-mode case. One must simply evaluate more integrals.

IV. MULTIMODE PROBLEMS

A. Path integral

It is also possible to apply these techniques to problems involving more than one mode. If one is dealing with N modes the propagator becomes a function of $2N$ complex variables. In particular we have

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=\langle\vec{\alpha}_f\,|\,U(t_f,t_i)\,|\,\vec{\alpha}_i\rangle, \tag{34}$$

where $\vec{\alpha}_i$ and $\vec{\alpha}_f$ are N-component vectors with components denoted by $\alpha_1^{(i)},\alpha_2^{(i)},\ldots,\alpha_N^{(i)}$ (similarly for $\vec{\alpha}_f$), and

$$|\vec{\alpha}_i\rangle=|\alpha_1^{(i)}\rangle\otimes|\alpha_2^{(i)}\rangle\otimes\cdots\otimes|\alpha_N^{(i)}\rangle.$$

There is also a path-integral representation for the N-mode propagator. One has

$$\begin{aligned}K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)&=\int\mathscr{D}[\vec{\alpha}(\tau)]e^{iS}\\&=\int\mathscr{D}[\alpha_1(\tau)]\cdots\int\mathscr{D}[\alpha_N(\tau)]e^{iS},\end{aligned} \tag{35}$$

where

$$iS=\int_{t_i}^{t_f} d\tau\left\{\sum_{n=1}^{N}\tfrac{1}{2}(\dot{\alpha}_n^*\alpha_n-\alpha_n^*\dot{\alpha}_n)-iH(\vec{\alpha}^*,\vec{\alpha};\tau)\right\}, \tag{36}$$

$\vec{\alpha}(t_i)=\vec{\alpha}_i$, $\vec{\alpha}(t_f)=\vec{\alpha}_f$, and if $H(a_1^\dagger,\ldots,a_n^\dagger,a_1,\ldots,a_n;\tau)$ is the normally ordered Hamiltonian for the system

$$H(\vec{\alpha}''^*,\vec{\alpha}';\tau)=\langle\vec{\alpha}''\,|\,H(a_1^\dagger,\ldots,a_n^\dagger,a_1,\ldots,a_n;\tau)\,|\,\vec{\alpha}'\rangle/\langle\vec{\alpha}''\,|\,\vec{\alpha}'\rangle. \tag{37}$$

B. Quadratic Hamiltonian

If the Hamiltonian is quadratic in $a_1,\ldots,a_N$ and $a_1^\dagger,\ldots,a_N^\dagger$ one can again explicitly evaluate the path integral. We express the Hamiltonian as

$$H=\sum_{i=1}^{N}\sum_{j=1}^{N}[\omega_{ij}(t)a_i^\dagger a_j+f_{ij}(t)a_ia_j+f_{ij}^*(t)a_i^\dagger a_j^\dagger] \tag{38}$$

and we assume that f has been chosen so that $f_{ij}(t)=f_{ji}(t)$. The detailed calculation of the propagator for this Hamiltonian is performed in Appendix B. We find that

$$\begin{aligned}K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=\exp\Big\{&-2i\int_{t_i}^{t_f} d\tau\,\mathrm{Tr}[X(\tau)f(\tau)]-\tfrac{1}{2}[(\vec{\alpha}_f^*)\cdot\vec{\alpha}_f+(\vec{\alpha}_i^*)\cdot\vec{\alpha}_i]+(\vec{\alpha}_f^*)^TY(t_f)\vec{\alpha}_i\\&+(\vec{\alpha}_f^*)^TX(t_f)\vec{\alpha}_f^*-i\int_{t_i}^{t_f} d\tau\,\vec{\alpha}_i^TY^T(\tau)f(\tau)Y(\tau)\vec{\alpha}_i\Big\}.\end{aligned} \tag{39}$$

In the above equation $X(t)$ and $f(\tau)$ are $N\times N$ symmetric matrices. The elements of $f(t)$ are simply the functions $f_{ij}(t)$ which appear in the Hamiltonian. The matrix $X(t)$ satisfies the equation

$$\frac{dX}{dt}=-i(\omega X+X\omega+f^*+4XfX), \tag{40}$$

where $\omega(t)$ is an $N\times N$ matrix whose elements are $\omega_{ij}(t)$, and $X(t_i)=0$. The $N\times N$ matrix $Y(t)$ is given by

$$Y(t)=T\exp\left\{-i\int_{t_i}^{t_f} d\tau[\omega(\tau)+4X(\tau)f(\tau)]\right\}. \tag{41}$$

The superscript T appearing on some of the vectors and matrices in Eq. (39) denotes transpose.

C. Parametric amplifier

The parametric amplifier with a classical pump field is a system which has been much studied in quantum optics.[12] Here we would like to use the formulas developed in the preceding section to find the propagator for this system.

The Hamiltonian we wish to consider is

$$H=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2 +\kappa(e^{i\omega_3 t}a_1 a_2+e^{-i\omega_3 t}a_1^\dagger a_2^\dagger)\,, \tag{42}$$

where $\omega_3=\omega_1+\omega_2$. The matrices $\omega(t)$ and $f(t)$ are

$$\omega(t)=\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix},\qquad f(t)=\tfrac{1}{2}\kappa e^{i\omega_3 t}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{43}$$

Considering first the equation for $X(t)$, Eq. (40), we find that

$$X(t)=-\tfrac{1}{2}ie^{-i\omega_3 t}\tanh[\kappa(t-t_i)]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{44}$$

Rather than solve for $Y(t)$, we instead solve for the vector

$$\vec{u}(t)=Y(t)\vec{\alpha}_i\,. \tag{45}$$

The vector $\vec{u}$ satisfies the equation

$$\frac{d\vec{u}}{dt}=-i(\omega\vec{u}+4Xf\vec{u})\,, \tag{46}$$

where $\vec{u}(t_i)=\vec{\alpha}_i$. One finds that

$$\vec{u}(t)=\operatorname{sech}[\kappa(t-t_i)]\begin{pmatrix}e^{-i\omega_1(t-t_i)}\alpha_1^{(i)}\\ e^{-i\omega_2(t-t_i)}\alpha_2^{(i)}\end{pmatrix}. \tag{47}$$

The final result for the propagator is then

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i)=[\operatorname{sech}\kappa(t_f-t_i)]\exp\Bigg\{-\tfrac{1}{2}[(\vec{\alpha}_f^{\,*})\cdot\vec{\alpha}_f+(\vec{\alpha}_i^{\,*})\cdot\vec{\alpha}_i]-\tfrac{1}{2}ie^{-i\omega_3 t_f}\tanh[\kappa(t_f-t_i)](\vec{\alpha}_f^{\,*})^T\sigma_1\vec{\alpha}_f^{\,*} +\operatorname{sech}[\kappa(t_f-t_i)](\vec{\alpha}_f^{\,*})^T\begin{pmatrix}e^{-i\omega_1(t_f-t_i)} & 0\\ 0 & e^{-i\omega_2(t_f-t_i)}\end{pmatrix}\vec{\alpha}_i -\tfrac{1}{2}ie^{i\omega_3 t_i}\tanh[\kappa(t_f-t_i)]\vec{\alpha}_i^{\,T}\sigma_1\vec{\alpha}_i\Bigg\}\,, \tag{48}$$

where $\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

V. CONCLUSION

We have shown how a formalism incorporating coherent-state propagators and path integrals can be of use in the consideration of certain problems in nonlinear optics. Here we concentrated on the formalism itself and certain basic results for the path integrals. These are necessary steps toward the development of approximation schemes for more complicated systems. It is in these approximations that the promise of these techniques lies.

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APPENDIX A

According to Eq. (21) the propagator $K(\alpha_f,t_f;\alpha_i,t_i)$ corresponding to the Hamiltonian (24) is given by

$$K(\alpha_f,t_f;\alpha_i,t_i)=\lim_{n\to\infty}\left[\frac{1}{\pi}\right]^n\int\cdots\int\left[\prod_{j=1}^{n}d^2\alpha_j\right]e^{iS_n}\,, \tag{A1}$$

where

$$iS_n=\sum_{j=1}^{n+1}[-\tfrac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+(1-i\epsilon\omega_j)\alpha_j^*\alpha_{j-1}-i\epsilon f_{j-1}\alpha_{j-1}^2-i\epsilon f_j^*\alpha_j^{*2}-i\epsilon g_{j-1}\alpha_{j-1}-i\epsilon g_j^*\alpha_j]\,. \tag{A2}$$

The α_j integrations in Eq. (A1) are lengthy but straightforward. The resulting equation is

$$K(\alpha_f,t_f;\alpha_i,t_i)=\lim_{n\to\infty}\frac{1}{\left[\prod_{l=1}^{n}(1+4i\epsilon f_l X_l)^{1/2}\right]} \times\exp\left[\sum_{j=0}^{n}\left\{i\epsilon\left[\left[\frac{f_jZ_j^2+g_jZ_j-i\epsilon g_j^2X_j}{1+4i\epsilon f_jX_j}\right]+\left[\frac{f_jY_j^2}{1+4i\epsilon f_jX_j}\right]\alpha_i^2 +\left[\frac{2f_jY_jZ_j+g_jY_j}{1+4i\epsilon f_jX_j}\right]\alpha_i\right]\right\}+X_{n+1}\alpha_f^{*2}+Y_{n+1}\alpha_i\alpha_f^*+Z_{n+1}\alpha_f^*\right]\,, \tag{A3}$$

where X_j, Y_j, and Z_j satisfy the following recursion relations:

$$Z_j=-i\epsilon f_j^*+\frac{(1-i\epsilon\omega_j)^2X_{j-1}}{1+4i\epsilon f_{j-1}X_{j-1}}\,, \tag{A4}$$

$$Y_j=\frac{(1-i\epsilon\omega_j)Y_{j-1}}{1+4i\epsilon f_{j-1}X_{j-1}}\,, \tag{A5}$$

$$Z_j=-i\epsilon g_j^*+\frac{(1-i\epsilon\omega_j)(Z_{j-1}-2i\epsilon g_jX_{j-1})}{1+4i\epsilon f_{j-1}X_{j-1}}\,, \tag{A6}$$

with $X_0=Z_0=0$ and $Y_0=1$. On taking the limit $n\to\infty$, we obtain

$$\prod_{l=1}^{n}(1+4i\epsilon f_lX_l)^{1/2}\to\exp\left[2i\int_{t_i}^{t_f}d\tau\,f(\tau)X(\tau)\right]\,, \tag{A7}$$

$$\sum_{j=0}^{n}\left[\frac{-i\epsilon(f_jZ_j^2+g_jZ_j-i\epsilon g_j^2X_j)}{1+4i\epsilon f_jX_j}\right]\to -i\int_{t_i}^{t_f}d\tau\,Z(\tau)[f(\tau)Z(\tau)+g(\tau)]\,, \tag{A8}$$

$$\sum_{j=0}^{n}\left[\frac{-i\epsilon f_jY_j^2}{1+4i\epsilon f_jX_j}\right]\to -i\int_{t_i}^{t_f}d\tau\,f(\tau)Y^2(\tau)\,, \tag{A9}$$

$$\sum_{j=0}^{n}\left[\frac{-i\epsilon(2f_jY_jZ_j+g_jY_j)}{1+4i\epsilon f_jX_j}\right]\to -i\int_{t_i}^{t_f}d\tau[2f(\tau)Y(\tau)Z(\tau)+g(\tau)Y(\tau)]\,, \tag{A10}$$

$$X_{n+1},Y_{n+1},Z_{n+1}\to X(t_f),Y(t_f),Z(t_f)\,, \tag{A11}$$

and, in view of the recursion relations (A4)–(A6), the functions $X(t)$, $Y(t)$, and $Z(t)$ satisfy the differential equation

$$\frac{dX}{dt} = -2i\omega(t)X - 4if(t)X^2 - if^*(t) , \tag{A12}$$

$$\frac{dY}{dt} = -i[\omega(t)+4f(t)X(t)]Y , \tag{A13}$$

$$\frac{dZ}{dt} = -i[\omega(t)+4f(t)X(t)]Z - i[g^*(t)+2g(t)X(t)] , \tag{A14}$$

where $X(t_i)=Z(t_i)=0$ and $Y(t_i)=1$.

On substituting from Eqs. (A7)–(A11) into Eq. (A3), we obtain

$$\begin{aligned} K(\alpha_f,t_f;\alpha_i,t_i) = \exp\Bigg[& -i\int_{t_i}^{t_f} d\tau[2f(\tau)X(\tau)+f(\tau)Z^2(\tau)+g(\tau)Z(\tau)] - \tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2) \\ & + Y(t_f)\alpha_f^*\alpha_i + X(t_f)(\alpha_f^*)^2 - i\alpha_i^2\int_{t_i}^{t_f} d\tau f(\tau)Y^2(\tau) \\ & - i\alpha_i\int_{t_i}^{t_f} d\tau[g(\tau)+2f(\tau)Z(\tau)]Y(\tau) + Z(t_f)\alpha_f^* \Bigg] . \end{aligned} \tag{A15}$$

Equations (A13) and (A14) can be integrated and the resulting solutions for $Y(t)$ and $Z(t)$ are

$$Y(t) = \exp\left[-i\int_{t_i}^{t} d\tau[\omega(\tau)+4f(\tau)X(\tau)]\right] , \tag{A16}$$

$$Z(t) = -i\int_{t_i}^{t} d\tau\, g^*(\tau)[1+2X(\tau)]\exp\left[-i\int_{\tau}^{t} d\tau'[\omega(\tau')+4f(\tau')X(\tau')]\right] , \tag{A17}$$

where $X(t)$ is determined by solving Eq. (A12) subject to $X(t_i)=0$.

APPENDIX B

We would like to compute the propagator for the system governed by the Hamiltonian given by Eq. (38). As in the one-mode case we have that the propagator is given by

$$K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i) \lim_{n\to\infty}\left[\frac{1}{\pi^N}\right]^n \int d\vec{\alpha}_1 \cdots d\vec{\alpha}_n e^{iS_n} , \tag{B1}$$

where $d\vec{\alpha}_j = d^2\alpha_1^{(j)} d^2\alpha_2^{(j)} \cdots d^2\alpha_N^{(j)}$ and

$$iS_n = \sum_{l=1}^{n+1} \{ -\tfrac{1}{2}[(\vec{\alpha}_l^*)\cdot\vec{\alpha}_l + (\vec{\alpha}_{l-1}^*)\cdot\vec{\alpha}_{l-1}] + (\vec{\alpha}_l^*)\cdot\vec{\alpha}_{l-1} - i\epsilon[(\vec{\alpha}_l^*)^T\omega_l\vec{\alpha}_{l-1} + \vec{\alpha}_{l-1}^T f_{l-1}\vec{\alpha}_{l-1} + (\vec{\alpha}_l^*)^T f_l^*(\vec{\alpha}_l^*)]\} . \tag{B2}$$

In the above equation $\vec{\alpha}^T$ designates the transpose of $\vec{\alpha}$ and $f_l = f(t_l)$ is an $N\times N$ matrix where $t_l = t_i + l\epsilon$.

To perform the integrations it is necessary to split each $\alpha_j^{(l)}$ into real and imaginary parts. That is, for each l we must go from a N-dimensional space, C^N (of which $\vec{\alpha}_l$ is a member), to a $2N$-dimensional space. It is best to view this space as a tensor product space $C^N\otimes C^2$. If $\eta_i\in C^N$ is the vector whose ith component is 1 and whose other components are 0, and $v_j\in C^2$ is the vector whose jth component is 1 and whose other component is 0, then $\vec{\alpha}\in C^N \to z\in C^N\otimes C^2$, where

$$z = \sum_{j=1}^{N} (x_j\eta_j\otimes v_1 + y_j\eta_j\otimes v_2) \tag{B3}$$

and the components of $\vec{\alpha}$ are $\alpha_j = x_j + iy_j$. It is then possible to express the action as

$$iS_n = -\sum_{l=1}^{n} z_l^T M_l z_l + \sum_{l=1}^{n+1} z_l^T L_l z_{l-1} - \tfrac{1}{2}[(\vec{\alpha}_f^*)\cdot\vec{\alpha}_f + (\vec{\alpha}_i^*)\cdot\vec{\alpha}_i] - i\epsilon[\vec{\alpha}_i^T f_i\vec{\alpha}_i + (\vec{\alpha}_f^*)^T f_f^*\vec{\alpha}_f^*] , \tag{B4}$$

where $M_l = I + i\epsilon(f_l\otimes\gamma_1 + f_l^*\otimes\gamma_2)$, $L_l = (I_N - i\epsilon\omega_l)\otimes\mu$, I is the identity on $C^N\otimes C^2$, I_N is the identity on C^N, and

$$\mu = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} , \quad \gamma_1 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} , \quad \gamma_2 = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} . \tag{B5}$$

We now want to do the integrations starting with $l=1$, then going to $l=2$ and so on. To do this we make use of the formula for the integral (assuming that it exists)

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n e^{-\vec{x}^T A\vec{x} + \vec{y}^T\vec{x}} = \frac{\pi^{n/2}}{(\det A)^{1/2}} e^{(1/4)\vec{y}^T A^{-1}\vec{y}} , \tag{B6}$$

where A is a symmetric $n\times n$ matrix and $\vec{y}$ is an n-component vector. Using this formula to do the $l=1$ integration we pick up a factor of

$$\pi^N(\det M_1)^{-1/2}\exp(\tfrac{1}{4}z_0^T L_1^T M_1^{-1} L_1 z_0)$$

and terms in the exponent which are linear and quadratic in z_2. We can express the part of the action containing z_2 (after having done the $l=1$ integration) as

$$-z_2^T M_2' z_2 + z_3^T L_2 z_2 + v_2^T z_2 , \tag{B7}$$

where

$$M_2' = M_2 - \tfrac{1}{4}L_2 M_1^{-1} L_2^T \tag{B8}$$

and

$$v_2 = \tfrac{1}{4}[L_2 M_1^{-1} L_1 z_0 + L_2(M_1^{-1})^T L_1 z_0] . \tag{B9}$$

In general, if one has done $l-1$ of the integrations the part of the action containing z_l can be expressed as

$$-z_l^T M_l' z_l + z_{l+1}^T L_l z_l + v_l^T z_l , \tag{B10}$$

where M_l' and v_l obey the recurrence relations

$$M_{l+1}' = M_{l+1} - \tfrac{1}{4}L_{l+1}(M_l')^{-1}L_{l+1}^T , \tag{B11}$$

$$v_l = \tfrac{1}{4}[L_{l+1}(M_l'^{-1})^T + L_{l+1}(M_l'^{-1})]v_l . \tag{B12}$$

Note also that each integration contributes a factor of

$$\pi^N(\det M_l')^{-1/2}\exp(\tfrac{1}{4}v_l^T M_l'^{-1} v_l) .$$

One can show from the above recursion relations that it is possible to express M_l' and v_l in the form

$$M_l' = M_l - X_l\otimes\gamma_2 , \quad v_l = u_l\otimes\hat{e}_1 ,$$

where $\hat{e}_1 = (1/\sqrt{2})(v_1 - iv_2)$ and, to first order in ϵ, X_l, and u_l obey the recursion relations

$$X_{l+1} = X_l - i\epsilon(\omega_{l+2}X_l + X_l\omega_{l+2} + f_{l+1}^* + 4X_l f_{l+1} X_l) , \tag{B13}$$

$$u_{l+1} = u_l - i\epsilon(\omega_{l+1}u_l + 4X_{l-1}f_l u_l) . \tag{B14}$$

Upon taking the $\epsilon\to 0$ limit these equations become

$$\frac{dX}{dt} = -i(\omega X + X\omega + f^* + 4XfX) , \tag{B15}$$

$$\frac{du}{dt} = -i(\omega u + 4Xfu) , \tag{B16}$$

where $X(t_i)=0$ and $u(t_i)=\sqrt{2}\,\vec{\alpha}_i$.

Upon performing all n integrations we find that

$$\begin{aligned} K(\vec{\alpha}_f,t_f;\vec{\alpha}_i,t_i) = \lim_{n\to\infty}\prod_{j=1}^{n}\frac{1}{(\det M_j')^{1/2}} & \times\exp\Bigg[-\tfrac{1}{2}[(\vec{\alpha}_i^*)\cdot\vec{\alpha}_i + (\vec{\alpha}_f^*)\cdot\vec{\alpha}_f] - z_f^T(M_{n+1}' - I)z_f + v_{n+1}^T z_f \\ & + \tfrac{1}{4}\sum_{l=1}^{n} v_l^T(M_l'^{-1})v_l - i\epsilon[\vec{\alpha}_i^T f_i\vec{\alpha}_i + (\vec{\alpha}_f^*)^T f_f^*\vec{\alpha}_f^*]\Bigg] . \end{aligned} \tag{B17}$$

We now take the limit $n\to\infty$ and find that

$$\prod_{l=1}^{n}\frac{1}{(\det M')^{1/2}} \to \exp\left[-2i\int_{t_i}^{t_f} d\tau\,\mathrm{Tr}[X(\tau)f(\tau)]\right] , \tag{B18}$$

$$-z_f^T(M_{n+1}' - I)z_f \to (\vec{\alpha}_f^*)^T X(t_f)\vec{\alpha}_f^* , \tag{B19}$$

$$v_{n+1}^T z_f \to \frac{1}{\sqrt{2}}(\vec{\alpha}_f^*)^T u(t_f) , \tag{B20}$$

$$\frac{1}{4}\sum_{l=1}^{n} v_l^T (M_l^{-1}) v_l \to -\frac{1}{2} i \int_{t_i}^{t_f} d\tau\, u^T(\tau) f(\tau) u(\tau) . \tag{B21}$$

We can reexpress the terms involving $u(\tau)$ by defining a matrix

$$Y(t) = T \exp\left[-i \int_{t_i}^{t} d\tau [\omega(\tau) + 4X(\tau) f(\tau)] \right] \tag{B22}$$

and noting that

$$u(t) = \sqrt{2}\, Y(t) \vec{\alpha}_i , \tag{B23}$$

so that $K(\vec{\alpha}_f, t_f; \vec{\alpha}_i, t_i)$ is given by the expression in Eq. (48).

One can check that this expression is correct by observing that $K(\vec{\alpha}_f, t_f; \vec{\alpha}_i, t_i)$ satisfies the equation

$$\begin{aligned} i\frac{\partial}{\partial t} K(\vec{\alpha}, t; \vec{\beta}, t_i) &= (\vec{\alpha} \,|\, H(t) U(t,t_i) \,|\, \vec{\beta}) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \left[\omega_{ij}\alpha_i^* \left[\frac{\partial}{\partial \alpha_j^*} + \frac{1}{2}\alpha_j \right] + f_{ij} \left[\frac{\partial}{\partial \alpha_i^*} + \frac{1}{2}\alpha_i \right] \left[\frac{\partial}{\partial \alpha_j^*} + \frac{1}{2}\alpha_j \right] + f_{ij}^* \alpha_i^* \alpha_j^* \right] K(\vec{\alpha}, t; \vec{\beta}, t_i) \end{aligned} \tag{B24}$$

and verifying that, indeed, the expression given by Eq. (48) does satisfy this equation.

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Path-integral approach to the quantum theory of the degenerate parametric amplifier

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The quantum theory of the degenerate parametric amplifier is usually treated in the parametric approximation where the pump field is treated classically. In this paper we present a fully quantized theory of this nonlinear optical device using a path-integral approach. A perturbation series, the first term of which corresponds to the parametric approximation, is employed to evaluate explicitly the coherent-state propagator. The question of the validity of the parametric approximation is considered and the conditions under which this approximation is justified are elucidated. Finally, certain correlation functions for the signal-mode operators are calculated that are needed to study squeezed states. It is shown that the quantum nature of the pump field tends to decrease the squeezing.

I. INTRODUCTION

The quantum statistical properties of the radiation produced by a degenerate parametric amplifier have recently received renewed attention.[1-5] Theoretical predictions indicate that under the proper conditions one should be able to produce light in both squeezed and antibunched states. Both types of states are nonclassical in nature. It has also been shown that squeezed states can be useful in the detection of very weak signals.[6] A device which can produce such states is, therefore, of some interest.

The degenerate parametric amplifier is a device which provides a nonlinear coupling between two modes of the radiation field.[7] The first, the pump mode, has a frequency of 2ω, while the second, the signal mode, has a frequency ω. The quantum theory of this device is usually treated in the so-called parametric approximation. In this approximation the pump mode is treated classically, i.e., replaced by a c-number, so that a single-mode Hamiltonian is obtained which is quadratic in the field operators. The problem can then be solved without further approximation. It should be noted that the parametric approximation neglects two effects. First, it ignores quantum fluctuations in the pump mode. Second, by treating the pump mode as a fixed c-number it also ignores depletion of this mode.

In this paper we will show that the parametric approximation can be derived from the first term of a perturbation series for the propagator of this system. Examination of the next term in the series allows us both to calculate corrections to the parametric approximation and to set bounds on its region of validity. We then use the lowest-order correction to the propagator to calculate corrections to both the intensity and squeezing of the signal mode. The perturbation series itself is derived from a path-integral representation for the propagator of this system. In a previous paper we presented a formalism for applying path integrals[8] to certain problems in nonlinear optics. Here we employ that formalism. The path-integral approach is useful because it allows one to see more clearly than the canonical approach the connection between the classical and quantum dynamics of the system.

II. PERTURBATION SERIES FOR PROPAGATOR

The Hamiltonian for a degenerate parametric amplifier is given by (we use units in which $\hbar = 1$)

$$H = \omega a^\dagger a + 2\omega b^\dagger b + \kappa (a^{\dagger 2} b + a^2 b^\dagger) , \tag{2.1}$$

where a ($a^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators for the signal and pump modes, respectively, and κ is a coupling constant which depends upon the second-order susceptibility tensor of the medium which mediates the interaction. In the parametric approximation the pump mode is treated classically so that b is replaced by $\beta_0 e^{-2i\omega t}$ where β_0 is the amplitude of the pump mode. The resulting Hamiltonian is

$$H_p = \omega a^\dagger a + \kappa (\beta_0 e^{-2i\omega t} a^{\dagger 2} + \beta_0^* e^{2i\omega t} a^2) . \tag{2.2}$$

The propagator for this Hamiltonian was calculated in Ref. 8 and is given by (where β_0 is assumed to be real)

$$G(\alpha_f, t_f, \alpha_i, t_i) = (\alpha_f \,|\, U_p(t_f, t_i) \,|\, \alpha_i) \tag{2.3}$$

$$\begin{aligned} &= [\operatorname{sech}[2\kappa\beta_0(t_f - t_i)]]^{1/2} \\ &\quad \times \exp[-\tfrac{1}{2}(|\alpha_i|^2 + |\alpha_f|^2) + \alpha_f^* \alpha_i e^{-i\omega(t_f - t_i)} \operatorname{sech}[2\kappa\beta_0(t_f - t_i)] \\ &\qquad - \tfrac{1}{2} i (\alpha_f^*)^2 e^{-2i\omega t_f} \tanh[2\kappa\beta_0(t_f - t_i)] - \tfrac{1}{2} i \alpha_i^2 e^{2i\omega t_i} \tanh[2\kappa\beta_0(t_f - t_i)]] . \end{aligned} \tag{2.4}$$

Here $U_p(t_f,t_i)$ is the time-development transformation corresponding to H_p and $|\alpha\rangle$ is a coherent state with amplitude α.

The propagator for the Hamiltonian given by Eq. (2.1) is given by

$$K(\alpha_f,\beta_f,t_f;\alpha_i,\beta_i,t_i)=\langle\alpha_f,\beta_f|e^{-iH(t_f-t_i)}|\alpha_i,\beta_i\rangle\ , \tag{2.5}$$

where $|\alpha,\beta\rangle=|\alpha\rangle\otimes|\beta\rangle$, i.e., the tensor product of a coherent state for the signal mode with amplitude α and a coherent state for the pump mode with amplitude β. It is also possible to express this propagator in terms of a path integral. We have that

$$K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\int\mathscr{D}[\alpha(\tau)]\int\mathscr{D}[\beta(\tau)]e^{iS}\ , \tag{2.6}$$

where

$$iS=\int_0^t d\tau[\tfrac{1}{2}(\dot{\alpha}^*\alpha-\alpha^*\dot{\alpha})+\tfrac{1}{2}(\dot{\beta}^*\beta-\beta^*\dot{\beta})-iH(\alpha,\alpha^*;\beta,\beta^*)]\ , \tag{2.7}$$

$$H(\alpha,\alpha^*;\beta,\beta^*)=\omega|\alpha|^2+2\omega|\beta|^2+\kappa[(\alpha^*)^2\beta+\alpha^2\beta^*] \tag{2.8}$$

and the paths $\alpha(\tau)$ and $\beta(\tau)$ are such that $\alpha(t)=\alpha_f$, $\beta(t)=\beta_f$, $\alpha(0)=\alpha_i$, and $\beta(0)=\beta_i$.

It is not possible to evaluate the expression appearing in Eq. (2.6) exactly and we, therefore, resort to a perturbation expansion.[9] The first term of this expansion gives the contribution to the propagator corresponding to a classical description of the pump field; that is, if we retain only this term and make a further approximation which corresponds to letting the pump mode propagate in time as if there were no interaction, then we obtain the results given by the parametric approximation. We can calculate corrections by calculating the next term in the perturbation series and by refining the freely-propagating-pump-mode approximation.

Because the Hamiltonian given by Eq. (2.1) has only linear terms in b and $b^\dagger$ appearing in the interaction it is possible to perform the integration over the paths $\beta(\tau)$ by using the results in Ref. 8 for an arbitrary quadratic Hamiltonian. We find that

$$K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\int\mathscr{D}[\alpha(\tau)]\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\times e^{iS_0+iS_1}\ , \tag{2.9}$$

where

$$iS_0=\int_0^t d\tau\{\tfrac{1}{2}(\dot{\alpha}^*\alpha-\alpha^*\dot{\alpha})-i\omega|\alpha|^2-i\kappa[(\beta_f^*e^{-2i\omega t})e^{2i\omega\tau}\alpha^2+\beta_i e^{-2i\omega\tau}(\alpha^*)^2]\}\ , \tag{2.10}$$

$$iS_1=-\kappa^2\int_0^t d\tau_2\int_0^{\tau_2}d\tau_1 e^{-2i\omega(\tau_2-\tau_1)}[\alpha^*(\tau_2)\alpha(\tau_1)]^2\ . \tag{2.11}$$

We have split the action into two parts, S_0 containing terms of zeroth and first order in κ, and S_1 containing only terms of second order in κ. We assume that the interaction is weak so that S_1 is small.

We now expand the propagator in Eq. (2.9) in a power series in S_1:

$$\begin{aligned}K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)&=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\sum_{n=0}^{\infty}\frac{1}{n!}\int\mathscr{D}[\alpha(\tau)]e^{iS_0}(iS_1)^n\\&\simeq K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)+K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\ ,\end{aligned} \tag{2.12}$$

where

$$K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\int\mathscr{D}[\alpha(\tau)]e^{iS_0}\ , \tag{2.13}$$

$$K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\int\mathscr{D}[\alpha(\tau)]e^{iS_0}(iS_1)\ . \tag{2.14}$$

Before evaluating $K^{(0)}$ let us note the following. The exponential factor appearing in both $K^{(0)}$ and $K^{(1)}$ has a magnitude given by

$$|\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]|=[\exp(-|\beta_f-\beta_i e^{-2i\omega t}|^2)]^{1/2}\ , \tag{2.15}$$

so that it is peaked about the value $\beta_f=e^{-2i\omega t}\beta_i$. This simply corresponds to free propagation of the pump mode, i.e., if there were no interactions and at $t=0$ the pump mode were in a coherent state with amplitude β_i, then at time t it would be in a coherent state with amplitude $e^{-2i\omega t}\beta_i$. If we replace β_f in iS_0 by $e^{-2i\omega t}\beta_i$ we find that (again assuming that β_0 is real)

$$iS_0\to\int_0^t d\tau\{\tfrac{1}{2}(\dot{\alpha}^*\alpha-\alpha^*\dot{\alpha})-i\omega|\alpha|^2-i\kappa\beta_0[(\alpha^*)^2e^{-2i\omega\tau}+\alpha^2e^{2i\omega\tau}]\}\ . \tag{2.16}$$

This is just the action for the signal mode in the parametric approximation (corresponding to the Hamiltonian H_p). If the path integral appearing in Eq. (2.13) is a slowly varying function of β_f then this replacement is justified and we can approximate $K^{(0)}$ by

$$K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\simeq\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\times G(\alpha_f,t;\alpha_i,0)\ , \tag{2.17}$$

where β_0 in the expression for G [Eq. (2.4)] is set equal to β_i. This expression for $K^{(0)}$ will reproduce all of the results of the parametric approximation. We can calculate corrections to this approximation by doing two things. First, we evaluate $K^{(1)}$ where we set $\beta_f=e^{-2i\omega t}\beta_i$ in the path integral appearing in Eq. (2.14). Second, we must calculate corrections to the approximation implied by Eq. (2.17) for $K^{(0)}$. We will discuss the validity of the approximations we have made in Sec. IV.

Let us now evaluate $K^{(0)}$ and $K^{(1)}$. We can find $K^{(0)}$ in the same way in which we found G in Ref. 8. We have that

$$K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\times G^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\ , \tag{2.18}$$

where

$$G^{(0)}(\alpha_f,\beta_f,t_2;\alpha_i,\beta_i,t_1)=\{\operatorname{sech}[2\sqrt{\kappa_1\kappa_2}(t_2-t_1)]\}^{1/2}\exp[-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)+A_{21}\alpha_i^2+B_{21}(\alpha_f^*)^2+C_{21}\alpha_f^*\alpha_i]\ , \tag{2.19}$$

$$A_{jl}=-\frac{1}{2}i\left(\frac{\kappa_2}{\kappa_1}\right)^{1/2}e^{2i\omega t_l}\tanh[2\sqrt{\kappa_1\kappa_2}(t_j-t_l)]\ , \tag{2.20a}$$

$$B_{jl}=-\frac{1}{2}i\left(\frac{\kappa_1}{\kappa_2}\right)^{1/2}e^{-2i\omega t_j}\tanh[2\sqrt{\kappa_1\kappa_2}(t_j-t_l)]\ , \tag{2.20b}$$

$$C_{jl}=e^{-i\omega(t_j-t_l)}\operatorname{sech}[2\sqrt{\kappa_1\kappa_2}(t_j-t_l)]\ , \tag{2.20c}$$

and

$$\kappa_1=\kappa\beta_i,\quad\kappa_2=\kappa\beta_f^*e^{-2i\omega t}\ . \tag{2.21}$$

In the above we assume that $t_j\geq t_l$ and we define $t_0=0$. We must also specify which branch of the square-root function is to be chosen. It should be chosen so that $\sqrt{\kappa_1/\kappa_2}\sqrt{\kappa_1\kappa_2}=\kappa_1$.

The evaluation of $K^{(1)}$ is complicated and the details of the calculation are given in Appendix A. We will be interested in the case in which the signal mode is initially in the vacuum state. This means that we will be interested in the propagator for $\alpha_i=0$. The resulting expression is

$$\begin{aligned}K^{(1)}(\alpha_f,\beta_f,t;0,\beta_i,0)&=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)\\&\quad\times[g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-i\omega t}\alpha_f^*)^2+g_0(t)]\ ,\end{aligned} \tag{2.22}$$

where

$$g_4(t)=\frac{-\kappa^2}{\eta_0^2}\frac{1}{8}[(\eta_0t)^2\operatorname{sech}^4(\eta_0t)+2(\eta_0t)\tanh(\eta_0t)\operatorname{sech}^2(\eta_0t)-\tanh^2(\eta_0t)-2\tanh^2(\eta_0t)\operatorname{sech}^2(\eta_0t)]\ , \tag{2.23a}$$

$$g_2(t)=-i\frac{\kappa^2}{\eta_0^2}\frac{3}{4}\{-(\eta_0t)^2\operatorname{sech}^2(\eta_0t)\tanh(\eta_0t)+(\eta_0t)[\tfrac{5}{3}\operatorname{sech}^2(\eta_0t)-\tfrac{1}{3}]-\tfrac{4}{3}\tanh(\eta_0t)+2\tanh^3(\eta_0t)\}\ , \tag{2.23b}$$

$$g_0(t)=-\frac{\kappa^2}{\eta_0^2}\frac{1}{8}\{(\eta_0t)^2[3\operatorname{sech}^2(\eta_0t)-1]+(\eta_0t)4\tanh(\eta_0t)-6\tanh^2(\eta_0t)\}\ , \tag{2.23c}$$

$$\eta_0=2\kappa\beta_i\ . \tag{2.23d}$$

It follows from Eq. (2.12) that Eqs. (2.18) and (2.22) give us an explicit expression for the propagator $K(\alpha_f,\beta_f,t;0,\beta_i,0)$ that contains the quantum corrections to the parametric approximation. The correlation functions for the field operator can be evaluated from the propagator. In Sec. III we calculate the correlation functions that are needed to study the intensity and the squeezing of the signal mode.

III. CORRECTIONS TO CORRELATION FUNCTIONS AND "SQUEEZING" OF THE SIGNAL MODE

The propagator K is closely related to the Q representation of the radiation field and, hence, can be used directly to evaluate expectation values of antinormally ordered products of creation and annihilation operators.[8] For the case of interest in which the pump mode is initially in the state $|\beta_i\rangle$ and the signal mode is initially in the vacuum states we have

$$Q(\alpha_f,\beta_f,t)=\frac{1}{\pi^2}|K(\alpha_f,\beta_f,t;0,\beta_i,0)|^2\ . \tag{3.1}$$

The two correlation functions which we wish to calculate, $\langle a^\dagger(t)a(t)\rangle$ and $\langle[a(t)]^2\rangle$, can therefore be expressed as

$$\langle a^\dagger(t)a(t)\rangle=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f|K(\alpha_f,\beta_f,t;0,\beta_i,0)|^2\times|\alpha_f|^2-1\ , \tag{3.2}$$

$$\langle[a(t)]^2\rangle=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f\,|K(\alpha_f,\beta_f,t;0,\beta_i,0)|^2\times(\alpha_f^*)^2\ . \tag{3.3}$$

The correlation function $\langle a^\dagger(t)a(t)\rangle$ is just the intensity of the signal mode and examination of it will allow us to see how this mode grows with time. Calculation of the correlation function $\langle[a(t)]^2\rangle$ allows us to examine the squeezing of the signal mode.

In a squeezed state, the fluctuations in one quadrature are smaller than the standard quantum limit. The fluctuations are increased in the conjugate one so that the uncertainty relation is not violated. Squeezing is a genuinely quantum-mechanical feature of the radiation field. It has been predicted that a number of nonlinear optical systems will generate such states.[10–19]

We define Hermitian dimensionless amplitudes

$$a_1=\tfrac{1}{2}ae^{i(\omega t-\pi/4)}+\text{H.c.}\ , \tag{3.4a}$$

$$a_2=\frac{1}{2i}ae^{i(\omega t-\pi/4)}+\text{H.c.} \tag{3.4b}$$

For initial vacuum state of the pump mode we obtain the following formulas for the variances of the amplitudes a_1 and a_2:

$$\Delta a_1^2=\tfrac{1}{4}+\tfrac{1}{2}\langle a^\dagger(t)a(t)\rangle+\tfrac{1}{2}\text{Im}\{\langle[a(t)]^2\rangle e^{2i\omega t}\}\ , \tag{3.5a}$$

$$\Delta a_2^2=\tfrac{1}{4}+\tfrac{1}{2}\langle a^\dagger(t)a(t)\rangle-\tfrac{1}{2}\text{Im}\{\langle[a(t)]^2\rangle e^{2i\omega t}\}\ . \tag{3.5b}$$

It is clear that we need to evaluate the correlation functions given in Eqs. (3.2) and (3.3) to study the squeezing in the variables a_1 and a_2.

In order to calculate the lowest-order approximation to the correlation functions $\langle a^\dagger(t)a(t)\rangle$ and $\langle a^2(t)\rangle$ we first substitute $K^{(0)}$ for K in Eqs. (3.2) and (3.3) and make use of the freely-propagating-pump approximation to evaluate the β_f integral. This yields

$$\langle a^\dagger(t)a(t)\rangle=\frac{1}{\pi}\int d^2\alpha_f\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2\times|\alpha_f|^2-1$$

$$=\sinh^2(\eta_0 t)\ , \tag{3.6}$$

$$\langle[a(t)]^2\rangle=\frac{1}{\pi}\int d^2\alpha_f\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2\alpha_f^2$$

$$=-ie^{-2i\omega t}\sinh(\eta_0 t)\cosh(\eta_0 t)\ . \tag{3.7}$$

These are the results which one obtains from the parametric approximation.

In order to calculate corrections to the above expressions we need (i) to improve the freely-propagating-pump approximation and (ii) to include the effects of $K^{(1)}$. It is clear how to do the latter as $K^{(1)}$ has been calculated in Sec. II. The idea behind the former is as follows. In making the freely-propagating-pump approximation we assumed that $G^{(0)}$ was a constant as a function of β_f in a neighborhood of $\beta_f=e^{-2i\omega t}\beta_i$. We can correct this by taking into account some of the variation of $G^{(0)}$ as a function of β_f in this region. This can be done by expanding in a power series in $\delta\beta_f=\beta_f-\beta_i e^{-2i\omega t}$. It turns out that the convenient quantity to expand is $\int d^2\alpha_f G^{(0)}$ multiplied by either α_f^2 or $|\alpha_f|^2$ (where we choose α_f^2 if we are evaluating $\langle a^2\rangle$ and $|\alpha_f|^2$ if we are evaluating $\langle a^\dagger a\rangle$) because we can do the α_f integration exactly. We then expand these quantities up to second order in $\delta\beta_f$ and then perform the β_f integration. The linear and quadratic terms in $\delta\beta_f$ give corrections to the freely-propagating-pump approximation. The details of these calculations are given in Appendix B. We obtain

$$\langle a^\dagger(t)a(t)\rangle=\sinh^2(\eta_0 t)+\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[2\sinh^2(\eta_0 t)+1]+\eta_0 t[2\sinh(\eta_0 t)\cosh(\eta_0 t)]-3\sinh^4(\eta_0 t)-3\sinh^2(\eta_0 t)\}\ , \tag{3.8}$$

$$\langle[a(t)]^2\rangle=-ie^{-2i\omega t}\sinh(\eta_0 t)\cosh(\eta_0 t)-ie^{-2i\omega t}\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[2\sinh(\eta_0 t)\cosh(\eta_0 t)]+\eta_0 t[2\sinh^2(\eta_0 t)+2]-3\sinh^3(\eta_0 t)\cosh(\eta_0 t)-2\sinh(\eta_0 t)\cosh(\eta_0 t)\}\ . \tag{3.9}$$

The fluctuations in the conjugate variables $a_1(t)$ and $a_2(t)$ are obtained on substituting from Eqs. (3.8) and (3.9) in Eqs. (3.5):

$$\Delta a_1^2=\tfrac{1}{4}e^{-2\eta_0 t}+\frac{\kappa^2}{2\eta_0^2}\{(\eta_0 t)^2e^{-2\eta_0 t}-\eta_0 t(e^{-2\eta_0 t}+1)+[3\sinh^2(\eta_0 t)+2]\sinh(\eta_0 t)e^{-\eta_0 t}-\sinh^2(\eta_0 t)\}\ , \tag{3.10}$$

$$\Delta a_2^2=\tfrac{1}{4}e^{2\eta_0 t}+\frac{\kappa^2}{2\eta_0^2}\{(\eta_0 t)^2e^{2\eta_0 t}+\eta_0 t(e^{2\eta_0 t}+1)-[3\sinh^2(\eta_0 t)+2]\sinh(\eta_0 t)e^{\eta_0 t}-\sinh^2(\eta_0 t)\}\ . \tag{3.11}$$

Equations (3.8), (3.10), and (3.11) give us the lowest-order quantum corrections to the parametric approximation for the quantities $\langle a^\dagger(t)a(t)\rangle$, Δa_1^2, and Δa_2^2. In Table I, we have calculated Δa_1^2 as a function of $\eta_0 t$ for different values of β_i. It is clear that the quantum fluctuations in the pump mode tend to decrease the squeezing in the signal mode.

As we will see in Sec. IV our values for Δa_1^2 will be

TABLE I. Calculated values of Δa_1^2 as a function of $\eta_0 t$ for different values of β_i.

	Δa_1^2 (10^4)			
$\eta_0 t$	Parametric approx.	$\beta_i=1000$	$\beta_i=100$	$\beta_i=10$
0.0	2500.00	2500.00	2500.00	2500.00
0.2	1675.80	1675.80	1675.80	1675.82
0.4	1123.32	1123.32	1123.32	1123.49
0.6	752.986	752.986	752.991	753.561
0.8	504.741	504.741	504.756	
1.0	338.338	338.339	338.373	
1.2	226.795	226.796	226.865	
1.4	152.025	152.026	152.157	
1.6	101.906	101.908	102.140	
1.8	68.3093	68.3133	68.7075	
2.0	45.7891	45.7956		
2.2	30.6933	30.7038		
2.4	20.5744	20.5909		
2.6	13.7914	13.8170		
2.8	9.24466	9.28390		
3.0	6.19688	6.25665		

good approximations to the actual values as long as $\eta_0 t$ is of order one or less and $\exp(2\eta_0 t)\ll\beta_i$. For values of $\eta_0 t$ which satisfy these conditions we find that the corrections to the parametric approximation are of the order of 1%. If one considers values of $\eta_0 t$ beyond the range specified by these conditions one finds that Δa_1^2 reaches a minimum and then starts increasing. This type of behavior is not unexpected because as the pump becomes depleted and loses its coherent-state character its phase becomes less well defined. This results in a decrease in the squeezing of the signal mode. An analysis with a classical pump with phase noise shows this explicitly.[5] For the case of a quantum-mechanical pump mode our results provide, at best, an indication of this type of behavior as we are extrapolating our results beyond their range of validity. Finally, we note that the minimum uncertainty relation $\Delta a_1\Delta a_2=\frac{1}{4}$ which holds for the signal mode in the parametric approximation is now no longer satisfied. The quantization of the pump mode removes the minimum uncertainty characteristic of the signal mode.

IV. DISCUSSION OF APPROXIMATIONS

In this section we would like to consider a number of the approximations which were made in Secs. II and III. First we will examine some limitations on the validity of the perturbation expansion itself. We will then consider the conditions under which the approximation implied by Eq. (2.17) is reasonable. Finally, we will examine under what conditions $K^{(0)}$ can be used to give an accurate evaluation of correlation functions.

An examination of the expressions we have obtained for $K^{(0)}$ and $K^{(1)}$ shows that they cannot be valid for all values of β_i and β_f. Both of these variables occur in the arguments of the functions sech and tanh. Both of these functions have singularities on the imaginary axis so that for certain values of β_i and β_f, $K^{(0)}$ and $K^{(1)}$ have essential singularities. This implies that for these values the perturbation expansion given in Eq. (2.12) does not make sense. There is, however, a more stringent requirement on β_i and β_f: The integrals which must be performed to compute the terms of the series, e.g., those in Eq. (A4), must converge. This restricts the range of values which β_i and β_f can assume.

In the determination of these restrictions we will work with the variables κ_1 and κ_2 [see Eq. (2.21)] rather than with β_i and β_f directly. Let us assume that κ_1 is real and positive. We then find a range of values of κ_2 for which the above-mentioned integrals converge. In Appendix C we show that the following region satisfies this requirement. Let $\kappa_2=|\kappa_2|e^{i\theta}$ and define $\sigma(\theta)$ as

$$\sigma(\theta)=4\left|\frac{\cosh[\pi/2s(\theta)]-1}{\cosh[\pi/2s(\theta)]+1}\right|\ , \tag{4.1}$$

where $s(\theta)=\tan(\theta/2)$. We define the region R as

$$R=\{\kappa_2\mid\ -\theta_{\max}\le\theta\le\theta_{\max}\ \text{and}\ \tfrac{1}{2}\{\sigma(\theta)-2-[\sigma^2(\theta)-4]^{1/2}\}\le|\kappa_2/\kappa_1|-1\le\tfrac{1}{2}\{\sigma(\theta)-2+[\sigma^2(\theta)-4]^{1/2}\}\} \tag{4.2}$$

and picture it in Fig. 1. The angle $\theta_{\max}$ is the angle for which $\sigma(\theta_{\max})=2$, i.e., the angle for which the inequality in Eq. (4.2) gives $0\le|(\kappa_2/\kappa_1)|-1\le0$. We find from Eq. (4.1) that $\theta_{\max}=0.46\pi$. If $\kappa_2\in R$ then the necessary integrals will converge. Unless this is true our perturbation series will not be justified.

The next thing which we would like to consider is the freely-propagating-pump approximation. We noted before that Eq. (2.17) would be a good approximation for $K^{(0)}$, at least in the region of interest where the Gaussian factor is

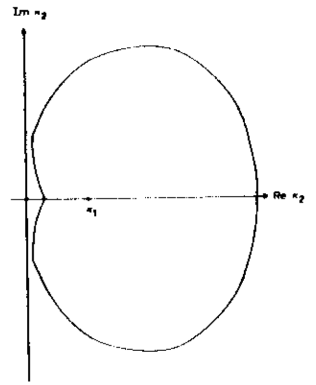

FIG. 1. Region R in complex κ_2 plane.

not small, if $G^{(0)}$ is a slowly varying function of β_f. We now want to determine when this is the case. Let us first define

$$\delta\beta_f=\beta_f-e^{-2i\omega t}\beta_i \tag{4.3}$$

and note that the exponential factor in Eq. (2.17) starts to drop off rapidly for $\delta\beta_f\sim 1$. Examining $G^{(0)}$ now, we see that if $\delta\beta_f\sim 1$ then the deviations in $\tanh(2\sqrt{\kappa_1\kappa_2}t)$ and $\mathrm{sech}(2\sqrt{\kappa_1\kappa_2}t)$ will be small if $\kappa t\ll 1$, and the deviation in $\sqrt{\kappa_1/\kappa_2}$ and $\sqrt{\kappa_2/\kappa_1}$ is of order $|\delta\beta_f/\beta_i|$ and so will be small if $|\beta_i|\gg 1$. These factors are, however, multiplied by $(\alpha_f^*)^2$ and α_i^2. Therefore, $G^{(0)}$ will be a slowly varying function of β_f if

$$|\alpha_i|^2\kappa t\ll 1\,, \tag{4.4a}$$

$$|\alpha_i|^2/|\beta_i|\ll 1\,, \tag{4.4b}$$

$$|\alpha_f|^2\kappa t\ll 1\,, \tag{4.4c}$$

$$|\alpha_f|^2/|\beta_i|\ll 1\,. \tag{4.4d}$$

We now want to discuss the calculation of correlation functions. Let us again assume that β_i is real and take it to be fixed. We define the function $f(\alpha_f,\beta_f,t;\alpha_i,\beta_i)$ by

$$K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)f(\alpha_f,\beta_f,t;\alpha_i,\beta_i)\,. \tag{4.5}$$

The region $S(t)$ in which we would expect the approximate propagator $K^{(0)}$ to be close to the actual propagator K is just

$$S(t)=\{(\alpha_i,\alpha_f,\beta_f)\mid\ \beta_f\in R \text{ and } |f(\alpha_f,\beta_f,t;\alpha_i,\beta_i)|\ll 1\}\,, \tag{4.6}$$

i.e., a point $(\alpha_i,\alpha_f,\beta_f)$ is in $S(t)$ if β_f is in R and α_i, α_f, and β_f are such that $|f|$ is small. If a point $(\alpha_i,\alpha_f,\beta_f)$ is in $S(t)$ we are justified in neglecting $K^{(1)}$ in comparison to $K^{(0)}$ but for the parametric approximation to hold we need also that the freely-propagating-pump approximation be valid. That is, we require that conditions (4.4) be satisfied. Therefore, we are interested in a region $S'(t)$ where

$$S'(t)=\{(\alpha_i,\alpha_f,\beta_f)\mid\ (\alpha_i,\alpha_f,\beta_f)\in S(t) \text{ and Eqs. (4.4) are satisfied}\}\,. \tag{4.7}$$

When calculating the correlation functions for the signal mode one encounters an expression of the form[8]

$$\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\alpha_i\int d^2\beta_f P(\alpha_i)|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2\times(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\,, \tag{4.8}$$

where the initial state of the system is given by

$$\rho=\int d^2\alpha_i P(\alpha_i)|\alpha_i,\beta_i\rangle\langle\alpha_i,\beta_i| \tag{4.9}$$

and $P(\alpha_i)$ is the P representation for the signal mode at $t=0$. If the "function" $P(\alpha_i)|K|^2$ is small outside of $S'(t)$ and falls off rapidly enough then we can accurately approximate Eq. (4.8) by confining the integration to $S'(t)$ and replacing K by the expression on the right-hand side of Eq. (2.17), at least for sufficiently small values of n_j and m_j where $j=1,2$. This replacement of K by the approximate expression given in Eq. (2.17) is nothing but the parametric approximation. We need to find, then, some sort of measure of the extent to which $P(\alpha_i)|K|^2$ is concentrated in $S'(t)$ and some information on the falloff properties of $|K|^2$.

Let us now consider a measure of the extent to which $P(\alpha_i)|K|^2$ is concentrated on $S'(t)$. The propagator K obeys the identity

$$1=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f\int d^2\alpha_i|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2\times P(\alpha_i)\,. \tag{4.10}$$

If we assume that $P(\alpha_i)$ is positive semidefinite (or the limit of positive semidefinite functions) then the quantity

$$\mu(t)=\frac{1}{\pi^2}\int\int_{S'(t)}\int d^2\alpha_f d^2\beta_f d^2\alpha_i|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2\times P(\alpha_i) \tag{4.11}$$

will provide a good indication of the extent to which the region in which $P(\alpha_i)|K|^2$ is contained in $S'(t)$. If $\mu(t)$ is close to 1 then $P(\alpha_i)|K|^2$ can be considered to be well concentrated in $S'(t)$.

It is possible to simplify the expression appearing on the right-hand side of Eq. (4.11). First, because of the definition of $S'(t)$ we have that

$$\mu(t)\simeq\frac{1}{\pi^2}\int\int_{S'(t)}\int d^2\alpha_f d^2\beta_f d^2\alpha_i\times|K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2 P(\alpha_i)\,. \tag{4.12}$$

We can go still further because the conditions for the freely-propagating-pump approximation hold. We can perform the β_f integration with the result that

$$\mu(t)\simeq\frac{1}{\pi}\int\int_{M(t)} d^2\alpha_f d^2\alpha_i|\times G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)|^2 P(\alpha_i)\,, \tag{4.13}$$

where

$$M(t)=\{(\alpha_f,\alpha_i)\mid\ (\alpha_i,\alpha_f,e^{-2i\omega t}\beta_i)\in S'(t)\}\,.$$

If $1-\mu(t)\ll 1$, then it is possible to simplify the expression (4.8). We have that

$$\begin{aligned}\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\alpha_i\int d^2\beta_f P(\alpha_i)|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\\ \simeq\frac{1}{\pi^2}\int\int_{S'(t)}\int d^2\alpha_f d^2\alpha_i d^2\beta_f P(\alpha_i)|K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\\ \simeq\frac{1}{\pi^2}\int\int_{M(t)} d^2\alpha_f d^2\alpha_i P(\alpha_i)|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i;\alpha_i,\beta_i,0)|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\,.\end{aligned} \tag{4.14}$$

Because $\mu(t)$ is close to 1 we have that $P(\alpha_i)|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)|^2$ is concentrated in $M(t)$ and is, therefore, small outside this region. If it also falls off rapidly enough outside of $M(t)$ then we can extend the α_i and α_f integrations over the entire complex plane without much error. Our final approximation to expression (4.8) is then

$$\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\alpha_i P(\alpha_i)|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\,. \tag{4.15}$$

If one substitutes expression (4.15) in the calculation of correlation functions one will obtain the results given by the parametric approximation. This is because $G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)$ is just the propagator for the Hamiltonian given in Eq. (2.2).

In the preceding discussion we had to assume that $P(\alpha_i)|K|^2$ fell off rapidly outside of $S'(t)$ in order for the parametric approximation to be valid. Proving this is difficult, but it is possible to provide some much weaker results which at least give some idea of the behavior of $P(\alpha_i)|K|^2$. For simplicity let us consider the case $P(\alpha_i)=\delta^2(\alpha_i)$. We are then interested in the properties of $K(\alpha_f,\beta_f,t;0,\beta_i,0)$. One can then show that for any integer $n\geq 1$ there exist constants $c_n(\beta_i)$ and $d_n(\beta_i)$ such that

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\leq\frac{c_n(\beta_i)}{|\alpha_f|^n}\,, \tag{4.16a}$$

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\leq\frac{d_n(\beta_i)}{|\beta_f|^n} \tag{4.16b}$$

so that $|K|$ falls off faster than any power of $|\beta_f|$ or $|\alpha_f|$. This is demonstrated in Appendix D. Because

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\leq 1\,, \tag{4.17}$$

inequalities (4.16) only really start providing useful information when $|\alpha_f|$ and $|\beta_f|$ are sufficiently large to make the right-hand sides less than 1. In general this will happen when α_f and β_f are far outside of $S'(t)$. Therefore, while inequalities (4.16) do tell us that $|K|$ falls off rapidly they do not really provide us with as much information as we would like. Therefore, the assumption that if $P(\alpha_i)|K|^2$ is well concentrated in $S'(t)$ [$\mu(t)$ close to 1], then the contribution to the integral in expression (4.8) from outside $S'(t)$ is small, must remain an assumption. The behavior of $|K|$ indicated by inequality (4.16) indicates, however, that it is a plausible one.

Finally, let us give some general conditions under which $\mu(t)$ is close to 1. We will consider the case β_i real and positive and the signal mode initially in the vacuum state, i.e., $P(\alpha_i)=\delta^{(2)}(\alpha_i)$. An examination of the expression for $f(\alpha_f,\beta_f,t;0,\beta_i)$ for the case $\kappa_2\in R$ (see Appendix E) shows that $|f(\alpha_f,\beta_f,t;0,\beta_i)|\ll 1$ if

$$\begin{aligned}&1/\beta_i\ll 1,\quad |\alpha_f|^2/\beta_i\ll 1\,,\\ &\kappa t\ll 1,\quad (\kappa t/\beta_i)|\alpha_f|^4\ll 1\,.\end{aligned} \tag{4.18}$$

These conditions determine $S(t)$. If we now impose the requirement that Eqs. (4.4) must also be satisfied we find that a point $(\alpha_i,\alpha_f,\beta_f)$ is in $S'(t)$ if $\beta_f\in R$, $1/|\beta_i|\ll 1$, and

$$\kappa t\ll 1,\quad |\alpha_f|^2/|\beta_i|\ll 1,\quad |\alpha_f|^2\kappa t\ll 1\,. \tag{4.19}$$

We now use these results in Eq. (4.13) to obtain

$$\mu(t)\simeq\frac{1}{\pi}\int_L d^2\alpha_f|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2\,, \tag{4.20}$$

where

$$L=\{\alpha_f\mid\ |\alpha_f|^2\ll|\beta_i| \text{ and } |\alpha_f|^2\kappa t\ll 1\}\,,$$

and we have assumed that $\kappa t\ll 1$ and $1/|\beta_i|\ll 1$.

It is possible to derive a more convenient condition than Eq. (4.20) if we note that

$$\begin{aligned}|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)|^2\\ =\exp[-|x_f+y_f\tanh(\eta_0 t)-x_i\,\mathrm{sech}(\eta_0 t)|^2\\ -|y_f\,\mathrm{sech}(\eta_0 t)+x_i\tanh(\eta_0 t)-y_i|^2]\,,\end{aligned} \tag{4.21}$$

where $\alpha_i=x_i+iy_i$ and $\alpha_f=e^{-i\omega t}(x_f+iy_f)$. From this expression we see that $|G^{(0)}|$ is peaked at

$$\alpha_f=e^{-i\omega t}[\alpha_i\cosh(\eta_0 t)-i\alpha_i^*\sinh(\eta_0 t)] \tag{4.22}$$

and that this peak has a width given roughly by $\cosh(\eta_0 t)$. If $\alpha_i=0$ this peak will lie within the disc-shaped region in the α_f plane given by

$$D=\{\alpha_f \mid \ |\alpha_f| \lesssim \cosh(\eta_0 t) \sim e^{\eta_0 t}\} \ . \tag{4.23}$$

If $D \subseteq L$ then $\mu(t)$ will be approximately 1. This will be the case when

$$e^{2\eta_0 t} \ll |\beta_i| \, , \quad e^{2\eta_0 t}\kappa t \ll 1 \ . \tag{4.24}$$

Let us summarize our conclusions. In order for the parametric approximation to give accurate values for correlation functions it must be the case that $\mu(t)$ be close to one. In the case in which the signal mode is initially in the vacuum state this condition will be satisfied if

$$1/|\beta_i| \ll 1 \ , \tag{4.25a}$$

$$\kappa t \ll 1 \ , \tag{4.25b}$$

$$\kappa t e^{4\beta_i \kappa t} \ll 1 \ , \tag{4.25c}$$

$$e^{4\beta_i \kappa t} \ll \beta_i \ . \tag{4.25d}$$

There is a certain amount of redundancy in these conditions. For example, if Eqs. (4.25a) and (4.25d) are satisfied then Eq. (4.25b) follows as a consequence. We also note that if Eqs. (4.25a) and (4.25d) are satisfied and the condition that $\kappa t \beta_i$ be of order one or less is also satisfied then Eqs. (4.25b) and (4.25c) follow as consequences. This is in contrast to ordinary perturbation theory which is valid only for times such that $\kappa t \beta_i \ll 1$ so that the parametric approximation represents a definite improvement over the perturbative result.

V. CONCLUDING REMARKS

We have presented a fully quantum-mechanical theory of the degenerate parametric amplifier using a path-integral representation of the coherent-state propagator. We have developed a perturbation series for this propagator, the first term of which, under certain conditions, corresponds to the parametric approximation. We studied the effect of the quantum fluctuations of the pump mode on the squeezing of the signal mode and showed that these fluctuations not only reduce the squeezing but also that the minimum uncertainty relation does not hold. Finally we examined the conditions under which the parametric approximation will be valid.

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APPENDIX A

In order to calculate $K^{(1)}$ we must first evaluate the path integral appearing in Eq. (2.14):

$$\int \mathcal{D}[\alpha(\tau)]e^{iS_0}(iS_1) = -\kappa^2 \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-2i\omega(t_2-t_1)}F(t_1,t_2) \ , \tag{A1}$$

where

$$F(t_1,t_2)=\int \mathcal{D}[\alpha(\tau)]e^{iS_0}[\alpha^*(t_2)\alpha(t_1)]^2 \ . \tag{A2}$$

The path integral in the above equation can be evaluated by making use of the following rule: If $t_2 > t' > t_1$ and $f(\alpha(t'))$ is a function of the path $\alpha(\tau)$ at the time t', then[9]

$$\int \mathcal{D}[\alpha(\tau)]e^{iS_0}f(\alpha(t')) = \frac{1}{\pi}\int d^2\alpha' G^{(0)}(\alpha_f,\beta_f,t_2;\alpha',\beta_i,t') \times f(\alpha')G^{(0)}(\alpha',\beta_f,t';\alpha_i,\beta_i,t_1) \ , \tag{A3}$$

where $G^{(0)}$ is the propagator corresponding to S_0 and is given by Eq. (2.19). Application of this rule twice gives us that

$$F(t_1,t_2)=\frac{1}{\pi^2}\int d^2\alpha_1 \int d^2\alpha_2 G^{(0)}(\alpha_f,\beta_f,t;\alpha_2,\beta_i,t_2)G^{(0)}(\alpha_2,\beta_f,t_2;\alpha_1,\beta_i,t_1)G^{(0)}(\alpha_1,\beta_f,t_1;\alpha_i,\beta_i,0)(\alpha_2^*)^2\alpha_1^2 \ . \tag{A4}$$

It should be noted, though we have not explicitly indicated it, that $F(t_1,t_2)$ depends upon α_f, β_f, α_i, β_i, and t as well as on t_1 and t_2. Evaluation of the integrals in Eq. (A4) is lengthy but straightforward. Upon performing them we find that

$$\begin{aligned} F(t_1,t_2)=G^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0) & \\ \times \Bigg\{ \frac{1}{D_1^2 D_2^2}(2B_{10}C_{21})^2 \Bigg\{ & \left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2+4C_{30}A_{32}\alpha_f^*\alpha_i+4A_{32}(A_{30}-A_{20})\alpha_i^2\right]^2 \\ & +12A_{32}\left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2+4C_{30}A_{32}\alpha_f^*\alpha_i+4A_{32}(A_{30}-A_{20})\alpha_i^2\right]+12A_{32}^2 \Bigg\} \\ & +\frac{4}{D_1 D_2^2}B_{10}C_{20}\alpha_i(C_{32}\alpha_f^*+2A_{32}C_{20}\alpha_i) \\ & \times \left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2+4C_{30}A_{32}\alpha_f^*\alpha_i+4A_{32}(A_{30}-A_{20})\alpha_i^2+6A_{32}\right] \\ & +\frac{1}{D_2}\left[\frac{1}{D_1^2}C_{10}^2\alpha_i^2+\frac{2}{D_1}B_{10}\right] \\ & \times \left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2+4C_{30}A_{32}\alpha_f^*\alpha_i+4A_{32}(A_{30}-A_{20})\alpha_i^2+2A_{32}\right]\Bigg\} \ , \end{aligned} \tag{A5}$$

where

$$D_1=1-4A_{21}B_{10} \ , \tag{A6}$$

$$D_2=1-4A_{32}B_{20} \ , \tag{A7}$$

and $t_3=t$ and $t_0=0$. We will be interested in the case in which the signal mode is initially in the vacuum state. This means that we need only consider the propagator of the system when $\alpha_i=0$ which results in a considerable simplification of Eq. (A5). We also note that the same exponential factor which appears in $K^{(0)}$, i.e., $\exp[-\frac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]$, also appears in $K^{(1)}$. Therefore, we can approximate $K^{(1)}$ by

$$K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0) \cong -\kappa^2\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}] \times \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-2i\omega(t_2-t_1)}F(t_1,t_2)\big|_{\beta_f=e^{-2i\omega t}\beta_i} \ , \tag{A8}$$

where we have assumed that $F(t_1,t_2)$ is a slowly varying function of β_f. If we now calculate $F(t_1,t_2)$ under these two restrictions, i.e., $\alpha_i=0$ and $\beta_f=e^{-2i\omega t}\beta_i$, we find that

$$\begin{aligned} F(t_1,t_2)=G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)e^{2i\omega(t_2-t_1)} & \\ \times(-\operatorname{sech}^2(\eta_0 t)\cosh^2(\eta_0 t_2)\sinh^2(\eta_0 t_1)(e^{-i\omega t}\alpha_f^*)^4 & \\ +i\operatorname{sech}^2(\eta_0 t)\cosh(\eta_0 t_2)\sinh(\eta_0 t_1) & \\ \times\{6\operatorname{sech}(\eta_0 t)\sinh(\eta_0 t_1)\sinh[\eta_0(t-t_2)]-\cosh[\eta_0(t_2-t_1)]\}(e^{-i\omega t}\alpha_f^*)^2 & \\ +\operatorname{sech}(\eta_0 t)\sinh(\eta_0 t_1)\sinh[\eta_0(t-t_2)]\{3\operatorname{sech}(\eta_0 t)+\sinh(\eta_0 t_1)\sinh[\eta_0(t-t_2)]-\cosh[\eta_0(t_2-t_1)]\}) \ , & \end{aligned} \tag{A9}$$

where $\eta_0=2\kappa\beta_i$ (again β_i is assumed real). Before proceeding we note the identities

$$D_1=\operatorname{sech}[\eta_0(t_2-t_1)]\cosh(\eta_0 t_2)\operatorname{sech}(\eta_0 t_1) \ , \tag{A10a}$$

$$D_2=\operatorname{sech}[\eta_0(t-t_2)]\cosh(\eta_0 t)\operatorname{sech}(\eta_0 t_2) \tag{A10b}$$

which were of use in deriving Eq. (A9) from Eq. (A5).

In order to complete our calculation of $K^{(1)}$ we must perform the time integrations appearing in Eq. (A8). On doing so we obtain Eqs. (2.22) and (2.23) of the text.

APPENDIX B

We first consider the improvement of the freely-propagating-pump approximation. The contribution of $K^{(0)}$ to $\langle a^\dagger(t)a(t)\rangle$ is

$$\langle a^\dagger(t)a(t)\rangle \cong \frac{1}{\pi^2}\int d^2\beta_f \int d^2\alpha_f \exp(-|\beta_f-e^{-2i\omega t}\beta_i|^2)|G^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)|^2|\alpha_f|^2-1 \ . \tag{B1}$$

We evaluate the α_f integral first; this can be done exactly. We then expand the result in terms $\delta\beta_f=\beta_f-e^{-2i\omega t}\beta_i$, i.e.,

$$\frac{1}{\pi}\int d^2\alpha_f |G^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)|^2|\alpha_f|^2=c_1^{(0)}+c_1^{(1)}\delta\beta_f+c_1^{(1)*}\delta\beta_f^*+c_1^{(2)}\delta\beta_f^2+c_2^{(2)}|\delta\beta_f|^2+c_1^{(2)*}(\delta\beta_f^*)^2 \ , \tag{B2}$$

where $c_i^{(j)}$ is a function of β_i and t. We recall that if inequalities (4.4) are satisfied, then $|G^{(0)}|$ is a slowly varying function of β_i. Therefore, we expect $c_1^{(1)}$ and $c_{1,2}^{(2)}$ to be small if these inequalities are satisfied. We can now evaluate the integral. The terms linear in $\delta\beta_f$ give no contribution, and the terms proportional to $c_1^{(2)}$ integrate to zero as well. The $c_1^{(0)}$ term is just given by Eq. (3.6) while the term proportional to $c_2^{(2)}$ represents a correction to this. It is this term which is the lowest-order correction to the freely-propagating-pump approximation. We find that

$$\frac{1}{\pi}\int d^2\beta_f c_2^{(2)}(\beta_i,t)e^{-|\delta\beta_f|^2}|\delta\beta_f|^2=\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[\tfrac{3}{2}\sinh^2(\eta_0 t)+\tfrac{1}{2}]-(\eta_0 t)[6\sinh^3(\eta_0 t)\cosh(\eta_0 t)+3\cosh(\eta_0 t)\sinh(\eta_0 t)] + \tfrac{15}{4}\sinh^6(\eta_0 t)+\tfrac{21}{4}\sinh^4(\eta_0 t)+\tfrac{3}{2}\sinh^2(\eta_0 t)\} \ . \tag{B3}$$

In the case of $\langle[a(t)]^2\rangle$ the calculation is carried out in the same way. Now one has

$$\frac{1}{\pi}\int d^2\alpha_f |G^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)|^2\alpha_f^2=d_1^{(0)}+d_1^{(1)}\delta\beta_f+d_1^{(1)*}\delta\beta_f^*+d_1^{(2)}(\delta\beta_f)^2+d_2^{(2)}|\delta\beta_f|^2+d_1^{(2)*}(\delta\beta_f^*)^2 \ . \tag{B4}$$

Upon performing the β_f integration we find that only the terms proportional to $d_1^{(0)}$ and $d_2^{(2)}$ contribute. The $d_1^{(0)}$ term yields the result in Eq. (3.7) while the $d_2^{(2)}$ term yields

$$\frac{1}{\pi}\int d^2\beta_f d_2^{(2)}(\beta_i,t)e^{-|\delta\beta_f|^2}|\delta\beta_f|^2$$

$$= -ie^{-2i\omega t}\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[\tfrac{5}{2}\sinh^2(\eta_0 t)\tanh(\eta_0 t)+\tfrac{5}{2}\tanh(\eta_0 t)]$$

$$-\eta_0 t[6\sinh^4(\eta_0 t)+4\sinh^2(\eta_0 t)+\tfrac{3}{2}\sinh^2(\eta_0 t)\tanh^2(\eta_0 t)+\tfrac{3}{2}\tanh^2(\eta_0 t)]$$

$$+[\tfrac{15}{4}\sinh^6(\eta_0 t)+\tfrac{27}{4}\sinh^4(\eta_0 t)+3\sinh^2(\eta_0 t)]\tanh(\eta_0 t)\} . \qquad (B5)$$

We now want to briefly discuss some of the assumptions underlying this approximation. We are assuming that inequalities (4.24) are satisfied so that the range of integration in Eq. (4.6) should be restricted to $S'(t)$. Now, because these inequalities are satisfied the freely-propagating-pump approximation is valid so that only the integration region in which $|\delta\beta_f|\sim 1$ and $\alpha_f\in L$ is important. If inequalities (4.24) are satisfied, then for $|\delta\beta_f|\sim 1$, $|G^{(0)}|^2|\alpha_f|^2$ decreases rapidly outside of L and its integral over the entire complex plane converges. Therefore, we can, with little error, extend the α_f integration to the entire complex plane. We expand the resulting expression about $\delta\beta_f=0$ in order to take into account the variation of $\int d^2\alpha_f|G^{(0)}|^2|\alpha_f|^2$ with $\delta\beta_f$ in the neighborhood $|\delta\beta_f|\sim 1$. This expansion, when multiplied by $\exp(-|\delta\beta_f|^2)$, decreases rapidly away from the region in which $|\delta\beta_f|\sim 1$. We can, therefore, again extend the integration to the entire complex plane. The results of this procedure are exhibited in the preceding paragraph.

Next we consider the effects of $K^{(1)}$ on $\langle a^\dagger(t)a(t)\rangle$ and $\langle[a(t)]^2\rangle$. This is done by evaluating the following integrals:

$$\langle a^\dagger(t)a(t)\rangle|_{K^{(1)}}=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f K^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)[K^{(1)}(\alpha_f,\beta_f,t;0,\beta_i,0)]^*|\alpha_f|^2+\text{c.c.}$$

$$=\int d^2\alpha_f|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2[g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-i\omega t}\alpha_f^*)^2+g_0(t)]|\alpha_f|^2+\text{c.c.} , \qquad (B6)$$

$$\langle[a(t)]^2\rangle|_{K^{(1)}}=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f K^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)[K^{(1)}(\alpha_f,\beta_f,t;0,\beta_i,0)]^*\alpha_f^2+\text{c.c.}$$

$$=\int d^2\alpha_f|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2[g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-i\omega t}\alpha_f^*)^2+g_0(t)]\alpha_f^2+\text{c.c.} \qquad (B7)$$

In writing Eqs. (B6) and (B7), we have substituted for $K^{(0)}$ and $K^{(1)}$ from Eqs. (2.18) and (2.22), respectively. Furthermore we have made the substitution $\beta_f=\exp(-2i\omega t)\beta_i$ in the integrands following our earlier discussion. The integrals in Eqs. (B6) and (B7) are rather lengthy but straightforward. On carrying them out, we obtain

$$\langle a^\dagger(t)a(t)\rangle|_{K^{(1)}}=\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[-\tfrac{3}{2}\cosh^2(\eta_0 t)]+\eta_0 t[5\sinh(\eta_0 t)\cosh(\eta_0 t)+6\sinh^3(\eta_0 t)\cosh(\eta_0 t)]$$

$$-\tfrac{15}{4}\sinh^6(\eta_0 t)-\tfrac{33}{4}\sinh^4(\eta_0 t)-\tfrac{9}{2}\sinh^2(\eta_0 t)\} , \qquad (B8)$$

$$\langle[a(t)]^2\rangle|_{K^{(1)}}=-ie^{-2i\omega t}\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[2\sinh(\eta_0 t)\cosh(\eta_0 t)-\tfrac{5}{2}\sinh^2(\eta_0 t)\tanh(\eta_0 t)-\tfrac{5}{2}\tanh(\eta_0 t)]$$

$$+\eta_0 t[6\sinh^4(\eta_0 t)+6\sinh^2(\eta_0 t)+\tfrac{3}{2}\sinh^2(\eta_0 t)\tanh^2(\eta_0 t)+\tfrac{3}{2}\tanh^2(\eta_0 t)+2]$$

$$-[\tfrac{15}{4}\sinh^6(\eta_0 t)+\tfrac{39}{4}\sinh^4(\eta_0 t)+8\sinh^2(\eta_0 t)+2]\tanh(\eta_0 t)\} . \qquad (B9)$$

We now add the contributions to $\langle a^\dagger(t)a(t)\rangle$ and $\langle[a(t)]^2\rangle$ due to the freely-propagating-pump approximation, and [illegible]rrections to it (Eqs. (3.6), (B2), (B3), and (B8) for $\langle a^\dagger(t)a(t)\rangle$ and Eqs. (3.7), (B4), (B5), and (B9) for $\langle[a$ [illegible] Ve then obtain Eqs. (3.8) and (3.9) of the text.

APPENDIX C

In this appendix we would like to examine the convergence of the integrals which occur in the perturbation series, i.e., integrals of the type which occur in Eq. (A4). All of these integrals are of the form

$$I_0=\int dx_2\int dy_2\exp[a_2x_2^2+b_2y_2^2+a_1x_2+b_1y_2+c_1x_2y_2]x_2^ny_2^m , \qquad (C1)$$

where

$$a_2=-1+B_{21}+A_{32} ,$$
$$b_2=-1-B_{21}-A_{32} , \qquad (C2)$$
$$c_1=2i(A_{32}-B_{21}) ,$$

and A_{ij}, B_{ij}, and C_{ij} are defined by Eqs. (2.20). The coefficients a_1 and b_1 can also be expressed in terms of A_{ij}, B_{ij}, and C_{ij} but are not relevant to convergence considerations. The integral I_0 will converge if

$$\text{Re}\left[(x_2\,y_2)\begin{bmatrix}a_2 & \tfrac{1}{2}c_1\\ \tfrac{1}{2}c_1 & b_2\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}\right]<0 \qquad (C3)$$

for all values of x_2 and y_2. This will be the case if both of the eigenvalues of the real, symmetric matrix $\underline{A}$ given by

$$\underline{A}=\text{Re}\begin{bmatrix}a_2 & \tfrac{1}{2}c_1\\ \tfrac{1}{2}c_1 & b_2\end{bmatrix}$$

are negative. This is equivalent to the two conditions

$$\text{Tr}\underline{A}<0,\ \ \det\underline{A}>0 . \qquad (C4)$$

Substituting from Eq. (C2) we obtain

$$\text{Tr}\underline{A}=-2,\ \ \det\underline{A}=1-|B_{21}+A_{32}^*|^2 . \qquad (C5)$$

As can be seen the trace condition is satisfied automatically so that we are left with the condition

$$1>|B_{21}+A_{32}^*|^2 . \qquad (C6)$$

Let us define

$$\xi_1=\tanh(2\sqrt{\kappa_1\kappa_2}\tau_1),\ \ \xi_2=\tanh(2\sqrt{\bar{\kappa}_1\kappa_2}\tau_2) . \qquad (C7)$$

We then have that inequality (C6) will be satisfied for all values of κ_1 and κ_2 such that the inequality

$$1>\tfrac{1}{4}|\kappa_1/\kappa_2|\,|\xi_1|^2+\tfrac{1}{4}|\kappa_2/\kappa_1|\,|\xi_2|^2-\tfrac{1}{2}\text{Re}(\xi_1\xi_2) \qquad (C8)$$

is satisfied for all values of τ_1 and τ_2 greater than zero. That is, if inequality (C8) is satisfied for some specific values of κ_1 and κ_2, and all values of $\tau_1>0$ and $\tau_2>0$, then these values of κ_1 and κ_2 will be such that inequality (C6) is also satisfied. Therefore we want to examine inequality (C8).

Before doing so, however, we need to place a bound on $|\tanh z|$ for z on the line

$$L=\{z\,|\ \ z=re^{i\theta},\ \theta\text{ fixed and }|\theta|<\pi/2,\ \ r\ge 0\} .$$

We have that

$$|\tanh z|=\left[\frac{\cosh(2x)-\cos(2y)}{\cosh(2x)+\cos(2y)}\right]^{1/2} , \qquad (C9)$$

where $z=x+iy$. This expression achieves a maximum on L when $\pi/4<y<\pi/2$, i.e., $\pi/4u<x<\pi/2u$, where $u=\tan\theta$. Therefore, on L we have

$$|\tanh z|\le\left[\frac{\cosh(\pi/2u)+1}{\cosh(\pi/2u)-1}\right]^{1/2}\equiv m(\theta) . \qquad (C10)$$

Let us now assume that κ_1 is real and positive, $\arg\kappa_2=\theta_0$, and that $|\theta_0|<\pi$. Then we have that $\arg\sqrt{\kappa_1\kappa_2}=\tfrac{1}{2}\theta_0$. This then implies that

$$0\le|\xi_1|\le m(\tfrac{1}{2}\theta_0),\ \ \text{Re}\xi_1\ge 0$$
$$0\le|\xi_2|\le m(\tfrac{1}{2}\theta_0),\ \ \text{Re}\xi_2\ge 0 . \qquad (C11)$$

If we let $x=|\kappa_2/\kappa_1|$ we see that

$$\frac{1}{4}\left[x+\frac{1}{x}\right]m(\tfrac{1}{2}\theta_0)^2>\tfrac{1}{4}x|\xi_1|^2+\frac{1}{4}\left[\frac{1}{x}\right]|\xi_2|^2-\tfrac{1}{2}\text{Re}(\xi_1\xi_2) \qquad (C12)$$

so that inequality (C8) is satisfied if

$$x+\frac{1}{x}<\frac{4}{m(\tfrac{1}{2}\theta_0)^2}\equiv\sigma(\theta_0) . \qquad (C13)$$

The function $(1/x)+x$ has a minimum of 2 for $x>0$ so that we must have $\sigma(\theta_0)\ge 2$. The angle for which $\sigma(\theta_0)=2$ is $\theta_{max}=0.46\pi$. For all angles $|\theta_0|\le\theta_{max}$ we have $\sigma(\theta_0)\ge 2$. Inequality (C13) is satisfied, then, if $|\theta_0|\le\theta_{max}$ and

$$\tfrac{1}{2}[\sigma-2-(\sigma^2-4)^{1/2}]\le x-1\le\tfrac{1}{2}[\sigma-2+(\sigma^2-4)^{1/2}] \qquad (C14)$$

which is the condition given in the text.

APPENDIX D

Here we would like to show that the propagator falls off more rapidly than any power of $|\alpha_f|$ or $|\beta_f|$. In order to do this we first note that the operator

$$M=2b^\dagger b+a^\dagger a \qquad (D1)$$

commutes with the Hamiltonian. Therefore, the Hilbert space for the problem splits into the direct sum of the Hilbert spaces $\mathscr{H}_m$ on which M has the eigenvalue m. If $\psi\in\mathscr{H}_m$, then $\exp(-itH)\psi\in\mathscr{H}_m$. Let P_m be the projection operator onto $\mathscr{H}_m$. We then have that

$$e^{-itH}|\alpha_i,\beta_i\rangle=\sum_{m=0}^{\infty}\psi_m(t) , \qquad (D2)$$

where

$$\psi_m(t)=P_me^{-itH}|\alpha_i,\beta_i\rangle . \qquad (D3)$$

Because $[M,H]=0$ the norm of $\psi_m(t)$ is independent of time.

The power bounds are obtained from the inequality

$$|\alpha_f^r\beta_f^s\langle\alpha_f,\beta_f|e^{-itH}|\alpha_i,\beta_i\rangle| = |\langle\alpha_f,\beta_f|(a^\dagger)^r(b^\dagger)^s e^{-itH}|\alpha_i\beta_i\rangle| \le \|(a^\dagger)^r(b^\dagger)^s e^{-iHt}|\alpha_i,\beta_i\rangle\| . \quad \text{(D4)}$$

This is really all that is necessary to obtain bounds of the form given in Eq. (4.16) as the right-hand side of inequality (D4) is independent of α_f and β_f. It is useful, however, to examine this expression a little more closely. We see that

$$(a^\dagger)^r(b^\dagger)^s\psi_m(t)\in\mathscr{H}_{m+r+2s} \quad \text{(D5)}$$

so that

$$((a^\dagger)^r(b^\dagger)^s\psi_{m'}(t)|(a^\dagger)^r(b^\dagger)^s\psi_{m''}(t))=0 \quad \text{(D6)}$$

if $m''\neq m'$. Therefore

$$\|(a^\dagger)^r(b^\dagger)^s e^{-iHt}|\alpha_i,\beta_i\rangle\| = \left[\sum_{m=0}^{\infty}\|(a^\dagger)^r(b^\dagger)^s\psi_m(t)\|^2\right]^{1/2} . \quad \text{(D7)}$$

Let us now consider the case $\alpha_i=0$ and $s=0$. We then have that

$$\|\psi_m(0)\|=e^{-|\beta_i|^2/2}\frac{|\beta_i|^{m/2}}{\sqrt{(m/2)!}} . \quad \text{(D8)}$$

Because there can be at most m signal-mode photons in $\mathscr{H}_m$ we have

$$\|(a^\dagger)^r\psi_m(t)\|\le\left[\frac{(r+m)!}{m!}\right]^{1/2}\|\psi_m(0)\| . \quad \text{(D9)}$$

This then, provides the bound

$$\|(a^\dagger)^r e^{-itH}|0,\beta_i\rangle\|\le e^{-|\beta_i|^2/2}\left[\sum_{l=0}^{\infty}\frac{|\beta_i|^{2l}}{l!}\frac{(2l+r)!}{(2l)!}\right]^{1/2}$$
$$\le e^{-|\beta_i|^2/2}\left[\frac{d^r}{d|\beta_i|^r}(|\beta_i|^r e^{|\beta_i|^2})\right]^{1/2}$$
$$\le\left[\sum_{l=0}^{r}\binom{r}{l}\frac{r!}{(r-l)!}|\beta_i|^{r-l}(-i)^{r-l}H_{r-l}(i|\beta_i|)\right]^{1/2} , \quad \text{(D10)}$$

where $H_n(x)$ is the nth Hermite polynomial. Combining inequalities (D4) and (D9) gives

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\le\frac{1}{|\alpha_f|^r}\left[\sum_{l=0}^{r}\binom{r}{l}\frac{r!}{(r-l)!}|\beta_i|^{r-l}(-i)^{r-l}H_{r-l}(i|\beta_i|)\right]^{1/2} . \quad \text{(D11)}$$

A similar derivation for the case $\alpha_i=0$ and $r=0$ gives

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\le\frac{1}{|\beta_f|^s}[s!L_s(-|\beta_i|^2)]^{1/2} , \quad \text{(D12)}$$

where L_s is the sth Laguerre polynomial. We note that for large $|\beta_i|$

$$\left[\sum_{l=0}^{r}\binom{r}{l}\frac{r!}{(r-l)!}|\beta_i|^{r-l}(-i)^{r-l}\times H_{r-l}(i|\beta_i|)\right]^{1/2}\sim(\sqrt{2}|\beta_i|)^r , \quad \text{(D13)}$$

$$[s!L_s(-|\beta_i|^2)]^{1/2}\sim|\beta_i|^s \quad \text{(D14)}$$

so that the bounds (D11) and (D12) start being useful (i.e., the right-hand sides become less than 1) for $|\alpha_f|\sim|\beta_i|$ and $|\beta_f|\sim|\beta_i|$.

APPENDIX E

In this appendix we want to find the conditions on α_f, β_i, and t so that $|f(\alpha_f,\beta_f,t;0,\beta_i)|\ll1$ for $\kappa_2 = \kappa\beta_f^* e^{-2i\omega t}\in R$. In Eqs. (2.22) and (2.23) we have given an expression for $K^{(1)}$ from which $f(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i)$ can be immediately derived, i.e.,

$$f(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i) = g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-2i\omega t}\alpha_f^*)^2+g_0(t) , \quad \text{(E1)}$$

where $g_4(t)$, $g_2(t)$, and $g_0(t)$ are given by Eq. (2.23). By going back through the derivation of $K^{(1)}$ we can find an expression for $f(\alpha_f,\beta_f,t;0,\beta_i)$, i.e., for the case in which β_f is not equal to $\exp(-2i\omega t)\beta_i$. The result is

$$f(\alpha_f,\beta_f,t;0,\beta_i)=\left[\frac{\kappa_1}{\kappa_2}\right]\tilde{g}_4(t)(e^{-i\omega t}\alpha_f^*)^4 +\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2}\tilde{g}_2(t)(e^{-i\omega t}\alpha_f^*)^2+\tilde{g}_0(t) , \quad \text{(E2)}$$

where $\tilde{g}_j(t)$ is just $g_j(t)$ with η_0 replaced by $2\sqrt{\kappa_1\kappa_2}$.

We now note that for $\kappa_2\in R$, κ_1/κ_2 is of order 1 and

$$|\tanh(2\sqrt{\kappa_1\kappa_2}t)|\le\sqrt{2} ,$$
$$|\mathrm{sech}(2\sqrt{\kappa_1\kappa_2})|\le1 ,$$
$$|(2\sqrt{\kappa_1\kappa_2}t)^2\mathrm{sech}^4(2\sqrt{\kappa_1\kappa_2}t)|\le3 ,$$
$$|(2\sqrt{\kappa_1\kappa_2}t)^2\mathrm{sech}^2(2\sqrt{\kappa_1\kappa_2}t)|\le3 . \quad \text{(E3)}$$

It can then be seen that

$$|f(\alpha_f,\beta_f,t;0,\beta_i)| \le\left|\frac{\kappa_1}{\kappa_2}\right||\tilde{g}_4(t)||\alpha_f|^4 +\left|\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2}\right||\tilde{g}_2(t)||\alpha_f|^2+|\tilde{g}_0(t)| , \quad \text{(E4)}$$

where

$$\left|\frac{\kappa_1}{\kappa_2}\right||\tilde{g}_4(t)|\sim\frac{\kappa^2}{\eta_0^2}[\eta_0t+O(1)] ,$$
$$\left|\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2}\right||\tilde{g}_2(t)|\sim\frac{\kappa^2}{\eta_0^2}[\eta_0t+O(1)] , \quad \text{(E5)}$$
$$|\tilde{g}_0(t)|\sim\frac{\kappa^2}{\eta_0^2}[(\eta_0t)^2+\eta_0t+O(1)] ,$$

where we have used Eqs. (E3) and the fact that for $\kappa_2\in R$, $(\eta_0/2\sqrt{\kappa_1\kappa_2})\sim1$. Putting Eqs. (E4) and (E5) together we find that $|f|$ is small if

$$1/\beta_i\ll1,\quad|\alpha_f|^2/\beta_i\ll1 ,$$
$$\kappa t\ll1,\quad\kappa t/\beta_i|\alpha_f|^4\ll1 . \quad \text{(E6)}$$

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Photon statistics of a free-electron laser

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A fully quantized theory of the free-electron laser in the small-signal regime is presented which allows for a calculation of the photon statistics. For an initial vacuum, we find photon antibunching if the electron momentum is below resonance. We conjecture that, in general, the free-electron laser preserves coherent states only in the absence of gain.

I. INTRODUCTION

Historically, the first explanation[1] of the gain mechanism of a free-electron laser (FEL) invoked quantum mechanics. Although Planck's constant $\hbar$ dropped out of the final expression for the gain indicating that it should be derivable from a classical approach, this was supposed to be very difficult for a long time, since the first approach[1] relied crucially on quantum recoil corrections to the frequencies of emitted photons for which there is no classical analog. There is now general agreement that all essential features of the FEL can be understood in terms of classical concepts.[2] This excludes, of course, the problem of the photon statistics of the FEL and, consequently, the very question of whether or not the FEL is a laser in the sense that it radiates a coherent state. This question albeit interesting in itself is by no means purely academic. The well-known example of multiphoton ionization of atoms[3] shows that the photon statistics of an intense monochromatic light beam can be of vital importance with respect to its interaction with matter. A general solution to this problem requires a fully quantized approach. In this paper we are far from solving the problem of the photon statistics of a free-electron *laser*, instead when speaking about a FEL we actually mean a free-electron amplifier in the small-signal cold-beam noncollective regime. No attempt has been made yet to investigate the photon statistics of a free-electron laser above threshold.

Quantum descriptions of the FEL often start from the Bambini-Renieri Hamiltonian,[4] which specifies the FEL (in the context of the Weizsäcker-Williams approximation) in a moving frame in which the frequencies of the laser and the wiggler coincide. In this frame resonance occurs when the electron is at rest, hence the electron can be treated nonrelativistically. This paper relies on a reformulation of this approach in the interaction picture in contrast to the Schrödinger or Heisenberg picture which are usually applied.[5,6]

In the interaction picture, the time-evolution operator of an electron-laser photon state is given by the time-ordered exponential of the transformed interaction Hamiltonian. If the electron momentum operator is treated as a classical c number, the problem reduces to that of a classical current interacting with a quantized radiation field. There is, however, no gain in this approximation due to the neglect of the electron quantum recoil. In an earlier approach to the same problem[7,8] this had been remedied by introducing the recoil corrections (as obtained from energy-momentum conservation) by hand into the detuning parameter, which is the only quantity to depend significantly on these very small corrections. By means of this procedure, one obtains in a very simple way all basic results of FEL theory.[8] In spite of its success, this *ad hoc* approach is not completely satisfactory. We replace it here by expanding the exact time-evolution operator up to first order in the recoil which is sufficient to describe the small-signal regime. To our knowledge, this is then the only fully quantized treatment of the FEL, which does not resort at some stage to the classical equations of motion in order to infer gain.

In Sec. II, we derive the time-evolution operator in the above-mentioned linear recoil approximation. In Sec. III, we employ it to compute gain, spread, and the photon statistics in terms of eigenstates of the photon number. If the FEL starts from the field vacuum, the resulting final state of the radiation field is bunched, antibunched, or coherent depending upon whether the electron momentum is $p>0$, $p<0$, or $p=0$, respectively. We suggest that, in general, the FEL preserves coherent states only inasmuch as gain is zero or can be neglected. This is equivalent to the startling conclusion that the FEL is a laser in the sense that it produces a coherent state only if it is not a laser in the sense that it does not amplify. In Sec. IV we compare our present results with the earlier mentioned semiphenomenological approach.[7,8] The latter turns out to be perfectly justified if the initial radiation field is either sufficiently intense or in the vacuum state. We finally relate our work to Refs. 5 and 6.

II. TIME-EVOLUTION OPERATOR

We start with the one-electron nonrelativistic Hamiltonian which describes the FEL in the so-called Bambini-Renieri frame.[4] In this moving frame, the laser and wiggler frequency coincide with $\omega = ck/2$. The Hamiltonian is given by

$$H = H_0 + H_1 , \tag{1a}$$

$$H_0 = \frac{p^2}{2m} + \hbar\omega(a_L^\dagger a_L + a_W^\dagger a_W) , \tag{1b}$$

$$H_1 = i\hbar g(a_L^\dagger a_W e^{-ikz} - a_W^\dagger a_L e^{ikz}) . \tag{1c}$$

Here $a_L(a_L^\dagger)$ and $a_W(a_W^\dagger)$ are photon annihilation (creation) operators which represent the laser field and the wiggler field, respectively, in the Weizsäcker-Williams approximation, p and z the electron's momentum and coordinate with $[z,p]=i\hbar$, m is a renormalized electron mass, and the coupling constant g is given by

$$g = \left[\frac{4\pi c}{kV}\right] r_0 , \tag{2}$$

where r_0 is the classical electron radius and V is the quantization volume.

In the interaction picture, H_1 transforms to

$$H_I(t) = e^{iH_0t/\hbar} H_1 e^{-iH_0t/\hbar}$$

$$= i\hbar g(e^{-it(\hbar k^2+2kp)/2m} A^\dagger a_W - \text{c.c.}) , \tag{3}$$

where in analogy with Ref. 5, we introduced the operator

$$A = a_L e^{ikz} \tag{4}$$

with the properties

$$[A, A^\dagger] = 1, \quad A^\dagger A = a_L^\dagger a_L . \tag{5}$$

In deriving Eq. (3), we used the following relations:

$$e^{i\omega t a^\dagger a} a e^{-i\omega t a^\dagger a} = a e^{-i\omega t} \quad (a = a_L, a_W) \tag{6a}$$

$$e^{ip^2t/2m\hbar} e^{-ikz} e^{-ip^2t/2m\hbar} = e^{-ikz} e^{it(\hbar k^2 - 2kp)/2m} . \tag{6b}$$

The time-evolution operator for the electron-photon state is given by

$$S(T/2, -T/2) = \mathscr{T} \exp\left[-\frac{i}{\hbar}\int_{-T/2}^{T/2} dt\, H_I(t)\right] , \tag{7}$$

where $\mathscr{T}$ is the Dyson time-ordering operator and the symmetric integration has been chosen by convenience. The interaction time $T = L/c$ is specified by the wiggler length L. Equation (7) as it stands can only be evaluated in perturbation theory. This is due to the time-ordering prescription as well as the appearance of the operator p in Eq. (3). We are now trying to get rid of both difficulties by expanding $S(T/2, -T/2)$ around some c-number average value p_0 which will be specified afterwards. Hence we write

$$S(T/2, -T/2) = S_0(T/2, -T/2) + S_1(T/2, -T/2) + \cdots , \tag{8a}$$

$$S_0(t_2, t_1) = \mathscr{T} \exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, H_I(t)\right]\Bigg|_{p=p_0} , \tag{8b}$$

$$S_1(T/2, -T/2) = \frac{1}{\hbar}\int_{-T/2}^{T/2} dt\, S_0(T/2, t)\left[(p - p_0)\frac{\partial}{\partial p}[-iH_I(t)]\Big|_{p=p_0}\right] S_0(t, -T/2) . \tag{8c}$$

Here $S_0(T/2, -T/2)$ is the time-evolution operator in the classical recoilless approximation for the electron current. It has been shown earlier[7] that, in this approximation, the photon distribution function exhibits a Poisson distribution if initially no laser field is present, and that $S_0(T/2, -T/2)$ preserves coherent states. There is, however, no gain in this approximation since the quantum recoil of the emitted photons, which is responsible for the gain mechanism in the free-electron laser,[1] is not taken into account. The quantum recoil is accounted for up to first order by $S_1(T/2, -T/2)$. Owing to this linear approximation we are henceforth restricted to the small-signal regime. Note that in the expansion we were carefully respecting the time ordering. The square bracket in Eq. (8c) is a symbolic notation: the correct order of the operators must be inferred from Eq. (3) [see Eq. (17) below].

From now on, we will take the semiclassical limit of the wiggler field, i.e., we will set

$$a_W^\dagger \approx a_W \approx \sqrt{N_W} . \tag{9}$$

This limit is reasonable because the quantum nature of the wiggler field is a mathematical device only and no quantum effects of it can have a physical meaning. With this, we obtain from Eq. (3) (for $p = p_0$),

$$[H_I(t'), H_I(t'')] = 2ig^2 N_W \sin[\beta(t' - t'')] , \tag{10}$$

where

$$\beta = \frac{k^2\hbar}{2m} + \frac{kp_0}{m} . \tag{11}$$

The commutator of the interaction Hamiltonian for $p = p_0$ at different times is therefore a purely imaginary c number. Under this condition it can be shown[9] that the time-ordering operator merely introduces a phase:

$$S_0(t_2, t_1) = e^{i\theta(t_2, t_1)} \exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} dt'\, H_I(t')\right]\Bigg|_{p=p_0} , \tag{12a}$$

$$i\theta(t_2, t_1) = -\frac{1}{2\hbar^2}\int_{t_1}^{t_2} dt' \int_{t_1}^{t'} dt'' [H_I(t'), H_I(t'')] . \tag{12b}$$

It can easily be shown that $S_0(t_2, t_1)$ is unitary and satisfies the group property

$$S_0(t_1, t_2) S_0(t_2, t_3) = S_0(t_1, t_3) . \tag{13}$$

On substituting from Eq. (3) in Eq. (12a) and applying the Baker-Hausdorff formula, we get

$$S_0(t_2, t_1) = e^{i\theta(t_2, t_1)} e^{j^*(t_2, t_1)A^\dagger} e^{-j(t_2, t_1)A}$$
$$\times e^{(-1/2)|j(t_2, t_1)|^2} , \tag{14}$$

where

$$j(t_2, t_1) = g\sqrt{N_W}\int_{t_1}^{t_2} dt\, e^{i\beta t} = \frac{g\sqrt{N_W}}{i\beta}(e^{i\beta t_2} - e^{i\beta t_1}) . \tag{15}$$

It is evident from Eq. (15) that

$$j(T/2, -T/2) \equiv j(T) = \frac{2g\sqrt{N_W}}{\beta}\sin(\beta T/2) = j^*(T) . \tag{16}$$

Equation (14) provides us with an explicit expression for the time-evolution operator $S_0(T/2, -T/2)$ in the classical recoilless approximation.

Next we derive an expression for the lowest order correction $S_1(T/2, -T/2)$. On substituting for $H_I(t)$ in Eq. (8c), we obtain

$$S_1(T/2, -T/2)$$
$$= \frac{-ig\sqrt{N_W}k}{m}\int_{-T/2}^{T/2} dt\, t S_0(T/2, t)[(p - p_0)A^\dagger e^{-i\beta t} + A(p - p_0)e^{i\beta t}] S_0(t, -T/2)$$
$$= -\frac{ig\sqrt{N_W}k}{m} S_0(T/2, -T/2)$$
$$\times \int_{-T/2}^{T/2} dt\, t e^{-i\beta t}(\{p - p_0 - \hbar k[|j(t, -T/2)|^2 + j^*(t, -T/2)A^\dagger + j(t, -T/2)A]\}[A^\dagger + j(t, -T/2)]$$
$$-[A + j(T/2, t)]\{p - p_0 - \hbar k[|j(T/2, t)|^2 + j^*(T/2, t)A + j(T/2, t)A^\dagger]\}) . \tag{17}$$

In deriving Eq. (17) we have used the following commutation relations:

$$[A, S_0(t_2, t_1)] = j^*(t_2, t_1) S_0(t_2, t_1) , \tag{18a}$$

$$[A^\dagger, S_0(t_2, t_1)] = j(t_2, t_1) S_0(t_2, t_1) , \tag{18b}$$

$$[p, A] = \hbar k A, \quad [p, A^\dagger] = -\hbar k A^\dagger , \tag{18c}$$

$$[p, S_0(t_2, t_1)] = -\hbar k S_0(t_2, t_1)[j^*(t_2, t_1)A^\dagger + j(t_2, t_1)A + |j(t_2, t_1)|^2] , \tag{18d}$$

as well as the group property [Eq. (14)].

Since $|j|^2 << 1$,[10] Eq. (17) can be somewhat simplified. Applying the square bracket in Eq. (17) to a state $|\bar{p},N\rangle$, the resulting state is a superposition of states $|\bar{p},N\rangle$, $|\bar{p}\mp\hbar k,N\pm 1\rangle$ and $|\bar{p}\mp 2\hbar k,N\pm 2\rangle$. With the choice of p_0 specified in Sec. III, the eigenvalue $\bar{p}-p_0$ never vanishes. We can then safely neglect $|j(t,-T/2)|^2$ and $|j(T/2,t)|^2$ in Eq. (17). Moreover, it turns out that the underlined j's never contribute significantly except when multiplied with $p-p_0$. The resulting expression for $S(T/2,-T/2)$ is then

$$S(T/2,-T/2)=S_0(T/2,-T/2)\Bigg\{1-\frac{ig\sqrt{N_W}k}{m}\int_{-T/2}^{T/2}dt\, te^{-i\beta t}\times(\{p-p_0-\hbar k[j^*(t,-T/2)A^\dagger+j(t,-T/2)A]\}A^\dagger - A\{p-p_0-\hbar k[j^*(T/2,t)A+j(T/2,t)A^\dagger]\} + (p-p_0)[\underline{j}(t,-T/2)-\underline{j}(T/2,t)])\Bigg\} . \quad (19)$$

The last term proportional to $p-p_0$ is negligible for $N>>1$. It can easily be shown that $S(T/2,-T/2)$ as given by Eq. (19) is unitary up to the order of k/m.

$S(T/2,-T/2)$ apparently depends on the choice of p_0. We are now going to show that up to the order of k/m it is actually independent of p_0. According to Eqs. (15) and (11) we have

$$\frac{\partial j(T/2,-T/2)}{\partial p_0}=\frac{k}{m}\frac{\partial j(T/2,-T/2)}{\partial\beta}$$

and hence using Eq. (18b)

$$\frac{\partial S_0(T/2,-T/2)}{\partial p_0}=\frac{k}{m}\frac{\partial j(T/2,-T/2)}{\partial\beta}\times S_0(T/2,-T/2)(A^\dagger-A) , \quad (20)$$

where we have neglected the derivative of the phase $i\theta(T/2,-T/2)$ since it contributes only to higher orders. Calculating then the derivative $\partial S(T/2,-T/2)/\partial p_0$ from Eq. (19), $\partial S_0(T/2,-T/2)/\partial p_0$ cancels against the derivative of the integrand thus leaving us with

$$\frac{\partial S(T/2,-T/2)}{\partial p_0}=O\left|\frac{k}{m}\right|^2 . \quad (21)$$

III. PHOTON STATISTICS

We shall first consider an initial number state $|\bar{p},N\rangle$ which satisfies

$$p\,|\bar{p},N\rangle=\bar{p}\,|\bar{p},N\rangle , \quad (22a)$$

$$A\,|\bar{p},N\rangle=\sqrt{N}\,|\bar{p}+\hbar k,N-1\rangle , \quad (22b)$$

$$A^\dagger|\bar{p},N\rangle=\sqrt{N+1}\,|\bar{p}-\hbar k,N+1\rangle . \quad (22c)$$

Exploiting the arbitrariness of the expansion parameter p_0, we fix it by

$$p_0=\bar{p}-\tfrac{1}{2}\hbar k . \quad (23)$$

This will provide us with the most symmetric explicit results. Moreover, the parameter β then reads

$$\beta=k\bar{p}/m , \quad (24)$$

so that resonance at $\bar{p}=0$ becomes explicitly obvious. It then follows that

$$\langle\bar{\bar{p}},n\,|S(T/2,-T/2)|\bar{p},N\rangle$$
$$=\langle n\,|S_0|N\rangle+\frac{ig\sqrt{N_W}\hbar k^2}{m}\int_{-T/2}^{T/2}dt\,te^{-i\beta t}\{\tfrac{1}{2}\sqrt{N+1}\langle n\,|S_0|N+1\rangle+\tfrac{1}{2}\sqrt{N}\langle n\,|S_0|N-1\rangle$$
$$+(N+\tfrac{1}{2})[j(t,-T/2)-j(T/2,t)]\langle n\,|S_0|N\rangle$$
$$+j^*(t,-T/2)\sqrt{(N+1)(N+2)}\langle n\,|S_0|N+2\rangle$$
$$-j^*(T/2,t)\sqrt{N(N-1)}\langle n\,|S_0|N-2\rangle\} . \quad (25)$$

Here we used the abbreviations $S_0=S_0(T/2,-T/2)$, $\bar{\bar{p}}=\bar{p}+(N-n)\hbar k$, and $|l\rangle=|\bar{p}-(l-N)\hbar k,l\rangle$.

The photon-distribution function for the radiation field is then given by

$$P(n)=|\langle\bar{\bar{p}},n\,|S(T/2,-T/2)|\bar{p},N\rangle|^2$$
$$=|\langle n\,|S_0|N\rangle|^2-\frac{\hbar k^2}{m}\frac{\partial j(T)}{\partial\beta}[\sqrt{N+1}\langle n\,|S_0|N+1\rangle+\sqrt{N}\langle n\,|S_0|N-1\rangle$$
$$-j(T)\sqrt{N(N-1)}\langle n\,|S_0|N-2\rangle+j(T)\sqrt{(N+1)(N+2)}\langle n\,|S_0|N+2\rangle]\langle N\,|S_0^\dagger|n\rangle . \quad (26)$$

It can be shown[7,8] that

$$\langle n\,|S_0|N\rangle=\left|\frac{N!}{n!}\right|^{1/2}e^{i\theta(T/2,-T/2)}e^{(-1/2)j^2(T)}j^{n-N}(T)L_N^{n-N}[j^2(T)] , \quad (27)$$

where the L_N^{n-N} are Laguerre polynomials. In view of Eq. (27), $\langle\alpha\,|S_0|\beta\rangle\langle\gamma\,|S_0^\dagger|\delta\rangle$ is real, which has been used in deriving Eq. (26). Owing to the unitarity of $S(T/2,-T/2)$, $P(n)$ should be properly normalized at least up to the order of k/m. Actually we find as a consequence of $S_0S_0^\dagger=1$,

$$\sum_{n=0}^{\infty}P(n)=1 .$$

The first term in Eq. (26) corresponds to the photon distribution in the absence of quantum recoil.[7,8] For $N=0$,[11] it yields the earlier mentioned Poisson statistics. The second and third term are responsible for gain, as will be shown below. They destroy Poisson statistics even for $N=0$. One can also easily convince oneself that the $P(n)$ for $N=0$ are *not* the first-order expansion of a Poisson distribution with a different mean value: for $N=0$, Eqs. (26) and (27) yield

$$P(n)=\frac{1}{n!}e^{-j^2(T)}j^{2n}(T)\left\{1-\frac{\hbar k^2}{mj(T)}\frac{\partial j(T)}{\partial\beta}[n^2-(2n+1)j^2(T)+j^4(T)]\right\} , \quad (28a)$$

whereas the shifted Poisson distribution is

$$\bar{P}(n)=\frac{1}{n!}e^{-(j^2(T)+\epsilon)}[j^2(T)+\epsilon]^{2n}$$
$$=\frac{1}{n!}e^{-j^2(T)}j^{2n}(T)\left\{1-\epsilon+\frac{2n\epsilon}{j^2(T)}+\cdots\right\} . \quad (28b)$$

Here ϵ might be specified by Eq. (29) below, according to

$$\epsilon=-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta} .$$

Obviously, the discrepancy between Eqs. (28a) and (28b) is considerable. This leads us to conjecture that the FEL radiates or preserves a coherent state only inasmuch as gain can be neglected.

A further interesting observation can be made when comparing Eqs. (25) and (26). The expression in curly braces in Eq. (25) is, for $N>>1$, proportional to N, whereas the term in square brackets in Eq. (26) is only proportional to $\sqrt{N}$ since the last two terms almost cancel for $N>>1$. This indicates that the phase of $\langle\bar{\bar{p}},n\,|S(T/2,-T/2)|\bar{p},N\rangle$ reacts much earlier to increasing laser-field strengths than its modulus, i.e., the applicability of the first-order recoil approximation depends upon the quantity to be calculated. Gain and spread (and all higher moments) can be calculated from $P(n)$; the expectation value of the field as well as two-time field-correlation functions, however, would incorporate the phase.

The photon-distribution function $P(n)$ allows for the calculation of all the moments,

$$\langle n^r\rangle=\sum_{n=0}^{\infty}n^rP(n) .$$

When investigating the (anti-) bunching properties of the emitted radiation, however, we will find that extensive cancellations erase all leading terms. Hence Eq. (16), which is based on the already approximated Eq. (19), is insufficient and we have to return to Eq. (17). We then find using the commutation relations (18a)–(18d):

$$\langle n\rangle=\langle\bar{\beta},N\,|\,S(T/2,-T/2)^{\dagger}A^{\dagger}AS(T/2,-T/2)\,|\,\bar{\beta},N\rangle$$
$$=N+j^2(T)-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}(2N+1)+\delta\,, \quad (29)$$

$$\langle n^2\rangle=\langle\bar{\beta},N\,|\,S(T/2,-T/2)^{\dagger}(A^{\dagger}A)^2S(T/2,-T/2)\,|\,\bar{\beta},N\rangle$$
$$=[N+j^2(T)]^2+j^2(T)(2N+1)-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}[4N^2+2N+1+4j^2(T)(2N+1)]$$
$$+[4N+2j^2(T)+1]\delta\,, \quad (30a)$$

$$\Delta n^2=\langle(n-\langle n\rangle)^2\rangle=j^2(T)(2N+1)-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}[1+2j^2(T)(2N+1)]+(2N+1)\delta\,, \quad (30b)$$

where

$$\delta=ig\sqrt{N_W}\hbar k^2j(T)/m\int_{-T/2}^{T/2}dt\,te^{-i\beta t}[2|j(T/2,t)|^2+2|j(t,-T/2)|^2-j^2(T/2,t)-j^2(t,-T/2)]\,.$$

The second term in Eq. (29) represents spontaneous emission. For $N>>1$ it is negligible with respect to the third term, which is the usual gain expression. Via Madey's theorem[12] this is related to the first term in the spread (30b). Inasmuch as $N\sim\hbar^{-1}$, all terms in Eq. (29), except the one in the factor $2N+1$, contribute as classical terms to the quantity $\hbar\omega\langle n\rangle$ [notice that $j(T)\sim\hbar^{-1/2}$ in view of Eq. (15)]. This includes the last term which we would not have obtained from the approximate Eqs. (19) or (26). It gives corrections to spontaneous emission and is negligible for all N. This quantity δ is also negligible in Eq. (30a). In the spread (30b), however, due to extensive cancellations the second and the third term, which involves δ, are of comparable magnitude. Notice that the spread is increased for positive and decreased for negative gain.

From Eqs. (29) and (30), we have for $N=0$

$$\Delta n^2-\langle n\rangle=-\frac{2\hbar k^2}{m}j^3(T)\frac{\partial j(T)}{\partial\beta}\,, \quad (31)$$

where δ has cancelled. Hence the radiation field which evolves by spontaneous emission is bunched for $\beta>0$, i.e., if the electron momentum is above resonance ($\bar{\beta}>0$), antibunched for $\beta<0$ ($\bar{\beta}<0$), and in a coherent state for $\beta=0$ ($\bar{\beta}=0$). This is a genuine quantum effect which cannot be obtained by any classical analysis. Intuitively we can understand the phenomenon of photon antibunching in a FEL by noting first that the classical current leads to a coherent state of the field, i.e., a Poisson distribution function. The effect of recoil for $\beta<0$ is to remove "bunches" of photons from the coherent state, thus leading to a narrower distribution function. The situation here is therefore similar to the multiphoton absorption process in atoms,[13] where photon antibunching has also been predicted.

The present analysis is based on an initial vacuum state. We conjecture that similar results concerning photon antibunching would be obtained for an arbitrary initial coherent state. A careful analysis of this problem within the framework of a many-electron theory remains to be carried out.

If we try to obtain corresponding results for a coherent state we run into the same difficulties which are already inherent in Ref. 5. Let us first take an initial electron field coherent state[5] $||\alpha\rangle$ with $A||\alpha\rangle=\alpha||\alpha\rangle$. The lowest-order contribution to the gain,

$$\langle\alpha||S(T/2,-T/2)^{\dagger}A^{\dagger}AS(T/2,-T/2)||\alpha\rangle$$
$$=|\alpha+j|^2+\cdots\,,$$

is strongly phase dependent, and the same occurs to all higher orders. This is not surprising since in contrast to a state $|\bar{\beta},N\rangle=|\bar{\beta}\rangle|N\rangle$, in which both the field and the electron are uniformly distributed in space, an electron field coherent state contains inbuilt correlations which reflect the classical initial conditions. Before reasonable results for an ensemble of electrons can be obtained, the phase of the coherent state must be averaged over, analogously to the averaging over classical initial conditions.[14] Generally, such an averaging procedure will not preserve a coherent state. The necessity of averaging does not occur in case of a state $|\bar{p}N\rangle$.

Alternatively we might consider the amplification of a field-coherent state, i.e., $|in\rangle=|p\rangle|v\rangle$ with $a_L|v\rangle=v|v\rangle$. Since

$$A|p\rangle|v\rangle=v|p+\hbar k\rangle|v\rangle\,,$$

in view of the orthogonality of electron states with different momenta, only terms with equal numbers of A's and $A^{\dagger}$'s survive. Hence we are essentially back to the results for photon number states. This conclusion, however, depends crucially on the orthogonality relation $\langle p\,|\,p+\hbar k\rangle=0$, which involves the quantum recoil. To use it in the zeroth-order term where recoil has been neglected otherwise, does not seem to be consistent. Moreover, making use of this orthogonality requires an extremely monochromatic electron beam.

We showed that starting from the field vacuum $N=0$, due to the presence of gain, the FEL does not radiate a coherent state. We believe this suggests that the FEL also conserves coherent states (be it field-coherent states or some averaged electron field coherent states) only inasmuch as gain is neglected.

IV. DISCUSSION

If the quantum recoil is neglected, we are left with the simple model of a classical current interacting with a quantized radiation field which is exactly solvable. This leads to the photon statistics as established in Eq. (27). The most essential features of the FEL, however, gain and electron bunching, have dropped out of this approximation. In an earlier approach[7,8] this had been remedied by reintroducing the recoil by hand into the detuning parameter. This procedure yielded correct results for gain, spread, and all other basic properties of FEL's. We are now going to compare our present exact first-order calculation of the photon statistics with the former semiphenomenological approach.

With the just-mentioned procedure[7,8] we have, instead of Eq. (26),

$$\tilde{P}(n)=\left|1+(n-N)\Delta z\frac{\partial}{\partial z}\right|\,|\langle n\,|\,S_0\,|\,N\rangle|^2\,, \quad (32)$$

where we proceeded as indicated in Eqs. (23) and (24) of Ref. 7 or Eqs. (2.16) and (2.18) of Ref. 8. In our present notation

$$z=j(T)^2,\quad \Delta z=-\frac{\hbar k^2}{m}j(T)\frac{\partial j(T)}{\partial\beta}\,. \quad (33)$$

Using the zeroth-order matrix element (27), which is common to both approaches, doing the derivative indicated in Eq. (32), and re-expressing $\tilde{P}(n)$ in terms of matrix elements of S_0 we obtain

$$\tilde{P}(n)=|\langle n\,|\,S_0\,|\,N\rangle|^2+\Delta z(n-N)\left\{\left[\frac{n-N}{z}-1\right]\langle n\,|\,S_0\,|\,N\rangle-2\langle n\,|\,S_0\,|\,N-1\rangle\right\}\langle N\,|\,S_0^{\dagger}\,|\,n\rangle\,. \quad (34)$$

To compare with Eq. (26) we evaluate the matrix element $\langle\bar{p},n\,|\,[p,S_0]\,|\,\bar{p},N\rangle$ [Eq. (18d)] which yields the relation

$$(n-N)\langle n\,|\,S_0\,|\,N\rangle=j(T)(\langle n\,|\,S_0\,|\,N-1\rangle\sqrt{N}+\langle n\,|\,S_0\,|\,N+1\rangle\sqrt{N+1})+j^2(T)\langle n\,|\,S_0\,|\,N\rangle\,. \quad (35)$$

It is consistent with our earlier approximations to drop the last term on the right-hand side of Eq. (35), which then can also be used to simplify Eqs. (25) and (26). Introducing Eq. (35) in Eq. (34) we obtain

$$\tilde{P}(n)=|\langle n\,|\,S_0\,|\,N\rangle|^2+\frac{\Delta z}{j(T)}\langle N\,|\,S_0^{\dagger}\,|\,n\rangle[\sqrt{N}\langle n\,|\,S_0\,|\,N-1\rangle+j(T)\sqrt{(N+1)(N+2)}\langle n\,|\,S_0\,|\,N+2\rangle$$
$$-j(T)\sqrt{N(N-1)}\langle n\,|\,S_0\,|\,N+2\rangle+j(T)\langle n\,|\,S_0\,|\,N\rangle]\,. \quad (36)$$

This differs from Eq. (26) only by the presence of the last term. It is this term which destroys unitarity so that $\sum_{n=0}^{\infty}\tilde{P}(n)\neq 1$. The last term can be safely neglected for $N>>1$. For small N inspection of the explicit form [Eq. (27)] of $\langle n\,|\,S_0\,|\,N\rangle$ shows that it only contributes significantly for $n=N$. Hence all moments calculated by means of $\tilde{P}(n)$ instead of $P(n)$ are reliable for $N>>1$ as well as $N=0$. This justifies the semiphenomenological approach[7,8] for all cases of interest. It also shows that the latter cannot be trusted whenever recoil related modifications of $\langle n\,|\,S_1(T/2,-T/2)\,|\,N\rangle$ become important, since the process of introducing the recoil by hand fails to reproduce Eq. (25).

Our work differs from Ref. 5 mainly by using the interaction instead of the Schrödinger picture. In Ref. 5 quantum fluctuations of the momentum operator are neglected by approximating

$$p^2=[\langle p\rangle+(p-\langle p\rangle)]^2\approx 2p\langle p\rangle-\langle p\rangle^2 . \quad (37)$$

If $\langle p\rangle$ is assumed to be constant, the resulting Hamiltonian no longer allows for gain. This is easily demonstrated by calculating $\langle N \mid \exp(iHt)A^\dagger A \exp(-iHt) \mid N\rangle$ with the Hamiltonian approximated according to Eq. (37). Hence in Ref. 5, $\langle p\rangle$ is assumed to be time dependent and to be given by a classical trajectory. One is then left with an explicitly time-dependent Hamiltonian, which is, moreover, ambiguous since the classical trajectories behave completely different depending on the classical initial conditions.[14] Since this procedure cannot be considered to be a consistent quantum-mechanical approach, conclusions drawn from it regarding genuine quantum-mechanical entities such as the evolving photon statistics do not seem to be reliable.

Our linear recoil approximation is similar to Eq. (37); we apply it, however, to the interaction picture-time evolution operator and not to the complete Hamiltonian. Up to that final expansion, the exact H_0 has been used. Speaking in terms of quantum-mechanical perturbation theory we have approximated the vertices, but not the propagators. The importance of retaining "quantum fluctuations" in the momentum is also evident from a semiclassical treatment; see Eq. (10) of Ref. 15 or Eq. (3.7) of Ref. 16. If within the mentioned equations the second-order terms are dropped, the gain is lost.

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[10] In view of Eqs. (15) and (2),

$$|j(T)|^2\le g^2N_wT^2=4\pi(r_0L/k)^2(N_w/V)(1/V) ,$$

which is proportional to the energy density N_w/V of the wiggler field as well as the inverse quantization volume. The latter must be chosen larger than the actual volume of the system, which is Ld^2 with $d\sim 1$ cm. With the numerical parameters of the Stanford experiments we then have $|j(T)|^2\lesssim 10^{-4}$. For arbitrary t_1 and t_2, $|j(t_2,t_1)|^2$ is of the same order of magnitude. Note that, although $N_w >> 1$ is required, the limit $N_w\to\infty$ corresponding to an infinite wiggler field is excluded due to our first-order expansion.

[11] In principle, the Bambini-Renieri frame which equates the laser and the wiggler frequency, restricts spontaneous emission to just one mode. If we want to consider different modes, we have to introduce a different frame in each case, which is then mainly reflected in different values of $\bar{p}$. Hence, by varying $\bar{p}$, we can still cover all modes of spontaneous emission (in axial direction).

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