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Donaldson invariants in Algebraic Geometry

L. Göttsche

Mathematics Section
The Abdus Salam I.C.T.P.
P.O. Box 586
Strada Costiera 11
34014 Trieste
Italy

Abstract

In these lectures I want to give an introduction to the relation of Donaldson invariants with algebraic geometry: Donaldson invariants are differentialble invariants of smooth compact 4-manifolds X, defined via moduli spaces of anti self-dual connections. If X is an algebraic surface, then these moduli spaces can for a suitable choice of the metric be identified with moduli spaces of stable vector bundles on X. This can be used to compute Donaldson invariants via methods of algebraic geometry and has lead to a lot of activity on moduli spaces of vector bundles and coherent sheaves on algebraic surfaces.

We will first recall the definition of the Donaldson invariants via gauge theory. Then we will show the relation between moduli spaces of anti self-dual connections and moduli spaces of vector bundles on algebraic surfaces, and how this makes it possible to compute Donaldson invariants via algebraic geometry methods. Finally we concentrate on the case that the number b_+ of positive eigenvalues of the intersection form on the second homology of the 4-mainfold is odd. In this case the Donaldson invariants depend on the metric (or in the algebraic geometric case on the polarization) via a system of walls and chambers. We will study the change of the invariants under wall-crossing, and use this in particular to compute the Donaldson invariants of rational algebraic surfaces.

Keywords: Donaldson invariants, moduli spaces of sheaves.

Contents

1 Introduction

Donaldson invariants have played an important role in the study and classification of compact differentiable 4-manifolds X. Discrete invariants of 4-manifolds are the fundamental group $\pi_1(X)$ and the intersection form on $H_2(X,\mathbb{Z})$. If X is simply connected, then the homotopy type of X is essentially determined by the intersection form. Friedman showed that X is determined up to homeomorphism by its homotopy type.

In order to attempt to make a differentiable classification ones needs additional invariants. The Donaldson invariants are defined in terms of moduli spaces of anti self-dual connections on differentiable bundles on X. If X is an algebraic surface, then theses moduli spaces can be identified with moduli spaces of stable bundles on X. This makes it possible to apply methods of algebraic geometry to compute the Donaldson invariants. In fact, because of this, for a long time most of the calculations of Donaldson invariants have been carried out in the case of algebraic surfaces. On the other hand the Donaldson invariants have provided a lot of interest for the study of moduli spaces of vector bundles and coherent sheaves on algebraic surfaces.

Some results obtained with Donaldson invariants are:

1. Algebraic surfaces are essentially indecomposable: If an algebraic surface X is the connected sum $X = Y\#Z$ of two 4 manifolds, then either Y or Z must have negative definite intersection form. An example, where this happens, is if X is the blowup of Y in a point.

2. The differentiable classification of elliptic surfaces.

3. The Kodaira dimension of an algebraic surface is a differentiable invariant.

Recently the Seiberg-Witten invariants have appeared, which are also defined via gauge theory, but are often easier to compute [W],[D2]. A number of conjectures from Donaldson theory were immediately proved e.g.

1. The plurigenera of algebraic surfaces are differentiable invariants.

2. The generalized Thom conjecture: Let X be an algebraic surfaces, then each smooth algebraic curve in X minimizes the genus of embedded 2-manifolds in its homology class.

Conjecturally the Donaldson- and Seiberg-Witten invariants are very closely related and in particular the Donaldson invariants can be computed in terms of the Seiberg-Witten invariants.

Since the appearence of the Seiberg-Witten invariants the interest in the Donaldson invariants has become a bit less, but there is still a large number of interesting open questions.

2 Definition and properties of the Donaldson invariants

In this lecture we define the Donaldson invariants via gauge theory and state some of their most important properties. We prefer here to formulate everything in terms of vector bundles which should be more familiar to the audience instead of principal bundles, which would be more natural.

2.1 Moduli spaces of connections

Let X be a smooth simply connected compact oriented 4-manifold. Let P be a principal $SU(2)$ or $SO(3)$ bundle on X. The Donaldson invariants are defined via intersection theory on a moduli space of anti selfdual connnections on P.

$SU(2)$ bundles on X are classified by the second Chern class $c_2(P)$. $SO(3)$ bundles on X are classified by the second Stiefel-Whitney class $w_2(P) \in H^2(X, \mathbb{Z}/2)$ and the first Pontrjagin class $p_1(P) \in H^4(X, \mathbb{Z})$.

In the $SU(2)$-case the moduli space of anti self dual connections on P can be identified with the moduli space of anti selfdual connections on the associated complex vector bundle on E with first Chern class $c_1 = 0$. In the $SO(3)$-case (after choosing a lift $c_1 \in H^2(X, \mathbb{Z})$ of $w_2(P)$ it corresponds to a moduli space of Hermitian Yang-Mills connections on the associated complex vector bundle with Chern classes c_1, c_2 (with $c_1^2 - 4c_2 = p_1(P)$). For simplicity we will in the following concntrate on the $SU(2)$-case.

Let E be a rank 2 complex differentiable vector bundle on X. We fix a hermitian metric h on E. (That is for each $x \in X$ we have a hermitian inner product on the fibre $E(x)$, varying smoothly with x.) We denote by $\Omega^i(E)$ the space of $\mathbb{C}^\infty$ sections of $E \otimes \Lambda^i T_X^*$.

A hermitian connection on E is a connection $D : \Omega^0(E) \to \Omega^1(E)$, which is compatible with h. (That D is a connection means that it is a linear map satisfying the Leibnitz rule $D(f \cdot s) = d(f \otimes s) + f \cdot D(S)$ and that D is compatible with the metric means furthermore that $d(h(s_1, s_2)) = h(D(s_1), s_2) + h((s_1, D(s_2)).)$ D is called *reducible* if E is the direct sum $L_1 \oplus L_2$ of two line bundles, and $D = D_1 \oplus D_2$ with D_i a connection on L_i.

We write $\mathcal{A}(E)$ for the space of hermitian connections on E, (which are trivial on $det(E)$ in the case $c_1(E) = 0$ and equal to a fixed connection on $det(E)$ otherwise). $\mathcal{A}^*(E) \subset \mathcal{A}(E)$ is the subspace of irreducible connections.

The *gauge group* $\mathcal{G}$ is the set of $\mathbb{C}^\infty$ automorphisms of E, which are compatible with h and act as the identity on $det(E)$. $\mathcal{G}$ acts on $\mathcal{A}$ and $\mathcal{A}^*$ via

$$\begin{array}{ccc} \Omega^0(E) & \xrightarrow{D} & \Omega^1(E) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ \Omega^0(E) & \xrightarrow{\alpha(D)} & \Omega^1(E) \end{array}$$

Let $\mathcal{B}(E) := \mathcal{A}(E)/\mathcal{G}$, $\mathcal{B}^*(E) := \mathcal{A}^*(E)/\mathcal{G}$.

2.2 ASD connections

We assume in this part that the first Now fix a Riemannian metric g on X. It gives rise to a Hodge star operator

$$*_g : \Lambda^2 T_X^* \to \Lambda^2 T_X^*, \quad *_g^2 = 1.$$

We write Λ_+ for the $(+1)$-eigenbundle and Λ_- for the (-1)-eigenbundle.

Definition 2.1 For $D \in \mathcal{A}^*(E)$, let $F(D) = D \circ D \in \Omega^2(End(E))$ be it's curvature. F is called anti self-dual (ASD), if

$$*F(D) = -F(D).$$

In other words, writing $F(D) := F_-(D) + F_+(D)$, with $F_-(D)$ a section of $\Lambda_-(End(E))$ (and similarly for $F_+(D)$), the condition is $F_+(D) = 0$.

We write $N_g(E)$ for the moduli space

$$\{ \text{ ASD-connections on } E\}/\mathcal{G} \subset \mathcal{B}^*(E).$$

In the case $c_1(E) \neq 0$ we have instead to take the moduli space of Hermitian Yang-Mills connections on E, because only these correspond to the moduli space of ASD connections on the corresponding principal bundle.

The differentiable type of E is determined by its Chern classes $c_1(E)$, and $c_2(E)$. Therefore we also write $N_g(c_1, c_2)$ for $N_g(E)$.

If D is an ASD (or Hermitian Yang Mills in case $c_1(E) \neq 0$ connection on E, then by Chern-Weil theory

$$4c_2(E) - c_1^2(E) = -p_1(E) = \frac{1}{4\pi^2} = \int_X tr(F(D) \wedge F(D)) = \int_X \|F_-(D)\|^2 > 0.$$

Let $b_+(X)$ be the number of positive Eigenvalues of the intersection form on $H_2(X, \mathbb{R})$. We write

$$k := (c_2 - c_1^2/4)(E)$$

Then we have the following generic smoothness result:

Theorem 2.2 *If g is generic, then $N_g(E)$ is a smooth manifold of dimension*

$$2d = 8k - 3(1 + b_+(X)).$$

For a generic path g_t of metrics, the corresponding parameterized moduli space is smooth.

Furthermore $N_E(g)$ is orientable. The orientation depends on the choice of an orientation of a maximal dimensional subspace $H_+^2(X, \mathbb{R})$ of $H^2(X, \mathbb{R})$ on which the intersection form is positive definite.

2.3 Relations to holomorphic vector bundles

Assume here, that $c_1(E) = 0$. Let X be a projective algebraic surface. Let H be an ample divisor. Let $g(H)$ be the corresponding Hodge metric and ω the Kähler form. We write $\Lambda^{p,q}$ for the bundle of (p, q) forms. Then we get

$$\Lambda_+ = \Re(\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R}\omega$$
$$\Lambda_- = \omega^\perp \qquad \text{in } \Lambda^{1,1}.$$

We can write $D := \partial_D + \overline{\partial}_D$, where $\partial_D : \Omega^0(E) \to \Omega^{1,0}(E)$ and $\overline{\partial}_D : \Omega^0(E) \to \Omega^{0,1}(E)$. So we get

$$F(D) = \partial_D^2 + (\partial_D\overline{\partial}_D + \overline{\partial}_D\partial_D) + \overline{\partial}_D^2.$$

D is ASD if

1. $\overline{\partial}_D^2 = 0.$

2. $\partial_D^2 = 0$ and $F(D) \wedge \omega = 0.$

1. means that $\overline{\partial}_D$ defines a holomorphic structure on E. 2. implies after some work that $(E, \overline{\partial}_D)$ is μ-stable with respect to H.

Recall that a vector bundle E of rank 2 on an algebraic surface X is called μ-stable (slope stable) with respect to an ample divisor H, if

$$Hc_1(L) < \frac{Hc_1(E)}{2}$$

for all locally free subsheaves L of rank 1 of E. We denote by $M_H^X(c_1, c_2)^\mu$ the moduli space of μ-stable rank 2 bundles on X with Chern classes c_1 and c_2. We have motivated that (at least in case $c_1 = 0$) there is map

$$\Psi : N_{g(H)}(c_1, c_2) \to M_H^X(c_1, c_2)^\mu.$$

In fact this map exists for any c_1 and furthermore we get:

Theorem 2.3 *(Donaldson) Ψ is a homeomorphism.*

This will give a relation between the Donaldson invariants (which we will define via moduli spaces of ASD connections) and moduli of vector bundles.

2.4 Uhlenbeck compactification

We want to define the Donaldson invariants as intersection numbers on $N_g(E)$ which is usually not compact. We have therefore to compactify.

Theorem 2.4 *Let $(A_i)_i$ be a sequence in $N_g(E)$. After passing to a subsequence we obtain: There is a finite collection of points $p_1, \dots p_l \in X$ with multiplicities $n_1, \dots, n_l > 0$, such that up to gauge transformation $A_i|_{X \setminus \{p_1,\dots,p_l\}}$ converges to an ASD connection A_∞. A_∞ can be extended to an ASD connection on a vector bundle E' with*

$$det(E') = det(E), \quad c_2(E) = c_2(E') + \sum_{i=1}^{l} n_i.$$

This leads to the Uhlenbeck compactification:

Theorem 2.5 *There exists a topology on*

$$\coprod_{n \geq 0} N_g(c_1, c_2 - n) \times X^{(n)}.$$

such that the closure $\overline{N}_g(c_1, c_2)$ is compact. (Here $X^{(n)}$ denotes the n-th symmetric power of X).

2.5 Definition of the invariants

We write $H^*(X) := H^*(X, \mathbb{Q})$ and $H_*(X) = H_*(X, \mathbb{Q})$. If on $X \times N_g(E)$ there exists universal bundle $\mathcal{E}$ with a universal connection $\mathcal{D}$ with $\mathcal{D}|_{X \times \{D\}} = D$, then we can define the μ-map as follows.

$$\mu : H_*(X) \to H^*(N_g(E)); \quad \mu(\alpha) = -\frac{1}{4} p_1(\mathcal{E})/\alpha.$$

Here $-\frac{1}{4}p_1(\mathcal{E}) = c_2(\mathcal{E}) - c_1(\mathcal{E})^2/4$, and the slant product $-\frac{1}{4}p_1(\mathcal{E})/\alpha$ means: write

$$-\frac{1}{4}p_1(\mathcal{E}) = \sum_i \beta_i \otimes \gamma_i, \quad \beta_i \in H^*(X), \quad \gamma_i \in H^*(N_g(E)).$$

Then

$$-\frac{1}{4}p_1(\mathcal{E})/\alpha = \sum_i \langle \beta_i, \alpha \rangle \gamma_i.$$

If the universal bundle does not exist, its endomorphism bundles $End(\mathcal{E})$ will still exist, and we can define define μ by replacing $c_2(\mathcal{E}) - c_1^2/4(\mathcal{E})$ by $-c_2(End(\mathcal{E}))/4$.

It can be shown that $\mu(\alpha)$ extends over the Uhlenbeck compactification $\overline{N}_g(E)$. For generic g, $\overline{N}_g(E)$ is a stratified space with smooth strata, and the submaximal stratum has codimension at least 4. Therefore $\overline{N}_g(E)$ has a fundamental class.

Now let $d := 4c_2 - c_1^2 - \frac{3}{2}(1 + b_+(X))$ and write $d = l + 2m$. Let $\alpha_1, \dots \alpha_l \in H_2(X)$ and let $p \in H_0(X)$ be the class of a point. Then we define the *Donaldson invariant*

$$\Phi^{X,g}_{c_1,d}(\alpha_1 \cdot \ldots \cdot \alpha_l \cdot p^m) := \int_{[\overline{N}_g(E)]} \mu(\alpha_1) \cup \ldots \cup \mu(\alpha_l) \cup \mu(p)^m.$$

More generally let $A_*(X) := Sym^*(H_2(X) \oplus H_0(X))$. This is graded by giving degree $(2 - i/2)$ to elements in $H_i(X)$. We denote by $A_d(X)$ the part of degree d, By linear extension we get $\Phi^{X,g}_{c_1,d} : A_d(X) \to \mathbb{Q}$ and

$$\Phi^{X,g}_{c_1} := \sum_{d \geq 0} \Phi^{X,g}_{c_1,d} : A_*(X) \to \mathbb{Q}.$$

By definition the Donaldson invariants depend on the choice of the metric g. We have however

Theorem 2.6 *1. If $b_+(X) > 1$, then $\Phi^{X,g}_{C,d}$ is independent of the generic metric g.*

2. If $b_+(X) = 1$, then $\Phi^{X,g}_{C,d}$ depends only on the chamber of the period point of g.

We will discuss walls and chambers later. The result means that the Donaldson invariants are really invariants of the differentiable structure of X. In case $b_+(X) > 1$, we can therefore drop the g from our notation.

The argument for showing the theorem is that one connects two generic metrics by a generic path in order to make a cobordism. Reducible connections occur in codimension $b_+(X)$, so they make no problem for $b_+(X) > 1$, but can disconnect the path for $b_+(X) = 1$.

2.6 Structure theorems

It is often useful to look at generating functions for the Donaldson invariants. For $a \in H_2(X)$ and $\alpha \in A_*(X)$ and a variable z we write

$$\Phi^X_C(\alpha e^{az}) := \sum_{n \geq 0} \Phi^X_C(\alpha a^n/n!) z^n.$$

Definition 2.7 A 4-manifold X is *of simple type* if

$$\Phi^X_C(\alpha(p^2 - 4)) = 0$$

for all $\alpha \in A_*(X)$ and all $C \in H^2(X, \mathbb{Z})$.

Many 4-manifolds like $K3$ surfaces and complete intersections are known to be of simple type, and it is possible that all simply connected 4- manifolds are.

Theorem 2.8 *Let X be a simply connected 4-manifold of simple type. Then there exist so called basic classes $K_1, \ldots, K_l \in H^2(X, \mathbb{Z})$ and rational numbers $\alpha_1(C), \ldots \alpha_l(C)$, such that for all $a \in H_2(X)$*

$$\Phi^X_C(e^{at}(1 + p/2)) = e^{(a \cdot a)t^2/2} \sum_{i=1}^{l} \alpha_i(C) e^{\langle K_i, a \rangle t}.$$

(Here $(a \cdot a)$ denotes the quadratic form on $H_2(X)$).

3 Algebro-geometric definition of Donaldson invariants

Let X be a simply connected algebraic surface, and let H be an ample divisor on X. We denote by $M := M_H^X(C, c_2)$ the moduli space of (Gieseker) H-semistable rank 2 torsion free coherent sheaves $\mathcal{F}$ on X with $c_1(\mathcal{F}) = C$ and $c_2(\mathcal{F}) = c_2$. We want to relate M to the Uhlenbeck compactification $N := N_{g(H)}(C, c_2)$. Here $g(H)$ is the Fubini study metric associated to H. As the Donaldson invariants are defined in terms of the Uhlenbeck compactification, this allows us to compute them on the moduli space M of sheaves.

The steps of the argument are as follows:

1. Introduce the determinant bundles $L_1(D)$ on M.
2. Construct sections of $L_1(D)^{\otimes n}$ for $n >> 0$, show that the corresponding linear system is base point free, thus giving a morphism
$$\Psi : M \to \mathbb{P}(H^0(M, L_1(D)^{\otimes n})^\vee).$$
3. Show that $Im(\Psi)$ is homeomorphic to N.
4. Apply this to the computation of the Donaldson invariants.

3.1 Determinant line bundles

We will assume for simplicity that there is a universal sheaf $\mathcal{E}$ over $X \times M$ (for instance, this is the case if H is general and either C is not divisible by 2 or $C^2/4 + c_2$ is odd.

For a coherent sheaf $\mathcal{F}$ on $X \times M$ we can form
$$det(p_{2*!}(\mathcal{F})) := \bigotimes_j det(G_j)^{\otimes(-1)^j} \in Pic(M),$$
where the finite complex of locally free sheaves
$$0 \to G_l \to \ldots \to G_s \to 0$$
is quasiisomorphic to $Rp_{2*}(\mathcal{F})$.

Definition 3.1 Let $D \in |nH|$ be a smooth curve. For a general $E \in M$ let $\chi_1 := \chi(E|_D)$. Let $a \in X$ be a point. Then we put
$$L_1(nH) := det(p_{2*!}(\mathcal{E}|_{D\times M}))^{\otimes -2} \otimes det(\mathcal{E}|_{\{a\}\times M})^{\otimes \chi_1}.$$

Let M_D be the moduli space of semistable rank 2 vector bundles on D of degree $D \cdot C$. Assume for simplicity that also on $D \times M_D$ there is a universal sheaf $\mathcal{F}$. Let $F \in M_D$. Then we define
$$L_0 := det(p_{2*!}\mathcal{F})^{\otimes -2} \otimes det(\mathcal{E}|_{\{a\}\times M_D})^{\otimes \chi(F)}.$$

Remark 3.2 $L_1(nH)$ is independent of the choice of $\mathcal{E}$ (and also of D and a). This comes from the fact that for a line bundle λ on M we get

$$det(p_{2*!}(\mathcal{E}\otimes\lambda)) = \lambda^{\otimes -2\chi(E)} \otimes det(p_{2*!}(\mathcal{E})),$$

and therefore $L_1(nH)$ stays unchanged if we replace $\mathcal{E}$ by $\mathcal{E}\otimes\lambda$.

In fact we do not need the existence of $\mathcal{E}$ in order to define it. It is part of a more general formalism of determinant sheaves as it was explained in the lectures of Huybrechts and Lehn (see [LP], [H-L] where these line bundles are defined via descent from the corresponding Quot scheme).

In the same way we see that L_0 is independent of the choice of $\mathcal{F}$ and indeed we do not need the existence of $\mathcal{F}$ to define L_0.

3.2 Construction of sections of $L_1(nH)$

We have the following theorem

Theorem 3.3 *[D-N] L_0 is ample on M_D.*

Let $U(D) \subset M$ be the open subset of all sheaves E such that $E|_D$ is semistable. Thus for $E \in U(D)$, we get that $E|_D \in M_D$. We get therefore a rational map

$$j : M \dashrightarrow M_D,$$

which is defined on $U(D)$. By definition we see that

$$j^*(L_0) = L_1(nH), \quad \text{on } U(D).$$

As L_0 is ample, $L_0^{\otimes m}$ will have many sections. So we want to extend the pullbacks $j^*(s)$ of sections $s \in H^0(M_D, L_0^{\otimes m})$ to sections $\widetilde{s} \in H^0(M, L_1(nH))^{\otimes m}$. By Bogomolovs theorem ([H-L] p. 174) we have the following: For $n >> 0$ and all $E \in M$ the restriction $E|_D$ is semistable, unless E is not locally free over D. For $c_2 >> 0$ the general element in M is locally free. If $E \in M$ is not locally freem then its singularities occur in codimension 2. Therefore the condition that $E|_D$ is not locally free has codimension 1 in the locus of not locally free sheaves. Therefore the complement $M\setminus U(D)$ has codimension ≥ 2 in M. Furthermore M is normal. Therefore every $j^*(s)$ for $s \in H^0(M_D, L_0^{\otimes m})$ extends to $\widetilde{s} \in H^0(M, L_1(nH))^{\otimes m}$.

More precisely one can show the following ([Li], Prop. 2.5).

Lemma 3.4 *For every $s \in H^0(M_D, L_0^{\otimes m})$ the pullback $j^*(s)$ extends to $\widetilde{s} \in H^0(M, L(nH)^{\otimes m})$. Furthermore the vanishing locus of $\widetilde{s}$ is*

$$Z(\widetilde{s}) := \{E \in M \mid E|_D \text{ is semistable and } s(E|_D) = 0 \text{ or } E|_D \text{ is not semistable}\}$$

Now choose $m >> n >> 0$.

Proposition 3.5 *$H^0(M, L_1(nH)^{\otimes m})$ is base point free.*

Proof. Let $E \in M$. By the theorem of Mehta and Ramanathan (see [H-L] Thm. 7.2.1), we can find a smooth curve $D \in |nH|$ such that $E|_D$ is semistable. Choose $s \in H^0(M_D, L_0^{\otimes m})$, such that $s(E|_D) \neq 0$. Then $\widetilde{s}(E) \neq 0$. □

3.3 Uhlenbeck compactification

$L_1(nH)^{\otimes m}$ defines a morphism

$$\Psi : M \to \mathbb{P}(H^0(M, L_1(nH)^{\otimes m})^\vee).$$

Theorem 3.6 *$\Psi(M)$ is homeomorphic to the Uhlenbeck compactification N.*

For $E \in M$ we introduce the pair $(A(E), Z(E))$, where

1. If E is μ-stable, then

$$A(E) = E^{\vee\vee}, \qquad Z(E) = \sum_{p \in X} l(E^{\vee\vee}/E)_p \cdot p \in X^{(k)}.$$

2. If E is not μ-stable, we have the Harder-Narasimhan filtration

$$0 \to F \to E \to G \to 0$$

Where F and G are rank 1 sheaves with with $deg_H(F) = deg_H(G)$. We put

$$A(E) = F^{\vee\vee} \oplus G^{\vee\vee}, \qquad Z(E) = \sum_{p \in X} l(F^{\vee\vee} \oplus G^{\vee\vee}/F \oplus G)_p \cdot p \in X^{(k)}.$$

Claim: For $E_1, E_2 \in M$ we have $\Psi(E_1) = \Psi(E_2)$ if and only if $(A(E_1), Z(E_1)) = (A(E_2), Z(E_2))$. In other words $\Psi(M) = N$ as sets.

We want to check the claim in a special case. Assume $\Psi(E_1) = \Psi(E_2)$, where E_1 and E_2 are μ-stable. Take $D \in |nH|$ general, then $E_1|_D = E_2|_D$ (otherwise we can find a section $s \in H^0(M_D, L_0^{\otimes m})$, such that $0 = s(E_1|_D) \neq s(E_2|_D)$. Then $\widetilde{s}(E_1) = 0, \widetilde{s}(E_2) \neq 0$.) The exact sequence

$$Hom(E_1^{\vee\vee}, E_2^{\vee\vee}) \to Hom(E_1|_D, E_2|_D) \to H^1(Hom(E_1^{\vee\vee}, E_2^{\vee\vee}(-nH)) = 0$$

implies that $Hom(E_1^{\vee\vee}, E_2^{\vee\vee}) \neq 0$ and therefore $E_1^{\vee\vee} = E_2^{\vee\vee}$. Now assume $p \in Z(E_1)$ but $p \notin Z(E_2)$. Then we choose $D \in |nH|$ such that $p \in D$ and $E_2|_D$ is semistable. Then $E_1|_D$ is not semistable and therefore we can find $s \in H^0(M_D, L_0^{\otimes m})$, such that $\widetilde{s}(E_1) = 0$ and $\widetilde{s}(E_2) \neq 0$.

3.4 Donaldson invariants via algebraic geometry

Let again $M := M_H^X(C, c_2)$ be the moduli space of H-semistable sheaves. Assume that there is a universal sheaf $\mathcal{E}$ over $X \times M$. Write $d := 4c_2 - C^2 - 3(1 + p_g(X))$. Let $\nu : H_*(X) \to H^{4-*}(M)$ be defined by

$$\nu(a) := c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2/a,$$

(i.e. we write

$$c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2 := \sum_i \beta_i \otimes \gamma_i, \quad \beta_i \in H^*(X, \mathbb{Q}), \ \gamma_i \in H^*(M, \mathbb{Q}),$$

then

$$\nu(\alpha) = \sum_i \langle \beta_i, a \rangle \gamma_i.)$$

Again ν is independent of the choice of a universal sheaf, and, if no universal sheaf exists, ν can be defined without using it. We denote again by $A_d(X)$ the set of elements of degree d in $Sym^*(H_0(X) \oplus H_2(X))$, where the class p of a point in $H_0(X)$ is given weight 2 and a class in $H_2(X)$ is given weight 1. For $\alpha := a_1 \cdot \ldots \cdot a_k \in A_d(X)$, we define

$$\nu(\alpha) := \nu(a_1) \cup \ldots \cup \nu(a_k) \in H^{2d}(M).$$

and

$$\gamma_{C,d}^{X,H}(\alpha) := \int_M \nu(\alpha).$$

Theorem 3.7 *[M],[Li] Under the conditions specified below we have*

$$\Phi_{C,d}^{X,g(H)} = (-1)^{(C^2 + K_X C)/2} \gamma_{C,d}^{X,H}.$$

Conditions:

1. Locally free μ-stable sheaves are dense in M (otherwise replace M be the closure of the locus of locally free sheaves).

2. Every L in $Pic(S) \setminus \{0\}$ with $L \equiv C \bmod 2$ and $LH = 0$ satisfies $L^2 < -(4c_2 - C^2)$ (this means that H does not lie on a wall (see below).

3. $M_H^X(C, c_2)$ has dimension $4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)$ and

$$dim(M_H^X(C, n)) + 2(c_2 - n) < dim(M_H^X(C, c_2)), \quad \text{for all } n < c_2.$$

4. If $C \in 2H^2(X, \mathbb{Z})$ there is an extra condition e.g. for $\alpha \in Sym^d(H_2(X))$, the condition is $d > 2c_2 - C^2/2$.

The point is that $\Psi^*(\mu(\alpha)) = \nu(\alpha)$ and (up to different sign convention) $\Psi_*([M]) = N$. Then the theorem follows from the projection formula.

4 Flips of moduli spaces and wallcrossing for Donaldson invariants

Let X be a compact simply connected differentiable 4-manifold.

4.1 Walls and chambers

In the case $b_+(X) > 1$ the Donaldson invariants $\Phi^{X,g}_{C,d}$ are independant of the metric g (as long as it is generic).

Now assume $b_+(X) = 1$. In this case the Donaldson invariants will indeed depend on the metric g. Let $H^2(X,\mathbb{R})^+$ be the set of all $\alpha \in H^2(X,\mathbb{R})$ with $\alpha^2 > 0$. In fact they depend on g via a system of walls and chambers in $H^2(X,\mathbb{R})^+$.

We fix $C \in H^2(X,\mathbb{Z})$ and $d \in \mathbb{Z}_{\geq 0}$. The positive cone $H^2(X,\mathbb{R})^+/\mathbb{R}^+$ has two connected components Ω^+ and Ω^-. A homology orientation (i.e. the choice of an orientation on a maximal dimensional linear subspace of $H^2(X,\mathbb{R})$ on which the intersection form is positive definite) which is needed to define an orientation on the moduli space of ASD connections is equivalent to the choice of one of them, say Ω^+.

Definition 4.1 Let g be a Riemannian metric on X. The *period point* $\omega(g)$ is the point in Ω^+ defined by the one dimensional subspace of g-self-dual g-harmonic 2-forms in $H^2(X,\mathbb{R})$. I.e these are the harmonic two forms $\eta \in \Omega^2(X)$ with $*_g\eta = \eta$. An element $\xi \in H^2(X,\mathbb{Z}) + C/2$ is called of *type* (C,d) if

$$(d+3)/4 + \xi^2 \in \mathbb{Z}_{\geq 0}.$$

In this case

$$W^\xi := \{L \in \Omega^+ \mid \xi \cdot L = 0\}$$

is called the corresponding wall of type (C,d). The *Chambers of type* (C,d) are the connected components of complement of the walls of type (C,d) in Ω^+.

Theorem 4.2 *[K-M]*

1. *$\Phi^{X,g}_{C,d}$ depends only on the chamber (of type (C,d)) of $\omega(g)$.*
2. *For all ξ of type (C,d) there exists a linear map $\delta^X_{\xi,d} : A_d(X) \to \mathbb{C}$. such that*
$$\Phi^{X,g_1}_{C,d} - \Phi^{X,g_2}_{C,d} = \sum_{\xi\omega(g_2)<0<\xi\omega(g_2)} (-1)^{\xi - C/4)C}\delta^X_{\xi,d}.$$

4.2 Interpretation of the walls in algebraic geometry

Let X be a simply connected algebraic surface with geometric genus $p_g = 0$ (this is equivalent to $b_+(X) = (1+2p_g(X)) = 1$.) Let H be an ample divisor on X. Let $\mathcal{C}$ be the ample cone of X. We choose Ω^+ as the connected component of

$H^2(X,\mathbb{R})^+/\mathbb{R}^+$, which contains $\mathcal{C}$. Then the period point of the Fubini Study metric $g(H)$ is $\omega(g(H)) = \mathbb{R}^+H \in \mathcal{C}$.

Fix $C \in H^2(X,\mathbb{Z})$ and $c_2 \in \mathbb{Z}$, such that $d := 4c_2 - C^2 - 3$ is a nonnegative integer. By the previous lecture we can compute $\Phi^{X,g(H)}_{C,d}$ on $M^X_H(C,c_2)$. So we now need to know how $M^X_H(C,c_2)$ depends on H.

Let E be a torsion free rank 2 sheaf on X with Chern classes C and c_2. Let H_+ and H_- be two ample line bundles on X, and assume that E is Gieseker stable with respect to H_-, but Gieseker unstable with respect to H_+.

Then the Harder-Narasimhan filtration of E with respect to H_+ gives an exact sequence.

$$0 \to \mathcal{I}_{Z_1}(F) \to E \to \mathcal{I}_{Z_2}(G) \to 0,$$

where

1. The class $\xi := (F-G)/2$ satisfies

$$\xi H_- < 0 < \xi H_+.$$

2. $\mathcal{I}_{Z_1}$ and $\mathcal{I}_{Z_2}$ are the ideal sheaves of 0-dimensional subschemes $Z_1 \in X^{[n]}$ and $Z_2 \in X^{[n]}$. and $c_2(E) = FG + n + m$ or equivalently

$$c_2 - C^2/4 + \xi^2 = n + m \geq 0.$$

This means that ξ is a class of type (C,d) and there exists an ample line bundle H between H_+ and H_- with $\xi H = 0$. In other words ξ defines a wall of type (C,d) in $\mathcal{C}$.

Definition 4.3 Let $E^{n,m}_\xi$ be the set of all sheaves E lying in extensions

$$0 \to \mathcal{I}_{Z_1}(F) \to E \to \mathcal{I}_{Z_2}(G) \to 0$$

with $\xi := (F-G)/2$, $Z_1 \in X^{[n]}$, $Z_2 \in X^{[n]}$.

Then we conclude

1. $M^X_H(C,c_2)$ depends only on the chamber (of type (C,d)) of H. (in particular the Donaldson invariants are constant on each chamber).

2. $$M^X_{H_-}(C,c_2) \setminus M^X_{H_+}(C,c_2) \subset \coprod_{\xi H_- < 0 < \xi H_+} \; \coprod_{m+n = c_2 - C^2/4 + \xi^2} E^{n,m}_\xi.$$

We would like to say that $M^X_{H_+}(C,c_2)$ is obtained from $M^X_{H_-}(C,c_2)$, by removing the $E^{n,m}_\xi$ and replacing them by the $E^{m,n}_{-\xi}$ (for ξ classes of type (C,d) with $\xi H_- <$

$0 < \xi H_+$). This however is not quite true. The problem is that $E_{\xi}^{n,m}$ and $E_{-\xi}^{l,r}$ can intersect, i.e. we can have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{I}_{W_1}(G) & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathcal{I}_{Z_1}(F) & \longrightarrow & E & \longrightarrow & \mathcal{I}_{Z_2}(G) & \longrightarrow & 0 \\ & & & \searrow & \downarrow & & & & \\ & & & & \mathcal{I}_{W_2}(F) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

To deal with this, we need a finer notion of stability. We use: Gieseker stability is not invariant under tensorizing by a line bundle.

Assume H_- and H_+ are separated by a unique wall W^{ξ} with $\xi H_- < 0 < \xi H_+$. Let H lie between H_- and H_+ with $H\xi = 0$. If E is a torsion free H-semistable sheaf of rank 2 with Chern classes C and c_2. Then:

1. E is H_--semistable if and only if $E(l(H_- - H_+))$ is H-semistable for $l >> 0$.

2. E is H_+-semistable if and only if $E(l(H_+ - H_-))$ is H-semistable for $l >> 0$.

This gives us a finer notion of stability. By using a parabolic structure of length 1 (which essentially ammounts to tensorizing with a fractional power of $H_- - H_+$), we get moduli spaces

$$M_a, \quad a \in [-1, 1], \quad M_{-1} = M_{H_-}^X(C, c_2), \quad M_1 = M_{H_+}^X(C, c_2).$$

There are *miniwalls* $a_i \in [-1, 1]$ such that for all i and $0 < \epsilon << 1$

$$M_{a_i+\epsilon} = (M_{a_i-\epsilon} \setminus E_{\xi}^{n,m}) \amalg E_{-\xi}^{m,n}$$

for suitable m, n.

4.3 Flip construction

Definition 4.4 A class ξ of type (C, d) defines a *good* wall if W^{ξ} contains ample divisors and $2\xi + K_X$ and $-2\xi + K_X$ are not effective. In particular if $-K_X$ is effective, then all walls in the ample cone are good.

We want to describe the wall crossing through a good wall defined by ξ. Let b be a miniwall, as above, and let

$$M_- := M_{b-\epsilon}, \quad M_+ := M_{b+\epsilon} = (M_- \setminus E_{\xi}^{n,m}) \amalg E_{-\xi}^{m,n}.$$

We write

$$E_- := E_\xi^{n,m}, \quad E_+ := E_{-\xi}^{m,n}.$$

Let $T := X^{[n]}) \times X^{[m]}$, let $\mathbb{Z}_1, \mathbb{Z}_2 \subset S \times T$ be the two universal families e.g. $\mathbb{Z}_1 := \{(x, Z, W) \in X \times T \mid x \in Z\}$ and similar for $\mathbb{Z}_2$. Let q be the projection $S \times T \to T$. Write $C := F + G$, $\xi := F - G/2$ and write We write $\mathcal{F} := \mathcal{I}_{\mathbb{Z}_1}(F)$, $\mathcal{G} := \mathcal{I}_{\mathbb{Z}_2}(G)$, (these are sheaves on $X \times T$ and we supress the various pullbacks in the notation). Finally we write

$$\mathcal{A}_- := \mathrm{Ext}^1_q(\mathcal{F}, \mathcal{G}), \quad \mathcal{A}_+ := \mathrm{Ext}^1_q(\mathcal{G}, \mathcal{F}).$$

and denote the tautological line bundles on $\mathbb{P}(\mathcal{A}_-)$ and $\mathbb{P}(A_+)$ by τ_- and τ_+.

Lemma 4.5 *1. A_- is locally free of rank $-\xi(2\xi - K_S) + n + m - 1$.*

2. The tautological extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G}(\tau_-) \to 0$$

gives an isomorphism $\mathbb{P}(\mathcal{A}_-) \simeq E_-$.

3. $N_{E_-/M_-} = A_+(\tau_-)$.

Proof. 1. follows from Riemann-Roch, because the condition of a good wall implies $\mathrm{Hom}_q(\mathcal{F}, \mathcal{G}) = 0$ $\mathrm{Ext}^2_q(\mathcal{F}, \mathcal{G}) = 0$. 2. is then easy. For 3. we use that $T_{M_-} = \mathrm{Ext}^1_q(\mathcal{E}, \mathcal{E})$, where $\mathcal{E}$ is the universal sheaf on $X \times M_-$. Then one has to chase some diagrams. □

Part 3. of this lemma lets us hope that the blowup of M_- along E_- and the blowup of M_+ along E_+ might be the same. So let $\widetilde{M}_-$ be the blowup of M_- along E_- and $\widetilde{M}_+$ the blowup of M_+ along E_+. Let $D \simeq \mathbb{P}(A_-) \times_T \mathbb{P}(A_+)$ be the exceptional divisor. We see that $\mathcal{O}(D)|_D = \mathcal{O}_D(\tau_- + \tau_+)$.

Theorem 4.6 $\widetilde{M}_- = \widetilde{M}_+$.

Proof. Let $\mathcal{E}_-$ be the universal family on $X \times M_-$ (and denote by the same symbol its pullback to $\widetilde{M}_-$). Let $\mathcal{E}_+$ be the kernel of the composition $\mathcal{E}_- \to \mathcal{E}_-|_D \to \mathcal{G}(\tau_-)_D$, where $\mathcal{G}(\tau_-)_D$ is the pullback of $\mathcal{G}(\tau)$ from $S \times T$ to $S \times D$. So we define E_+ via elementary transform along the exceptional divisor D. Then we check that $\mathcal{E}_+$ is a $b_{+\epsilon}$ stable family. Thus $\mathcal{E}_+$ defines a morphism $\widetilde{M}_- \to M_+$. It is not difficult to check that it is the blowup along E_+. □

So we see that $M^X_{H_+}(C, c_2) = M_1$ is obtained from $M^X_{H_-}(C, c_2) = M_{-1}$ via a sequence of blow ups along smooth subvarieties of the form $E_\xi^{n,m}$ followed by a blowup of the exceptional divisor in another direction to $E_{-\xi}^{m,n}$.

4.4 Computation of the wallcrossing

Now we want to compute the wallcrossing terms $\delta^X_{\xi,d}$. For simplicity we restrict to $Sym_d(H_2(X))$. Let $a \in H_2(X)$. Let b run through the miniwalls corresponding to ξ and write $\widetilde{M}_b$ for the blowup $\widetilde{M}_{b-\epsilon} = \widetilde{M}_{b+\epsilon}$

$$\delta^X_{\xi,d}(a^d) = \pm \left(\int_{M^X_{H_+}(C,c_2)} \nu(a)^d - \int_{M^X_{H_-}(C,c_2)} \nu(a)^d \right) = \sum_b \int_{\widetilde{M}_b} (\nu_+(a)^d - \nu_-(a)^d).$$

Here $\nu_-(a) := (c_2(\mathcal{E}_-) - c_1(\mathcal{E}_-)^2/4)/a$, and similarly for ν_+.

Let us again put ourselves in the situation of the previous section: $\widetilde{M}_-$ is the blowup of M_- along E_- and $\widetilde{M}_+$ the blowup of M_+ along E_+, and D is the exceptional divisor.

Lemma 4.7 *1.* $\nu_+(a) - \nu_-(a) = -\langle \xi, a\rangle D$.

2. $\int_{\widetilde{M}_-}(\nu_+(a)^d - \nu_-(a)^d)$ *is the evaluation of a suitable (explicitely computable) cohomology class on* $X^{[n]} \times X^{[m]}$.

Proof. 1. Is an easy application of Riemann-Roch without denomiators (which tells how to compute the Chern classes of sheaves supported on subvarieties). 2.

$$\nu_+(a)^d - \nu_-(a)^d = (\nu_+(a) - \nu_-(a))(\nu_+(a)^{d-1} + \ldots + \nu_-(a)^{d-1})$$

is by 1. divisible by D. We can push the class from D down to T. □

Putting all this together and summing over all the miniwalls corresponding to a given wall ξ we obtain the following:

Theorem 4.8

$$\delta^X_{\xi,d}(a^d) = \pm \sum_{b=0}^{d} 2^b \binom{d}{b} \langle \xi, a\rangle^{d-b} \cdot$$
$$\int_{(X\sqcup X)^{[n]}} \alpha^b s_{2l-b}(\mathrm{Ext}^1_q(\mathcal{I}_{\mathbb{Z}_1}, \mathcal{I}_{\mathbb{Z}_2} \otimes (\mathcal{O}(-2\xi) \oplus \mathcal{O}(-2\xi + K_X)))).$$

Here p and q are the projections of $X \times (X \sqcup X)^{[n]}$ to X and $(X \sqcup X)^{[n]}$ respectively and $l := c_2 - C^2/4 + \xi^2$. $\mathbb{Z}_1$ and $\mathbb{Z}_2 \subset X \times (X \sqcup X)^{[n]}$ are the universal families, s_i denotes the i-th Segre class and $\alpha := q_*(p^*\alpha \cdot ([\mathbb{Z}_1] + [\mathbb{Z}_2]))$. So we are reduced to a (very complicated) intersection computation on the Hilbert scheme of points on X. The intersection theory of $X^{[n]}$ is in general not understood. It gets harder very fast as n grows. So in our case the difficulty of the computation depends on the number $l := c_2 - C^2/4 + \xi^2$. The intersection number above can be computed for l not too large, say $l \leq 3$. For $l = 0$ we get for instance

$$\delta^X_{\xi,d}(a^d) = \pm \langle \xi, a\rangle^d.$$

There is an alternative way of carrying out the final step of the computation, i.e. the computation in the cohomology ring of the Hilbert scheme of points. Assume X is a blowup of $\mathbb{P}_2$. On $\mathbb{P}_2$ we have actions of $\mathbb{C}^*$ with finitely many fixpoints. We can do the blowup in such a way that X still carries an action of $\mathbb{C}^*$ with finitely many fixpoints (it is enough to at each step only blow up fixpoints). This action lifts to an action on $Hilb^n(X)$, which still has only finitely many fixpoints. All the intersection numbers we have to compute for the wallcrossing are indeed intersection numbers of Chern classes of equivariant bundles for this action.

We can therefore apply the Bott residue formula. This allows us to compute the intersection numbers by looking at the weights of the action on the fibres of the equivariant bundles over the fixpoints. This gives an algorithm for computing the wallcrossing for rational surfaces. We used this in [E-G2] to compute the Donaldson invarinants of $\mathbb{P}_2$ of degree ≤ 50.

The fact that we can compute the Donaldson invariants of $\mathbb{P}_2$, where there are no walls might seem surprizing. We use the blowup formulas (see the next lecture) to be able to compute on the blowup of $\mathbb{P}_2$ is a point.

5 Wallcrossing and modular forms

Let X be a simply connected 4-manifold with $b_+(X) = 1$. In this lecture I want to give a generating function for the wallcrossing terms $\delta^X_{\xi,d}$.

5.1 Ingredients

There are several ingredients which have to be put together in order to compute the generating function.

(1) **Kotschik-Morgan conjecture.** In their paper [K-M], where they show that the the Donaldson invariants $\Phi^{X,g}_{C,d}$ depend only on the chamber of the period point of the metric g, they also make a conjecure about the structure of the wallcrossing terms $\delta^X_{\xi,d}$.

Conjecture 5.1 *[K-M] $\delta^X_{\xi,d}(a^d)$ is for $a \in H_2(X)$ a polynomial in $\langle \xi, a\rangle$ and a^2, whose coefficients depend only on ξ^2, d and the homotopy type of X.*

In a series of papers Fehan and Leness are working on a proof of this conjecture.

(2) **Blowup formulas.** The blowup formulas relate the Donaldson invariants of a 4-manifold X with those of the connected sum $\widehat{X} := X\#\overline{\mathbb{P}}_2$ of X with $\mathbb{P}_2$ with the opposite orientation. In the case that X is an algebraic surface, we can take $\widehat{X}$ to be the blowup of X in a point. In the case $b_+(X) = 1$, when the Donaldson invariants depend on the choice of a metric, we need to choose the

metric on $\widehat{X}$ to be very close to the pullback of a metric on X, in order to make the blowup formulas applicable. Let E be the class of the exceptional divisor, then $H^2(\widehat{X},\mathbb{R}) = H^2(X,\mathbb{R}) \oplus \mathbb{R}E$. We will identify $H^2(X,\mathbb{R})$ with the classes in $H^2(\widehat{X},\mathbb{R})$ orthogonal to E.

If $L \in H^2(X,\mathbb{R})^+$ is (a representative of) the period point of the metric g, we write

$$\Phi^{X,L}_{C,d} := \Phi^{X,g}_{C,d}.$$

For $C \in H^2(\widehat{X},\mathbb{Z})$, $H \in H^2(X,\mathbb{R})^+$, we write

$$\Phi^{\widehat{X},H}_{C,d} := \Phi^{\widehat{X},H-\epsilon E}_{C,d}$$

for $0 < \epsilon << 1$. (This will be independent of sufficiently small ϵ).

Theorem 5.2 *Let $C \in H^2(X,\mathbb{Z})$, $a \in H_2(X)$, $H \in H^2(X,\mathbb{R})^+$. We write $e \in H_2(\widehat{X},\mathbb{Z})$ for the Poincaré dual of E. Then*

1. $\Phi^{\widehat{X},H}_{C,d}(a^d) = \Phi^{X,H}_{C,d}(a^d)$.
2. $\Phi^{\widehat{X},H}_{C+E,d+1}(e\, a^d) = \Phi^{X,H}_{C,d}(a^d)$.
3. $\Phi^{\widehat{X},H}_{C,d}(e^2 a^{d-2}) = 0$.

This result holds also if $b_+ > 1$. More generally Fintushel and Stern [1] found generating functions for the blowup formulas: Let $p \in H_0(X)$ be the class of a point. Then there are power series

$$B(x,t) = \sum_k B_k(x)t^k$$
$$S(x,t) = \sum_k S_k(x)t^k$$

such that

$$\Phi^{\widehat{X},H}_{C}(a^d e^k) = \Phi^{X,H}_{C}(a^d B_k(p))$$
$$\Phi^{\widehat{X},H}_{C+E}(a^d e^k) = \Phi^{X,H}_{C}(a^d S_k(p))$$

$B(x,t)$ and $S(x,t)$ can be expressed in terms of elliptic functions e.g. $S(x,t) = e^{-t^2x/6}\sigma(t)$, where σ is the Weierstrass σ function.

(3) Vanishing results

Lemma 5.3 *Let X be a ruled surface. Let F be the class of a fibre and assume $CF = 1$. Let H be an ample divisor on X. Then $M^X_{F+\epsilon H}(C,c_2) = \emptyset$ for $0 < \epsilon << 1$. In particular $\Phi^{X,F+\epsilon H}_{C,d} = 0$ for $0 < \epsilon << 1$.*

More generally the folloing holds: Let $f : X \to C$ be a surjective morphism of an algebraic surface to a curve. Let F be the class of a fibre and let H be ample on X. Then a vector bundle E over X is semistable with respect to $F + \epsilon H$ if and only if the restriction of E to the generic fibre of f is semistable. This fact is also e.g. used by Friedman to study the Donaldson invariants of elliptic surfaces.

5.2 The result

Our aim is to show:

Theorem 5.4 *Let $a \in H_2(X)$ and let z be a variable. Then*

$$\delta^X_\xi(\exp(at)) = \mathrm{Coeff}_{q^0}\left[f(\tau)R(\tau)\theta(\tau)^{\sigma(X)}q^{-\xi^2/2}\exp\Big(-\frac{\langle\xi,a\rangle t}{f(\tau)} - \frac{a^2G(\tau)t^2}{f(\tau)^2}\Big)\right]$$

Here $\sigma(X)$ is the signature of X. For the rest of the notations I briefly review modular forms.

Review of modular forms

Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ be the complex upper half plane. For τ in $\mathbb{H}$ we denote $q := e^{2\pi i\tau}$. The group $SL_2(\mathbb{Z})$ acts of $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau + b}{c\tau + d}.$$

A function $g : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k for $SL_2(\mathbb{Z})$, if

$$g\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k f(\tau), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

and furthermore g has a q-development

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

To $\tau \in \mathbb{H}$ one can associate an elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and E_τ and $E_{\tau'}$ are isomorphic if and only if τ and τ' related by an element of $SL_2(\mathbb{Z})$. Therefore modular forms are related to moduli of elliptic curves.

One can also talk about modular forms for subgroups Γ of finite index of $SL_2(Z)$. In this case one requires the transformation behaviour only for the elements in Γ and the requirement on the q-development has to be modified.

All the functions appearing in the theorem are (related to) modular forms.

$$\Delta(\tau) := q\prod_{k=1}^{\infty}(1 - q^k)^{24}$$

is the discriminant, a modular form for $SL_2(\mathbb{Z})$.

$$\eta(\tau) = \Delta(\tau)^{1/24}$$

is the Dirichlet eta-function.

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2/2}$$

is the theta function for $\mathbb{Z}$.

$$G_2(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n$$

is an Eisenstein series and

$$e_3(\tau) := \frac{1}{12} + \sum_{n=1}^{\infty}\Big(\sum_{d|n,d \text{ odd}} d\Big)q^n,$$

the value of the Weierstrass $\wp$-function at one of the two-division points. We put

$$R(\tau) := \frac{\Delta(\tau)^2}{\Delta(\tau/2)\Delta(2\tau)},$$
$$f(\tau) := e^{-\pi i/4}\frac{\eta(\tau)^3}{\theta(\tau)},$$
$$G(\tau) := G_2(\tau) + e_3(\tau)/2.$$

As a corollary to this result we can compute the Donaldson invariants of the projective plane $\mathbb{P}_2$. Let $H \in H^2(\mathbb{P}_2, \mathbb{Z})$ be the hyperplane class, and let h be its Poincaré dual.

Corollary 5.5

$$\Phi_H^{\mathbb{P}_2,H}(\exp(ht)) = \mathrm{Coeff}_{q^0}\Bigg[f(\tau)R(\tau) \sum_{a\geq n>0}(-1)^{n+\frac{1}{4}}q^{\frac{1}{2}(a^2-(n-\frac{1}{2})^2)}\exp\Big(-\frac{(n+\frac{1}{2})t}{f(\tau)} - \frac{G(\tau)t^2}{f(\tau)^2}\Big)\Bigg].$$

There is a similar formula for $\Phi_0^{\mathbb{P}_2,H}$.

Proof. The blowup X of $\mathbb{P}_2$ is a point is a ruled surface, the class of the fibre is $F = H - E$. So we get $\Phi_H^{X,F+\epsilon H} = 0$ by the vanishing result above. On the other hand the blowup formulas give that $\Phi_H^{\mathbb{P}_2,H}(h^d) = \Phi_H^{X,H}(h^d)$, and the last can be computed by adding all the wallcrossing terms $\delta_{\xi,d}^X$ for all classes ξ of type (H,d) with $\xi H > 0 > \xi F$. □

5.3 Proof of the theorem

Now I want to sketch the proof of the theorem. The idea is a follows: We want to relate the wallcrossing on X and its blowup $\widehat{X}$. So fix $C \in H^2(X,\mathbb{Z})$ and let ξ defined the only wall of type (C,d) on X between H_- and H_+. Instead of directly applying the wallcrossing formula for the wall W^ξ, we can also instead first apply the blowup formulas, then cross all the walls between H_- and H_+ on $\widehat{X}$ and then apply the blowup formula again to get back to X. This gives us two different ways to compute the wallcrossing term δ_ξ^X, which will give us recursion formulas.

By definition we see that the classes η of type (C,d) on $\widehat{X}$ with $H_-\eta < 0 < H_+\eta$ are precisely the

$$\eta = \xi + nE, \quad n \in \mathbb{Z},\ n^2 \leq (d+3)/4 + \xi^2,$$

and the classes of type $(C+E, d+1)$ are precisely the

$$\eta = \xi + (n+1/2)E, \quad n \in \mathbb{Z},\ (n+1/2)^2 \leq (d+4)/4 + \xi^2.$$

We write

$$\delta_\xi^X := \sum_{d\geq 0} \delta_{\xi,d}^X.$$

Then together with the above discussion the blowup formulas give:

$$\begin{aligned}
\delta_\xi^X(a^d) &= \sum_{n\in\mathbb{Z}} \delta_{\xi+nE}^{\widehat{X}}(a^d)\\
\delta_\xi^X(a^d) &= \sum_{n\in\mathbb{Z}} (-1)^{n-1}\delta_{\xi+(n+1/2)E}^{\widehat{X}}(e\, a^d)\\
0 &= \sum_{n\in\mathbb{Z}} \delta_{\xi+nE}^{\widehat{X}}(e^2\, a^{d-2})
\end{aligned}$$

Now we use the Kotschick-Morgan conjecture. Let $X(b)$ be the blowup of X in b points. It allows us to write

$$\delta^{X(b)}(a^d/d!) = \sum_{l+2k=d} \frac{\langle \xi, a\rangle^l}{l!}\frac{(a\cdot a)^k}{k!}P(l,k,b,\xi^2),$$

for universal constants $P(l,k,b,w)$ for $l,k,b \in \mathbb{Z}$, $w \in \mathbb{Z}/4$. Then the Lemma implies

$$\begin{aligned}
P(l,k,b,w) &= \sum_{n\in\mathbb{Z}} P(l,k,b+1,w-n^2),\\
P(l,k,b,w) &= \sum_{n\in\mathbb{Z}} (-1)^n(n+1/2)P(l,k,b+1,w-(n+1/2)^2),\\
\sum_{n\in\mathbb{Z}} n^2P(l,k,b,w-n^2) &= 2\sum_{n\in\mathbb{Z}} P(l,k,b,w-n^2).
\end{aligned}$$

Now we put

$$\Lambda_X := \sum_{l,k,b,w} P(l,k,b,w) q^{w/2} \frac{L^l Q^k t^b}{l!k!b!},$$

for variables q, L, Q, t. Now in the generating function Λ_X all the information about the wallcrossing formulas on blowups of X is encoded. The formulas for the $P(l,k,b,w)$ translate into the following differential equations for Λ_X.

$$\theta(\tau)\frac{\partial}{\partial t}\Lambda_X = \Lambda_X,$$

$$\eta(\tau)^3 \frac{\partial}{\partial L}\frac{\partial}{\partial t}\Lambda_X = \Lambda_X,$$

$$2\theta(\tau)\frac{\partial}{\partial Q}\Lambda_X = \left(q\frac{\partial}{\partial q}\theta(\tau)\right)\frac{\partial^2}{\partial L^2}\Lambda_X.$$

These differential equations are trivial to solve: Writing

$$\lambda_X(q) := \Lambda_X(0,0,0,q)$$

we get

$$\Lambda_X = \exp\Big(-\frac{L}{f(\tau)} - \frac{Q\,G(\tau)}{f(\tau)^2} + \frac{t}{\theta}\Big)\lambda_X(q).$$

Finally we need to determine $\lambda_X(q)$. It is enough to do this in case $X = \mathbb{P}_1 \times \mathbb{P}_1$: For every simply connected 4-manifold with $b_+ = 1$ the blowup Y of X in two points is homotopy equivalent to the blowup of $\mathbb{P}_1 \times \mathbb{P}_1$ in a number of points. The Kotschick-Morgan conjecure says that the wallcrossing terms only depend on the homotopy type of X.

Let F and G be the fibres of the two projections of $\mathbb{P}_1 \times \mathbb{P}_1$ onto its factors. By the vanishing result we get

$$\Phi^{\mathbb{P}_1\times\mathbb{P}_1, F+\epsilon G}_{F+G,d} = \Phi^{\mathbb{P}_1\times\mathbb{P}_1, G+\epsilon F}_{F+G,d} = 0.$$

Therefore the sum of all the wallcrossing terms for all the walls between F and G has to vanish. This is enough to determine all the coefficients of $\Lambda_{\mathbb{P}_1\times\mathbb{P}_1}(q)$.

5.4 Further results

This result has later been used [G-Z] to prove structure theorems like those of Kronheimer and Mrowka for manifolds with $b_+ = 1$. These results work when one takes the limit of the Donaldson invariants $\Phi^{X,H}_C$ as H tends to a class F with $F^2 = 0$. We write $\Phi^{X,F}_C$ for this limit.

We get for instance the following: Let X be a rational elliptic surface (i.e. the blowup of $\mathbb{P}_2$ in the 9 points of intersection of two smooth cubics.). Let F be the class of a fibre. Then for all $a \in H_2(X)$ we get

$$\Phi^{X,F}_H(e^{at}(1+p/2)) = -\frac{e^{(a\cdot a)t^2/2}}{\cosh(\langle F,a\rangle t)}.$$

To prove such results, one has to sum over all walls between two classes F, G with $F^2 = G^2 = 0$. These sums organize themselves into thetafunctions and arguments with modular forms will give the result.

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