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THE SUPERRADIANT REGIME OF A FREE-ELECTRON LASER

R. BONIFACIO & F. CASAGRANDE
Dipartimento di Fisica dell'Università
Via Celoria, 16
20133 Milano
Italy

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R.Bonifacio and F.Casagrande

Dipartimento di Fisica dell'Università
Via Celoria 16, 20133 Milano, Italy.

In this paper we demonstrate that if a single-pass free-electron laser is operated with enough gain and the electron bunches are shorter than a suitable "cooperation length", the radiated power will be proportional to γ^2 and to the square of the number of electrons in the bunch.

1. Introduction

Since the first operation of a free-electron laser (FEL) in the mid seventies by Madey and coworkers[1], the tremendous potentiality of this device has made it the object of very intense theoretical and experimental investigations. In particular, many proposals aimed at raising the gain and improving the efficiency of the FEL process have been put forward. To quote an example, Kroll, Morton and Rosenbluth[2] suggested the use of tapered wigglers designed so to keep the energy of the electrons close to the resonance condition in spite of its decrease which corresponds with gain ; indeed, a tapered-wiggler FEL oscillator has been recently operated in the TRW-Stanford experiment[3].
Nowadays several FEL's have been operated both in the United States and in Europe, while other devices are expected to operate very soon[4]. Most of them work, or are planned to, in the infrared or in the visible. Furthermore, the possibility to extend FEL operation to the XUV and soft X-ray region has been taken into consideration more and more seriously[5]. In this case a high-gain is absolutely necessary to achieve oscillation, due to the high mirror losses at very short wavelengths, at least at the present status of technology. However, high-gain operation appears to be possible, even at such short wavelengths, for Compton FEL's operating in a single-pass configuration, thus circumventing the difficulty of the poor efficiency of available mirrors. This promising perspective, already under experimental investigation, is based on the cooperative instability of the travelling-wave-tube type[6] for an FEL process developing from noise. This is the <u>amplified spontaneous emission</u> regime, whose feasibility in the region below 1000 Å has been the object of detailed analysis by Pellegrini and Murphy[7].
On the basis of this same instability for a single-pass device, in this paper we develop the basic ideas concerning FEL operation in a quite different regime, the <u>superradiant</u> regime, in which the radiated power turns out to be proportional to the squared number of particles and to the square of the relativistic

kinematic factor γ^2. The conditions for the occurrence of this effect turn out to be that a suitably defined superradiant gain must be i) greater than one on a wiggler length, ii) smaller than one on a slippage distance. Only a short presentation of this topics, even if containing both a classical and a quantum treatment, has been previously included in ref.8.
We start in Section 2 with the introduction of the lossless FEL model, including the effects of detuning and space-charge, which is the proper framework for the description of the amplified spontaneous emission regime. Next, to develop the classical theory of FEL superradiance, in section 3 we generalize the model to include dissipative effects and define a "radiation suppression" condition or short-bunch limit in which these effects are dominant, contrary to the long-bunch limit considered in amplified spontaneous emission, where they are neglibible. In the superradiant regime the field variables can be adiabatically eliminated ; this elimination is performed in section 4 and leads to the derivation of the expression for the radiated power, which exhibits the cooperative superradiant nature of the process. In Section 5 the conditions under which the phenomenon should be observed are stated in terms of the relevant physical parameters and also elucidated in terms of a properly defined superradiant gain. Numerical results are discussed in section 6, whereas section 7 contains some concluding remarks.

2. The conservative model and the amplified spontaneous emission regime

The FEL process takes place inside a helical wiggler of length L_w where a beam of relativistic electrons interacts with a radiation field (laser field) $\vec{E}_L$ and a transverse spatially periodic magnetostatic field (wiggler field) $\vec{B}_w$

$$\begin{aligned} \vec{E}_L(z,t) &= E_0(z,t)\left[\sin(k_L z-\omega_L t+\phi_0)\hat{x}+\cos(k_L z-\omega_L t+\phi_0)\hat{y}\right] \\ \vec{B}_w(z) &= B_w\left[-\sin(k_w z)\hat{x}+\cos(k_w z)\hat{y}\right], \end{aligned} \qquad (1)$$

where $\omega_L = ck_L$; $\omega_w = ck_w = 2\pi c N_w/L_w$, N_w being the number of wiggler periods ; $E_0(z,t)$ is a slowly-varying amplitude, whereas B_w is taken real and constant since the wiggler field $\vec{B}_w$ plays the role of a pump field.
Under the assumptions that all electrons are injected in the wiggler parallel to the magnetic field axis (cold-beam approximation) and with the same energy $\gamma_0 m_0 c^2$, and that during the interaction the electron energies do not vary appreciably, namely

$$(\gamma_i-\gamma_0)/\gamma_0 \ll 1 \qquad (i=1,\dots,N) \qquad (2)$$
$$\gamma_0 = \left(1-(\beta_\parallel)_0^2\right)^{-1/2},$$

the time evolution of the system is ruled by the following set of classical, relativistic differential equations of the travelling-wave-tube type in dimensionless form [6-9] :

$$\begin{cases} d\theta_i/d\tau = \eta_i & (i=1,\dots,N) \quad (3a) \\ d\eta_i/d\tau = -\left(a+i\bar{\sigma}b\right)\exp(i\theta_i)+c.c. & (i=1,\dots,N) \quad (3b) \\ da/d\tau = b+i\delta a & (3c) \end{cases}$$

In eqs. (3) the electron variables are

$$\begin{aligned} \theta_i &= (k_L+k_w)z_i-(\omega_L+2\tilde{\omega}_w\rho\delta)t \\ \eta_i &= (1/\rho)\,(\gamma_i-\gamma_0)/\gamma_0 \end{aligned} \qquad (4)$$

whereas a is a complex field amplitude

$$a = \frac{i E_0 \exp\left[i(\phi_0+2\tilde{\omega}_w\rho\delta)t\right]}{\left[4\pi(N/V)\rho\,\gamma_0 m_0 c^2\right]^{1/2}} \qquad (5)$$

defined so that $\rho|a|^2$ is the ratio of field energy to electron beam energy, where

$$\rho = \left(\frac{\mathbb{K}}{4}\frac{\Omega_p}{\tilde{\omega}_w}\right)^{2/3}$$
$$\mathbb{K} = eB_w/m_o c\,\omega_w$$
$$\Omega_p = \left(4\pi e^2 N/V m_o \gamma_o^{\,3}\right)^{1/2} \qquad (6)$$
$$\tilde{\omega}_w = (\gamma_R^{\,2}/\gamma_o^{\,2})\,\omega_w$$
$$\delta = \frac{1}{\rho}\,\frac{\gamma_o-\gamma_R}{\gamma_R} \quad ;$$

ρ is the basic strength parameter (or Pierce parameter), $\mathbb{K}$ the wiggler parameter, Ω_p the relativistic plasma frequency, δ the detuning parameter, and the electron resonant energy $\gamma_R m_o c^2$ is such that

$$\omega_L = 2\omega_w\gamma_R^{\,2}/(1+\mathbb{K}^2) \qquad (7)$$

A relevant collective dynamical parameter in eqs.(3) is the electron bunching parameter

$$b = \langle \exp(-i\theta)\rangle = N^{-1}\sum_{j=1}^{N}\exp(-i\theta_j) \qquad (8)$$

defined in the range $0 \le |b| \le 1$, where $|b| = 1$ (0) corresponds with complete bunching (unbunching) of the particles. Space-charge effects are included in eq. (3b) via the parameter

$$\bar{\sigma} = 4\rho\,(1+\mathbb{K}^2)/\mathbb{K}^2. \qquad (9)$$

Furthermore, the time is scaled as

$$\tau = 2\tilde{\omega}_w\,\rho\, t. \qquad (10)$$

From eqs. (3a-c) one easily obtains the conservation law

$$\langle\eta\rangle + |a|^2 = \text{const.} \qquad (11)$$

i.e., total energy conservation

$$m_o c^2\sum_{i=1}^{N}(\gamma_i-\gamma_o) + \left(|E_o|^2/4\pi\right)V = \text{const.} \qquad (11')$$

Apparently, space-charge does not contribute to the global energy balance (11') ; this is due to the very nature of space-charge forces, which are internal forces for the many-electron system. Eq. (11') shows that in this conservative model electrons and field exchange energy ; due to this process, under suitable conditions the electrons transfer a fraction of their energy to the field which experiences gain. If the field does not vary appreciably during the interaction, one finds the well-known small signal gain[10]. In the small signal regime the electrons radiate almost independently and do not bunch appreciably. Hence the small signal gain is proportional to the number N of radiating electrons.
The possibility of a substantially higher efficiency in the FEL process is due to the existence of an instability for eqs. (3).
Let us consider the following initial condition :

$$\eta_i(0) = 0 \;,\quad a(0) = 0 \;,\quad b(0) = 0, \qquad (12)$$

namely, at the start of the process the monokinetic electrons are distributed in space so that the bunching parameter is vanishing and no field excitation is present. In this situation, for suitable values of the detuning parameter δ and space-charge parameter $\bar{\sigma}$ [6,9] the system is unstable and starts radiating with an intensity exponentially growing in time. This is the regime of amplified spontaneous emission (ASE). After this initial stage of exponential gain, saturation effects become dominant so that condition (2) can be violated and the analysis requires more general equations than eqs. (3).
An FEL operating in the ASE regime is a single-pass device in which the time evolution lasts for a time interval equal to the transit time in the wiggler L_w/c, that is from $\tau = 0$ to the scaled interaction time

$$\tau_i = 4\pi\,\frac{\gamma_R^{\,2}}{\gamma_o^{\,2}}\,\rho\, N_w. \qquad (13)$$

As discussed in the introduction, since this regime does not require mirrors, it appears to be a natural candidate for operation at very short wavelengths.

Because the scaled field intensity $|a|^2$ reaches peak values $|a|_p^2$ on the order of unity, from eqs. (5),(6) it follows at once that the radiated peak intensity is proportional to ρN, that is scales as $N^{4/3}$; due to electron cooperation in the process, one finds a higher dependence on electron density than the linear one in the small signal regime.

Operation in this regime requires a high-quality electron beam, like that provided by storage rings. Both energy spread and emittance are controlled by the strength parameter ρ ; typical values of ρ required for operation in the XUV region are on the order of 10^{-3} [6,9]. Furthermore, the overall efficiency of the process is ruled by ρ as well ; in fact , from eqs. (11) and (6) one sees immediately that the fraction of beam energy transferred to the field is proportional to ρ :

$$\frac{|a_0|^2-|a|^2}{|a_0|^2} = \frac{\langle \eta \rangle}{|a_0|^2} \propto \rho . \qquad (14)$$

3. The dissipative model

In this section we introduce another scheme for high-gain operation in a single-pass Compton-FEL, which is based on the same instability of ASE, but where the emitted radiation can reach an even higher power level, namely it is proportional to the squared number of electrons N^2. We call it superradiant (SR) regime.

In simple intuitive terms, the limitation on the (very high) power level in the case of ASE can be explained as follows : because the particles travel at a velocity very close to that of light, the energy exchange between particles and photons can occur during the whole interaction time ; hence, a photon emitted by an electron can be reabsorbed by another one before it may leave the wiggler. The idea underlying our proposal is that this reabsorption should be greatly reduced if the photons could escape as fast as possible from the electron beam. This "radiation suppression" [11] mechanism should allow to reach a very high saturation power, since saturation effects should be counteracted

here by quickly "extracting" the photons. In order that this extraction can occur, first of all we need a short-pulse regime for the injected electron beam. Second, we have to generalize the model equations (3a,b,c) to describe the slippage effect, i.e. the escape of photons from an electron bunch of length L_e . The simplest way to do it is by introducing a loss term in the field eq. (3c) ; thus, we replace eq. (3c) by the more general equation

$$da/d\tau = b + (i\delta - \bar{k})a \qquad (3c')$$

containing a linear damping term $-\bar{k}a$, where the scaled field amplitude damping constant is

$$\bar{k} = \frac{k}{2\tilde{\omega}_w \rho} = \frac{1}{2\tilde{\omega}_w \rho}\,\frac{c\,(1-(\beta_\parallel)_0)}{L_e} \simeq \frac{\lambda_L}{4\pi\rho L_e} . \qquad (15)$$

The expression (15) of the photon decay rate can be derived as follows. Let $\Delta t' = L_e'/c$ be the transit time of photons in the electron bunch in the reference frame moving with the (initial) electron velocity. A Lorentz transform to the laboratory frame yields the expression for the same time interval

$$\Delta t = \frac{L_e}{c\,(1-(\beta_\parallel)_0)} \simeq \frac{2\gamma_0^2 L_e}{c} . \qquad (16)$$

We assume a decay constant for the field amplitude $k \cong (\Delta t)^{-1}$, which leads to expression (15) for the dimensionless parameter $\bar{k}$. However, an alternative way to introduce this parameter will be discussed in the next section.

From eqs. (3a,b) and (3c') we obtain a dissipation law

$$\frac{d}{d\tau}\left(\langle \eta \rangle + |a|^2\right) = -2\bar{k}|a|^2 \qquad (17)$$

which generalizes the conservation law (11) for $\bar{k} \neq 0$.

Damping can be neglected in eq.(3c') in the limit

$$\bar{k}^{-1} \gg \tau_i \qquad (18)$$

which corresponds with a condition on the electron bunch length

$$L_e \gg (1-(\beta_\parallel)_0)L_w \simeq N_w \lambda_L \qquad (18')$$

ASE has been defined [6-9] in the long-bunch limit (18) or (18'). In this limit the slippage effect can be neglected, i.e. the velocity difference between the electrons and the photons during the transit in the wiggler is completely irrelevant.

On the other hand, we define a condition of radiation suppression in which the time scale of incoherent decay is much faster than the interaction time so that dissipation is quite important :

$$\bar{K}^{-1} \ll \tau_i , \tag{19}$$

namely the short-bunch limit

$$L_e \ll (1-(\beta_\parallel)_0) L_w \simeq N_w \lambda_L . \tag{19'}$$

We shall show in the next section that condition (19) (or (19')) is a necessary condition for the occurrence of SR effects.

4. The superradiant regime

In the short-bunch limit (19) we can adiabatically eliminate the field variables from our working equations (3a,b,c'). In practice, this amounts to dropping the time derivative of the complex field amplitude a in eq. (3c'), thus obtaining the adiabatic expression

$$a = \frac{\bar{K} + i\delta}{\bar{K}^2 + \delta^2}\, b , \tag{20}$$

which is a good approximation to a after a short initial transient on the order of $\bar{K}^{-1}$. By taking the squared modulus of both sides of eq. (20) and with the help of eqs. (5) and (6), we derive the following expression of the power radiated out of the electron bunch in the adiabatic elimination regime :

$$\begin{aligned} P(t) &= \frac{|E_0|^2(t)}{4\pi}\, c\, S_e \\ &= 4\gamma_0^2 \frac{B_w^2 c}{4\pi} \frac{\lambda_w^2}{S_e} \frac{1}{1+(\delta/\bar{K})^2} \left(r_e N |b|(t)\right)^2, \end{aligned} \tag{21}$$

where S_e is the electron bunch cross section and $r_e = e^2/m_0 c^2$ is the classical radius of the electron. It is apparent from

parameter b reaches values on the order of one, the radiated power exhibits a dependence on N^2, that is we find superradiance. We stress that no preparation of the initial state of the particles has been introduced, rather the dynamics of the system is such that the electrons self-organize to radiate cooperatively. In this sense this phenomenon should be called more properly superfluorescence[12].

Eq. (21) admits a transparent physical interpretation. For the sake of simplicity, let us consider the resonant case $\delta = 0$. In the electron rest frame the process can be described as a Thomson backscattering of the "pseudoradiation" field, that is the magnetic field as it is felt by the particles, into the laser field. In the laboratory frame this pseudoradiation field has an amplitude B_{pr} and a wavelength λ_{pr} such that

$$B_{pr} \lambda_{pr} = B_w \lambda_w . \tag{22}$$

Hence eq. (21) can be rewritten on resonance in the form

$$P = 4\gamma_0^2 \frac{B_{pr}^2 c}{4\pi} \frac{\lambda_{pr}^2}{S_e} \left(r_e N |b|\right)^2 \tag{23}$$

Equation (23) is just the power radiated by the electrons in a coherent Thomson scattering of the pseudoradiation field : $(B_{pr}^2/4\pi)c$ is the pseudoradiation intensity, $(r_e N |b|)^2$ the coherent Thomson cross section, λ_{pr}^2/S_e a diffraction solid angle, and $4\gamma_0^2$ a kinematic factor due to the extremely fast motion of the radiating charges with respect to the observer. Also note that the peak power in the ASE regime exhibits a dependence on $\gamma_0^{4/3}$.

An alternative derivation of the power formula relates the radiated power to the decrease in the energy of the electrons :

$$P(t) = -\gamma_0^2 m_0 c^2 \sum_{i=1}^{N} d\gamma_i(t)/dt . \tag{24}$$

The result (24) follows at once from the balance eq. (17). In fact, in the adiabatic elimination regime $d|a|^2/d\tau$ is negligible with respect to $2\bar{K}|a|^2$ and eq. (17) can be approximated by

$$d\langle\eta\rangle/d\tau = -2\bar{K}|a|^2 \tag{25}$$

In terms of the original variables eq. (25) reads

$$m_0c^2\sum_{i=1}^{N}\dot{\gamma}_i = -\frac{1}{\gamma_0^2}\frac{|E_0|^2}{4\pi}c\,S_e \equiv -\frac{1}{\gamma_0^2}P(t), \tag{26}$$

that is eq. (24). Again the factor γ_0^2 is due to the relativistic motion of the radiating electrons. Note that eq. (26) can be written as

$$m_0c^2\sum_{i=1}^{N}\dot{\gamma}_i = -2k\left(|E_0|^2/4\pi\right)V_e \equiv -2k\,\mathcal{E}_{e.m.} \tag{27}$$

where $\mathcal{E}_{e.m.}$ is the energy of the electromagnetic field, $V_e = L_eS_e$ is the electron bunch volume and k is the field amplitude decay rate introduced in eq. (15).
Eq. (27) offers an alternative way to introduce this parameter which characterizes our dissipative model. In fact, from eqs. (26), (27) it follows that the field intensity decays in time due to the escape of photons from the electron bunch, at a rate $2k = c/\gamma_0^2L_e$, which correctly reduces to $2k = c/L_e$ in the nonrelativistic approximation, while corresponds exactly to the amplitude decay rate introduced in eq. (15).
Relations (26) and (27) can be derived also by calculating the energy flux directed to the observer leaving the bunch cross section per unit time in the reference frame moving with the electrons, where the photons escape from a fixed volume, and then transforming to the laboratory frame.
One more comment concerns the space-charge effect. We recall that in the ASE regime space-charge sets a limitation on the otherwise infinite instability range of the detuning parameter δ. In this sense it has a stabilizing effect. This is true also in more general terms, because unstable higher-order harmonics become stable due to the presence of space-charge force[9].
In the SR regime, the net effect of space-charge vanishes in all global equations such as eqs. (21), (24) or (27). This is not at all surprising, because it holds for balance equations of the lossless model such as eq. (16). However, it is interesting to see that in the SR regime the space-charge force can be cancelled even at the level of single-particle dynamics by a proper choice of the detuning. To show this, we substitute the adiabatic expression (20) for the field amplitude into eq. (3b), which becomes

$$\frac{d\eta_i}{d\tau} = -\frac{2}{N}\left[\frac{\bar{K}}{\bar{K}^2+\delta^2}\sum_{j=1}^{N}\cos(\theta_i-\theta_j) - \left(\frac{\delta}{\bar{K}^2+\delta^2}+\bar{\sigma}\right)\sum_{j=1}^{N}\sin(\theta_i-\theta_j)\right]. \tag{28}$$

In the r.h.s. of eq. (28) we distinguish a dissipative contribution, with a Lorentzian weight $\bar{K}/(\bar{K}^2+\delta^2)$, and a dispersive contribution, which vanishes when we sum up over all the N particles.
The point here is that since $\bar{\sigma} \ll \rho$ (eq. (9)), where typically $\rho \ll 1$, the dispersive contribution including the space-charge force can be dropped in the dynamical equation for each electron by choosing small negative values of detuning

$$\delta^2 \ll \bar{K}^2, \qquad \delta/\bar{K}^2 \simeq -\bar{\sigma} \tag{29}$$

5. The conditions for FEL superradiance

We have shown that the condition of radiation suppression (19) is a necessary condition for SR effects. However, another severe limitation concerns the time scale of the phenomenon, which must be shorter than the transit time in the wiggler ; this obvious requirement gives a threshold condition which adds to the radiation suppression condition.
A simple approximate evaluation of the superradiant time scale can be given as follows. Since it turns out that the equations of motion linearized around the unstable equilibrium condition (12) are valid nearly up to the first peak of the emitted radiation[6], we approximate the time evolution up to the first SR peak by

extrapolating the linear stage evolution. The growth rate in time of the exponential regime gives us the SR time scale.
Let us introduce the electron position deviations

$$\tilde{\theta}_i = \theta_i + \theta_i^{0}, \qquad (i=1,\dots,N) \tag{30}$$

where we assume that the initial values are such that

$$\sum_{j=1}^{N} \exp(-im\theta_j^{0}) = 0 \qquad (m=1,2,\dots) \tag{31}$$

If we define the electron collective dynamical variables [13]

$$\Theta = \sum_{j=1}^{N} \exp(-i\theta_j^{0})\,\tilde{\theta}_j \,, \quad \mathcal{P} = \sum_{j=1}^{N} \exp(-i\theta_j^{0})\,\eta_j \,, \tag{32}$$

the linearized equations of motion in the SR regime take the simple form

$$d^2\Theta/d\tau^2 = d\mathcal{P}/d\tau = (i/\bar{K})\,\Theta \tag{33}$$

According to eq. (33), the quantity $\Theta(\tau)$ diverges exponentially in time with a growth rate τ_s, that is the dimensionless SR time scale:

$$\Theta(\tau) \propto \exp(\tau/\tau_s) \,, \qquad \tau_s = \sqrt{\bar{K}} \,. \tag{34}$$

Now we can state the conditions for the occurrence of SR behaviour. Namely, the escape time of photons $\tau_e \geq \bar{K}^{-1}$ (eq. (15) or (27)), the SR rate τ_s (eq.(34)) and the interaction time τ_i (eq.(13)) must obey the inequalities

$$\tau_e \ll \tau_s \ll \tau_i \,, \qquad i.e. \quad 1 \ll \bar{K}^{3/2} \ll \bar{K}\tau_i \tag{35}$$

Let us consider first the two inequalities (35) separately (we take $\gamma_0 \simeq \gamma_R$ from now on in this section). The condition of radiation suppression (or adiabatic elimination limit), $\tau_s \gg \tau_e$ or $\bar{K}^{3/2} \gg 1$, implies (at least) that $\sqrt{\bar{K}} > 1$. Since the threshold condition can be written as $\sqrt{\bar{K}} \ll \tau_i$, conditions (35) imply $1 < \sqrt{\bar{K}} \ll \tau_i$, that is

$$1 < \bar{K} \ll \tau_i^2 \,. \tag{36}$$

Inequalities (36) can be rephrased in terms of lengths as follows:

$$L_T \ll L_e < L_c \tag{37}$$

where the cooperation length L_c and the threshold length L_T are defined

$$L_c = \lambda_L/4\pi\rho \,, \qquad L_T = L_c/\tau_i^2 \tag{37'}$$

Using definitions (37'), inequalities (35) can be rephrased in terms of times:

$$k^{-1} \ll t_s \ll L_w/c \tag{38}$$

where t_s is the SR growth rate

$$t_s = \frac{1}{k}\left(\frac{L_c}{L_e}\right)^{3/2} . \tag{38'}$$

Conditions (35) or (37) can be understood in terms of a suitably defined gain. Consistently with the result (34), let us define a superradiant gain G_s over the wiggler length L_w

$$G_s = \tau_i/\tau_s = \sqrt{L_e/L_T} \,. \tag{39}$$

From definition (39) we see immediately that the threshold condition, $\tau_s \ll \tau_i$ (eq.(35)) or $L_T \ll L_e$ (eq.(37)), is a condition of high-gain over the wiggler length, $G_s \gg 1$. Furthermore, the threshold length L_T (eq.(37')) turns out to be that length of the electron bunch such that the total integrated SR gain G_s is equal to one.
On the other hand, if we consider a SR slippage time, namely the photon escape time (eq.(16)) which is also the effective interaction time in the short-bunch limit (19'), we can define a SR gain g_s^{sl} over a slippage time (or over a slippage distance)

$$g_s^{sl} = (G_s/L_w)\,2\gamma_0^2 L_e \tag{40}$$

where G_s/L_w is the SR gain per unit length. Using the expressions of G_s, τ_i, τ_s and the relations $\tau_s = \sqrt{\bar{k}} = \sqrt{L_e/L_e}$, we easily obtain

$$g_s^{\alpha} = \tau_e/\tau_s = (L_e/L_c)^{3/2}. \qquad (41)$$

On imposing $g_s^{\alpha} \ll 1$, we immediately get the radiation suppression condition, $\tau_e \ll \tau_s$ (eq.(35)) or $L_e < L_c$ (eq.(37)). Therefore this condition means low-gain over one slippage distance. Moreover, the cooperation length L_c (eq.(37')) can be defined as that bunch length such that the SR gain over a slippage distance, g_s^{α}, is equal to one.

Remarkably, this statement on the cooperation length holds also in the ASE regime. In fact, the integrated gain in that regime G_{ASE} is

$$G_{ASE} = 4\pi \rho N_w \equiv \tau_i \qquad (42)$$

Again, the gain calculated on the same slippage time (or distance) as in eq. (40) is such that when it is set equal to one, the bunch length L_e turns out to be equal to L_c, with the latter quantity still defined as in eq. (37').

It is easy to see that the number of independent parameters can be reduced to three, namely the strength parameter ρ, the number of wiggler periods N_w and the ratio of bunch length to laser wavelength L_e/λ_L. Just to give a numerical example, conditions (35) or (37) are satisfied, at least in a weak sense although not as strong inequalities, by the following set of parameters :

$$L_e/\lambda_L = 100 \quad (\text{e.g.: } L_e = 1\,\text{mm}, \ \lambda_L = 10\,\mu) \qquad (43)$$
$$N_w = 200 \quad ; \quad \rho = 8\times 10^{-4}.$$

6. Numerical results

The time evolution equations (3a,b,c') have been numerically integrated with an initial condition very close to the unstable equilibrium condition (12), namely for a monokinetic electron beam and in the absence of photons, but with a small value of the bunching parameter ($|b|_0 = 10^{-3}$). The aim of this numerical investigation was to test the very basic predictions of the theory, namely the occurrence of superradiant emission of radiation and self-bunching of the electrons. To put in evidence the main features of the physics involved, we have fixed the system on resonance, putting $\delta = 0$ in the evolution equations, and neglected space-charge, consistently with the assumption of a small value of the strength parameter ρ (recall that the space-charge parameter $\bar{\sigma}$ in eq. (3b) is proportional to ρ (eq. (9)).

While in the case of no field damping ($\bar{k} = 0$) we recover the ASE dynamics, for $\bar{k} \neq 0$ we have found evidence of SR emission proportional to N^2, corresponding with very high peak values of the bunching parameter $|b|_p$ in a narrow range $0.74 \le |b|_p \le 0.77$ for $1 \le \bar{k} \le 10$. As the value of the field damping constant $\bar{k}$ increases, these peaks occur at longer and longer times τ_p, i.e. the lethargy of the system increases with $\bar{k}$. For instance, for $\bar{k} = 0$ the peak occurs at $\tau_p \simeq 10$, for $k = 3$ at $\tau_p \simeq 20$, for $\bar{k} = 8$ at $\tau_p \simeq 30$. However, and this is another relevant point, the adiabatic elimination (SR) regime is obtained, within a reasonably good approximation, already for $\bar{k} = 3$. In fact in this case after a short initial transient the numerical values of $|a|^2$ and of the modulus squared of the adiabatic expression (19), i.e. (for $\delta = 0$), $|b|^2/\bar{k}^2$, differ by no more than 10%, while the peaks of $|a|^2(\tau)$ and $|b|(\tau)$ occur nearly simultaneously. This result is in agreement with the radiation suppression condition $\bar{k}^{3/2} \gg 1$ (eq.(35)). Note also that when the adiabatic elimination approximation is valid ($\bar{k} \ge 3$), the nearly constant peak value of the bunching parameter yields immediately the scaling law for the peak intensity $|a|_p^2 \propto \bar{k}^{-2}$.

In a second set of runs we have fixed $\bar{k} = 3$, a value sufficient to describe SR, and investigated the effects of detuning by varying the parameter δ. We have considered the values $\delta = \pm 1$, such that $\delta^2 \ll \bar{k}^2$, and $\delta = \pm 3$, such that $|\delta| = \bar{k}$. For both values $\delta = \pm 1$, no oscillation occurs in the dynamics up to the first peak, like in the resonant case. The ratio

$$|a|_p^2 \Big/ \left[|b|^2 / (\bar{k}^2 + \delta^2) \right]_p \tag{44}$$

of intensity to its adiabatic approximation (eq.(19)) calculated at peak values, which is 90% for $\delta = 0$, decreases to 84% for $\delta = -1$ whereas increases to 107% for $\delta = 1$. Hence a small positive (negative) detuning improves (gets worse) the accuracy of adiabatic approximation. On the other hand, for $\delta = -1$ ($\delta = +1$) the SR peak is shifted to shorter (longer) times by roughly 10% with respect to the case $\delta = 0$.
For $\delta = \pm 3$ we have different pictures. In fact, for $\delta = 3$ we find a sensible initial oscillation of both the field intensity and the bunching parameter instead of the monothonically growing time evolution. This causes a dramatic increase in the SR time scale ; e.g., the peak of superradiation is delayed from (roughly) $\tau = 20$ for $\delta = 0$, to $\tau = 35$ for $\delta = 3$. Furthermore, the ratio (44) becomes as big as 135%. On the contrary, for large negative detuning $\delta = -3$ there is an initial oscillation but it is tiny ; peak times and accuracy of adiabatic approximation are very close to the case $\delta = -1$. However, the absolute values of the peaks get depressed ; e.g., the peak intensity is lowered from 6×10^{-2} for $\delta = 0$, to 2.8×10^{-2} for $\delta = -3$. In conclusion, a small detuning appears to be compatible with SR behaviour ; a small negative (positive) detuning can even shorten the time scale (improve the accuracy of the adiabatic approximation). On the contrary, a large positive detuning is equivalent to introduce very high losses, i.e., it increases the lethargy in a dramatic way, whereas a large negative detuning reduces the impressive level of radiated power.

7. Concluding remarks

We have outlined the classical theory of FEL superradiance, a nice example of cooperative phenomenon in which N relativistic electrons, interacting with a radiation field and a strong magnetic pump field, self-bunch and radiate proportional to N^2 and to γ_0^2. We have demomnstrated that this phenomenon occurs if the following conditions are simultaneously satisfied : the SR gain (39) is i) larger than one over a wiggler length, ii) smaller than one over a slippage distance.
Here we discuss some aspects which remain to be investigated, both of specific and of more general nature.
Among the more specific aspects, we mention the effects of energy spread and emittance which should be incorporated in the initial conditions for eqs. (3a,b,c'). Also, the space-charge contribution in eq.(3b) should be kept in the numerical analysis, and higher-order harmonics eventually included. Moreover, one can drop the restriction $(\gamma_i - \gamma_0)/\gamma_0 \ll 1$ (eq.(2)) and consider evolution equations more general than eqs.(3).
From a more general viewpoint, the validity of the inclusion in the FEL evolution equations of a linear damping term to describe dissipative effects, which works in atomic SR[12], should be checked by numerical integration of the evolution equations without damping but with no approximation on the field equation, that is a first-order partial differential equation with respect to both time and space. Furthermore, the analysis should be extended to take into account diffraction and multidimensional effects, as considered e.g. by Scharlemann, Sessler and Wurtele[14] and by Moore[15].
Last but not least, a quantum analysis is important due to the relevant role played by fluctuations. Like in the ASE regime, large initial intensity fluctuations may lead to large fluctuations of the peak times (delay times). In general, the problem of the initial level of noise is relevant for FEL design and is the object of intense investigations[5,16]. We have first performed a quantum analysis of the ASE regime including the derivation of

photonstatistics[17], and next extended such analysis to the SR regime[8]. However, the overall effect of noise, the different contributions to noise and their relative weight are a topics which surely deserves further investigations.

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