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WINTER COLLEGE ON LASERS, ATOMIC AND MOLECULAR PHYSICS

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Topical Meeting on the Free Electron Laser

INSTABILITY THRESHOLD, QUANTUM INITIATION AND PHOTONSTATISTICS IN HIGH-GAIN FREE ELECTRON LASERS

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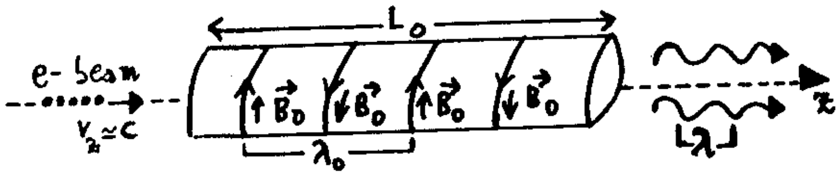


SINGLE-PASS DEVICE

$$\begin{cases} \vec{B}_0 = B_0\left[-\sin(k_0 z)\hat{x} + \cos(k_0 z)\hat{y}\right] \\ L_0 = N_0\lambda_0\ ;\quad \omega_0 = 2\pi c/\lambda_0 = ck_0\ ;\quad \mathbb{K} = eB_0/m_0 c\omega_0 \end{cases}$$

STRONG, CONSTANT HELICAL WIGGLER

$$\begin{cases} \vec{E} = E_0\left[\sin(kz-\omega t+\phi_0)\hat{x} + \cos(kz-\omega t+\phi_0)\hat{y}\right] \\ \omega = ck \end{cases}$$

ONE RADIATION MODE

ON RESONANCE

$$\omega = 2\omega_0 \frac{\gamma_R^2}{1+\mathbb{K}^2}$$

$$\left\{ \gamma_R = (1-\beta_R^2)^{-1/2}\ ;\quad \beta_R = \frac{v_R}{c} = \frac{k}{k+k_0} \right\}$$

RELEVANT (DIMENSIONLESS) VARIABLES :

ELECTRON-FIELD PHASE: $\psi_i = (k+k_0)z_i - \omega t$

ELECTRON ENERGY: $\gamma_i = \dfrac{E_i}{m_0 c^2}$

FIELD AMPLITUDE: $A \propto E_0 \quad \left[|A|^2 = \dfrac{|E_0|^2 V / 4\pi}{\rho N \gamma_0 m_0 c^2} \right]$

DEFINE $\quad \theta_i = \psi_i - \delta\bar{\tau} \qquad p_i = \dfrac{1}{\rho}\dfrac{\gamma_i}{\gamma_0}$

$\delta = \dfrac{1}{\rho}\dfrac{\gamma_0 - \gamma_R}{\gamma_0}$ DETUNING PARAMETER

$\bar{\tau} = 2\omega_0 \rho t \qquad \left[0 \le t \le \dfrac{L_0}{c} \rightarrow 0 \le \bar{\tau} \lesssim 4\pi\rho N_0 = \bar{\tau}_i \right]$

$\rho = \left(\dfrac{K}{4}\dfrac{\Omega_p}{\omega_0} \right)^{2/3}$ STRENGTH (PIERCE) PARAMETER

$\Omega_p = \left(\dfrac{4\pi e^2 N}{m_0 V \gamma_0^3} \right)^{1/2}$ RELATIVISTIC PLASMA FREQUENCY

• EVOLUTION EQUATIONS $\quad$ [BONIFACIO - PELLEGRINI - NARDUCCI Opt. Commun. 50, 373 (1984)]

$$\begin{cases} \dfrac{d\theta_i}{d\bar{\tau}} = \dfrac{1}{2\rho}\left(1 - \dfrac{1}{\rho^2 p_i^2}\right) \\[2mm] \dfrac{dp_i}{d\bar{\tau}} = -\dfrac{1}{\rho}\left(A \dfrac{e^{i\theta_i}}{p_i} + c.c. \right) \\[2mm] \dfrac{dA}{d\bar{\tau}} = \dfrac{1}{\rho}\sum_j \dfrac{e^{-i\theta_j}}{p_j} + i\delta A \end{cases}$$

CONSTANT OF MOTION

$$\frac{1}{N}\sum_i p_i + |A|^2 = \text{const.} \rightarrow m_0 c^2 \sum_i \gamma_i + \frac{|E_0|^2}{4\pi} V = \text{const.}$$

ELECTRON+FIELD ENERGY CONSERVATION

$$\frac{\gamma_i - \gamma_0}{\gamma_0} \ll 1 \quad , \quad \frac{\gamma_0 - \gamma_R}{\gamma_0} \ll 1$$

$$\frac{1}{\rho p_i} = \frac{\gamma_0}{\gamma_i} \simeq 1$$

$$\frac{1}{2\rho}\left(1 - \frac{1}{\rho^2 p_i^2}\right) \simeq \frac{1}{\rho}\frac{\gamma_i - \gamma_0}{\gamma_0} \equiv \bar{\eta}_i$$

$$\begin{cases} d\theta_i / d\bar{\tau} = \bar{\eta}_i \\ d\bar{\eta}_i / d\bar{\tau} = A e^{i\theta_i} + c.c. \\ dA / d\bar{\tau} = N^{-1} \sum_j e^{-i\theta_j} + i\delta A \end{cases} \qquad (I)$$

• SELF-AMPLIFIED SPONTANEOUS EMISSION

$$(II) \quad \begin{cases} A(0) = 0 & \text{NO FIELD EXCITATION} \\ \gamma_i(0) = \gamma_0 \rightarrow \bar{\eta}_i(0) = 0 & \text{MONOKINETIC} \\ \sum_j e^{-i\theta_j} = 0 & \text{UNBUNCHED} \end{cases} \Big\rangle \text{ ELECTRON BEAM}$$

$$\Updownarrow$$

$b_0 = 0 \qquad b \equiv \dfrac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j} \simeq \dfrac{1}{N_\lambda}\sum_{j=1}^{N_\lambda} e^{-i\theta_j}$ BUNCHING PARAMETER $\quad 0 \le |b| \le 1$

$b_0 \simeq 0$

$b_0 \simeq 1$

(II) ≡ EQUILIBRIUM CONDITION FOR SYSTEM (I)

STABLE OR UNSTABLE ?

RESULTS OF THE LINEAR STABILITY ANALYSIS

$\delta = \delta_T = 3/2^{2/3} \simeq 1.89$: THRESHOLD FOR COLLECTIVE INSTABILITY

$\delta > \delta_T$: STABLE REGIME

WEAKLY COUPLED PENDULA $\rightarrow$ MADEY'S SMALL SIGNAL GAIN

AT ALL TIMES $b(\tau) \simeq b(0) \simeq 0$

$\delta < \delta_T$: UNSTABLE REGIME

EXPONENTIAL GAIN : "COMPTON HIGH-GAIN REGIME"

AT THE PEAK $b_p \simeq 1$ (SELF-BUNCHING)

KROLL - MC MULLIN (1978) ; BERNSTEIN - HIRSCHFIELD ; VAINSHTEIN ;

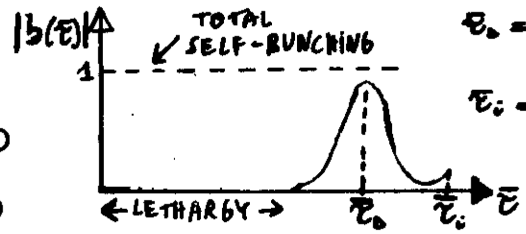
SPRANGLE - TANG - MANNHEIMER ; GOVER - SPRANGLE ; SHIH - YARIV ;

BRATMAN - GINZBURG - PETELIN ; DATTOLI - MARINO - RENIERI - ROMANELLI ;

BONIFACIO - NARDUCCI - PELLEGRINI - MURPHY :
- DROPPED RESTRICTION $(\gamma_i - \gamma_0)/\gamma_0 \ll 1$
- INCLUDED SPACE-CHARGE, HARMONICS, INITIAL SPREAD ...

NUMERICAL RESULTS

i)



τ_D = DELAY TIME (BUILD-UP TIME)

τ_i = INTERACTION TIME $= 4\pi\rho N_0$

THRESHOLD : $\tau_D \lesssim \tau_i$

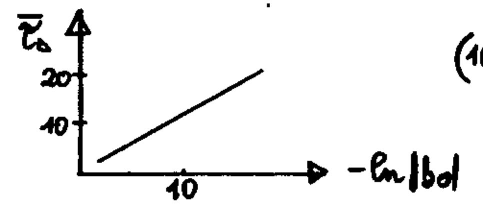
$$\begin{cases} A(0)=0 \; ; \; \eta_j(0)=0 \\ b(0) = \frac{1}{N_\lambda}\sum_j e^{-i\theta_j^0} \simeq 0 \end{cases}$$

IF $b(0)=0$ (UNSTABLE) EQUILIBRIUM CONDITION

CLASSICAL SIMULATION OF NOISE NECESSARY FOR THE START-UP OF THE PROCESS

$\Rightarrow$ TAKE SMALL $b_0 \neq 0$

ii)



$(10^{-8} \lesssim |b_0| \lesssim 10^{-1})$

$\bar{\tau}_D \propto -\ln|b_0|$

AS IF $|b(\tau)| = |b_0|\, e^{(\mathrm{Im}\,\bar{\lambda})\bar{\tau}}$ $\left\{\begin{array}{l}\bar{\lambda} \text{ ROOT THE CUBIC CHARACTERISTIC EQUATION} \\ \text{OBTAINED IN THE LINEAR STABILITY ANALYSIS}\end{array}\right\}$

BY IMPOSING : $|b(\bar{\tau}_D)| \simeq 1 \Rightarrow \bar{\tau}_D \simeq -\frac{1}{\mathrm{Im}\,\bar{\lambda}} \ln|b_0|$ $\left[\begin{array}{l}\text{IF } b_0 = 0 \\ \bar{\tau}_D \rightarrow \infty\end{array}\right]$

$\Rightarrow$ THE LINEAR ANALYSIS IS VALID UP TO THE FIRST PEAK OF EMITTED RADIATION

FIT FORMULA FROM THE NUMERICAL SOLUTION OF THE <u>NONLINEAR</u> EQUATIONS

$$\bar{\tau}_D \simeq -\frac{1}{\mathrm{Im}\,\bar{\lambda}} \ln|b_0| + 1$$

THRESHOLD CONDITION

$$\bar{\tau}_D \lesssim \bar{\tau}_i$$

IT DEPENDS CRITICALLY ON THE INITIAL NOISE LEVEL

QUESTION : WHAT IS $b_0 = \frac{1}{N_\lambda} \sum_{j=1}^{N_\lambda} e^{-i\theta_j}$?

IF θ_j^0 "LAW" NUMBERS : $\rightarrow b_0 = 0$ (UNBUNCHED BEAM)

IF θ_j^0 RANDOM QUANTITIES : $|b_0| \cong \sqrt{\langle |b_0|^2 \rangle}$, $|b_0|^2 = N_\lambda^{-2} \sum_{i,j} e^{i(\theta_i^0 - \theta_j^0)}$
(HEURISTIC ARGUMENT)

$= N_\lambda^{-1} + N_\lambda^{-2} \sum_{i \neq j} \cancel{e^{i(\theta_i^0 - \theta_j^0)}}$ (UNBUNCHED ELECTRONS)

$\rightarrow |b_0| = \frac{1}{\sqrt{N_\lambda}}$ (SHOT NOISE)

E.G. : SYSTEM ON RESONANCE

$|\delta| \to 0$, $\mathrm{Im}\,\bar\lambda = \sqrt{3}/2$.

DELAY TIME : $\bar\tau_D \simeq \frac{1}{\sqrt{3}} \ln N_\lambda + 1$

THRESHOLD : $\frac{1}{\sqrt{3}} \ln N_\lambda \leq 4\pi\rho N_0$

LINEAR QUANTUM THEORY NECESSARY (AND SUFFICIENT) TO EVALUATE b_0 AND $\bar\tau_D$

$\Rightarrow$ i) QUANTUM INITIATION

ii) COLLECTIVE INSTABILITY THRESHOLD (EVEN STARTING FROM ELECTRON POSITION AND MOMENTUM QUANTUM FLUCTUATIONS)

iii) PHOTONSTATISTICS (GLAUBER'S $P(\alpha)$)

$\{$ R. BONIFACIO – F.C. Opt. Commun. 50, 251 (1984)
JOSA B 2, 250 (1985)
Proceedings FEL Conference '84 (Castelgandolfo) $\}$

BEFORE QUANTUM TREATMENT, REWRITE THE CLASSICAL WORKING EQUATIONS (I)

$$\begin{cases} d\theta_i/d\bar\tau = \bar\eta_i \\ d\bar\eta_i/d\bar\tau = A\, e^{i\theta_i} + c.c. \\ dA/d\bar\tau = N^{-1}\sum_j e^{-i\theta_j} + i\delta A \end{cases} \qquad (I)$$

$\bar\eta_i = \frac{1}{\rho}\frac{\gamma_i - \gamma_0}{\gamma_0} \equiv \frac{1}{\rho}\eta_i$ $\qquad \eta_i = \frac{\gamma_i - \gamma_0}{\gamma_0}$

$A = \frac{\alpha}{\sqrt{\rho N}}$ $\qquad \alpha = \frac{i E_0\, e^{i(\phi_0 + \tilde\Delta\tilde\tau)}}{[(4\pi/V)\gamma_0 m_0 c^2]^{1/2}}$

$\bar\tau = 2\omega_0 \rho t \equiv \rho\tilde\tau$ $\qquad \tilde\tau = 2\omega_0 t$

$\delta = \frac{1}{\rho}\frac{\gamma_0 - \gamma_R}{\gamma_0} \equiv \frac{1}{\rho}\tilde\Delta$ $\qquad \tilde\Delta = \frac{\gamma_0 - \gamma_R}{\gamma_0}$

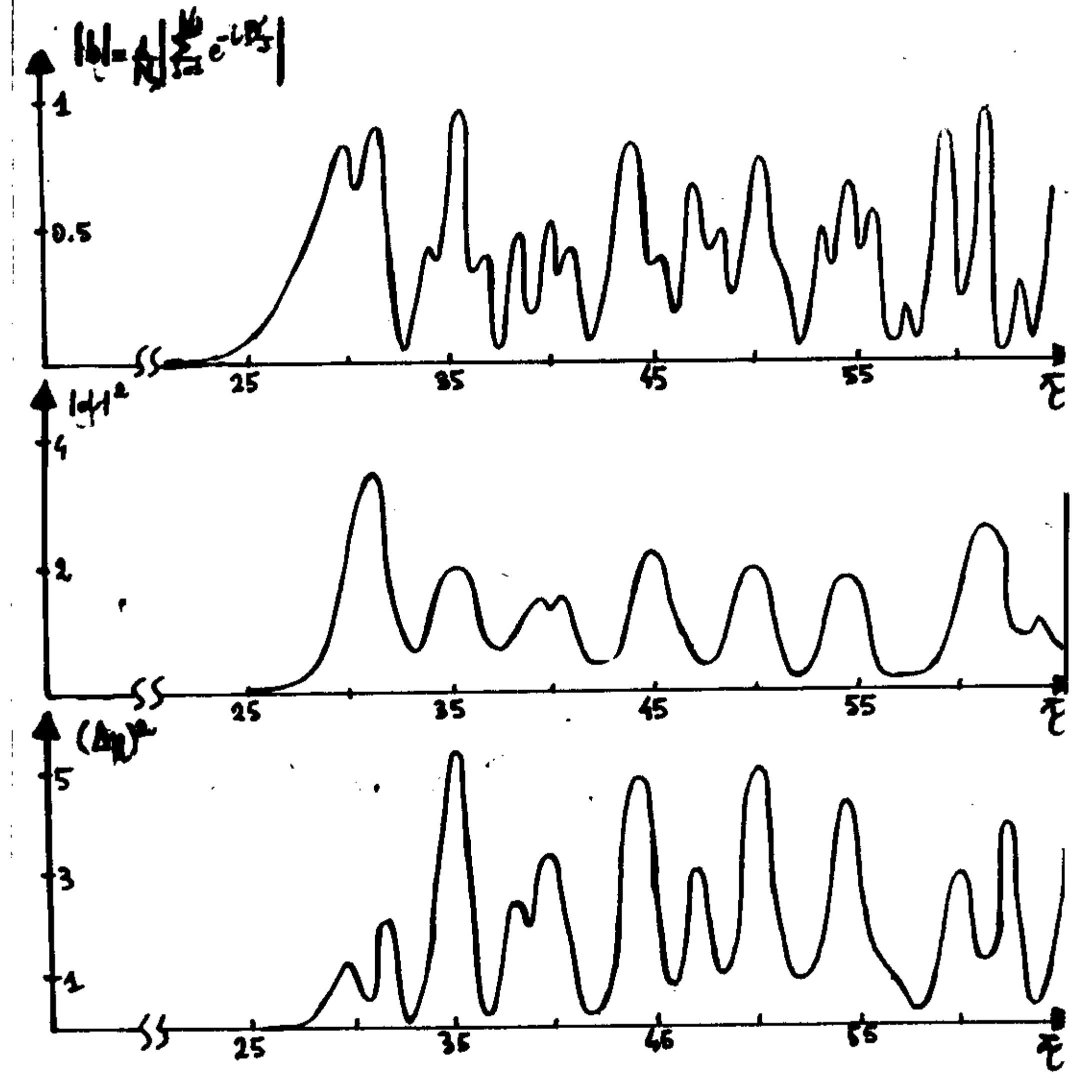
$\tilde w = \rho^{3/2}/\sqrt{N}$

$$\begin{cases} d\theta_i/d\tilde\tau = \eta_i \\ d\eta_i/d\tilde\tau = -\tilde w\left(\alpha\, e^{i\theta_i} + c.c\right) \\ d\alpha/d\tilde\tau = \tilde w \sum_j e^{-i\theta_j} + i\tilde\Delta\alpha \end{cases} \qquad (I')$$

HAMILTONIAN CHAOS IN NONLINEAR SYSTEM (I')

BONIFACIO – C. – CASATI Opt. Commun. 40, 219 ('82)
" in "Evolution of Order and Chaos" (Springer, '82)
" (+CELI) in "Coherence and Quantum Optics V" (Plenum, '84)

$\boxed{\delta > \delta_T}$ $[N_\lambda = 4]$



NON COOPERATIVE BEHAVIOUR		COOPERATIVE BEHAVIOUR
UNBUNCHED ELECTRONS	$\Longrightarrow$	SELF-BUNCHED ELECTRONS
WEAK RADIATION		STRONG STIMULATED EMISSION
$\delta \lesssim \delta_T$	$\delta \simeq \delta_T$	$\delta > \delta_T$

QUESTION:

TRANSITION FROM NONCOOPERATIVE TO COOPERATIVE BEHAVIOUR $\longleftrightarrow$ TRANSITION FROM ORDERED TO CHAOTIC MOTION?

BY ELIMINATION OF THE FIELD VARIABLES, ONE OBTAINS A CLASSICAL HAMILTONIAN SCHEME FOR ELECTRON DYNAMICS

$$H(q_i, p_i) = \sum_i \frac{p_i^2}{2} + \frac{w}{2}\sqrt{1-\langle p\rangle}\sum_i \sin q_i$$

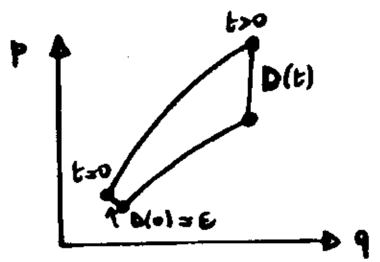
$$\begin{cases} \dfrac{dq_i}{d\tau} = \dfrac{\partial H}{\partial p_i} = p_i - \dfrac{w}{4}\dfrac{\langle \sin q\rangle}{\sqrt{1-\langle p\rangle}} & (i = 1, 2, \dots, N_\lambda) \\[2ex] \dfrac{dp_i}{d\tau} = -\dfrac{\partial H}{\partial q_i} = -\dfrac{w}{2}\sqrt{1-\langle p\rangle}\cos q_i & (i = 1, 2, \dots, N_\lambda) \end{cases}$$

$$\left\{ \langle \theta \rangle = \frac{1}{N_\lambda}\sum_{j=1}^{N_\lambda} \theta_j \right\}$$

$W_T \simeq 1.54$

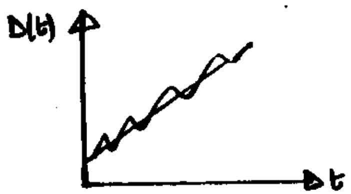
CHAOS

CRITERION OF LOCAL INSTABILITY OF MOTION



I) INTEGRABLE SYSTEMS

$D(t) = \varepsilon + \alpha t$ (on average)

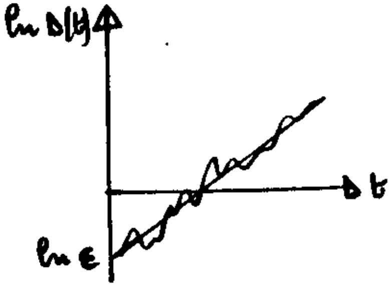


(e.g. G. Casati, B.V. Chirikov, J. Ford – Phys. Lett. 77 A, 91 (1980))

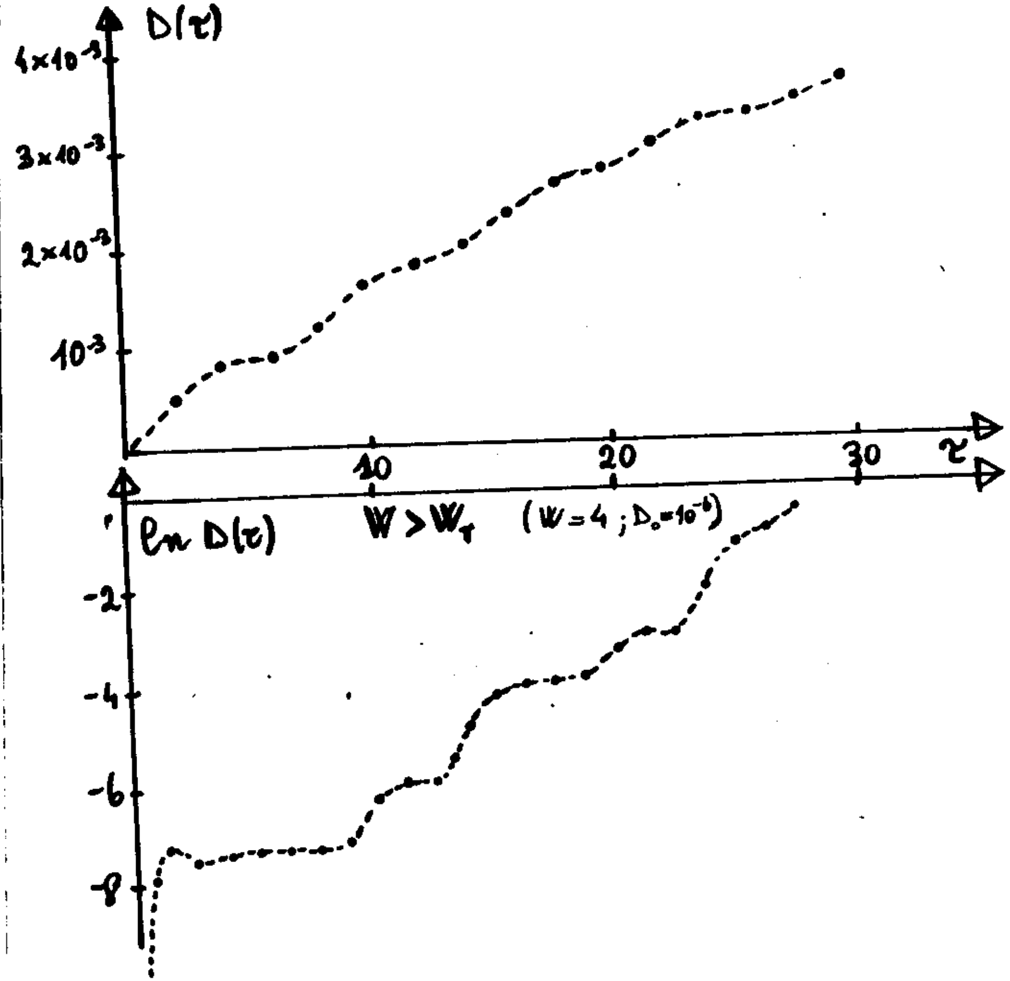
II) CHAOTIC SYSTEMS

$D(t) \simeq \varepsilon\, e^{ht}$, $\ln D(t) = \ln \varepsilon + ht$

$h \equiv$ Maximum Ljapounov exponent $\Rightarrow$ positive Kolmogorov–Sinai entropy



$W < W_T$ ($W = 1.53$; $D_0 = 10^{-6}$) ($W_T \simeq 1.54$)



POINCARÉ MAPS

$\boxed{N=2}$

4 COORDINATES : $q_1, p_1 ; q_2, p_2$

$H(q_1, p_1 ; q_2, p_2) = E = \text{const.}$

$p_2 = p_2(q_1, p_1 ; q_2)$

$\Rightarrow$

3 INDEPENDENT COORDINATES : q_1, p_1, q_2

POINCARÉ MAP (SURFACE OF SECTION)

$(q_1, p_1) \leftrightarrow q_2 = \pi$

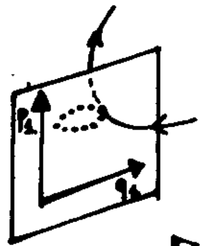
INTERSECTION POINTS OF THE ORBIT

$q_1(t), p_1(t), q_2(t)$

WITH THE PLANE (q_1, p_1) AT $q_2 = \pi$

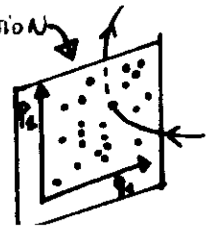
$\Downarrow$

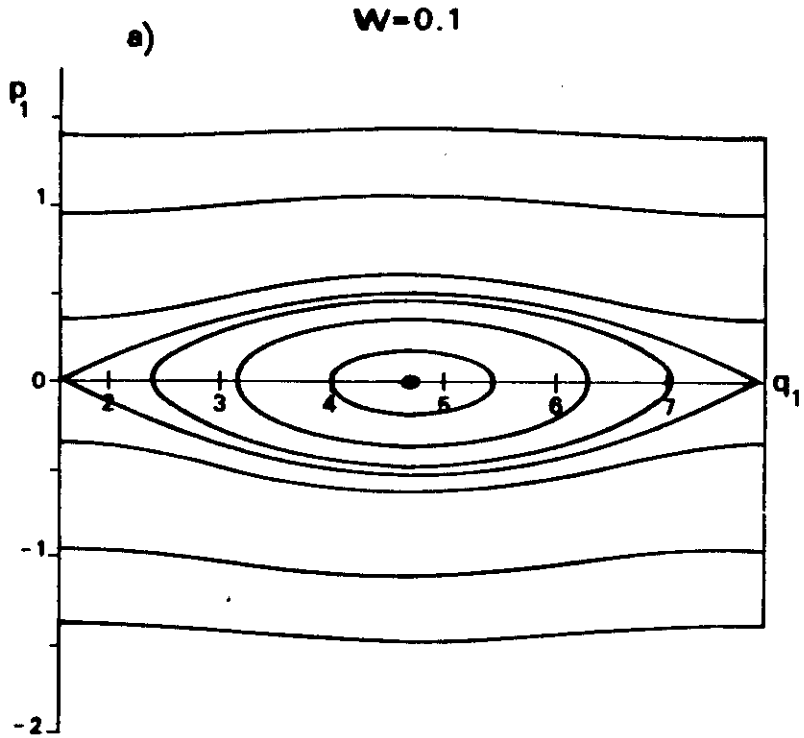
INFORMATION ABOUT THE (FULL) DYNAMICS



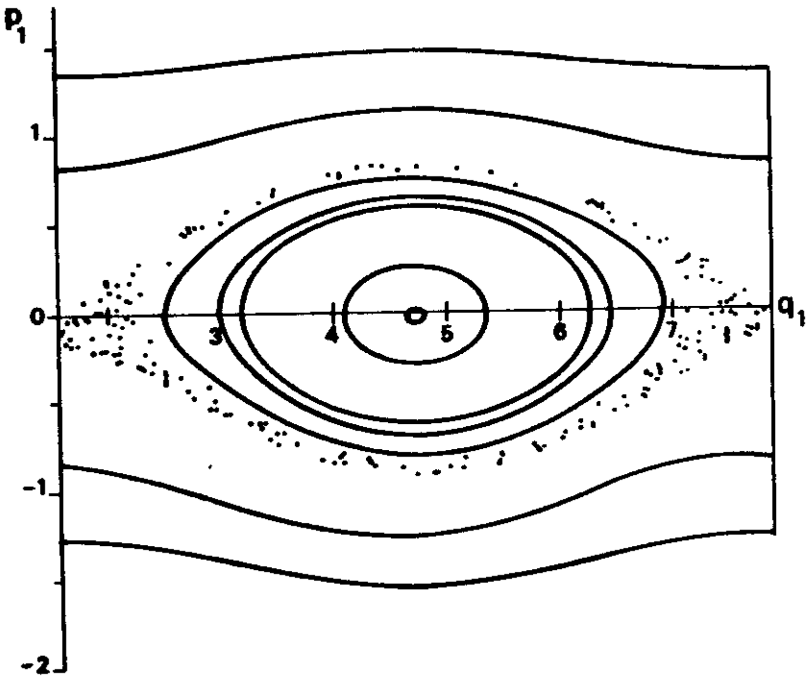
INTERSECTION POINTS ALONG DEFINITE LINES $\leftrightarrow$ ORDERED MOTION

RANDOM INTERSECTION POINTS $\leftrightarrow$ CHAOTIC MOTION

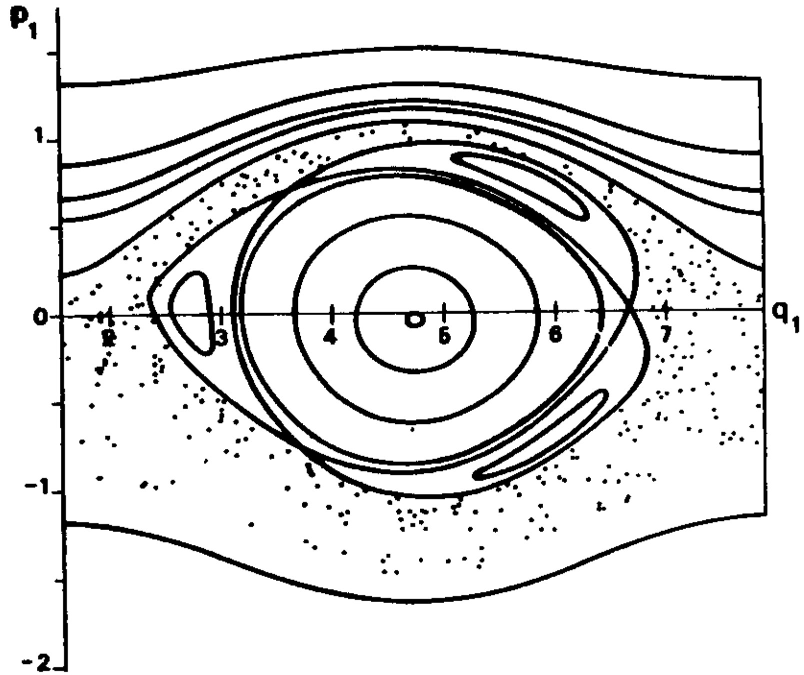


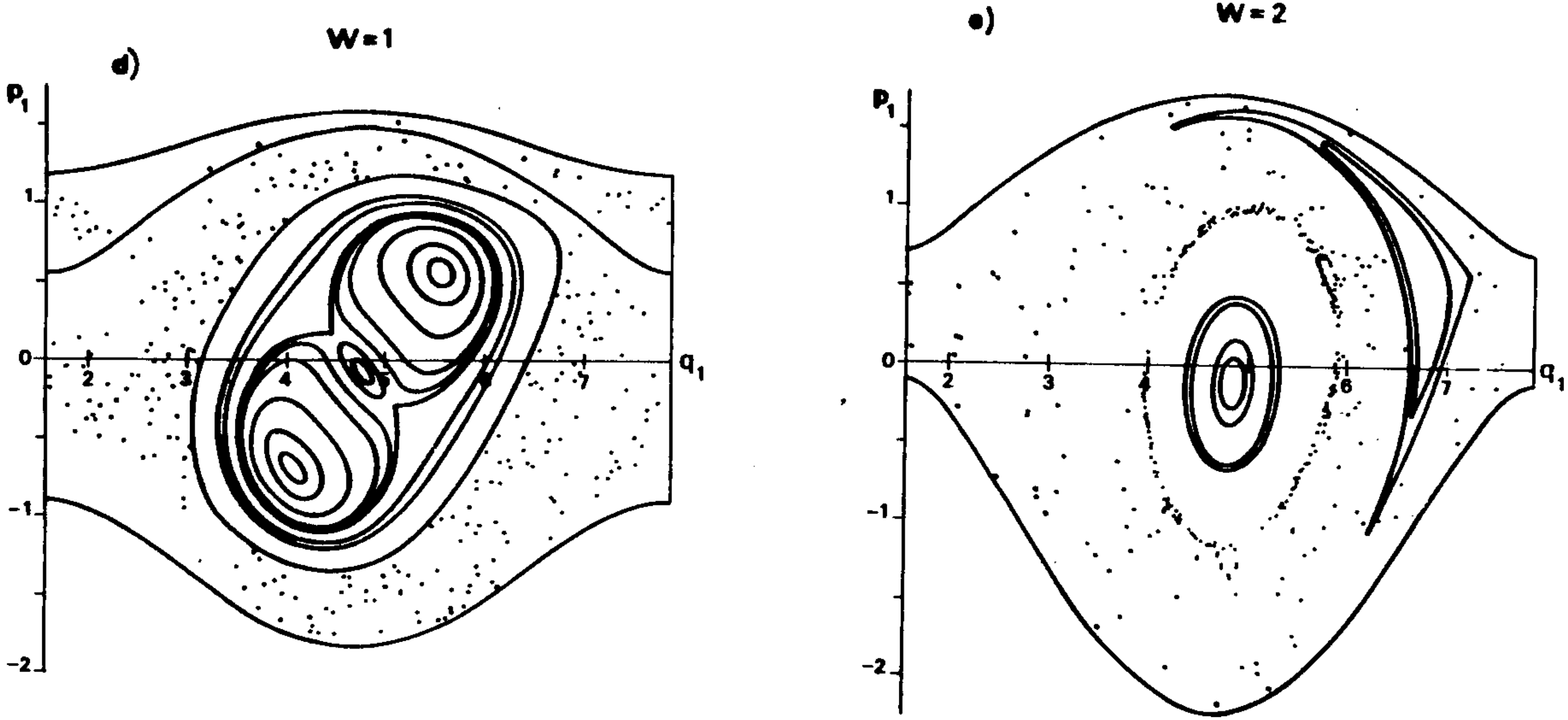


b) W = 0.3



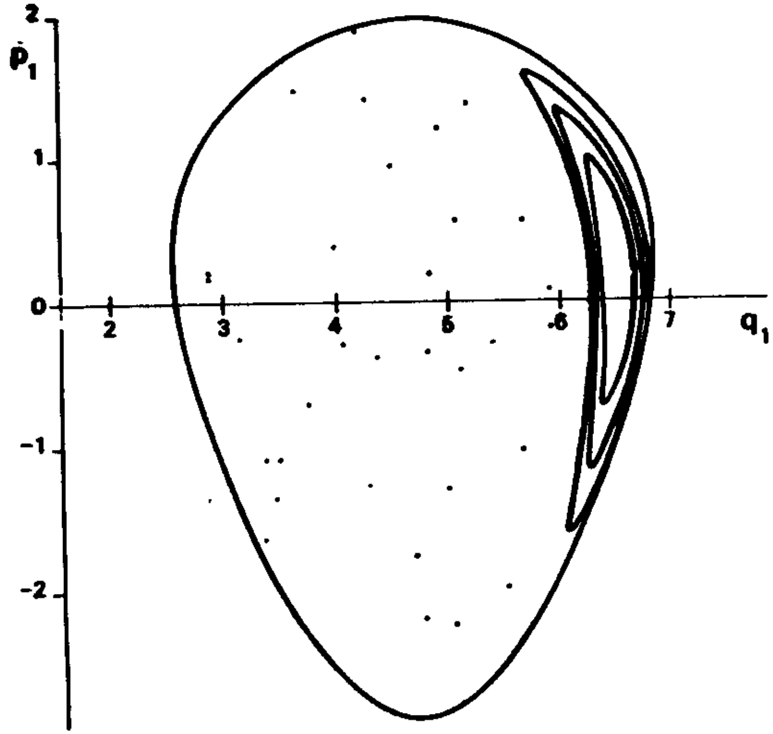
c) W = 0.5




d)
W = 1
p_1
1
0
-1
-2
2
3
4
5
6
7
q_1
e)
W = 2
p_1
1
0
-1
-2
2
3
4
5
6
7
q_1

W = 4

f)



18. EQUATIONS (I') ARE THE CLASSICAL LIMIT OF THE FOLLOWING QUANTUM EQUATIONS

$$\begin{cases} d\vartheta_i/d\tau = \bar{p}_i \\ d\bar{p}_i/d\tau = -w\,(a\,e^{i\vartheta_i} + \text{h.c.}) \\ da/d\tau = w\sum_j e^{-i\vartheta_j} + i\Delta a \end{cases} \qquad \begin{aligned} [\vartheta_i, \bar{p}_j] &= i\delta_{ij} \\ [a, a^+] &= 1 \end{aligned}$$

THESE EQUATIONS CAN BE DERIVED AS HEISENBERG EQUATIONS FROM THE N-ELECTRON ONE-MODE QUANTIZED HAMILTONIAN, IN A REFERENCE FRAME MOVING AT THE MEAN ELECTRON VELOCITY $\langle \bar{p} \rangle_0 / m$:

$$H = \sum_{i=1}^{N} \bar{p}_i^2/2 + i w \left(a^+ \sum_j e^{-i\vartheta_j} - \text{h.c.} \right) - \Delta a^+ a$$

ALL THE (DIMENSIONLESS) VARIABLES AND PARAMETERS ARE EXPRESSED IN TERMS OF THE LABORATORY FRAME:

$$\left. \begin{aligned} \vartheta_i &= (k+k_0)\,z_i - \left(\frac{\omega}{2\omega_0}q + \Delta\right)\tau \\ \bar{p}_i &= \frac{p_i - \langle p \rangle_0}{\hbar(k+k_0)} \end{aligned} \right\} \quad [\vartheta_i, \bar{p}_j] = i\delta_{ij}$$

$$a = \frac{i E_0\, e^{i(\phi_0 + \Delta\tau)}}{[(4\pi/V)\hbar\omega]^{1/2}} \qquad \begin{aligned} &[a, a^+] = 1 \\ &\hbar\omega \langle a^+ a \rangle \leftrightarrow |E_0|^2 V/4\pi \end{aligned}$$

$$\tau = \frac{2\omega_0}{q} t \quad ; \quad \Delta = \frac{\langle p \rangle_0 - p_R}{\hbar(k+k_0)}$$

$$q = \frac{\gamma_0 \omega_0 c}{\hbar(k+k_0)} \quad ; \quad w = \left(\frac{ec}{\omega_0}\right)^2 \frac{B_0}{2} \left(\frac{\pi}{V(\hbar\omega)^3}\right)^{1/2} \equiv \left(\frac{\rho q}{\sqrt{N}}\right)^{3/2}$$

CONSTANT OF MOTION:

$$\sum_i \bar{p}_i + a^+ a = \text{const.} \qquad \text{(TOTAL MOMENTUM)}$$

QUANTUM EQUATIONS $\Longleftrightarrow$ CLASSICAL EQUATIONS

$\langle \theta_i \rangle$	θ_i
$\langle p_i \rangle$	$q\,\eta_i$
$\langle a \rangle$	$\sqrt{q}\,A$
τ	$q^{-1}\tilde{\tau}$
Δ	$q\,\tilde{\Delta}$
w	$q^{3/2}\tilde{w}$

$$q = \frac{\gamma_0 m_0 c}{\hbar(k+k_0)} = \frac{\gamma_0 m_0 c^2}{\hbar(\omega+\omega_0)} \simeq \frac{\gamma_0 m_0 c^2}{\hbar\omega}$$

$$\frac{1}{q} = \frac{\hbar\omega}{\gamma_0 m_0 c^2} \simeq \left(\frac{\Delta E}{E}\right)_{RECOIL}$$

(ELECTRON RECOIL IN THE EMISSION OF A PHOTON)

THE QUANTIZED HAMILTONIAN CAN BE DERIVED FROM FIRST PRINCIPLES
[BONIFACIO-C., in Proceedings FEL Conference '84 – Appendix A]

I) $H = c\sqrt{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 + m_0^2c^2}$ $\qquad \vec{A}(z,t) = \vec{A}_0(z) + \vec{A}_L(z,t)$

"COLD-BEAM" APPROXIMATION : $\vec{p}_\perp = 0$; $\vec{A}\cdot\vec{p} = \vec{A}_\perp \cdot \vec{p}_\parallel = 0$

1 D-MODEL : $\begin{cases} \gamma \to \gamma_\parallel = \gamma/\sqrt{1+K^2} \\ m_0 \to m = m_0\sqrt{1+K^2} \end{cases}$ $\qquad \gamma_\parallel m = \gamma m_0$

II) LORENTZ TRANSFORM TO THE "BAMBINI-RENIERI" REFERENCE FRAME, MOVING AT THE RESONANCE VELOCITY V_R : POTENTIAL TIME-INDEPENDENT

$v_0 \simeq V_R \simeq c$: NON RELATIVISTIC APPROXIMATION

III) QUANTIZATION OF THE ELECTRON VARIABLES AND THE RADIATION FIELD VARIABLES (+ GENERALIZATION FROM 1 e^- TO $N \gg 1$ e^-)

IV) GALILEO TRANSFORM TO A SYSTEM MOVING AT THE MEAN INITIAL ELECTRON VELOCITY $\langle p \rangle_0/m$ $\quad$ ($\langle p \rangle_0 \neq p_R \to$ DETUNING $\neq 0$)

V) ALL QUANTITIES EXPRESSED IN TERMS OF THE LABORATORY REFERENCE FRAME (AND MADE DIMENSIONLESS)

BASIC EQUATIONS OF THE LINEAR QUANTUM REGIME

RECALL

$$\begin{cases} d\theta_i/d\tau = p_i \\ dp_i/d\tau = -w\left(a e^{i\theta_i} + h.c.\right) \\ da/d\tau = w\sum_j e^{-i\theta_j} + i\Delta a \end{cases} \qquad (i)$$

DEFINE ELECTRON COLLECTIVE OPERATORS

$$\begin{cases} \Theta = i\left(\frac{\bar{\rho}}{N_\lambda}\right)^{1/2} \sum_{j=1}^{N_\lambda} e^{-i\theta_j} \\ P = \frac{1}{(\bar{\rho} N_\lambda)^{1/2}} \sum_{j=1}^{N_\lambda} e^{-i\theta_j} p_j \end{cases}$$

$\bar{\rho} \equiv \rho q$ $\qquad \left[\rho = \left(\frac{K}{4}\frac{\Omega_p}{\omega_0}\right)^{2/3}, \; q = \frac{\gamma_0 m_0 c}{\hbar(k+k_0)}\right]$

TAKE $\left\{a, \; p_i, \; \sum_j e^{-i\theta_j}\right\}$ SMALL QUANTITIES

(AT $t=0$, $\langle a \rangle_0 = \langle p_i \rangle_q = \sum_j e^{-i\langle\theta_j\rangle_0} = 0$)

NEGLECT HIGHER-ORDER QUANTITIES $\quad a^\dagger \sum_j e^{-2i\theta_j}, \; \sum_j e^{-i\theta_j} p_j^2$

SYSTEM (i) $\longrightarrow$ LINEAR SYSTEM FOR COLLECTIVE OPERATORS

$$\begin{cases} d\Theta/d\bar{\tau} = P \\ dP/d\bar{\tau} = -a \\ da/d\bar{\tau} = -i\Theta + i\delta a \end{cases}$$

$$\left[\bar{\tau} = 2\omega_0 \rho t \; ; \quad \delta = \frac{1}{\rho}\frac{\gamma_0-\gamma_R}{\gamma_R}\right]$$

LOOK FOR SOLUTIONS $\quad \Theta(\bar\tau) = e^{i\lambda\bar\tau}\Theta_0 \quad \ldots$

$\rightarrow$ CUBIC CHARACTERISTIC EQUATION

$$\lambda^3 - \delta\lambda^2 + 1 = 0$$

$\rightarrow$ THREE ROOTS $\lambda_1, \lambda_2, \lambda_3$

e.g., ON RESONANCE $\delta = 0$

$\lambda_1 = -1 \; ; \lambda_{2,3} = \frac{1}{2}(1 \pm i\sqrt{3})$

$\rightarrow$

$$a(\bar\tau) = f_1(\bar\tau)\Theta_0 + i f_2(\bar\tau)P_0 - f_3(\bar\tau)a_0 \qquad f_{1,2}(0)=0\;,\; f_3(0)=-1$$

$$\Theta(\bar\tau) = h_1(\bar\tau)\Theta_0 + i h_2(\bar\tau)P_0 + h_3(\bar\tau)a_0 \qquad h_1(0)=1 \;\; h_{2,3}(0)=0$$

$$P(\bar\tau) = i k_1(\bar\tau)\Theta_0 + k_2(\bar\tau)P_0 + k_3(\bar\tau)a_0 \qquad k_{1,3}(0)=0,\; k_2(0)=1$$

$f_i\,;\, h_i\,;\, k_i$ FUNCTIONS OF $\lambda_1, \lambda_2, \lambda_3\,;\,\delta$

$\Downarrow$

GENERAL EXPRESSIONS FOR

MEAN PHOTON NUMBER : $\langle n\rangle(\bar\tau) = \langle a^+a\rangle(\bar\tau)$

ELECTRON BUNCHING :

$$\hat b = N^{-1}\sum_{j=1}^{N} e^{-i\theta_j} \simeq N_\lambda^{-1}\sum_{j=1}^{N_\lambda} e^{-i\theta_j}$$

$$\langle \hat b^+\hat b\rangle(\bar\tau) = \frac{1}{\rho N_\lambda}\langle \Theta^+\Theta\rangle(\bar\tau) \qquad \text{quantum analog of } |b|^2(\bar\tau)$$

$t=0$: (ELECTRON + FIELD) STATE = ELECTRON STATE $\otimes$ FIELD STATE

MEAN PHOTON NUMBER

$[a(\bar\tau) = f_1(\bar\tau)\Theta_0 + i f_2(\bar\tau)P_0 - f_3(\bar\tau)a_0 \quad ; \quad n = a^+a]$

$$\langle n\rangle(\bar\tau) = |f_1(\bar\tau)|^2\langle\Theta^+\Theta\rangle_0 + |f_2(\bar\tau)|^2\langle P^+P\rangle_0 + i\left[(f_1^* f_2)(\bar\tau)\langle\Theta^+P\rangle_0 - h.c.\right] + |f_3(\bar\tau)|^2\langle a^+a\rangle_0$$

$$\begin{cases} \langle n\rangle = \langle n\rangle_{sp} + \langle n\rangle_{st} \\ \langle n\rangle_{sp} = |f_1|^2\langle\Theta^+\Theta\rangle_0 + |f_2|^2\langle P^+P\rangle_0 + i\left(f_1^* f_2\langle\Theta^+P\rangle - h.c.\right) & \text{SPONTANEOUS} \\ \langle n\rangle_{st} \; |f_3|^2 n_0 & \text{STIMULATED} \end{cases}$$

$f_{1,2}(0)=0\;,\; f_3(0)=1 \;\Rightarrow\; \langle n\rangle(0) = \langle n\rangle_{sp}(0) + \langle n\rangle_{st}(0) = n_0$

IF $n_0 = 0$ THE SYSTEM STARTS RADIATING FROM NOISE

ELECTRON BUNCHING

$[\Theta(\bar\tau) = h_1(\bar\tau)\Theta_0 + i h_2(\bar\tau)P_0 + h_3(\bar\tau)a_0 \quad ; \quad \hat b^+\hat b = \frac{1}{\rho N_\lambda}\Theta^+\Theta]$

$$\langle\hat b^+\hat b\rangle(\bar\tau) = \frac{1}{\rho N_\lambda}\left\{|h_1(\bar\tau)|^2\langle\Theta^+\Theta\rangle_0 + |h_2(\bar\tau)|^2\langle P^+P\rangle_0 + i\left[(h_1^* h_2)(\bar\tau)\langle\Theta^+P\rangle_0 - h.c.\right] + |h_3(\bar\tau)|^2 n_0\right\}$$

STABILITY REGION : $\delta > \delta_T$

SMALL-SIGNAL GAIN RECOVERED

i) NEGLECT SPONTANEOUS CONTRIBUTION

$$a(\bar\tau) \simeq -f_3(\bar\tau)a_0 \quad , \quad \langle n\rangle(\bar\tau) \simeq |f_3(\bar\tau)|^2 n_0 \equiv \langle n\rangle_{st}$$

ii) TAKE THE LIMIT $\delta \gg 1$

$$\Rightarrow \quad \lambda_1 \simeq \delta\left(1-\frac{1}{\delta^3}\right) \quad ; \quad \lambda_{2,3} \simeq \pm\frac{1}{\sqrt{\delta}}\left(1 \pm \frac{1}{2\delta^{3/2}}\right)$$

iii) TAKE THE LIMIT $\bar\tau/\sqrt{\delta} \ll 1$

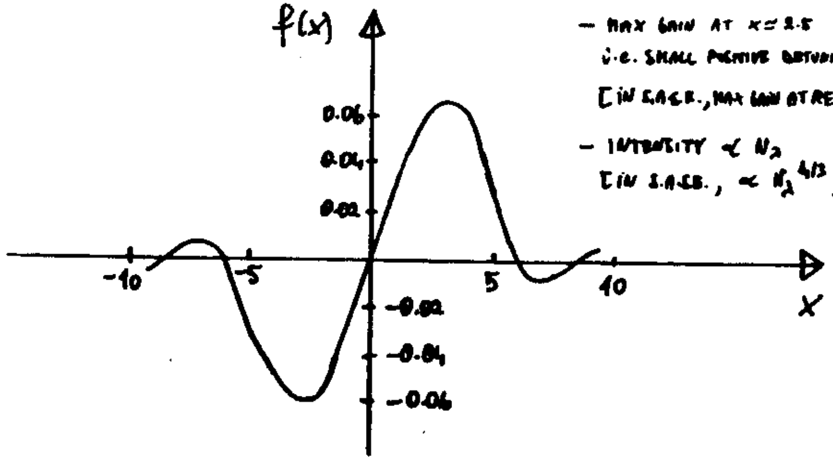
$$\Rightarrow \quad |f_3|^2(\bar\tau) \simeq 1 + \frac{4}{\delta^3}\left(1-\cos\delta\bar\tau - \frac{1}{2}\delta\bar\tau\sin\delta\bar\tau\right)$$

$$G_g = \frac{|a(\bar\tau)|^2-|a_0|^2}{|a_0|^2} = |f_3(\bar\tau)|^2-1 = \frac{4}{\delta^3}\left(1-\cos\delta\bar\tau-\frac{1}{2}\delta\bar\tau\sin\delta\bar\tau\right)$$

$\bar\tau = \bar\tau_i = 4\pi\rho N_0$

$x \equiv \delta\bar\tau_i = 4\pi N_0 \frac{\gamma_0-\gamma_R}{\gamma_R}$

$$f(x) = \frac{1}{x^3}\left(1-\cos x - \frac{1}{2}x\sin x\right)$$



– MAX GAIN AT $x \simeq 2.5$
i.e. SMALL POSITIVE DETUNING ($\gamma_0 > \gamma_R$)
[IN S.A.S.E., MAX GAIN AT RESONANCE]

– INTENSITY $\propto N_\lambda$
[IN S.A.S.E., $\propto N_\lambda^{4/3}$]

INSTABILITY REGION : $\delta < \delta_T$

SELF-AMPLIFIED SPONTANEOUS EMISSION

i) RESONANCE : $|\delta| \to 0$: $\begin{cases} \lambda^3+1=0 \\ \lambda_1 = -1 \ ; \ \lambda_{2,3} = \frac{1}{2}(1 \pm i\sqrt{3}) \end{cases}$

ii) "SUFFICIENTLY" LONG TIMES $\bar\tau \gg 1$: KEEP ONLY THE DIVERGENT CONTRIBUTIONS IN $f_i(\bar\tau), h_i(\bar\tau)$ $\left[\propto e^{i\lambda_3\bar\tau} = e^{(\sqrt{3}/2)\bar\tau}\right]$

$$|f_i(\bar\tau)|^2 \simeq |h_i(\bar\tau)|^2 \simeq \frac{1}{9}e^{\sqrt{3}\bar\tau} \qquad (i=1,2,3)$$

$$f_1^*(\bar\tau)f_2(\bar\tau) \simeq f_1(\bar\tau)f_2^*(\bar\tau) = -\frac{1}{18}e^{\sqrt{3}\bar\tau}$$

MEAN PHOTON NUMBER

$$\langle n\rangle(\bar\tau) = \frac{1}{9}\left[\langle\theta^+\theta\rangle_0 + \langle\beta^+\beta\rangle_0 - \frac{i}{2}\left(\langle\theta^+\beta\rangle_0 - \langle\beta^+\theta\rangle_0\right) + n_0\right]e^{\sqrt{3}\bar\tau}$$

ELECTRON BUNCHING

$$\langle\theta^+\theta\rangle(\bar\tau) = \langle n\rangle(\bar\tau)$$

$$\rightarrow \quad \langle\hat{b}^+\hat{b}\rangle(\bar\tau) = \frac{1}{N_\lambda\rho}\langle n\rangle(\bar\tau)$$

HOW TO EVALUATE $\langle\theta^+\theta\rangle_0, \langle\beta^+\beta\rangle_0, \langle\theta^+\beta\rangle_0, \langle\beta^+\theta\rangle_0$?

iii) WE ASSUME THAT AT TIME $t=0$ EACH ELECTRON IS DESCRIBED BY A MINIMUM UNCERTAINTY WAVE-PACKET

$$(\sigma_z)_i(0)(\sigma_p)_i(0) = \hbar/2$$

$$\left.\begin{aligned} \theta_i &= (k+k_0)z_i - \left(\frac{\omega}{2\omega_0}q+\Delta\right)t \\ \bar{p}_i &= \frac{p_i-\langle p\rangle_0}{\hbar(k+k_0)} \end{aligned}\right\} \Rightarrow (\sigma_\theta)_i(0)(\sigma_{\bar p})_i(0) \equiv \sigma_\theta\sigma_{\bar p} = \frac{1}{2}$$

$$\begin{cases} \langle \Theta^\dagger \Theta \rangle_0 = \bar\rho \left(1 - e^{-\sigma_\theta^2}\right) \\ \langle P^\dagger P \rangle_0 = \frac{1}{\bar\rho}\left(\sigma_{\bar p}^2 - \frac{1}{4} e^{-\sigma_\theta^2}\right) \\ \langle \Theta^\dagger P \rangle_0 = \frac{i}{2} e^{-\sigma_\theta^2}, \quad \langle P^\dagger \Theta \rangle_0 = -\frac{i}{2} e^{-\sigma_\theta^2}. \end{cases} \qquad \sigma_\theta \sigma_{\bar p} = 1/2$$

$$\langle \xi^\dagger \xi \rangle(\bar\tau) = \frac{1}{\bar\rho N_\lambda} \left\{ 1 - e^{-\sigma_\theta^2} + \frac{1}{\bar\rho^2}\left(\sigma_{\bar p}^2 - \frac{1}{4} e^{-\sigma_\theta^2}\right) + \frac{1}{\bar\rho}\left(\underline{\tfrac{1}{2} e^{-\sigma_\theta^2}} + n_0\right) \right\} e^{\sqrt{3}\bar\tau} = \frac{1}{N_\lambda \bar\rho} \langle n \rangle(\bar\tau)$$

SAME EXPRESSION BY TREATING $\theta_i, \bar p_i$ (in Θ, P) AS CLASSICAL STOCHASTIC VARIABLES WITH INDEPENDENT DISTRIBUTIONS WITH DISPERSIONS $\sigma_\theta, \sigma_{\bar p}$ NOT RELATED BY THE UNCERTAINTY RELATION $\sigma_\theta \sigma_{\bar p} = 1/2$, WITH THE EXCEPTION OF THE (PURELY QUANTUM-MECHANICAL) TERM $\frac{1}{2\bar\rho} e^{-\sigma_\theta^2}$

— o —

AT THE PEAK (IF OCCURS) OF EMITTED RADIATION

$$\langle \xi^\dagger \xi \rangle_p \simeq |b|_p^2 \simeq 1$$

$$\rightarrow \quad \langle n \rangle_p = \bar\rho N_\lambda \langle \xi^\dagger \xi \rangle_p \simeq \bar\rho N_\lambda \qquad \left[\bar\rho = \rho q = \rho \frac{\gamma_0 m_0 c}{\hbar(k+k_0)}\right]$$

$$\rightarrow \quad \bar\rho \simeq \frac{\langle n \rangle_p}{N_\lambda}$$ MAXIMUM NUMBER OF PHOTONS WHICH CAN BE RADIATED PER ELECTRONS

$$\langle \xi^\dagger \xi \rangle_p \simeq \frac{1}{\bar\rho N_\lambda} \langle n \rangle_p \simeq \frac{\hbar\omega \langle n \rangle_p}{\rho N_\lambda \gamma_0 m_0 c^2} \simeq \frac{|E_0|_p^2 V/4\pi}{\rho N_\lambda \gamma_0 m_0 c^2} \simeq 1$$

$\Rightarrow$

$\rho N_\lambda \gamma_0 m_0 c^2$ = MAXIMUM AMOUNT OF ENERGY WHICH CAN BE TRANSFERRED FROM THE ELECTRONS TO THE FIELD

$$\propto \rho N_\lambda \propto N_\lambda^{1/3} \cdot N_\lambda = N_\lambda^{4/3}$$

DISCUSSION IN TWO CASES: $\begin{cases} \text{A) "LOCALIZED-ELECTRON" LIMIT} \\ \text{B) "DELOCALIZED-ELECTRON" LIMIT} \end{cases}$

A) "LOCALIZED-ELECTRON LIMIT" ["HARMONIC-OSCILLATOR" LIMIT]

$$\sigma_\theta \ll 1 \quad \text{i.e.} \quad \sigma_z(0) \ll \frac{\lambda}{2\pi} \qquad \left[\theta_i = k z_i = \frac{2\pi}{\lambda} z_i\right]$$

AT $t=0$, UNBUNCHED BEAM WITH WELL LOCALIZED PARTICLES

$$\langle \xi^\dagger \xi \rangle(\bar\tau) \simeq \frac{1}{\bar\rho N_\lambda} \left\{ \sigma_\theta^2 + \frac{1}{\bar\rho^2}\sigma_{\bar p}^2 + \frac{1}{\bar\rho}\left(n_0 + \frac{1}{2}\right) \right\} e^{\sqrt{3}\bar\tau} = \frac{1}{N_\lambda \bar\rho} \langle n \rangle(\bar\tau)$$

IN THIS LIMIT

$$\sum_j e^{-i\theta_j} = \sum_j e^{-i(\langle\theta_j\rangle_0 + \delta\theta_j)} \qquad \{\langle \delta\theta_j^2 \rangle = \sigma_\theta^2\}$$

$$= \sum_j e^{-i\langle\theta_j\rangle_0}\left(\cancel{1} - i\delta\theta_j + \dots\right) \simeq -i \sum_j e^{-i\langle\theta_j\rangle_0} \delta\theta_j$$

HENCE

$$\Theta = i\sqrt{\frac{\bar\rho}{N_\lambda}} \sum_j e^{-i\theta_j} \longrightarrow \tilde\Theta = \sqrt{\frac{\bar\rho}{N_\lambda}} \sum_j e^{-i\langle\theta_j\rangle_0} \delta\theta_j$$

$$P = \frac{1}{\sqrt{\bar\rho N_\lambda}} \sum_j e^{-i\theta_j} \bar p_j \longrightarrow \tilde P = \frac{1}{\sqrt{\bar\rho N_\lambda}} \sum_j e^{-i\langle\theta_j\rangle_0} \bar p_j$$

COMMUTATION RULES

$$\begin{cases} [\tilde\Theta, \tilde P^\dagger] = i \\ [\tilde\Theta, \tilde P] = [\tilde\Theta, \tilde\Theta^\dagger] = [\tilde P, \tilde P^\dagger] = 0 \end{cases}$$

BY THE ANSATZ : $\sum_{j=1}^{N_0} e^{-im\langle\theta_j\rangle_0} = 0 \quad (m=1,2,\dots)$ ["classical" random distribution]

THE LINEAR EVOLUTION EQUATIONS FOR $\tilde\Theta, \tilde P, a$ ARE THE SAME AS FOR Θ, P, a :

$$\begin{cases} d\tilde\Theta/d\bar\tau = \tilde P \\ d\tilde P/d\bar\tau = -a \\ da/d\bar\tau = -i\tilde\Theta + i\delta a \end{cases} \qquad \left[\bar\tau = 2\omega_0 \rho t \ ; \ \delta = \frac{1}{\rho}\frac{\gamma_0 - \gamma_R}{\gamma_R} \right]$$

BUT NOW THESE EQUATIONS CAN BE DERIVED AS HEISENBERG EQUATIONS FROM HAMILTONIAN

$$\tilde H = \tilde P^+ \tilde P + a\tilde\Theta^+ + a^+\tilde\Theta - \delta a^+ a \qquad (*)$$

ONE CAN INTRODUCE TWO COLLECTIVE ELECTRON HARMONIC-OSCILLATOR OPERATORS

$$\begin{cases} a_1 = \frac{1}{\sqrt2}\left(\frac{\chi}{\sqrt\rho}\tilde\Theta + i\frac{\sqrt\rho}{\chi}\tilde P\right) \\ a_2 = \frac{1}{\sqrt2}\left(\frac{\chi}{\sqrt\rho}\tilde\Theta^+ + i\frac{\sqrt\rho}{\chi}\tilde P^+\right) \end{cases} \qquad \chi = (\sigma_{\bar p}/\sigma_\theta)^{1/2}$$

$$[a_1, a_1^+] = [a_2, a_2^+] = 1 \quad ; \quad [a_1, a_2] = 0$$

HAMILTONIAN (*) CAN BE WRITTEN

$$\tilde H = (\sigma_{\bar p}/\rho)^2 \left(a_2^+ a_2 + a_1^+ a_1 - a_1^+ a_2^+ - a_1 a_2\right) + \sqrt\rho\, \sigma_\theta \left[a^+(a_1 + a_2^+) + a(a_1^+ + a_2)\right] - \delta a^+ a$$

i.e., THE HAMILTONIAN OF THREE PARAMETRICALLY COUPLED HARMONIC OSCILLATORS a, a_1, a_2, INDEPENDENTLY OF THE VALUE OF N_λ.

CONSTANT OF MOTION : $a_1^+ a_1 - a_2^+ a_2 + a^+ a$

(COMPARE WITH : $\sum_i \bar p_i + a^+ a$ IN THE NONLINEARIZED CASE)

- THE INITIAL STATE OF THE ELECTRONS, WITH EACH ELECTRON IN A MINIMUM-UNCERTAINTY WAVE-PACKET, TURNS OUT TO BE THE VACUUM STATE FOR THE COLLECTIVE OPERATORS a_1, a_2 $[a_1|0\rangle = a_2|0\rangle = 0]$.
- IF THE RADIATION MODE IS INITIALLY IN A COHERENT STATE WITH AN AMPLITUDE $\delta\alpha_0$, THE "THREE-MODE" SYSTEM IS INITIALLY IN THE STATE

$$\rho_0 = |0, 0, \delta\alpha_0\rangle\langle\delta\alpha_0, 0, 0|$$

NORMAL-ORDERED CHARACTERISTIC FUNCTION

$$\chi_N(\xi, \xi^*, \tau) = \mathrm{Tr}\left\{\rho(\tau)\, e^{i\xi^* a^+}\, e^{i\xi a}\right\} = \mathrm{Tr}\left\{\rho(0)\, e^{i\xi^* a^+(\tau)}\, e^{i\xi a(\tau)}\right\}$$

$$a(\tau) = f_1(\tau)\Theta_0 + i f_2(\tau) P_0 - f_3(\tau) a_0 \quad ; \quad \Theta_0, P_0 \Rightarrow a_1, a_2 \ ; \Rightarrow$$

$$a(\tau) = g_1(\tau) a_1 + g_2(\tau) P_0 - f_3(\tau) a_0$$

$$\chi_N(\xi, \xi^*, \tau) = \langle 0\, 0\, \delta\alpha_0 | e^{i\xi^* g_1^* a_1^+} e^{i\xi^* g_2^* a_2} e^{-i\xi^* f_3^* a^+} e^{-i\xi f_3 a} e^{i\xi g_2 a_2^+} e^{i\xi g_1 a_1} |0\, 0\, \delta\alpha_0\rangle$$

i) $e^A e^B = e^B e^A e^{[A,B]} \qquad \left([A,[A,B]] = [B,[A,B]] = 0\right)$

$$e^{i\xi^* g_2^* a_2}\, e^{i\xi g_2 a_2^+} = e^{i\xi g_2 a_2^+}\, e^{i\xi^* g_2^* a_2}\, e^{-|\xi|^2 |g_2|^2}$$

ii) $f(a)|\alpha\rangle = f(\alpha)|\alpha\rangle \quad \rightarrow \quad a|\delta\alpha_0\rangle = \delta\alpha_0|\delta\alpha_0\rangle$

$$\chi_N(\xi, \xi^*, \tau) = e^{-i(\xi^* f_3^* \delta\alpha_0^* + \xi f_3 \delta\alpha_0)} e^{-|\xi|^2 |g_2|^2} \langle 0\, 0\, \delta\alpha_0 | e^{i\xi g_2 a_2^+} e^{i\xi^* g_2^* a_2} | 0\, 0\, \delta\alpha_0\rangle$$

$$\chi_N(\xi, \xi^*, \tau) = e^{-i(\xi^* f_3^*(\tau)\delta\alpha_0^* + \xi f_3(\tau)\delta\alpha_0)} e^{-|\xi|^2 |g_2(\tau)|^2}$$

$$|B_2|^2 = \bar{\xi}\,\sigma_\gamma^2 |f_1|^2 + \frac{\sigma_F^2}{\xi}|f_2|^2 - \frac{1}{2}\left(f_1^* f_2 + f_1 f_2^*\right) = \langle n\rangle_{sp}$$

$$\chi_N(\xi,\xi^*,\bar{\tau}) = e^{-i\left(\xi^* f_3^*(\bar\tau)\,\delta\alpha_0^* + \xi f_3(\bar\tau)\,\delta\alpha_0\right)}\, e^{-|\xi|^2\langle n\rangle_{sp}(\bar\tau)}$$

$$\equiv \chi_N^{coh}(\xi,\xi^*,\bar\tau)\cdot\chi_N^{incoh}(\xi,\xi^*,\bar\tau)$$

$\Rightarrow$ GLAUBER QUASIPROBABILITY DISTRIBUTION FUNCTION $\left\{\begin{matrix} \rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha \\ P(\alpha) = \mathcal{F}(\chi_N(\xi)) \end{matrix}\right\}$

i) $\chi_N^{coh}(\xi,\xi^*) = e^{-i\left(\xi^* f_3^*\delta\alpha_0^* + \xi f_3\delta\alpha_0\right)}$

$$P^{coh}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \chi_N^{coh}(\xi,\xi^*)\, e^{i(\xi\alpha + \xi^*\alpha^*)}\, d^2\xi = \underbrace{\delta^{(2)}(\alpha - f_3\delta\alpha_0)}_{\text{COHERENT STATE}}$$

ii) $\chi_N^{incoh}(\xi,\xi^*) = e^{-|\xi|^2\langle n\rangle_{sp}}$

$$P^{incoh}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \chi_N^{incoh}(\xi,\xi^*)\, e^{i(\xi\alpha+\xi^*\alpha^*)}\, d^2\xi = \underbrace{\frac{1}{\pi\langle n\rangle_{sp}}\, e^{-|\alpha|^2/\langle n\rangle}}_{\text{INCOHERENT STATE}}$$

iii) CONVOLUTION

$$P(\alpha) = \int P^{coh}(\beta)\, P^{incoh}(\alpha-\beta)\, d^2\beta$$
$$= \int \delta^{(2)}(\beta - f_3\delta\alpha_0)\frac{1}{\pi\langle n\rangle_{sp}}\, e^{-|\alpha-\beta|^2/\langle n\rangle_{sp}}\, d^2\beta$$

$$\Rightarrow P(\alpha,\alpha^*,\bar\tau) = \frac{1}{\pi\langle n\rangle_{sp}(\bar\tau)}\exp\left\{-\frac{|\alpha-\alpha(\bar\tau)|^2}{\langle n\rangle_{sp}(\bar\tau)}\right\}$$

$$\alpha(\bar\tau) = f_3(\bar\tau)\,\delta\alpha_0\,, \qquad \langle n\rangle_{st} = |\alpha(\bar\tau)|^2$$

$|\delta|\to 0\ ;\ \bar\tau \gg 1$:

$$\begin{cases} \langle n\rangle_{sp} = \frac{1}{9}\left\{\bar\xi\sigma_\gamma^2 + \frac{1}{\xi}\sigma_F^2 + \frac{1}{2}\right\} e^{\sqrt{3}\,\bar\tau} \\ \langle n\rangle_{st} = |\alpha(\bar\tau)|^2 = \frac{1}{9}|\delta\alpha_0|^2 e^{\sqrt{3}\,\bar\tau} \\ \langle n\rangle = \langle n\rangle_{sp} + \langle n\rangle_{st} \end{cases}$$

$$P(\alpha,\bar\tau) = \frac{1}{\pi\langle n\rangle_{sp}}\, e^{-\frac{|\alpha-\alpha(\bar\tau)|^2}{\langle n\rangle_{sp}}}$$

MEAN PHOTON NUMBER

$$\begin{cases} \langle n\rangle = \langle n\rangle_{sp} + \langle n\rangle_{st} \\ \langle n\rangle_{sp} \simeq \frac{1}{9}\left\{\bar\xi\sigma_\gamma^2 + \frac{1}{\xi}\sigma_F^2 + \frac{1}{2}\right\} e^{\sqrt{3}\,\bar\tau} \\ \langle n\rangle_{st} \simeq |\alpha(\bar\tau)|^2 = \frac{1}{9}|\delta\alpha_0|^2 e^{\sqrt{3}\,\bar\tau} \end{cases}$$

DISPLACED GAUSSIAN : SUPERPOSITION OF A CHAOTIC (GAUSSIAN) FIELD WITH A COHERENT FIELD

- IF $\delta\alpha_0 = 0$ (NO INITIAL FIELD EXCITATION)

  CHAOTIC FIELD [STABILITY REGION : Becker–Mc Iver, Phys. Rev. A 27, 1030 (1983)]

- ONLY IF $\langle n\rangle_{sp}$ NEGLIGIBLE $\rightarrow$ COHERENT FIELD

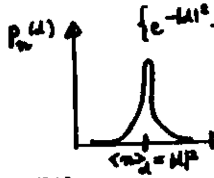
  $\Rightarrow$ ONLY FIRST-ORDER COHERENCE [SINGLE-MODE]

PHOTON NUMBER VARIANCE

$$\sigma^2(n) \equiv \langle n^2\rangle - \langle n\rangle^2 = \underbrace{\langle n\rangle_{st}}_{\text{COH.}} + \underbrace{\langle n\rangle_{sp}(\langle n\rangle_{sp}+1)}_{\text{INCOH.}} + \underbrace{2\langle n\rangle_{st}\langle n\rangle_{sp}}_{\text{MIXED}}$$

i) $\langle n\rangle_{st} \gg 1$, $\langle n\rangle_{sp} \ll 1$

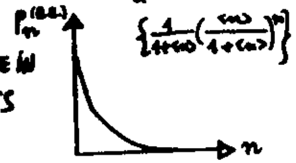
$\sigma^2(n) \simeq \langle n\rangle_{st}$ POISSON STATISTICS



$\left\{e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\right\}$

ii) $\langle n\rangle_{sp} \gg \langle n\rangle_{st}$

$\sigma^2(n) \simeq \langle n\rangle_{sp}(\langle n\rangle_{sp}+1)$ BOSE-EINSTEIN STATISTICS



$\left\{\frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n\right\}$

iii) $\langle n\rangle_{st} \gg \langle n\rangle_{sp} \gg 1$

$\sigma^2(n) \simeq 2\langle n\rangle_{st}\langle n\rangle_{sp}$ INTERMEDIATE STATISTICS

NO NONCLASSICAL EFFECTS IN THE STATISTICAL PROPERTIES OF RADIATION (PHOTON ANTIBUNCHING ; SQUEEZING)

- MANY-PARTICLE EFFECTS FULLY TAKEN INTO ACCOUNT
- ELECTRON VARIABLES AND FIELD VARIABLES TREATED SEPARATELY

DELAY TIME AND INSTABILITY THRESHOLD

$$\left\{ |\delta| \to 0 \,;\ \bar\tau \gg 1 \,;\ \oplus\ \sigma_\theta \ll 1 \right\}$$

$$\langle \hat b^\dagger \hat b\rangle(\bar\tau) = \frac{1}{N_\lambda \bar\rho} \langle n\rangle(\bar\tau) \simeq \frac{1}{9 N_\lambda \bar\rho}\left(\bar\rho\,\sigma_\theta^2 + \frac{\sigma_p^2}{\bar\rho} + \frac{1}{2} + n_0\right) e^{\sqrt{3}\,\bar\tau}$$

$$\langle \hat b^\dagger \hat b\rangle(\bar\tau_D) \simeq 1$$

$\to$

$$\bar\tau_D = \frac{1}{\sqrt{3}} \ln N_c$$

$$N_c = \frac{9 N_\lambda \bar\rho}{\bar\rho\,\sigma_\theta^2 + \frac{\sigma_p^2}{\bar\rho} + \frac{1}{2} + n_0}$$

ELECTRON CO... NUMBER

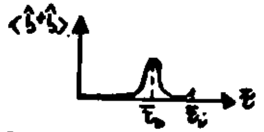
N_c MAXIMUM, WHEN $\sigma_\theta^2 = 1/2$: $\quad (N_c)_{MAX} = \dfrac{9 N_\lambda \bar\rho}{n_0 + 3/2}$

FOR EVOLUTION FROM NOISE $\quad (n_0 = 0)$

$$(\bar\tau_D)_{MAX} = \frac{1}{\sqrt{3}} \ln\left(6 N_\lambda \bar\rho\right)$$

ON IMPOSING

$$(\bar\tau_D)_{MAX} \lesssim \bar\tau_i$$



$\Rightarrow$ INSTABILITY THRESHOLD FOR S.A.S.E. STARTING FROM QUANTUM FLUCTUATIONS

$$\frac{1}{\sqrt{3}} \ln\left(6 N_\lambda \bar\rho\right) \lesssim 4\pi\rho N_0$$

Classical

$$\frac{1}{\sqrt{3}} \ln N_\lambda \lesssim 4\pi\rho N_0$$

B) "DELOCALIZED-ELECTRON" LIMIT ("CLASSICAL" LIMIT)

RECALL:

$$\langle \hat b^\dagger \hat b\rangle(\bar\tau) = \frac{1}{N_\lambda \bar\rho} \langle n\rangle(\bar\tau)$$

$$= \frac{1}{9 N_\lambda}\left\{1 - e^{-\sigma_\theta^2} + \frac{1}{\bar\rho^2}\left(\sigma_p^2 - \frac{1}{4} e^{-\sigma_\theta^2}\right) + \frac{1}{\bar\rho}\left(\frac{1}{2} e^{-\sigma_\theta^2} + n_0\right)\right\} e^{\sqrt{3}\,\bar\tau}$$

TAKE:

$$\bar\rho \gg 1 \qquad \text{i.e.,} \quad \frac{\rho\,\gamma_0 m_0 c^2}{\hbar\omega} \gg 1$$

IF THE PARTICLES ARE INITIALLY DELOCALIZED IN THE UNBUNCHED BEAM WITH LARGE POSITION FLUCTUATIONS $\left(\sigma_\theta \gtrsim 1\,,\ \exp(-\sigma_\theta^2) \simeq 0\right)$

$$\langle \hat b^\dagger \hat b\rangle(\bar\tau) \simeq \frac{1}{9 N_\lambda}\left(1 + \frac{n_0}{\bar\rho}\right) e^{\sqrt{3}\,\bar\tau}$$

- THE PURELY QUANTUM TERM $1/2$
- AND THE VARIANCES $\sigma_\theta^2,\ \sigma_p^2$

$\Big\}$ DISAPPEAR FROM $\langle \hat b^\dagger \hat b\rangle$ (or from $\langle n\rangle$)

"DELOCALIZED-PARTICLE" LIMIT $\approx$ "CLASSICAL" LIMIT

NOTE THAT IN THE CLASSICAL APPROXIMATION, $\langle \hat b^\dagger \hat b\rangle \simeq |b|^2$

AND FOR NEGLIGIBLE STIMULATED CONTRIBUTION, $\quad n_0/\bar\rho \ll 1$

$\to$

$$b_0 = \frac{1}{3}\,\frac{1}{\sqrt{N_\lambda}}$$

i.e., THE CLASSICAL "SHOT-NOISE" EVALUATION $\quad b_0 = 1/\sqrt{N_\lambda}$

TO WITHIN A FACTOR $1/3$ [due to the extrapolation backwards in time of the asymptotic results]

DELAY TIME

ON IMPOSING AGAIN

$$\langle \hat{b}^+ \hat{b} \rangle (\tau_D) \simeq 1$$

BUT NOW WITH

$$\langle \hat{b}^+ \hat{b} \rangle (\tau) \simeq \frac{1}{9 N_\lambda} \left(1 + \frac{n_0}{\bar{\rho}}\right) e^{\sqrt{3}\,\tau}$$

$\Longrightarrow$

$$\tau_D = \frac{1}{\sqrt{3}} \ln\left(\frac{9 N_\lambda \bar{\rho}}{\bar{\rho} + n_0}\right)$$

STARTING FROM NOISE $(n_0 = 0)$

$$(\tau_D)_{max} = \frac{1}{\sqrt{3}} \ln 9 N_\lambda \simeq \frac{1}{\sqrt{3}} N_\lambda \; +1$$

i.e., THE CLASSICAL FIT-FORMULA DERIVED QUANTUM-MECHANICALLY

GENERALIZATION OF THE CRITERION FOR THE NEGLECT OF QUANTUM EFFECTS

ON IMPOSING THAT

i) THE SPREAD OF A (FREE-ELECTRON) WAVE-PACKET IS NEGLIGIBLE WITH RESPECT TO A RADIATION WAVELENGTH : $(\Delta z)_t << \lambda$

OR THAT

ii) THE ELECTRON RECOIL IN THE EMISSION OF A PHOTON IS NEGLIGIBLE WITH RESPECT TO HOMOGENEOUS BROADENING : $\varepsilon \propto \left(\frac{\Delta E}{E}\right)_{RECOIL} << \left(\frac{\Delta E}{E}\right)_{HOMOG.}$

$\Longrightarrow$

$$\frac{1}{\varepsilon} = \frac{1}{(4\pi)^2} \frac{\gamma_R^3 \lambda^2}{\lambda_c L_0} >> 1 \qquad \left(\begin{array}{c} \lambda_c = \hbar/m_0 c \\ \text{REDUCED COMPTON WAVELENGTH} \end{array}\right)$$

{ RENIERI ; BENSON & MADEY ; COLSON ; GOVER ; MOORE ... ; DATTOLI ... ;
see Proceedings BROOKHAVEN '84 AND CASTELGANDOLFO '84 }

THIS "FREE-PARTICLE", "SMALL-RECOIL" CRITERION CAN BE WRITTEN

$$\frac{1}{\varepsilon} = \frac{9}{4\pi N_0} = \frac{\bar{\rho}}{G} >> 1 \qquad \left\{\begin{array}{l} G = 4\pi\rho N_0 \text{ (TOTAL) GAIN} \\ \bar{\rho} = \rho 9 \end{array}\right\}$$

HENCE THE LIMIT $\bar{\rho} >> 1$ MEANS

$$\frac{G}{\varepsilon} >> 1$$

- IF $G \lesssim 1$ (LOW-GAIN) $\quad \frac{G}{\varepsilon} >> 1 \Rightarrow \frac{1}{\varepsilon} >> 1$ (MORE SEVERE)
- IF $G >> 1$ (HIGH-GAIN) $\quad \frac{1}{\varepsilon} >> 1 \Rightarrow \frac{G}{\varepsilon} >> 1$ (LESS SEVERE)

FURTHER SUPPORT TO THE "CLASSICAL" LIMIT $\bar{\rho} \gg 1$

RECALL : $\bar{\rho}$ = MAXIMUM NUMBER OF PHOTONS WHICH CAN BE RADIATED PER ELECTRON

- IF $\bar{\rho} \ll 1$, THE ELECTRONS WILL RADIATE PRACTICALLY IN UNISON(?)
  $\rightarrow$ QUANTUM EFFECTS ARE EXPECTED TO BE RELEVANT
- IF $\bar{\rho} \gg 1$, THE SYSTEM WILL BEHAVE CLASSICALLY

— ♦ —

RECALL (GENERAL EXPRESSION)

$$\langle b^{+} b \rangle (\bar{\tau}) = \frac{1}{\bar{\rho} N_\lambda} \left\{ 1 - e^{-\sigma_\theta^2} + \underbrace{\frac{1}{\bar{\rho}^2}\left(\sigma_p^2 - \frac{1}{4} e^{-\sigma_\theta^2}\right)}_{\text{CONTRIBUTION FROM MOMENTUM FLUCTUATIONS}} + \frac{1}{\bar{\rho}}\left(\frac{1}{2} e^{-\sigma_\theta^2} + \mathrm{mo}\right) \right\} e^{\sqrt{3}\,\bar{\tau}}$$

NEGLIGIBLE WHEN

$$\frac{\bar{\rho}}{\sigma_p} \sim \bar{\rho}\,\sigma_\theta \equiv \frac{\rho}{\varepsilon}\sigma_\theta \gg 1 \quad , \text{ i.e. } \quad \frac{\varepsilon}{\rho\,\sigma_\theta} \ll 1 \quad (*)$$

HENCE IF $\rho\,\sigma_\theta \lesssim \varepsilon$ [e.g., $\sigma_\theta \ll 1$]

IT IS POSSIBLE TO BE IN THE LIMIT $\varepsilon \ll 1$ ("free-particle, small recoil")

BUT TO VIOLATE (*) ,

THAT IS, QUANTUM MOMENTUM FLUCTUATIONS CAN BE RELEVANT.