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ELEMENTS OF QUANTUM THEORY OF E.M. FIELD AND COHERENT STATES

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A generic component of the vector potential can be expanded in normal modes as:

$$A(\underline{r},t) = \sum_{K} q_{\underline{K}}(t)\, f_{\underline{K}}(\underline{r}) \qquad (1)$$

where:

$$\nabla^2 f_K(\underline{r}) = -K^2 f_{\underline{K}}(\underline{r}) \quad ; \quad K = |\underline{K}| \qquad (2)$$

and

$$(f_K , f_{K'}) = \delta_{KK'} \qquad (3)$$

In a rectangular geometry (L_x, L_y, L_z)

$$f_{\underline{K}}(\underline{r}) = e^{i\underline{K}\cdot\underline{r}} \quad , \quad K_x = \frac{2\pi}{L_x} n_x \; , \dots \qquad (4)$$

n_x, n_y and n_z are integer numbers.
Inserting 1 in the wave eq.

$$\frac{\partial^2 A}{\partial t^2} - c^2 \nabla^2 A = 0 \qquad (5)$$

one obtains the Harmonic oscillator equation

$$\boxed{\ddot q_{\underline{K}}(t) + c^2 K^2 q_{\underline{K}}(t) = 0} \qquad (6)$$

Similarly a generic component of $\underline{E}$ is:

$$E(\underline{r},t) = -\sum_{K} p_{\underline{K}}(t)\, f_K(\underline{r}) \qquad (7)$$

Since $E = -\frac{\partial A}{\partial t}$, (1) and (7) give:

$$\boxed{\dot q_K = p_K} \qquad (8)$$

Eqs. (6) and (8) can be obtained as Hamilton equation with the Hamiltonian

$$H = \frac{1}{2} \sum_{K} \left(p_K^2 + c^2 K^2 q_K^2 \right) \qquad (9)$$

$$\left\{ \begin{aligned} \dot q_K &= \frac{\partial H}{\partial p_K} = p_K & \qquad \ddot q_K = \dot p_K \\ \dot p_K &= -\frac{\partial H}{\partial q_K} = -c^2 K^2 q_K = \ddot q_K & \end{aligned} \right\} \qquad (9')$$

FIELD QUANTIZATION

Define the operators $\hat A$ and $\hat E$ as:

$$\hat A(\underline{r},t) = \sum_{K} \hat q_K\, f_K(\underline{r}) \qquad (10)$$

$$\hat E(\underline{r},t) = -\sum \hat p_K\, f_K(\underline{r}) \qquad (10')$$

where:

$$[q_K , p_{K'}] = i\hbar\, \delta_{KK'} \qquad (11)$$

The Heisenberg equations associated to:

$$\hat H = \frac{1}{2} \sum_{K} \left(\hat p_K + c^2 K^2 \hat q_K^2 \right) \qquad (H')$$

are:

$$\begin{aligned} \dot{\hat q}_K &= \hat p_K \\ \dot{\hat p}_K &= -c^2 K^2 q_K \end{aligned} \qquad (11'')$$

identical to (9').

CREATION AND ANNAHILATION OPERATORS

Define the operator (for each mode K)

$$\hat{a} = \frac{\omega \hat{q} + i\hat{p}}{\sqrt{2\hbar\omega}} \quad , \quad \hat{a}^{+} = \frac{\omega \hat{q} - i\hat{p}}{\sqrt{2\hbar\omega}} \quad (\omega_K = cK) \qquad (12)$$

$\therefore$

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a}^{+} + \hat{a}) \quad , \quad \hat{p} = \sqrt{\frac{\hbar\omega}{2}}\, i(\hat{a}^{+} - \hat{a}) \qquad (12')$$

From:

$[\hat{q}_K , \hat{p}_{K'}] = i\hbar\,\delta_{KK'}$ it follows

$$\boxed{[\hat{a}_K , \hat{a}^{+}_{K'}] = \delta_{KK'}} \qquad (13)$$

Inserting (12') into (9) we get:

$$\hat{H} = \frac{1}{2}\sum_K \hbar\omega_K \left(\hat{a}^{+}_K \hat{a}_K + \hat{a}_K \hat{a}^{+}_K \right) \qquad (14)$$

$$= \sum_K \hbar\omega_K \left(a^{+}_K a_K + 1/2 \right)$$

Where from (13) we have used $a_K a^{+}_K = 1 + a^{+}_K a_K$

The Hiesemberg eqs. assoc[i]ated to (14) are.

$$\dot{q}_K = \frac{1}{i\hbar}[q_K , H] = -i\omega_K q_K \qquad (15)$$

By (12') This is equivalent to (11).

PROPERTIES OF a and a^{+}

Given a and a^{+} such that

$$[a, a^{+}] = 1 \qquad (16)$$

it follows (see Messiah)

$$a^{+}a|m\rangle = m|m\rangle \qquad m = 0,1,2\dots \qquad (17)$$

with $\langle m'|m\rangle = \delta_{mm'}$

$$a^{+}|m\rangle = \sqrt{m+1}\,|m+1\rangle \quad \text{and} \quad a|m\rangle = \sqrt{m}\,|m-1\rangle \qquad (18)$$

Note that

$$a|0\rangle = 0 \qquad (19)$$

Eq.(19) defines the vaccum state

The $+1$ in $\sqrt{m+1}$ is Responsible for Spontaneous Emission. i.e. $a^{+}|0\rangle = |1\rangle$. From (18) iterating

$$|m\rangle = \frac{(a^{+})^{m}}{m!}|0\rangle \qquad (20)$$

relevance as the E.M. Field, Define.

$$|\{m_K\}\rangle = |m_1, m_2, \dots, m_K, \dots\rangle \quad \text{so that} \qquad (21)$$

$$a^{+}_K a_K |\{m_K\}\rangle = m_K |\{m_K\}\rangle \qquad (22)$$

Hence from (14) and (22) it follows

$$H|\{m_k\}\rangle = E_{\{m_k\}} \, |\{m_k\}\rangle$$

where

$$E_{\{m_k\}} = \sum \hbar \omega_k \left(m_k + \tfrac{1}{2} \right) \qquad (23)$$

Hence for each mode k the energy of the E.M. is m times the "one photon energy $\hbar\omega$ plus the zero point Energy $\frac{1}{2}\hbar\omega$.
From (18) we have $\langle m|a|m\rangle = \langle m|a^+|m\rangle = 0$ so that from (12) $\langle m|q|m\rangle = \langle m|p|m\rangle = 0$. Hence from eqs (10) it follows.

$$\langle E_0\rangle_m = \langle A_0\rangle_m = 0 \qquad (24)$$

Hence the photon number i.e. the energy is perfectly defined but the phase is undefined. Furthermore from (12')

$$\left.\begin{aligned} q^2 &= \frac{\hbar}{2\omega}\left[(a^+)^2 + a^2 + aa^+ + a^+a\right] \\ p^2 &= \frac{\hbar\omega}{2}\left[a^+a + aa^+ - a^2 - (a^+)^2\right] \end{aligned}\right\} \qquad (25)$$

Hence using (18) we have

$$\left.\begin{aligned} \langle q^2\rangle_m &= \frac{\hbar}{\omega}\left(m+\tfrac{1}{2}\right) \\ \langle p^2\rangle_m &= \hbar\omega\left(m+\tfrac{1}{2}\right) \end{aligned}\right\} \qquad (26)$$

This gives

$$\Delta_m(p)\,\Delta_m(q) = \hbar\left(m+\tfrac{1}{2}\right) \qquad (27)$$

where $\Delta(A) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$ is the mean square deviation. For a quasi classical state, we must demand $\Delta(p)\Delta(q) = \frac{1}{2}\hbar$ which is the minimum possible value.

Quasi classical state of a Harmonic oscillator.

Define a state $|\alpha\rangle$ such that

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad , \quad \langle\alpha|a^+ = \langle\alpha|\alpha^* \qquad (27)$$

These state is given by

$$|\alpha\rangle = \sum_{m=0}^{\infty} C_m |m\rangle \quad , \quad C_m = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^m}{\sqrt{m!}} \qquad (28)$$

In fact.

$$a|\alpha\rangle = \sum C_m a|m\rangle = \sum_{m=1}^{\infty} C_m \sqrt{m}\,|m-1\rangle = \sum_{m=0}^{\infty} C_{m+1}\sqrt{m+1}\,|m\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{m=0} \frac{\alpha^{m+1}}{\sqrt{(m+1)!}} \sqrt{m+1}\,|m\rangle = \alpha\, e^{-\frac{|\alpha|^2}{2}} \sum_{m=0} \frac{\alpha^m}{\sqrt{m!}}|m\rangle$$

$$= \alpha \sum C_m |m\rangle = \alpha|\alpha\rangle$$

Here we have used $a|m\rangle = \sqrt{m}\,|m-1\rangle$.

We have the property

$$\langle (a^+)^m a^m\rangle_\alpha = \langle\alpha|(a^+)^m a^m|\alpha\rangle = (\alpha^*)^m \alpha^m \qquad (29)$$

That is the mean value of a normally ordered product $(a^+)^m a^m$ is obtained just substituting $a \to \alpha$, $a^+ \to \alpha^*$ and this implies

$$\langle a\rangle_\alpha = \alpha \qquad \langle a^+\rangle_\alpha = \alpha^* \qquad (30)$$

$$\langle a^+a\rangle_\alpha = |\alpha|^2 \qquad (30')$$

Hence from (12')

$$\langle q \rangle = \alpha \, \mathrm{Re}\,\alpha \qquad \langle p \rangle = \alpha \, \mathrm{Im}\,\alpha \qquad (31)$$

Thus defining

$$|\{\alpha_K\}\rangle = |\alpha_1, \alpha_2 \ldots \alpha_K \ldots \rangle \qquad (32)$$

we have

$$\langle \hat{A} \rangle = \Sigma \langle q_K \rangle f_K \propto \Sigma (\mathrm{Re}\,\alpha_K) f_K$$ and similarly

$$\langle \hat{E} \rangle = - \Sigma \langle p_K \rangle f_K \propto \Sigma (\mathrm{Im}\,\alpha_K) f_K .$$

Furthermore,

$$\langle H \rangle = \Sigma_K \hbar \omega_K (\langle a_K^\dagger a_K \rangle + 1/2) = \Sigma_K \hbar \omega_K (|\alpha_K|^2 + 1/2) \qquad (33)$$

[one can show that eqs (30) and (30') are equivalent to (27)]

the probability of having n-photons is given by.

$$\boxed{p_n = |C_n|^2 = e^{-\bar{m}} \frac{\bar{m}^n}{n!} \qquad \bar{m} = |\alpha|^2} \qquad (34)$$

This is a Poisson distribution with mean value $\bar{m}$ and mean squar deviation:

$$\sigma^2 = \Sigma (n - \bar{m})^2 p_n = \bar{m} \qquad (35)$$

Hence

$$\frac{\sigma}{\bar{m}} = \frac{1}{\sqrt{\bar{m}}} \qquad (36)$$

Hence in a $|\alpha\rangle$ state with $|\alpha^2| = \bar{m} \gg 1$ the photon number is quite well defined

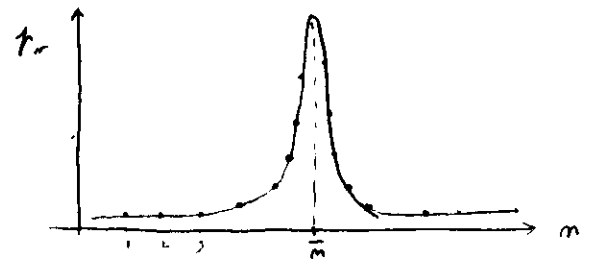

FIG. 1.

The photon statistics associated to a laser is with a good approximation is represented by FIG. 1.

CAMAPISON WITH THERMAL FIELD

A monochromatic thermal light gives a photon statistics determined by FIG 1.

$$p_n = \frac{1}{1+\bar{m}} \left(\frac{\bar{m}}{1+\bar{m}} \right)^n \qquad (37)$$

here $\bar{m}$ is the mean photon number. The mean square deviation σ is given by.

$$\sigma^2 = \bar{m}(1+\bar{m}) \qquad (38)$$

Thus, for $\bar{m} \ll 1$, coincide with (35) ~~[illegible]~~
$\bar{m} \gg 1$

$$\sigma^2 \simeq \bar{m}^2 \quad , \quad \frac{\sigma}{\bar{m}} = 1 \qquad (39)$$

in a coherent state $|\alpha\rangle$, $\frac{\sigma}{\bar{m}} = \frac{1}{\sqrt{\bar{m}}}$

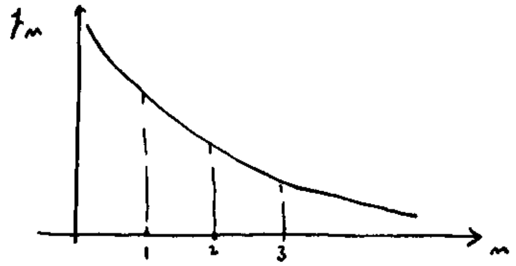

FIG. 2.

This demonstrates (37).

A monochromatic thermal field is a harmonic oscillator in thermal equilibrium. Hence.

$$\rho = \frac{e^{-H/KT}}{Z}, \quad H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \qquad (40)$$

$Z = Tr\, e^{-H/KT}$ is the partition function. Hence by definition

$$p_m = \langle m|\rho|m\rangle = \frac{e^{-E_m/KT}}{Z} = \frac{e^{-mx}}{\sum_{m=0}^{\infty} e^{-mx}} \quad \text{with } x = \frac{\hbar\omega}{KT}$$

Hence since $\sum e^{-mx} = (1-e^{-x})^{-1}$

$$p_m = (1-e^{-x})\,e^{-mx} \qquad (41)$$

Since

$$\bar{m} = \sum m p_m = \frac{1}{e^{x}-1} \quad \text{Planck's law.} \qquad (42)$$

$$\left(e^{-x} = \frac{\bar{m}}{1+\bar{m}}\right)$$

equation (41) is identical to (37).

Time Evolution of a Coherent state and Bloch, Nordsiek. Theorem

We now that a classical current generate a coherent field. If a classical current is coupled to the field we have.

$$H = \sum_K \hbar\omega_K (a_K^\dagger a + \tfrac{1}{2}) + i\hbar (a_K^\dagger J_K - a_K J_K) \qquad (43)$$

$$\dot{a}_K = \frac{1}{i\hbar}[a_K, H] = i\omega_K a_K + J_K \qquad (44)$$

Hence a classical current act as a forcing term for a_K. The solution of (44) is

$$a(t) = a\,e^{-i\omega t} + \alpha_1(t) \qquad (45)$$

where:

$$\alpha_1(t) = \int_0^t dt'\, J(t')\, e^{-i\omega(t-t')} \qquad (45')$$

for each mode K.

Suppose the initial state is $|\psi_0\rangle \equiv |\alpha_0\rangle$

$$a|\psi_0\rangle = \alpha_0|\psi_0\rangle \qquad (46)$$

Hence using (45)

$$a(t)|\psi_0\rangle = \alpha(t)|\psi_0\rangle \qquad (47)$$

with:

$$\alpha(t) = \alpha_0 e^{-i\omega t} + \alpha_1(t) \qquad (47')$$

Since $a(t) = U^\dagger(t)\, a\, U(t)$ where U is the time evolution operator, eq.(47) becomes.

$$U^\dagger a U|\psi_0\rangle = \alpha(t)|\psi_0\rangle$$

$$aU|\psi_0\rangle = \alpha(t)\, U|\psi_0\rangle, \quad \text{but } U|\psi_0\rangle = |\psi_t\rangle \quad \text{Hence:}$$

$$\alpha |\psi\rangle_t = \alpha(t)|\psi\rangle_t \qquad (48)$$

This equation by definition, says that $|\psi\rangle_t$ is a coherent state
to $\alpha(t)$ by (47')

In particular:

i) no current (free field evolution) ($J=0$).

$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

This is the free field oscillation at frequency ω, as for the classical field.

ii) $\alpha_0 = 0$ i.e. the initial state is the vaccum state. Hence $\psi(t)$ is a coherent field with $\alpha(t) = \alpha_1(t)$ determined by the current via eq.(45)'. Hence a classical current drives the field from the vaccume state to a coh. state with a Poisson distrib. for the photon number. This is the statement of the Bloch Nordsik Theorm in Q.E.D.

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