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H4.SMR/1150 - 26

Fifth Workshop on Non-Linear Dynamics
and Earthquake Prediction

4 - 22 October 1999

A Computer-Assisted Introduction to Nonlinear Dynamics, Fractals and Chaos

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Pioneers

Henri Poincaré (1854–1912)	F
George David Birkhoff (1884–1944)	US
A.M. Lyapunov (1857–1918)	R
A.N. Kolmogorov	R
E.N. Lorenz	US

Contemporaries

Benoit Mandelbrot	F
Vladimir I. Arnold	R
Yashi Sinai	R
Mitchell Feigenbaum	US
Stephen Smale	US

Books for Further Reading

General:

Baker, G.L. and J.P. Gollub: Chaotic Dynamics, Cambridge Univ. Press 1990 (181 pp., the best introduction)

Thompson, J.M.T. and H.B. Stewart: Nonlinear Dynamics and Chaos, Wiley 1986 (the best second book)

Hilborn, R.C.: Chaos and Nonlinear Dynamics, Oxford Univ. Press 1994 (a good third book)

Jackson, E.A.: Perspectives of Nonlinear Dynamics, 2 vols., Cambridge Univ. Press 1990 (very comprehensive and detailed, not difficult)

Abraham, R.A. and C.D. Shaw: Dynamics: The Geometry of Behavior, 2nd ed., Addison–Wesley 1992 (without mathematics, only drawings, very profound and simple, a masterpiece)

Gleick, J.: Chaos — Making a New Science, New York, 1987 (very readable and informative, a classic, no formulas)

Lorenz, E.N.: The Essence of Chaos, Univ. of Washington Press, Seattle 1993 (an unusual and wonderful book by a pioneer)

Mandelbrot B.B.: The Fractal Geometry of Nature, Freeman, New York 1982 (a classic, without formulas, highly stimulating, subjective, not easy)

Special Topics:

Herrmann, D.: Algorithmen für Chaos und Fraktale, Addison–Wesley 1994 (the best programs in Basic, Pascal and C)

Peitgen, H. and D. Saupe (eds.): The Science of Fractal Images, Springer Verlag 1998 (fractal terrain models)

Peitgen, H., H. Jürgens and D. Saupe: Chaos and Fractals: New Frontiers of Science, Springer Verlag 1992 (a didactic masterpiece)

Turcotte, D.L.: Fractals and Chaos in Geology and Geophysics, Cambridge Univ. Press 1992 (basic text for geophysical applications, treats also work of the Keilis–Borok school)

Keilis–Borok, V.I. (ed.): Intermediate–Term Earthquakes Prediction Models: Algorithms, Worldwide Tests, Physics of the Earth and Planetary Interiors, vol. 51 (1990)

Programs in	Basic:	*.*bas*
	C:	*.*c*
	C++:	*.*cpp*
	Mathematica:	*.*nb*

Part I

Fractals

1 Deterministic Fractals, Self–Similarity, and Fractal Dimension

1.1 Cantor Set

Remove middle third

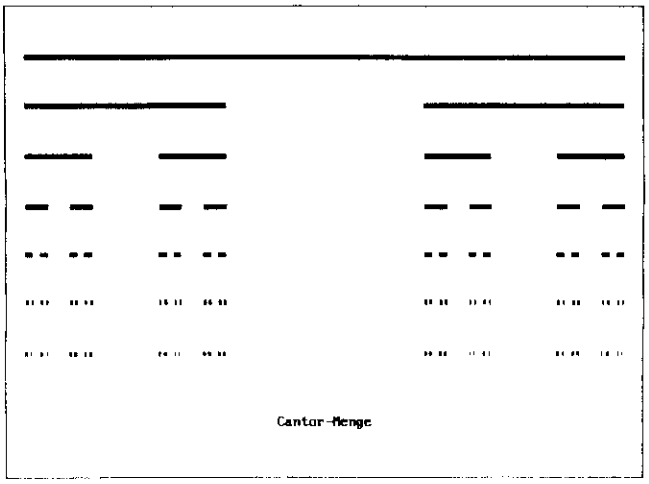


Figure 1: Cantor set

Cantor.bas
Cantquad.bas
Devil.bas

"Devil's staircase"

Basic throughout chaos theory.

1.2 Peano Curve, Hilbert Curve

Curves completely filling a square, Dimension $D = 2$ (!)

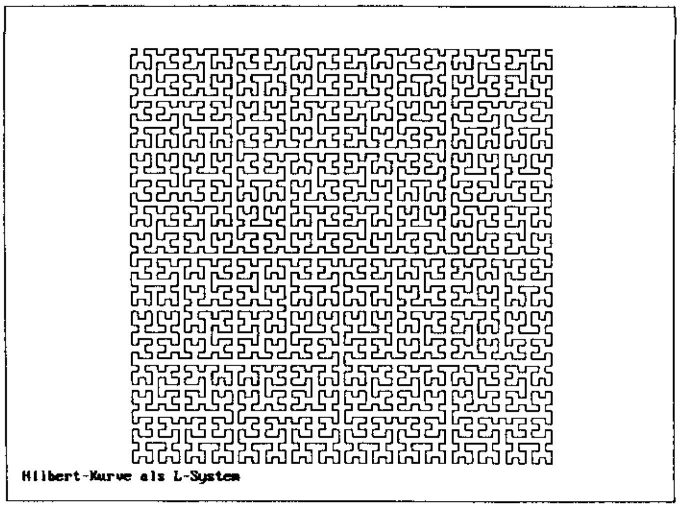


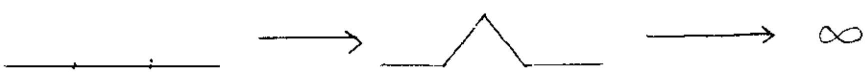
Figure 2: Hilbert curve

Hilbert.bas
Peano.bas

1.3 Koch Curve

Fractal Dimension $D = \ln 4/ln3 = 1.26 > 1$!

Koch.bas
Koch2,4,5.bas



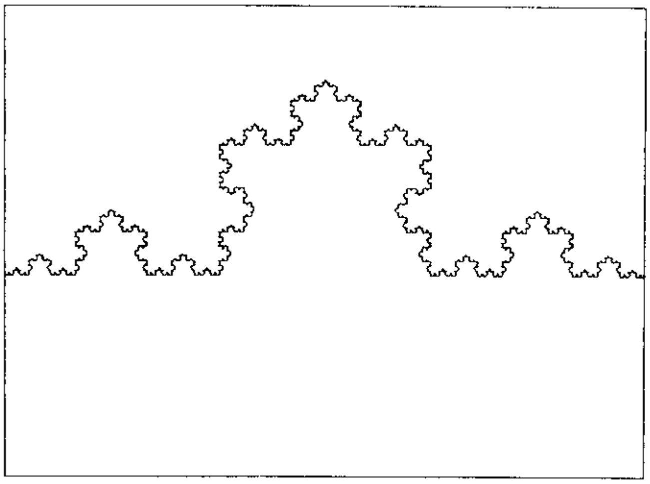


Figure 3: Koch curve

1.4 Sierpinski Triangle

$D = \ln 3 / \ln 2 = 1.58$

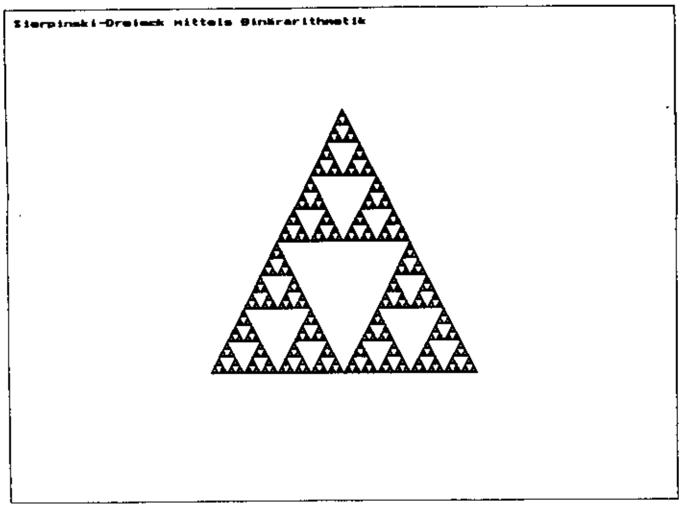


Figure 4: Sierpinski triangle

Sierpinski.bas (the simplest program possible, Peitgen et al., p. 135)
Sierpinski.c

1.5 Menger Carpet

Square instead of triangle, $D = \ln 8 / \ln 3 = 1.89$
Menger.bas

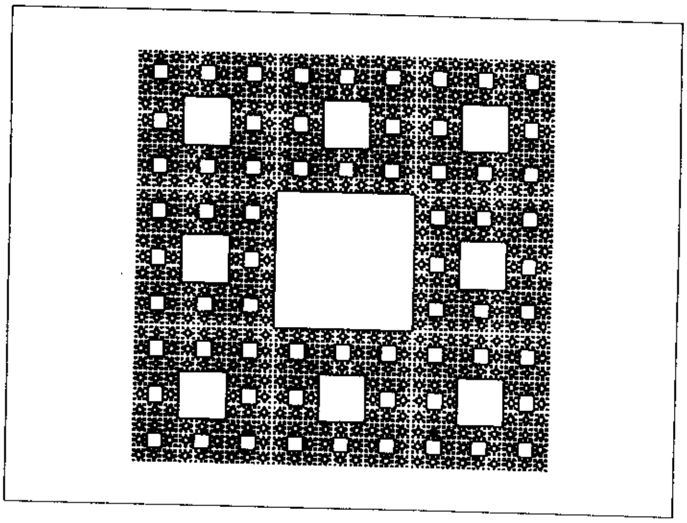


Figure 5: Menger carpet

1.6 Menger Sponge (Sierpinski Sponge)

Cube instead of square, fractal dimension $D = 2.73$
menger.nb

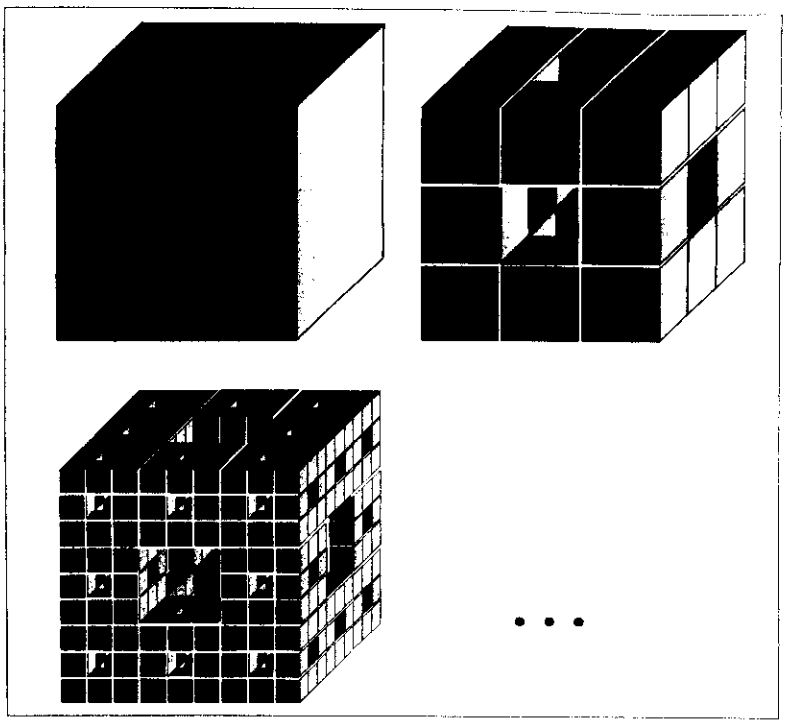


Figure 6: Menger sponge

2 Random Fractals

2.1 Random Walk

Brownian motion: Sum b in independent variables a

$$\begin{aligned} b_1 &= a_1 \\ b_2 &= a_1 + a_2 \\ b_3 &= a_1 + a_2 + a_3 \\ &\dots \end{aligned}$$

Mathematica-Command: FoldList

$vecA$ ... white noise (completely independent)
$vecB$ ... "brown noise" (random walk, dependent)
$vecC$ ... double summation (strongly dependent)

vec = vector ... discrete variable
continuous variables $A(t)$, $B(t)$, $C(t)$

$A(t)$ ... white noise, discontinuous

$B(t) = \int A(t)dt$ ... brown noise, continuous but not differentiable

$C(t) = \int B(t)dt = \iint A(t)dt\,dt$ ... differentiable

$A(t) \rightarrow B(t) \rightarrow C(t)$ increases smoothness

$A(t)$ ... Dimension $D = 2$ (fills an area)
$B(t)$ ... $D = 1.5$ (fractal)
$C(t)$ ... $D = 1$ (ordinary curve)

2.2 Spectral Methods

function $F(t)$
spectrum $S(f)$ $\quad f$ ... frequency

Fourier transformation:

$$S(f) = \int_0^{\infty} F(t) e^{2\pi i f t} dt \qquad \text{(FT)}$$

$$F(t) = \int_{-\infty}^{\infty} S(t) e^{-2\pi i f t} df \qquad \text{(inverse FT = IFT)}$$

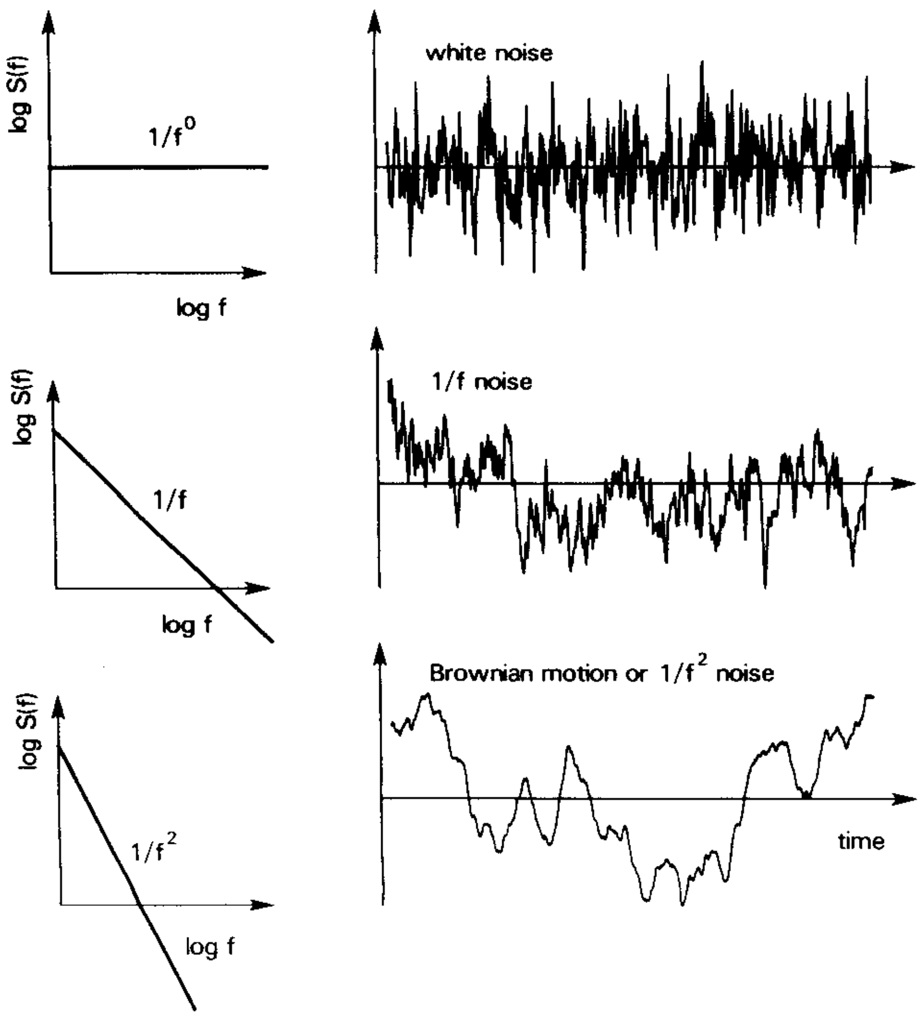
white noise $A(t) = X(t)$ spectrum (all frequencies occur equally)	$f^0 \equiv 1$
"colored noise": $Y(t)$, spectrum	$f^{-\beta}$
"brown" noise (Brownian movement)	f^{-2}

$0 \leq \beta \leq 4$

$\beta = 0$	white noise
$\beta = 1$	$1/f$ noise or "pink noise"
$\beta = 2$	"brown noise" ($\sim$ Brownian motion)
$\beta = 3$	"black noise"
$\beta = 4$	smooth curve ($\sim$ double summation)

$0 < \beta < 4$ fractal noise

Copy from Peitgen/Saupe p. 40



Samples of typical "noises", $V(t)$, the random variations of a quantity in time.
a. White noise, the most random.
b. $\frac{1}{f}$-noise, an intermediate but very commonly found type of fluctuation in nature, its origin is, as yet, a mystery.
c. Brownian motion or a random walk.
To the left of each sample is a graphical representation of the spectral density, $S_V(f)$, a measurement characterizing the time correlations in the noise.

Figure 7:

spectrum of white noise ... $S_w(f)$

spectrum of colored noise ... $S_c(f) = S_w(f)f^{-\beta}$

colored noise ... $F_c(t) = \int\limits_{-\infty}^{\infty} \left(S_w(f)f^{-\beta}\right) e^{-2\pi i f t} df$

principle extremely easy:

1. Compute spectrum of white noise by FT
2. Multiply this spectrum by $f^{-\beta}$
3. Compute colored noise by IFT

Practically: replace integrals by sums

Fast Fourier Transform = FFT

Brownian noise: $\beta = 2$
FFT instead of summation (sec. 1.2)

spectral dimension $\boxed{D = 2 - \beta/4}$

(Other definitions of spectral dimensions are also possible.)

white noise	$\beta = 0$	$D = 2$
brown noise	$\beta = 2$	$D = 1.5$
ordinary curve	$\beta = 4$	$D = 1$
pink noise ($1/f$)	$\beta = 1$	$D = 2.75$

ColoredProfiles.nb

3 Fractal Terrain Models

Spectral coefficients β as before

$0 \leq \beta \leq 4$

$\beta = 0$	...	two-dimensional white noise	$D = 3$
$0 < \beta < 4$	...	$2D$-fractal surface	$2 < D < 3$
$\beta = 2$	...	$2D$ Brownian surface	$D = 2.5$
$\beta = 4$	...	smooth surface	$D = 2$

dimension $\boxed{D = 3 - \beta/4}$ (3 instead of 2)

Same procedure:

0. Generate $2D$ white noise
1. Compute spectrum by $2D$-FFT

2. Multiply spectrum by $1/f^{\beta}$

3. Compute F by inverse FFT

Result: fractal surface of dimension $3 - \beta/4$

TerrainModels1.nb
TerrainModels2.nb

Part II

Nonlinear Dynamics

4 Lorenz Attractor

Butterfly with 2 somewhat "thick" wings.

From meteorology, describes turbulence in weather.

Equations (derivation in Turcotte, chapter 12)

$$\begin{array}{rcl@{\qquad}rcll} x' & = & -\sigma(x-y) & x & = & x(t) & \text{etc.} \\ y' & = & (r-z)x - y & x' & = & dx/dt & \text{etc.} \\ x' & = & xy - bz & \sigma & = & 3,\ r = 26.5,\ b = 1 & \text{Mathematica.} \\ & & & \sigma & = & 10,\ r = 28,\ b = 8/3 & \text{C} \end{array}$$

Nonlinearities zx and xy essential.

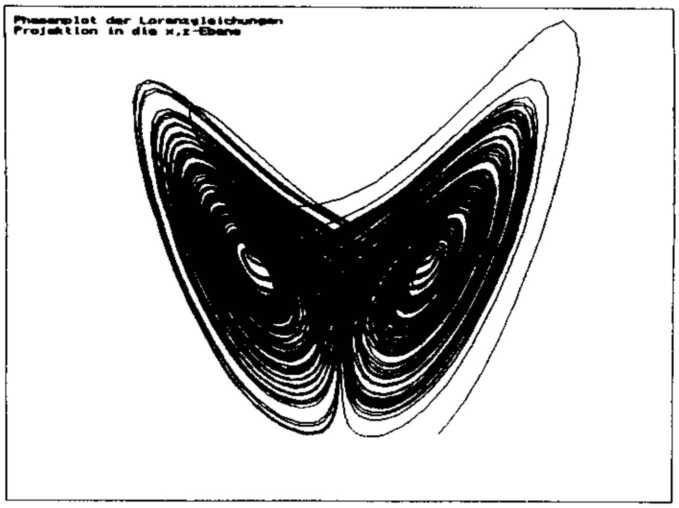


Figure 8: Lorenz Attractor

Lorenz.nb
Lorenz.c
Lorenz1.bas

Prediction soon becomes impossible ("butterfly effect"). Impossibility of weather fore-cast beyond 3–5 days.

Lorenz2.bas

"Chaos" or "turbulence":

1. **Intermittency:** quasiperiodicity with irregular bursts
2. **Positive Lyapunov exponents** λ

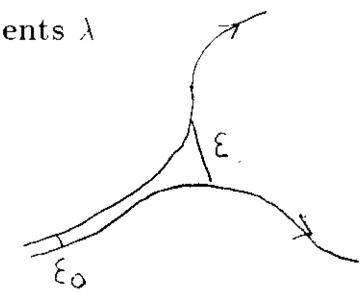


Figure 9:

$\epsilon = \epsilon_0^{\lambda t}$ exponential divergence of two neighboring trajectories

$\lambda > 0$ instability, divergence
$\lambda < 0$ stability, convergence

Intermit.c
Lyapunov.c
Lyapuno1.c

Computation of Lyapunov dimension (Kaplan–Yorke)

$\lambda_1 + \lambda_2 + \lambda_3 = -(\sigma + b + 1) = -(10 + 8/3 + 1) = -41/3$

- Program *Lyapuno1.c* gives $\lambda_1 = 0.882$
- $\lambda_2 = 0$ for every dissipative system without fixed point (Herrmann, pp. 86–88)
- $\lambda_3 = -\frac{41}{3} - 0.882 = -14.5$

Kaplan–Yorke Conjecture (Peitgen et al., pp. 709–743, Lorenz, p. 196)

fractal dimension $D = 2 - \frac{\lambda_1}{\lambda_3} = 2.06$

Dimension of Lorenz Attractor is $2.06 > 2$!!

"Strange attractor"

"Thick surface", cross–section = Cantor set (sec. 1.1).

Only for mathematicians:

Strange attractor = topological product of $1D$–interval (time) and Cantor set!

4.1 Geomagnetism

The Lorenz attractor occurs also in geomagnetism: model for the "core dynamo".

Magnetization of ocean floor:

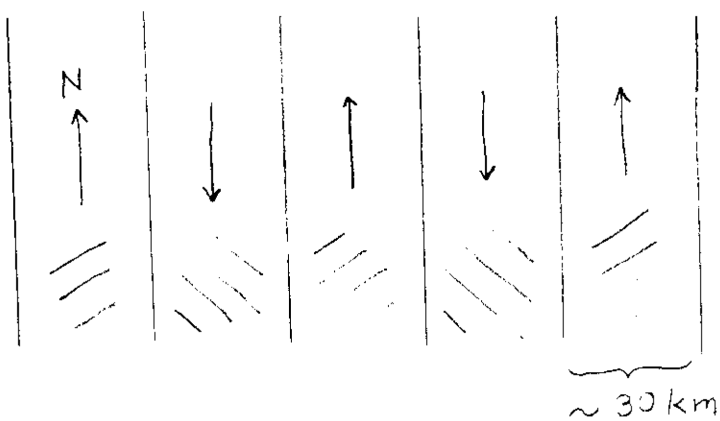


Figure 10:

Geomagnetic reversals about every 10^6 years.

Velocity of continental plates about

$$\frac{30 \times 10^3\text{m}}{10^6 \text{ years}} = 3\text{cm/year}$$

Differential equations from Turcotte, p. 165

Geodynamo.nb

4.2 Roessler Attractor

A second "strange attractor" with similar nonlinearities and fractal dimension

Roessler.c

5 Standard Map

Standard Map Poincaré–Birkhoff–Kolmogorov–Arnold–Boris Shirikov: Perturbation ($k > 0$) of a simple system ($k = 0$). Idealized Poincaré section of a continuous dynamic system. By using a Poincaré section, differential equations may be discretized:

$$\left.\begin{array}{lcll} dx & \doteq & x_{n+1} - x_n & \doteq \\ dt & = & 1 & = \\ \frac{dx}{dt} & \doteq & x_{n+1} - x_n & \doteq \end{array}\right\}!!$$

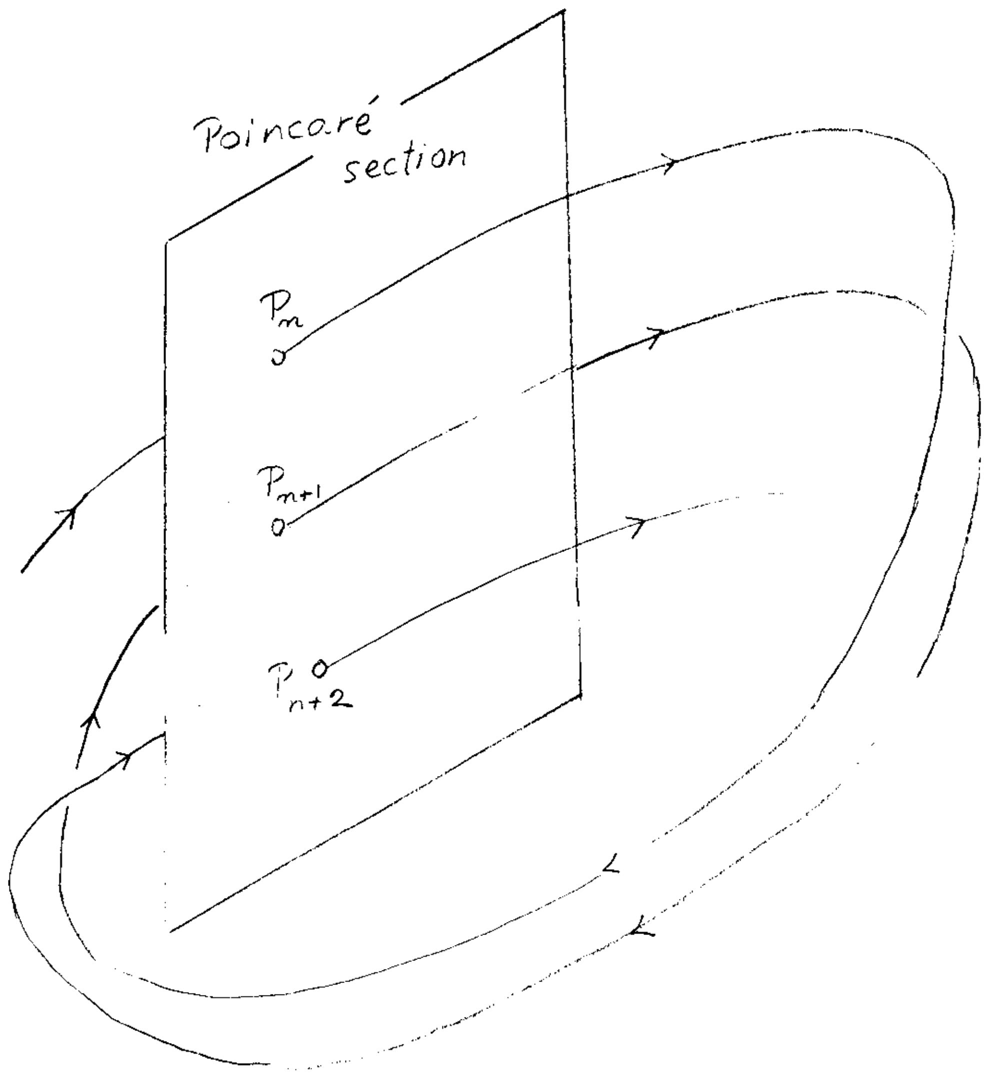


Figure 11:

Poincaré map $P_n \to P_{n+1} \to P_{n+2} \to \dots$

equations:

$$\begin{aligned} x_{n+1} &= x_n + y_n + k \sin x_n \\ y_{n+1} &= y_n + k \sin x_n \end{aligned}$$

$k = 0$ unperturbed
$k > 1$ perturbed

"nicest" $k = 1$ $k > 1 \to$ more and more chaotic

Shirikov.c (many randomly distributed trajectories)

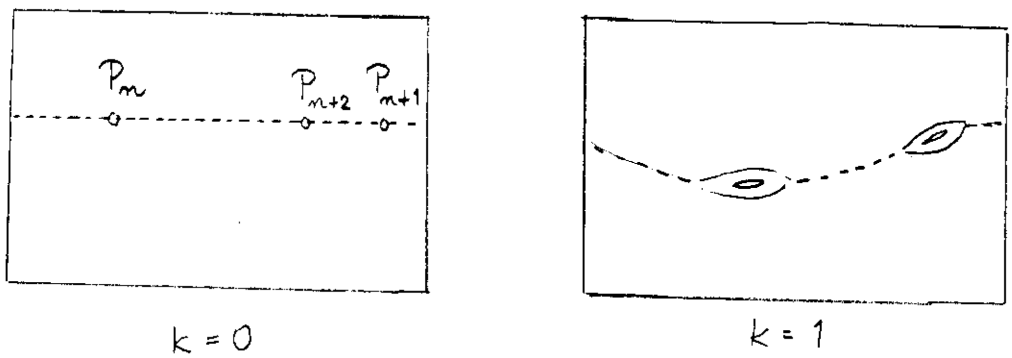


Figure 12:

$k = 0$ "trajectory" consists of Poincaré points of 3D trajectory. Poincaré points form a line.

$k = 1$ Poincaré points form a "thick line" with insular structures:

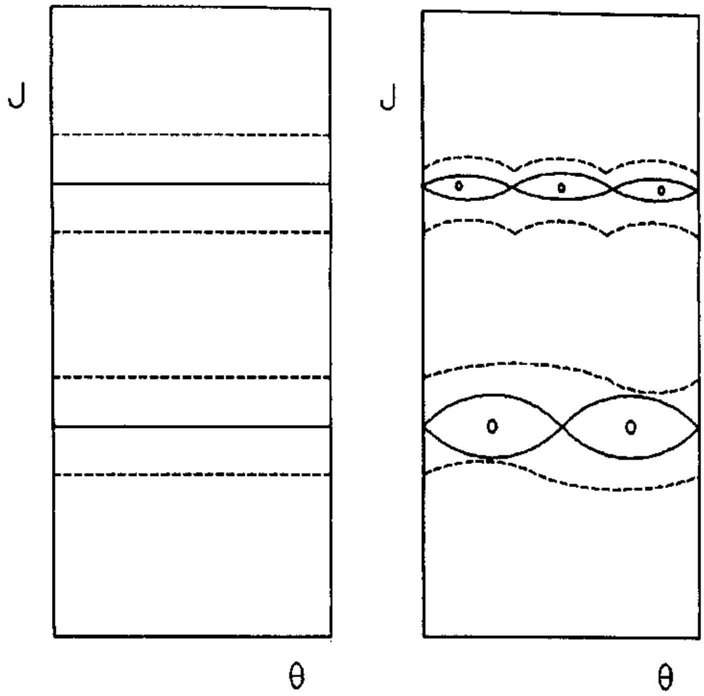


Figure 13:

Figure 14:

KAM Theorem (Kolmogorov – Arnold – Moser). For sufficiently small k, "most" trajectories remain intact; only a "minority" breaks up into insular structures (to $k = 1$).

With increasing k, more and more trajectories break up and become chaotic (e.g. $k \geq 1.5$).

Programs for 1 trajectory only

Shiriko0.c	$k = 0$
Shiriko1.c	$k = 0.5$
Shiriko2.c	$k = 1$
Shiriko3.c	$k = 1.5$

Play with different k until $k = 4$ (only one number must be changed in program!)

6 Map of Mira

figura mirabilis: wonderful picture (but Mira is the name of a scientist!).

Change of parameters a and b produce completely different pictures

$$\begin{aligned} x_{n+1} &= by_n + F(x_n) \\ y_{n+1} &= -x_n + F(x_{n+1}) \\ \text{with} & \\ F(x) &= ax - (1-a)\frac{2x^2}{1+x^2} \end{aligned}$$

Mira.cpp
Mira1-7.cpp

Play with different values of a and b.

7 Arnold's Cat Map and Sinai's Perturbation

Cat Map

Sinai.c

as perturbation of the cat map

More EXAMPLES for KAM Theory

Billiard.exe (Birkhoff)
3-Disk.exe Scattering on 3 disks

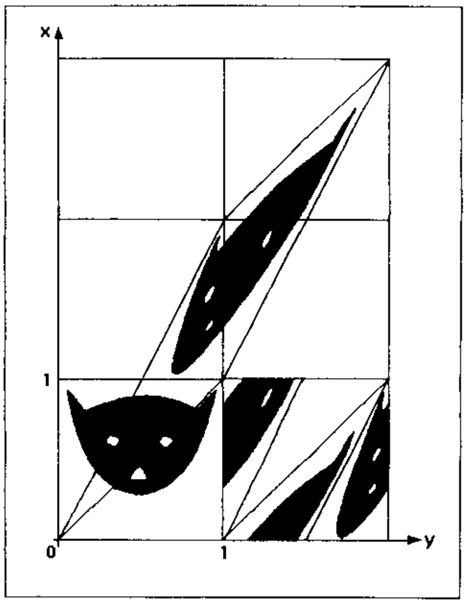


Figure 15: Cat Map

Part III

Stick–Slip Models for Earthquakes and Their Mathematical Environment

8 Slider–Block Models

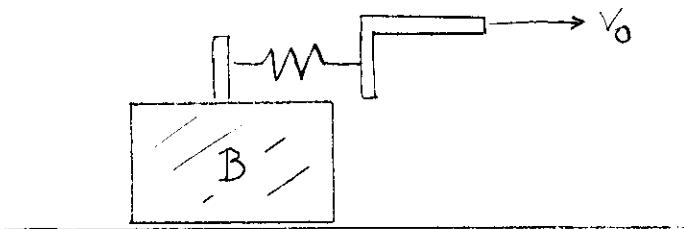


Figure 16:

Block B (representing a continental plate) is pulled with constant velocity v_0, with spring S between; there is friction between B and the floor. At first it sticks to the floor; the velocity v_0 is absorbed by the spring S. Only if the pull of the spring reaches a certain value F_s (static friction)

$$F_s = kx_{max} = kv_0 t_{max} \qquad (x = v_0 t) \tag{1}$$

(k ... spring constant, x_{max} ... maximum elongation) will the friction be overcome, and we shall have an earthquake. Ideally, B then moves without friction, performing an elastic "harmonic" motion (earthquake wave)

$$mx'' + kx = F_d = \text{const.} \qquad (x'' = d^2x/dt^2) \tag{2}$$

where F_d is the dynamic friction. With suitable values for the constants, we have

$$x = v_0 t \qquad \text{(constant plate motion)} \tag{3}$$

$$x'' + x = 1 \qquad \text{(harmonic earthquake wave)} \tag{4}$$

The appropriate solution of (3) is

$$x = 1 + \cos t \quad , \tag{5}$$

$$x' = -\sin t \quad . \tag{6}$$

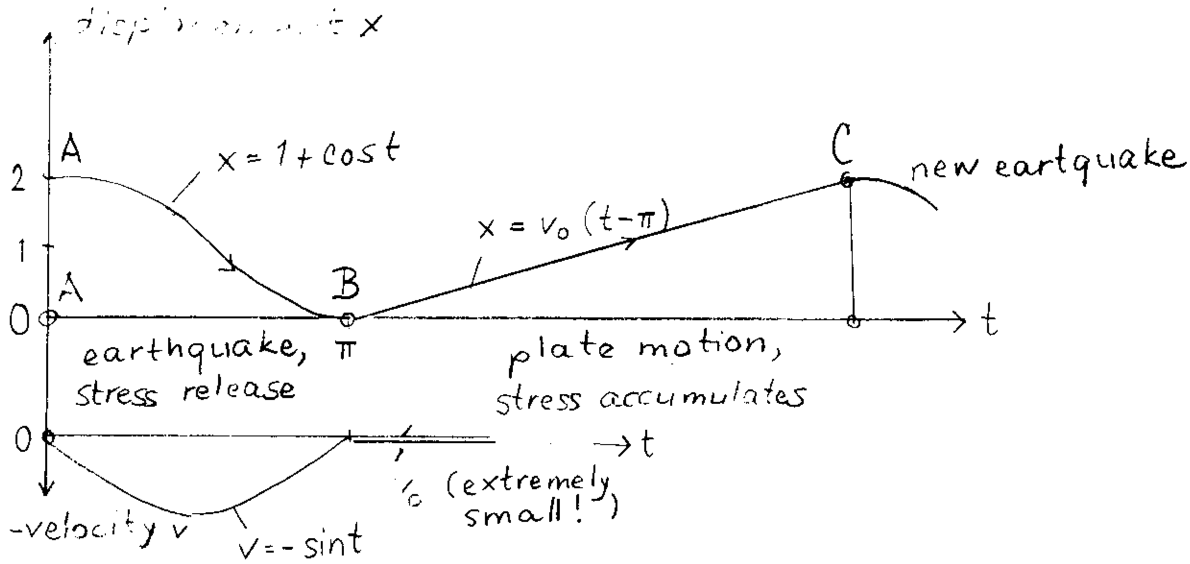


Figure 17:

Starting from point A with $x = 2$ and $v = 0$, after time $\pi = 3.14\ldots$ the point B with $x = 0$, $v = 0$ is reached. At rest: $v_B = 0$, $x_B = 0$, the static friction F_s takes hold again and moves the block with constant velocity,

$$x = v_0(t - \pi) \tag{7}$$

if time starts at A and reaches B at $t = \pi$. At D, the maximum elongation 2 ist reached again, slip ("earthquake") starts again with the harmonic motion (5). Thus the combined motion (earthquake release AB and plate motion BC) is periodic: $T = AC$. Very simplified indeed! Since v_0 and hence $\pi v_0 \doteq 3.14 v_0$ is very small ($AB << BC$), (7) is practically identical to (3).

It will be useful to consider the corresponding "phase diagram" in a xv-"phase plane".

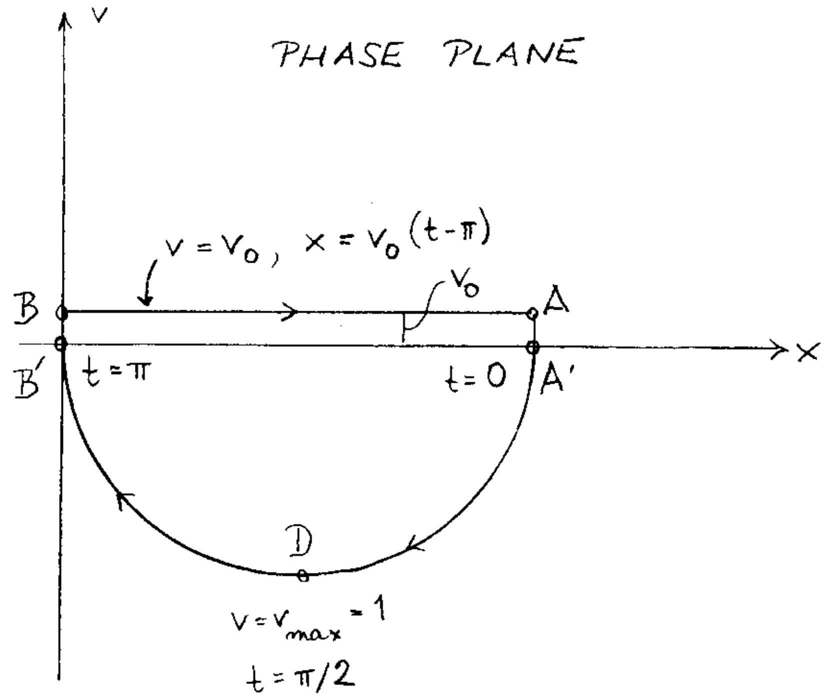


Figure 18:

Since $v_0 << 1$, A' and B' can practically be identified with A and B, respectively.

The motion starts at A (or rather A') with the sinusoidal release of the earthquake, reaching maximum (or rather minimum) velocity at D. It slows down, reaching $v = 0$ at B. Then we have again uniform plate motion (7). In this phase plane, the periodicity is evident: we have a cycle $AA'DB'BA$.

The uniform plate motion (7) satisfies the differential equation

$$x'' = 0 \qquad \text{(plate motion)} \quad . \tag{8}$$

Together with (4), we thus have

$$x'' + f(t)x = f(t) \tag{9}$$

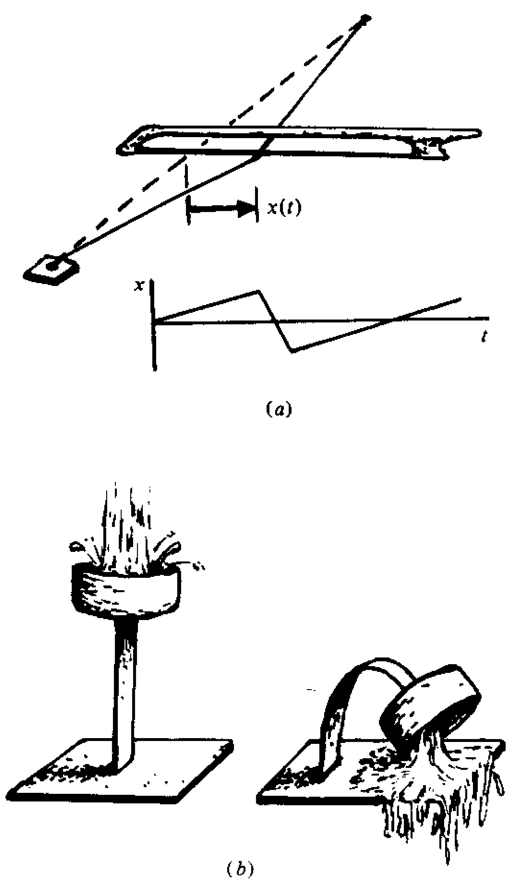
with the discontinuous "Heaviside function"

$$\begin{array}{lll} f(t) = 1 \quad , & 0 \leq t \leq \pi & \text{(earthquake wave)} \\ f(t) = 0 \quad , & \pi \leq t \leq T & \text{(plate motion)} \end{array} \tag{10}$$

Thus the innocent differential equation (9) is heavily discontinuous and thus strongly nonlinear! But remember the cycle $ADBA$.

9 Analogy with Violin Strings etc.

The problem is mathematically identical to producing a harmonic sound of a violin string by producing a uniform movement of the bow. The plate motion corresponds to the uniform stroke of the bow, and the earthquake to its harmonic sound ... See Fig. (a). Similar is the situation in Fig. (b) and in electromagnetic oscillations. Cf. Jackson vol. 1, pp. 290-291. Even the firing of neurons may be described by a similar equation.

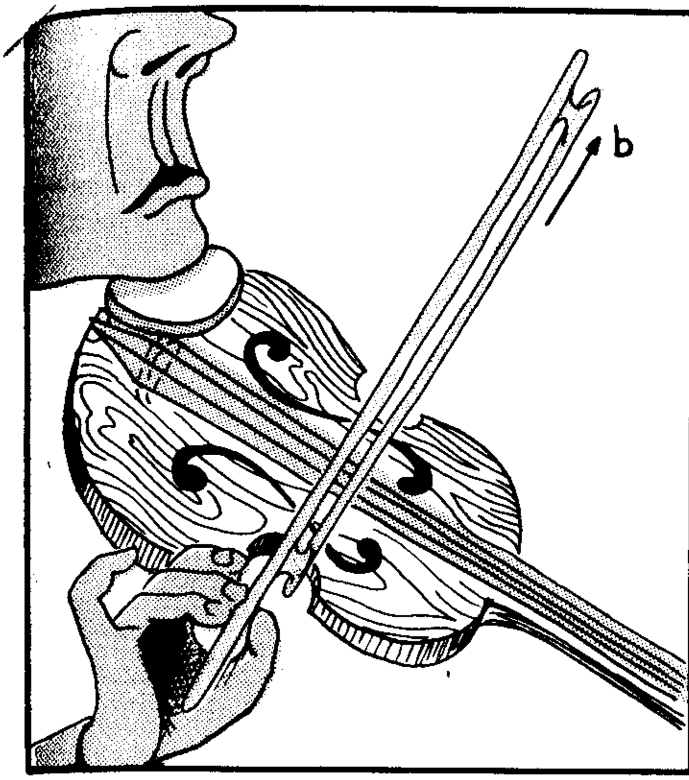


(a)

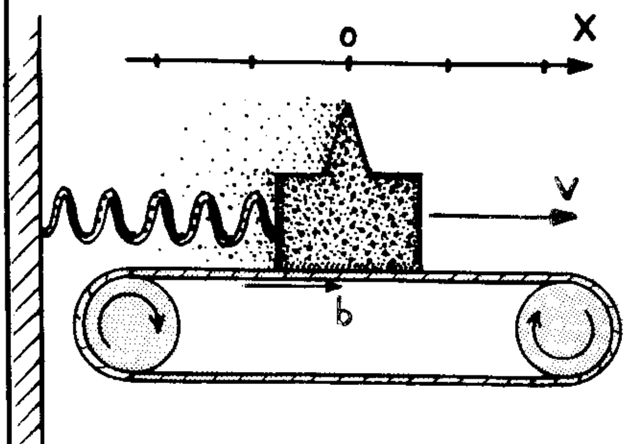
(b)

Figure 19:

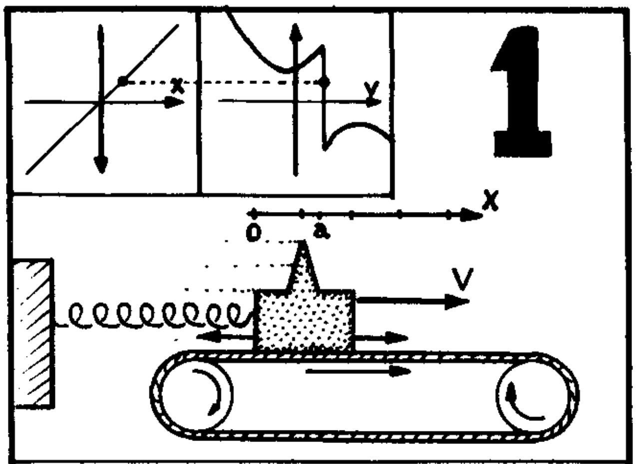
We copy p. 95 and 100–102 from Abraham and Shaw.



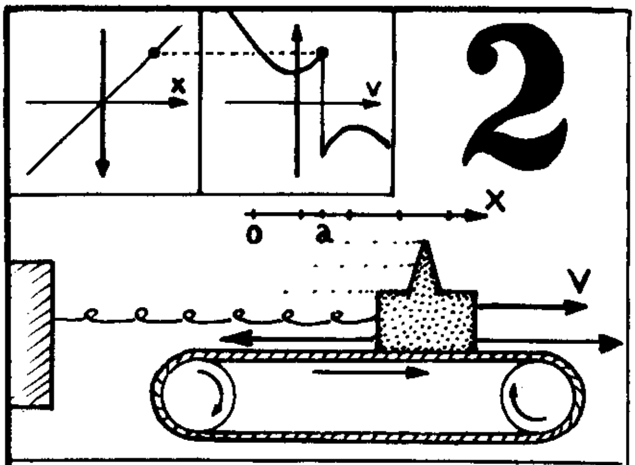
The violinist sustains the vibration by putting energy into the string with the bow. The friction of the bow on the string depends on the rate of bowing. We introduce a new symbol, *b*, to denote the rate of drawing the bow across the string.



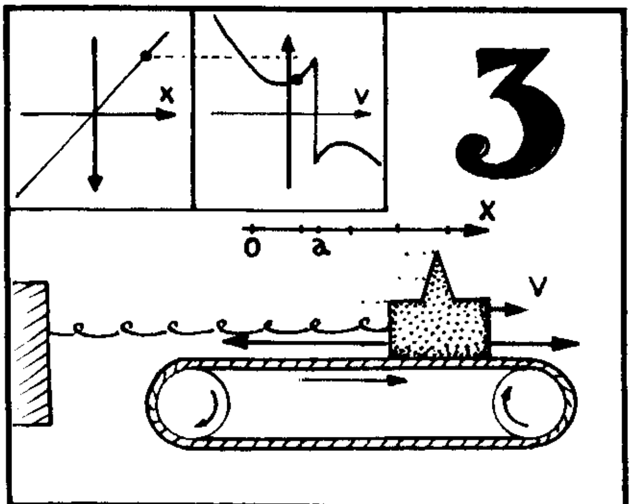
The spring model may be simply modified to include the action of the bow. Replace the tabletop on which the spring slides by a conveyor belt. This represents the bow. The weight, as before, represents the violin string.



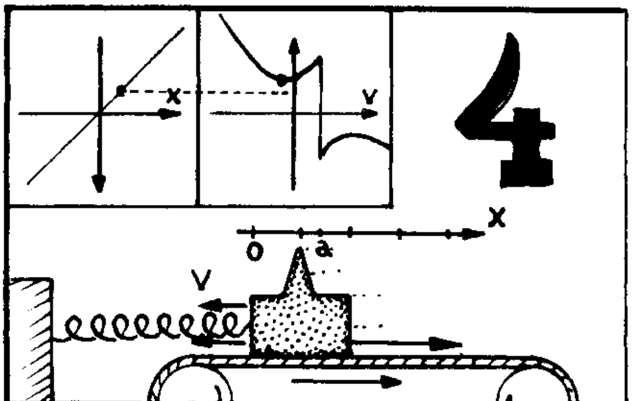
1. STUCK. The friction and spring forces are balanced and the weight is stuck to the belt. The weight is a little to the the right of zero (equilibrium of the spring) but not as far as the critical point.



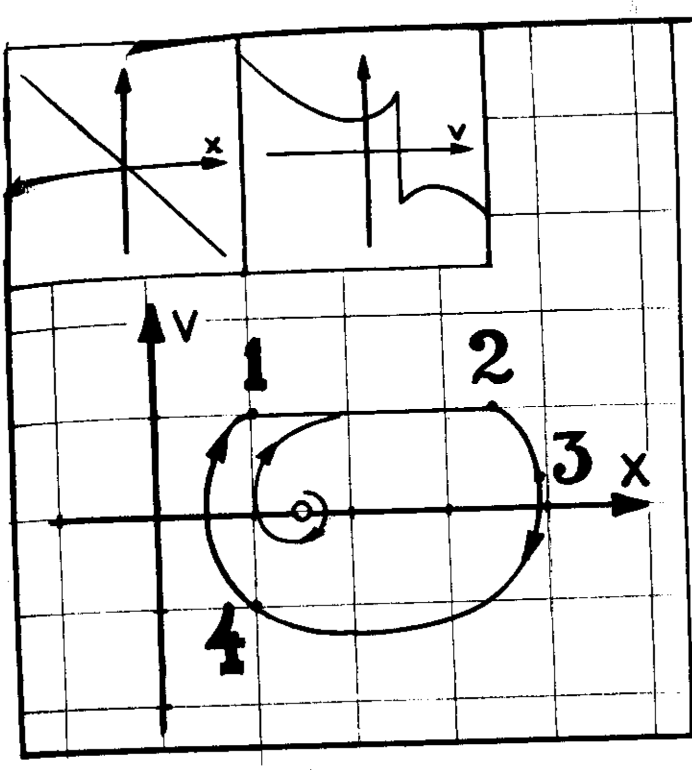
2. BEGINNING TO SLIP. The friction and spring forces are balanced, but larger, as the displacement increases. At the critical force for friction, slipping begins.



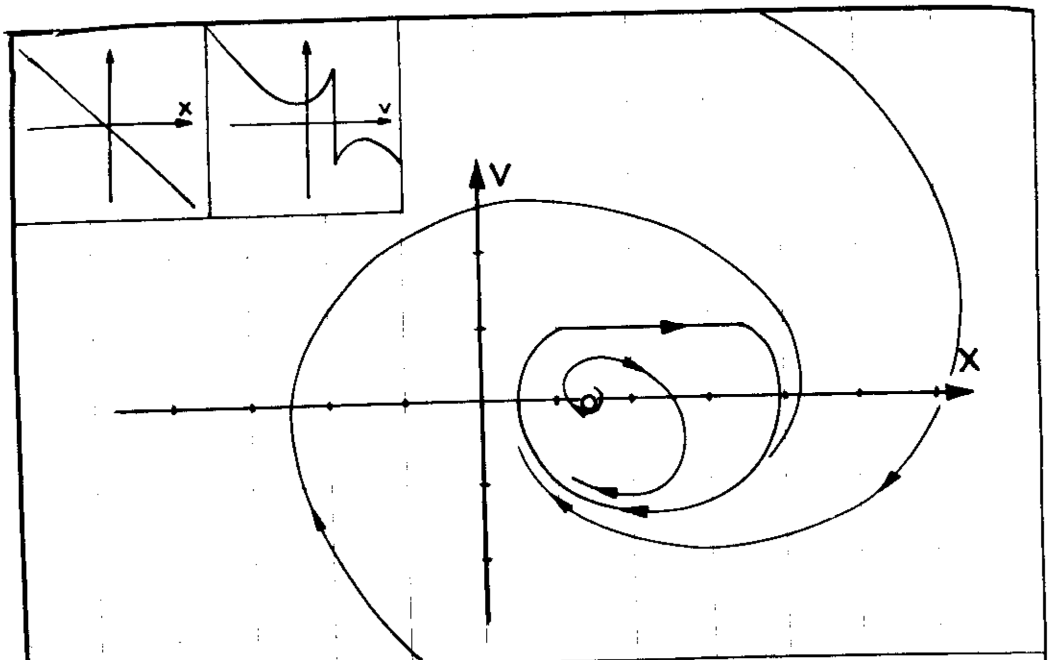
3. SLIPPING. When the velocity begins decreasing, while x is still increasing, then the sudden drop in the friction yields the tug of war to the spring. The acceleration is negative. Rightward motion slows to a halt, and the weight begins to move back to the left.



4. GRABBING. When the leftward motion has decreased the spring force to a value smaller than the slipping friction, the tug of the belt wins once more. Motion to the left slows, and the weight turns and begins once again to move to the right. When the velocity reaches the critical value (the red dot on the glitch) in the friction function, slippage is going to happen again (return to 1).



Here are the four stages in the cycle, located in the phase portrait. The rest of the limit cycle has been interpolated. The flat part at the top corresponds to the stuck phase. The repelling critical point is inside the cycle. Initial states near this repellor will spiral outward, clockwise, approaching the self-sustaining oscillation.



Returning to normal scale, here is the complete phase portrait for the dynamic model for the bowed violin string.

Figure 20:

By the way, if you don't have a violin, use a thin wine glass and rub the rim uniformly with your finger. A surprisingly pure sound will arise.

The program

Violin0.nb

ends with a failure: the appropriate differential equation cannot be integrated even with the powerful Mathematica function NDSolve[ ]. The reason is the discontinuity.

We therefore simplify the differential equation in

Violin.nb

to get a beautiful cycle (drawn in blue) corresponding to the cycle $ADBA$ in an earlier figure. This problem is similar to Rayleigh's equation.

10 Rayleigh Oscillator

The original Rayleigh equations are

$$x'' + a_1 x + a_3 x^3 + cx' = 0 \qquad (x' = dx/dt)$$

or

$$\begin{aligned} x' &= y \\ y' &= -x + ay - by^3 \quad . \end{aligned}$$

For special values of the constants we have

$$\begin{aligned} x' &= y \\ y' &= -x + y - y^3/3 \quad . \end{aligned}$$

Rayleigh.nb

Very similar to *Violin.nb* but simpler. Beautiful cycle.

11 Duffing–Ueda Attractor

The similar equation (Duffing)

$$x'' + ax' + cx + dx^3 = b\cos\omega t \qquad (x' = dx/dt)$$

or

$$\begin{aligned} x' &= y \\ y' &= -(cx + x^3 + ay) + b\cos\omega t \end{aligned}$$

for $a = 0.1$, $c = 0$, $b = 12$, $\omega = 1$

have a chaotic attractor (Duffing attractor, Ueda attractor, Japanese attractor) occurs in electromagnetic oscillators.

Ueda.c
Ueda.nb

12 Ikeda Attractor

Similar form but completely different mathematical structure.

Discrete difference equations (much faster!)

$$\begin{aligned} x_{n+1} &= a + b(x_n \cos t + y_n \sin t) \\ y_{n+1} &= b(x_n \sin t + y_n \cos t) \\ t &= c - d/\left(1 + x_n^2 + y_n^2\right) \end{aligned}$$

$a = 0.85$, $b = 0.9$, $c = 0.4$, $d = 9.0$

Occurs in laser optics ...

Ikeda.c

Remark: relation between differential and difference equations

$$\left.\begin{array}{lcll} dx & \doteq & x_{n+1} - x_n & \doteq \\ dt & = & 1 & = \\ \frac{dx}{dt} & \doteq & x_{n+1} - x_n & \doteq \end{array}\right\} !!$$

Cf. sec. 5.

13 A Ringshaped Laser Attractor

$$z_{n+1} = a + k z_n \exp(i|z_n|^2)$$

$$z = x + iy\,, \quad a = 5\,, \quad k = 0.2$$

14 More Nonlinear Equations

General simple mathematical background for earthquake theory.

14.1 Birkhoff–Shaw

Interesting because chaotic "cycle" attractor

$$\begin{aligned} x' &= 0.7y + 10x(0.1 - y^2) \qquad (x' = dx/dt \text{ as usual}) \\ y' &= -x + 0.25\sin(1.57t) \end{aligned}$$

Also called "Birkhoff's bagel" (Abraham and Shaw, pp. 275–282).

BirkhoffShaw.nb

A simple chaotic model for earthquake generation??

14.2 Duffing and Van der Pol

Only for theoretical interest as classical nonlinear differential equations

Duffing.nb

$$\begin{aligned} x' &= y \\ y' &= x - x^3 - \epsilon y + \gamma\cos(\omega t) \end{aligned}$$

Van-der-Pol.nb

$$x'' - \epsilon(1 - x^2)x' + x = k\cos(\omega t)$$

or

$$\begin{aligned} x' &= y \\ y' &= \epsilon(1 - x^2)y - x + k\cos(\omega t) \end{aligned}$$