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MULTIMODE QUANTUM THEORY OF THE FREE ELECTRON LASER OSCILLATOR

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MULTIMODE QUANTUM THEORY OF THE FREE ELECTRON LASER OSCILLATOR

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MOTIVATIONS FOR A MULTIMODE QUANTUM ANALYSIS

IN THE CONTEXT OF THE QUANTUM APPROACH TO THE FEL-OSCILLATOR DYNAMICS, THE STUDY OF THE EVOLUTION OF THE COUPLED LONGITUDINAL MODES IN THE FEL RESONATOR HAS A TWO-FOLD MOTIVATION :

a) THE OPERATION WITH A BUNCHED E-BEAM PRODUCES A PULSE STRUCTURE IN THE EMITTED RADIATION, WHICH CAN BE ADEQUATELY TREATED BY A MULTIMODE FORMALISM.

b) EVEN WITH A CONTINUOUS E-BEAM, SUCH AN ANALYSIS IS PROPER, FOR THE STARTING STATE (THE VACUUM STATE) CAN BE WRITTEN AS AN INFINITE SUPERPOSITION OF LONGITUDINAL MODES.

OUTLINE OF THE TALK

a) WE WRITE THE SCRÖDINGER EQUATION FOR THE COUPLED ELECTRON-FIELDS SYSTEM, IN THE FRAMEWORK OF MULTIMODE NON-RELATIVISTIC HAMILTONIAN.

b) WE SOLVE UP TO THE FIRST ORDER IN THE ELECTRON RECOIL THIS EQUATION, THUS GETTING THE PROBABILITY AMPLITUDE OF EMITTING ℓ_j- PHOTONS IN THE j-TH MODE

c) WE PROPOSE A SELF-CONSISTENT FORMALISM TO EVALUATE THE TIME EVOLUTION OF THE FEL RADIATION FIELD FROM THE QUANTUM TO THE CLASSICAL REGIME.

SINGLE-MODE ANALYSIS : SYNTHESIS OF THE RESULTS

FOR AN INITIAL VACUUM STATE, THE GAIN IS RESPONSIBLE FOR THE DEVIATIONS FROM THE TYPICAL COHERENCE AND STATISTICAL PROPERTIES OF THE CONVENTIONAL LASERS. IN FACT, WE FIND THAT

a) THE FEL STATES DO NOT EVOLVE INTO COHERENT GLAUBER ONES.

b) THE PROBABILITY OF EMITTING ℓ PHOTONS DOES NOT DISPERSE ACCORDING TO A POISSONIAN.

c) THE FEL STATES EXHIBIT GENUINE QUANTUM EFFECTS SUCH AS BUNCHING (OR ANTIBUNCHING) AND SQUEEZING

FOR AN ARBITRARY INITIAL COHERENT STATE, ALSO IN ABSENCE OF GAIN THE COHERENCE IS NOT PRESERVED.

FEL HAMILTONIAN PICTURE

IN A MOVING FRAME, WHERE THE ELECTRONS CAN BE DESCRIBED AS NON-RELATIVISTIC PARTICLES, THE QUANTUM MULTIMODE - SINGLE PARTICLE HAMILTONIAN READS

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hbar\omega_u\left(\hat{a}_u^+\hat{a}_u + \frac{1}{2}\right) + \sum_{s=1}^{N}\hbar\omega_s\left(\hat{a}_s^+\hat{a}_s + \frac{1}{2}\right) +$$
$$+ \hbar\sum_{s=1}^{N}\Omega_{s,u}\left\{\hat{a}_s^+\hat{a}_u \exp\left(-i\left[k_s+k_u\right]\hat{z}\right) + h.c.\right\} +$$
$$+ \hbar\sum_{s<j}\Omega_{s,j}\left\{\hat{a}_s^+\hat{a}_j \exp\left(-i\left[k_s+k_u\right]\hat{z}\right) + h.c.\right\}$$

WITH

N TOTAL NUMBER OF LONGITUDINAL MODES

$\hat{p}, \hat{z}$ ELECTRON MOMENTUM AND COORDINATE OPERATOR

$\hat{a}$ $(\hat{a}^+)$ PHOTON ANNIHILATION (CREATION) OPERATOR

k, ω FIELD WAVE NUMBER AND FREQUENCY

$\Omega_{s,u(j)} = \dfrac{2\pi c^2 r_0}{V\sqrt{\omega_s\omega_{u(j)}}}$ LASER - UNDULATOR (LASER - LASER) COUPLING CONSTANT

PHYSICAL INTERPRETATION OF THE VARIOUS TERMS IN THE FEL HAMILTONIAN

$$\hat{H}_o = \frac{p^2}{2m} + \hbar\omega_u\left(\hat{a}_u^+\hat{a}_u + \frac{1}{2}\right) + \sum_{s=1}^{N}\hbar\omega_s\left(\hat{a}_s^+\hat{a}_s + \frac{1}{2}\right)$$

ACCOUNTS FOR THE ELECTRON AND UNDULATOR AND LASER FIELDS ENERGY ;

$$\hat{H}_I = \hbar\sum_{s=1}^{N}\Omega_{s,u}\left\{\hat{a}_s^+\hat{a}_u \exp\left(-i\left[k_s + k_u\right]\hat{z}\right) + h.c.\right\}$$

$$+ \hbar\sum_{s<j}\Omega_{s,j}\left\{\hat{a}_s^+\hat{a}_j \exp\left(-i\left[k_s - k_j\right]\hat{z}\right) + h.c.\right\}$$

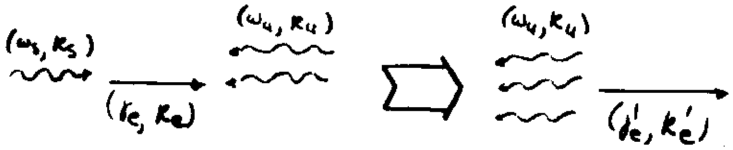
REPRESENTS THE LASER-UNDULATOR AND LASER-LASER INTERACTION IN THE PRESENCE OF THE ELECTRON

THE S-TH MODE - UNDULATOR INTERACTION CONSISTS OF TWO PROCESSES:

a) stimulated forward - scattering

$$k_e' = k_e - (k_s + k_u)$$

b) stimulated back-scattering



$$k_e' = k_e + (k_s + k_u)$$

THE LASER-LASER INTERACTION HAS A SIMILAR INTERPRETATION

LAWS OF CONSERVATION

THE FEL HAMILTONIAN ALLOWS FOR TWO LAWS OF CONSERVATION :

1. THE CONSERVATION OF THE TOTAL NUMBER OF PHOTONS

$$\hat{n}_u + \sum_{s=1}^{N}\hat{n}_s = n_u^{\circ} + \sum_{s=1}^{N} n_s^{\circ} \ ;$$

2. THE CONSERVATION OF THE TOTAL LINEAR MOMENTUM

$$\hat{p} + \hbar\sum_{s=1}^{N} k_s\hat{n}_s - \hbar k_u\hat{n}_u = p^{\circ} + \hbar\sum_{s=1}^{N} k_s n_s^{\circ} - \hbar k_u n_u^{\circ}$$

THEREFORE, IN THE REPRESENTATION

$$|k, \{n\}, n_u\rangle \qquad \{n\} \equiv (n_1, \dots, n_N)$$

THE TIME-EVOLUTION OF THE FEL STATE IS FULLY SPECIFIED BY THE SET OF INTEGERS

$$\{\ell\} \equiv (\ell_1, \dots, \ell_N)$$

THAT REPRESENT THE NUMBER OF PHOTONS EXCHANGED BY THE UNDULATOR INTO EACH LONGITUDINAL MODE

FEL STATE FUNCTION

ASSUMING

$$\boxed{n_s^\circ = 0 \qquad s = 1, \dots, N}$$

WE CAN EXPRESS THE FEL WAVE FUNCTION ψ AS

$$|\psi\rangle = \int dk_e\, g(k_e) \exp\left\{-i\tau T\left[\frac{\hbar k_e^2}{2m} + \omega_u\left(n_u^\circ + \frac{1}{2}\right) + \frac{1}{2}\sum_{s=1}^{N}\omega_s\right]\right\}$$

$$\cdot \sum_{\{\ell\}} C_{\{\ell\}}(k_e,\tau)\left|k_e - \sum_{s=1}^{N}(k_s + k_u)\ell_s, \{\ell\}, n_u^\circ - \sum_{s=1}^{N}\ell_s\right\rangle$$

WHERE

T = INTERACTION TIME

$\tau \equiv t/T$

k_e = ELECTRON WAVE VECTOR

$g(k_e)$ = ELECTRON WAVE FUNCTION IN MOMENTUM SPACE

$C_{\{\ell\}}(k_e,\tau)$ = COEFFICIENTS WHICH REPRESENT THE PROPABILITY AMPLITUDE OF A PHOTON DISTRIBUTION, SPECIFIED BY THE INTEGERS

$$\{\ell\} = (\ell_1, \dots, \ell_N)$$

FEL SCHRÖDINGER EQUATION

THE COEFFICIENTS $C_{\{\ell\}}(k_e,\tau)$ MUST SATISFY THE DIFFERENTIAL-DIFFERENCE EQUATION (SU_n RAMAN-NATH EQUATION)

$$i\frac{d}{d\tau}C_{\{\ell\}} = \sum_{j=1}^{N}\left(-\eta_j + \sum_{s=1}^{N}\epsilon_{s,j}\ell_s\right)\ell_j\, C_{\{\ell\}} +$$

$$+ \sum_{j=1}^{N}\bar{\Omega}_{j,u}\left\{\sqrt{(\ell_j+1)\left(n_u^\circ - \sum_{s=1}^{N}\ell_s\right)}\, C_{\{\ell;\ell_j+1\}} + \right.$$

$$\left. + \sqrt{\ell_j\left(n_u^\circ - \sum_{s=1}^{N}\ell_s + 1\right)}\, C_{\{\ell;\ell_j-1\}}\right\}$$

$$+ \sum_{s<j}\bar{\Omega}_{s,j}\left\{\sqrt{\ell_s(\ell_j+1)}\, C_{\{\ell,\ell_s-1;\ell_j+1\}} + \sqrt{(\ell_s+1)\ell_j}\, C_{\{\ell,\ell_s+1;\ell_j-1\}}\right\}$$

$$C_{\{\ell\}}(k_e, 0) = \prod_{s=1}^{N}\delta_{\ell_s,0}$$

WHERE

$$\bar{\Omega} = \Omega T$$

$$\{\ell; \ell_s \pm 1; \ell_j \pm 1\} \equiv (\ell_1, \dots, \ell_s \pm 1, \dots, \ell_j \pm 1, \dots, \ell_N)$$

SINGLE-MODE OPERATION

$$\eta_j = \text{DETUNING PARAMETER} = \left[\omega_u - \omega_j + (\omega_u + \omega_j)\frac{\hbar k_e}{mc}\right]T \Longrightarrow \eta = \frac{2\omega\hbar k_e}{mc}$$

$$\epsilon_{s,j} = \text{ELECTRON RECOIL PARAMETER} = \frac{\hbar}{2mc^2}(\omega_s + \omega_u)(\omega_j + \omega_u)T \Longrightarrow \epsilon = \frac{2\hbar\omega^2 T}{mc^2}$$

SIMPLIFICATION OF THE SU_n RAMAN-NATH EQUATION

IN ANY EXPERIMENTAL SITUATION

$$n_u^\circ \gg \sum_{s=1}^{N} \ell_s \quad ,$$

THEREFORE

a) n_u° CAN BE ASSUMED UNCHANGED DURING THE INTERACTION

b) THE TERMS CORRESPONDING TO THE LASER-LASER INTERACTION CAN BE NEGLECTED, BECAUSE

$$\Omega_{R_j} \equiv \bar{\Omega}_{j,u}\sqrt{n_u^\circ} \gg \bar{\Omega}_{j,s}$$

THUS, THE SU_n R-N EQUATION REDUCES TO

$$\begin{cases} i\dfrac{d}{d\tau} C_{\{\ell\}} = \displaystyle\sum_{s=1}^{N}\Big(-\eta_s + \boxed{\sum_{j=1}^{N} \varepsilon_{s,j}\,\ell_j}\Big)\ell_s\, C_{\{\ell\}} + \\ \quad + \displaystyle\sum_{s=1}^{N} \Omega_{R_s}\Big\{\sqrt{\ell_s+1}\, C_{\{\ell;\ell_s+1\}} + \sqrt{\ell_s}\, C_{\{\ell;\ell_s-1\}}\Big\} \\ C_{\{\ell\}}(k_e,0) = \displaystyle\prod_{s=1}^{N} \delta_{\ell_s,0} \end{cases}$$

THE CORRESPONDING EQ. IN THE SINGLE-MODE HYPOTHESIS IS

$$\begin{cases} i\dfrac{d}{d\tau} C_\ell = (-\eta + \boxed{\varepsilon\ell})\,\ell\, C_\ell + \\ \qquad + \Omega_R\Big\{\sqrt{\ell+1}\, C_{\ell+1} + \sqrt{\ell}\, C_\ell\Big\} \\ C_\ell(0) = \delta_{\ell 0} \end{cases}$$

RECOILLESS APPROXIMATION

IF

$$\varepsilon_{s,j} = 0$$

THE SOLUTION OF THE R-N EQ. CAN BE EXPRESSED AS PRODUCT OF N POISSON FUNCTIONS;

$$C_{\{\ell\}}(k_e,\tau) = \prod_{j=1}^{N} \frac{\alpha_j(\tau)^{\ell_j}}{\sqrt{\ell_j!}} \exp\Big\{\frac{i}{2}\eta_j \int_0^\tau |\alpha_j(\tau)|^2 d\tau' - |\alpha_j(\tau)|^2/2\Big\} \;,$$

WITH

$$\alpha_j(\tau) = (-i)\,\Omega_{R_j} \exp\big(i\eta_j \tau/2\big)\Big(\frac{\sin \eta_j\tau/2}{\eta_j/2}\Big)$$

AS A CONSEQUENCE

a) THE LONGITUDINAL MODES EVOLVE INDEPENDENTLY.

b) THE PROBABILITY OF EMITTING $(\ell_1,\dots,\ell_N)$ PHOTONS INTO THE VARIOUS LASER MODES FOR A GIVEN k_e IS

$$|C_{\{\ell\}}(k_e,\tau)|^2 = \prod_{j=1}^{N} \frac{1}{\ell_j!}\, |\alpha_j(\tau)|^{2\ell_j} \exp\big(-|\alpha_j(\tau)|^2\big) \;,$$

WHICH IMPLIES

$$\langle \ell_j \rangle = \int dk_e\, |g(k_e)|^2 |\alpha_j(\tau)|^2$$

$$|\alpha_j(\tau)|^2 = \Omega_{R_j}^2 \Big(\frac{\sin \eta_j\tau/2}{-}\Big)^2$$

c) THE FEL STATE

$$|\psi\rangle_{t=0} = \int dk_e\, g(k_e)\, \exp\left\{-iT\tau\left[\frac{\hbar k_e^2}{2m} + \omega_u\left(n_u^\circ + \frac{1}{2}\right)\right]\right\}\cdot$$

$$\cdot \prod_{j=1}^{N} \exp\left\{-\frac{i}{2}\left[\tau T\omega_j - \eta_j \int_0^{\tau} |\alpha_j(\tau')|^2 d\tau'\right] - \frac{|\alpha_j(\tau)|^2}{2}\right\}\cdot$$

$$\cdot \sum_{\ell_j} \frac{1}{\sqrt{\ell_j!}}\, \alpha_j(\tau)^{\ell_j} \left| k_e - (k_j + k_u)\ell_j, \ell_j \right\rangle$$

IS A COHERENT STATE, IN THE SENSE THAT

$$\hat{A}_j |\psi\rangle_{t=0} = \alpha_j(\tau)\, |\psi\rangle_{t=0} \qquad j = 1, \dots, N$$

WHERE

$$\hat{A}_j \equiv \hat{a}_j \exp\left[i(k_j + k_u)\hat{z}\right]$$

FIRST-ORDER SOLUTION OF THE R-N EQUATION

A FIRST-ORDER ANALYSIS OF THE R-N EQUATION IN TERMS OF THE ELECTRON RECOIL PARAMETERS $\varepsilon_{s,j}$ LEADS TO

$$C_{\{\ell\}}(k_e,\tau) = (-i)^{\ell_1} \cdots (-i)^{\ell_N} \exp\left\{\frac{i}{2}\sum_{j=1}^{N} \eta_j \left[\int_0^{\tau} |\alpha_j(\tau')|^2 d\tau' + \ell_j \tau\right]\right\}\cdot$$

$$\cdot \left\{ A_{\{\ell\}}(k_e,\tau) + D_{\{\ell\}}(k_e,\tau) \right\}$$

WHERE

$$\mathcal{L}_{\{\ell\}}(k_e,\tau) = \prod_{j=1}^{N} \frac{|\alpha_j(\tau)|^{\ell_j}}{\sqrt{\ell_j!}} \exp\left(-|\alpha_j(\tau)|^2/2\right)$$

$$A_{\{\ell\}}(k_e,\tau) = \left[1 + \sum_{j=1}^{N} \varepsilon_{jj} \frac{\partial}{\partial \eta_j} |\alpha_j(\tau)|^2\right] \mathcal{L}_{\{\ell\}}(k_e,\tau) +$$

$$+ \sum_{j,p} \varepsilon_{jp} \left\{ -\frac{\partial}{\partial \eta_j} |\alpha_j(\tau)| \left[(2\ell_p - \delta_{jp})\sqrt{\ell_j}\, \mathcal{L}_{\{\ell_j-1\}}(\tau) - \right.\right.$$

$$\left.\left. - (2\ell_p + \delta_{jp})\sqrt{\ell_j+1}\, \mathcal{L}_{\{\ell_j+1\}}(\tau)\right] + \dots \right\}$$

$$D_{\{\ell\}}(k_e,\tau) = \sum_{j=1}^{N} \varepsilon_{jp} \left\{ -\ell_j \ell_p \tau\, \mathcal{L}_{\{\ell\}}(k_e,\tau) + \right.$$

$$+ \frac{I}{2} |\alpha_j(\tau)| \left[(2\ell_p - \delta_{jp})\sqrt{\ell_j}\, \mathcal{L}_{\{\ell_j-1\}}(\tau) + \right.$$

$$\left.\left. + (2\ell_p + \delta_{jp})\sqrt{\ell_j+1}\, \mathcal{L}_{\{\ell_j+1\}}(\tau)\right] + \dots \right\}$$

SINGLE-MODE ANALYSIS

$$\boxed{\langle \varrho \rangle = |a(\tau)|^2 - \varepsilon \frac{\partial}{\partial \eta} |a(\tau)|^2 + O(\Omega_R^3)}$$

THE ABOVE EXPRESSION CONTAINS THE CONTRIBUTION OF

1. THE SPONTANEOUS EMISSION : $|a_j(\tau)|^2$

2. THE STIMULATED EMISSION, INDUCED BY VACUUM FIELD FLUCTUATIONS : $-\varepsilon_{j,j} \frac{\partial}{\partial \eta_j} |a_j(\tau)|^2$

AND

3. THE COUPLING BETWEEN LONGITUDINAL MODES :

$$\sum_{s=1}^{N} \varepsilon_{s,j}\, \Gamma(\eta_j, \eta_s, \tau) \sim O(\Omega_{R_j}^3)$$

b) THE FEL STATES DO NOT EVOLVE INTO COHERENT ONES

c) THE EXPRESSION

$$\langle \Delta \varrho_j^2 \rangle - \langle \varrho_j \rangle = -\varepsilon_{j,j} \int dk_e\, |g(k_e)|^2 \frac{\partial}{\partial \eta_j} |a_j(\tau)|^4$$

REVEALS THE ONSET OF PHOTON BUNCHING $(\eta_j > 0)$ OR ANTIBUNCHING $(\eta_j < 0)$.

THEREFORE

a) THE AVERAGE NUMBER OF PHOTONS IN THE J-TH MODE IS

$$\langle \varrho_j \rangle = \int dk_e\, |g(k_e)|^2 \Big[|a_j(\tau)|^2 - \varepsilon_{j,j} \frac{\partial}{\partial \eta_j} |a_j(\tau)|^2 + \sum_{s=1}^{N} \varepsilon_{s,j}\, \Gamma(\eta_j, \eta_s, \tau) \Big]$$

WHERE

$$\Gamma(\eta_j, \eta_s, \tau) = 2\Big[S(\tau, \eta_j, \eta_s) - \frac{2}{3} |a_s(\tau)|^2 \frac{\partial}{\partial \eta_j} |a_j(\tau)| \Big] |a_j(\tau)| + 2 |a_j(\tau)|^2 \frac{\partial}{\partial \eta_s} |a_s(\tau)|^2 - 2 |a_s(\tau)|^2 \frac{\partial}{\partial \eta_j} |a_j(\tau)|^2$$

$$\begin{aligned} S(\tau, \eta_j, \eta_s) = -\frac{2}{3} \Omega_{R_j} \Omega_{R_s}^2 \Big\{ & \frac{2\tau}{\eta_s^2 \eta_j} \cos \eta_j \tfrac{\tau}{2} \left[2 + \cos \eta_s \tau \right] + \\ & - \frac{2}{\eta_s^3 \eta_j} \sin \eta_s \tau \cos \tfrac{1}{2} \eta_j \tau - \frac{2\eta_j^3 + 2\eta_s^3 + 3\eta_j^2 \eta_s + 3\eta_s^2 \eta_j}{\eta_s^3 \eta_j^2 (\eta_s + \eta_j)^2} \sin \left(\tfrac{\eta_j}{2} + \eta_s \right) \tau \\ & + \frac{2\eta_j^2 + 2\eta_s^2 - \eta_s \eta_j}{\eta_j^2 \eta_s^3 (\eta_s - \eta_j)} \sin \left(\eta_s - \tfrac{\eta_j}{2} \right) \tau - \frac{\eta_s^3 + \eta_j^3 + \eta_j \eta_s^2}{\eta_s^3 \eta_j^3 (\eta_s - \eta_j)} \sin \tfrac{1}{2} \eta_j \tau + \\ & - \frac{4\eta_j \eta_s^3 + 5\eta_s \eta_j^3 - 9\eta_j^2 \eta_s^2 + \eta_j^4 - \eta_s^4}{\eta_s^3 \eta_j^3 (\eta_s + \eta_j)^2} \sin \tfrac{1}{2} \eta_j \tau \Big\} \end{aligned}$$

TIME EVOLUTION OF THE FEL RADIATION

AFTER THE m-th PASS, THE FEL STATE CAN BE REPRESENTED TO FIRST-ORDER IN $\varepsilon_{s,j}$ BY

$$|\psi\rangle_{m+1} = \sum_{\{n^\circ\}=0}^{\infty} \langle C^{m}_{\{n^\circ\}}(k_e,1)\rangle_{k_e} \int dk_e\, g(k_e) \cdot \exp\left\{-iT\tau\left[\frac{\hbar k_e^2}{2m} + \omega_u\left(n_u^\circ + \frac{1}{2}\right) + \sum_{j=1}^{N}\omega_j\left(n_j^\circ + \frac{1}{2}\right)\right]\right\} \cdot$$

$$\sum_{\{\ell\}=-\{n^\circ\}}^{\infty} C^{m+1}_{\{\ell\}}(k_e,\tau)\left|k_e - \sum_{j=1}^{N}\ell_j(k_j + k_u), \{n^\circ + \ell\}, n_u^\circ - \sum_{j=1}^{N}\ell_j\right\rangle$$

WHERE

$$\langle C^{m}_{\{n^\circ\}}(k_e,1)\rangle_{k_e} = \int dk_e\, |g(k_e)|^2\, C^{m}_{\{n^\circ\}}(k_e,1)$$

$\{n^\circ\} = (n_1^\circ, \ldots, n_N^\circ)$ REPRESENT THE PHOTON DISTRIBUTION CREATED IN THE m-th PASS.

THE COEFFICIENTS $C^{m+1}_{\{\ell\}}(k_e,\tau)$ SATISFY THE FOLLOWING EQUATION

$$i\frac{d}{d\tau} C^{m+1}_{\{\ell\}} = \sum_{j=1}^{N}\left(-\eta_j + \sum_{s=1}^{N}\varepsilon_{s,j}\,\ell_s\right)\ell_j\, C^{m+1}_{\{\ell\}} + \sum_{j=1}^{N}\Omega_{\ell_j}\left[\sqrt{n_j^\circ + \ell_j + 1}\; C^{m+1}_{\{\ell;\,\ell_j+1\}} + \sqrt{n_j^\circ + \ell_j}\; C^{m+1}_{\{\ell;\,\ell_j-1\}}\right\}$$

$$C^{m+1}_{\{\ell\}}(k_e,0) = \prod_{j=1}^{N}{}' \,\delta_{\ell_j,0}$$

WHOSE SOLUTION GIVES

$$\langle \ell_j\rangle^{m+1} = \langle \ell_j\rangle^{m} + \int dk_e\, |g(k_e)|^2\left[|a_j(\tau)|^2 - \varepsilon_{j,j}\frac{\partial}{\partial\eta_j}|a_j(\tau)|^2\left(1 + 2\langle \ell_j\rangle^{m}\right) + \sum_{s=1}^{N}\varepsilon_{s,j}\,\Gamma(\eta_j,\eta_s,\tau)\right]$$

IT CONTAINS

1. THE AVERAGE NUMBER OF PHOTONS AT m-th PASS $\langle \ell_j\rangle$

2. THE CONTRIBUTION OF THE STIMULATED EMISSION, INDUCED BY THE LASER RADIATION

$$-2\varepsilon_{jj}\,\langle \ell_j\rangle^{m}\frac{\partial}{\partial\eta_j}|a_j(\tau)|^2$$