united nations
ational, scientific
and cultural
organization

national atomic
energy agency

the
abdus salam
international centre for theoretical physics

SMR 1161/2

AUTUMN COLLEGE ON PLASMA PHYSICS

25 October - 19 November 1999

Matter and Light under Extreme Conditions. Low Temperatures Phase Space Cooling

Swapan CHATTOPADHYAY

Centre for Beam Physics
Berkeley, USA

Swapan Chattopadhyay
Berkeley, 1999

MATTER and LIGHT UNDER EXTREME CONDITIONS

Collective and Nonlinear Interactions of Lasers, Particle Beams and Plasmas

LECTURE III

LOW TEMPERATURES

Phase Space Cooling

Thursday

Oct. 28, 1999

Autumn College on Plasma Physics
Abdus Salam ICTP
Trieste, Italy.

Phase Space Cooling

The very word "cooling" implies a temperature. However we have to use "Temperature" with caution. It is mostly not what we normally mean by the equilibrium Maxwell-Boltzmann temperature, but rather a measure of the "average kinetic energy" of a system in "nonequilibrium" slowly evolving in time.

For this to make sense, we somehow have to go to a frame where all gross macroscopic motions have been transformed away and we are in a frame where the average center-of-mass motion is zero. Around this steady center-of-mass rest frame, we consider a system of particles — atoms, electrons, ions, ... — "confined" or "trapped" by an external field configuration into bounded oscillatory motion.

Examples:
- Atoms or ions in "Paul Trap" or "Laser Trap"
- Particles in Storage rings

We are thus concerned with bounded oscillations. All oscillations, linear or nonlinear, can be mathematically transformed to a normal form hamiltonian in suitable coordinate and momentum variables (x, p) such that the energy is:

$$\boxed{E = x^2 + p^2}$$

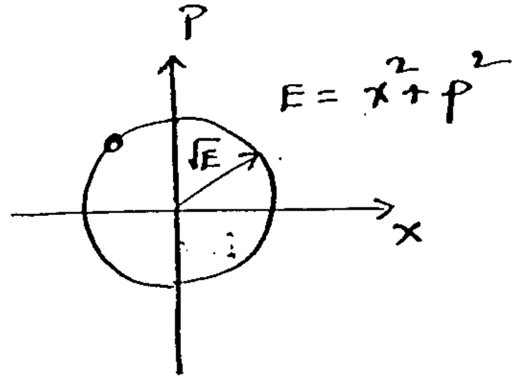
For a single oscillator, this is a circle in (x, p) phase-space. For a collection of a large number of oscillators with a distribution of energies from 0 to E, it fills up the circle with radius $\sqrt{E}$ in the phase space. For a single oscillator, the average energy is the same as total energy and

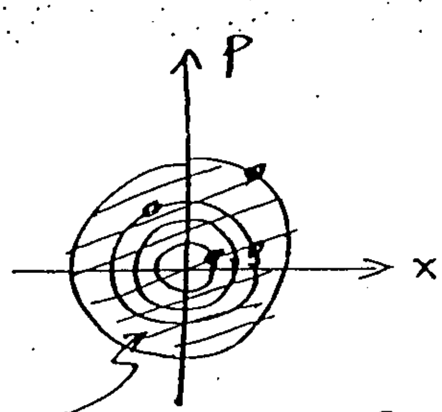
$$\boxed{\langle E \rangle = \langle x^2 + p^2 \rangle = E = \tfrac{1}{2} k_B T}$$

defines its oscillatory temperature.

For a collection of oscillators:

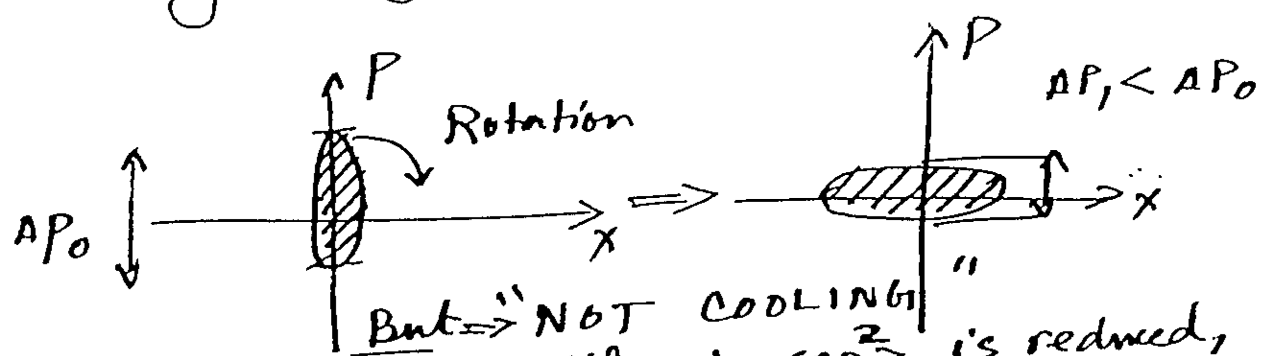
$$\boxed{\begin{aligned} \langle E \rangle = \langle x^2 + p^2 \rangle &= \langle a^2 \rangle \\ &= \frac{1}{\pi} \oint p\, dx \\ &= \text{Total Phase Volume} \\ &= \tfrac{1}{2} k_B T \end{aligned}}$$



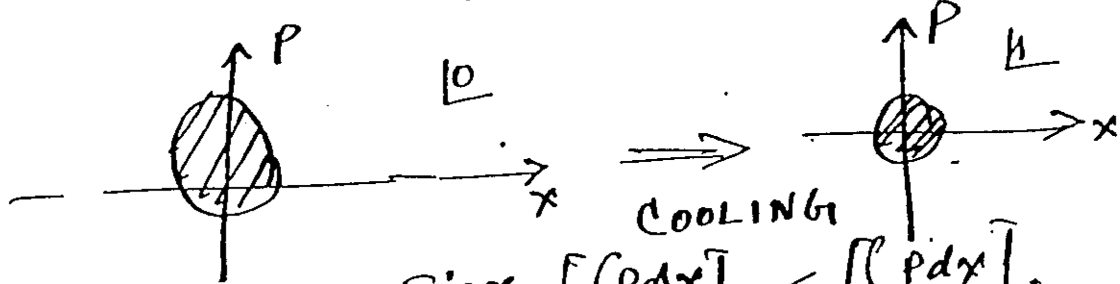


$$\int p\,dx = \pi \langle A^2 \rangle = \frac{\pi}{2}(k_B T).$$

Thus "temperature" is related to "phase-space volume" and we mean "phase-space" cooling or shrinking of "$\int p\,dx$" when we say cooling in general $\Rightarrow$ Net increase in phase space density.



<u>But</u> $\Rightarrow$ "NOT COOLING" although $\langle \Delta P^2 \rangle$ is reduced, $\int p\,dx$ = Constant.



COOLING

since $\left[\int p\,dx\right]_1 < \left[\int p\,dx\right]_0$

How to induce genuine phase-space cooling ??

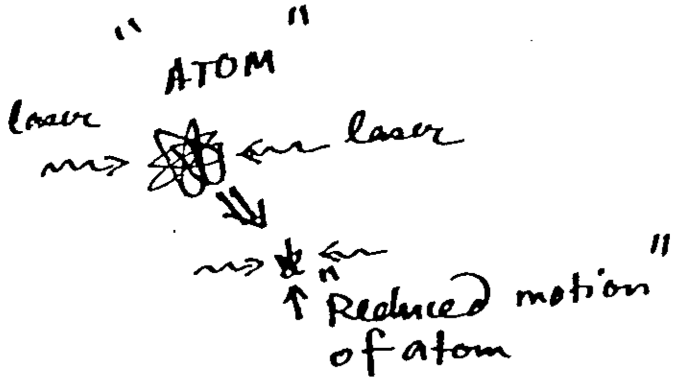
⇒ Introduce <u>dissipation</u> or damping into an otherwise conservative system.

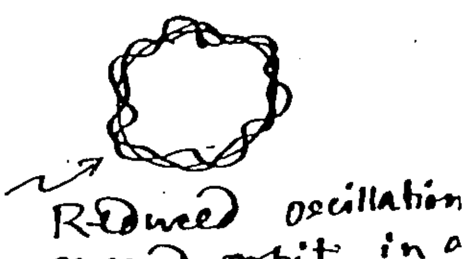
e.g. "<u>Synchrotron radiation</u>" ⇒ loss of energy ⇒ damping of oscillations as in a storage ring. DISCUSSED EARLIER

⇒ Introduce a "<u>velocity-dependent force</u>"

"<u>Laser Cooling</u>" of Atoms.

Maxwell's Demon "<u>Stochastic Cooling</u>" of Particles in a Storage Ring





Reduced oscillation around <u>orbit</u> in a storage ring.

TEMPERATURE SCALES

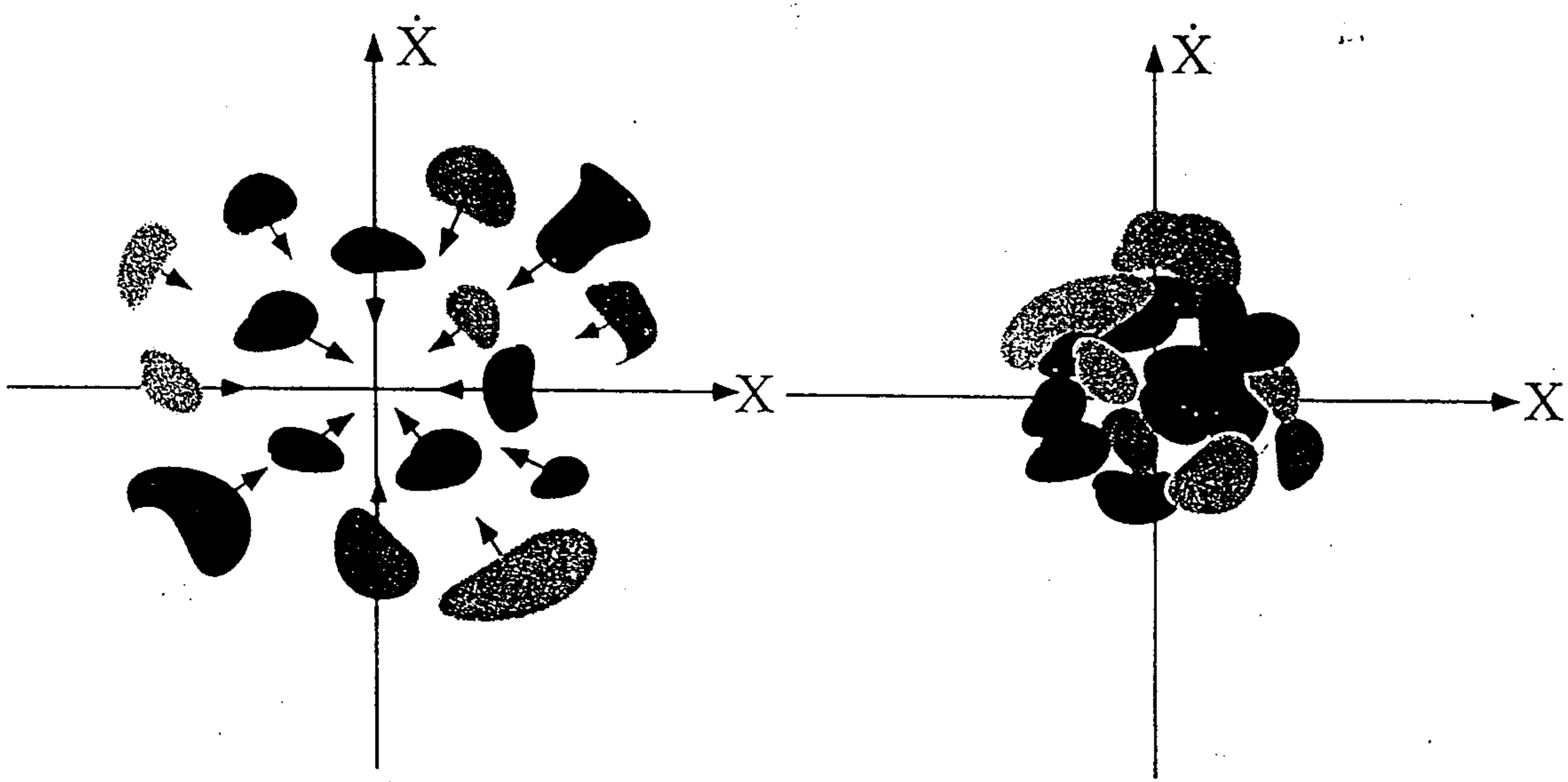
	← Surface of SUN
$3000\,^{\circ}K$	
$300\,^{\circ}K$	← Room Temp.
	← Liquid Nitrogen
$30\,^{\circ}K$	
$3\,^{\circ}K$	← Liquid He (1908)
$300\,m^{\circ}K$	← 3He cryostat
$30\,m^{\circ}K$	← Dilution Refrigerator (1965)
$3\,m^{\circ}K$	
$300\,\mu^{\circ}K$	← Doppler limit (1985)
$30\,\mu^{\circ}K$	
$3\,\mu^{\circ}K$	← Recoil limit (VSCPT, 1988)
$300\,n^{\circ}K$	← Evaporation (BEC, 1995)
$30\,n^{\circ}K$	← Raman Cooling (1996)
$3\,n^{\circ}K$	

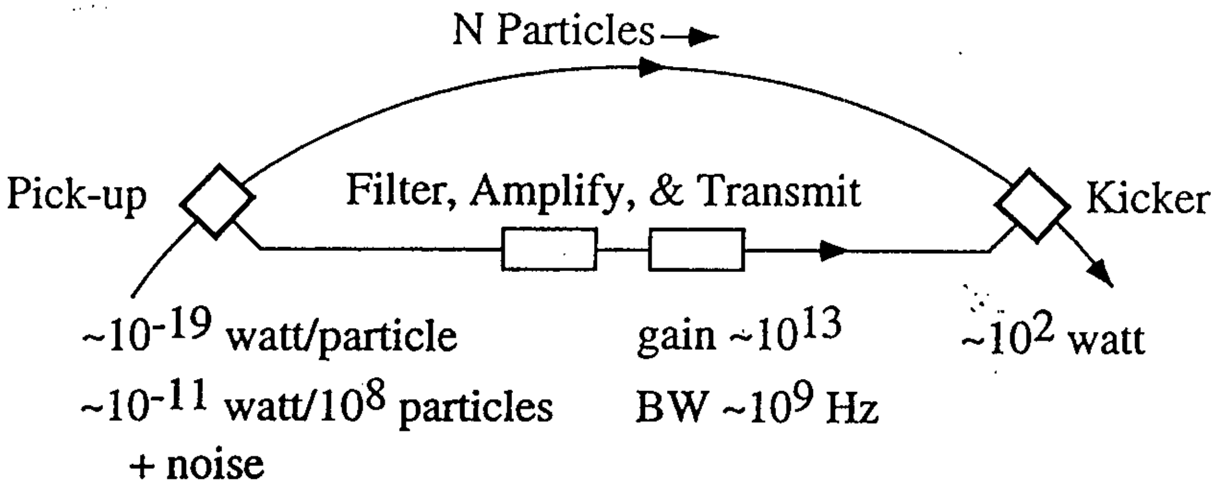
No fundamental low temp. limit

APPLICATIONS of COOLING:

- Atomic Clocks
- Atom Lithography
- Cold Non-neutral Plasmas
- Atom Interferometry
- Optical Tweezers
  - Biomedical applications
  - micromanipulation and force measurement
- Bose-Einstein Condensation
  - Atomic Laser i.e. based on Atom Beams
  - New Atomic Clocks
- Precision measurements and fundamental physics:
  - Parity Non-Conservation
  - T-violation
  - Particle-Antiparticle Studies
  - Exotic elementary particles

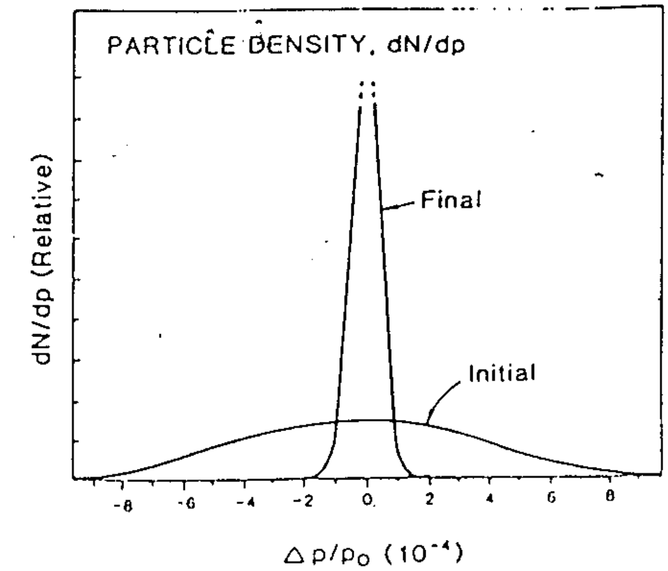
Phase-Space Cooling in Any One Dimension

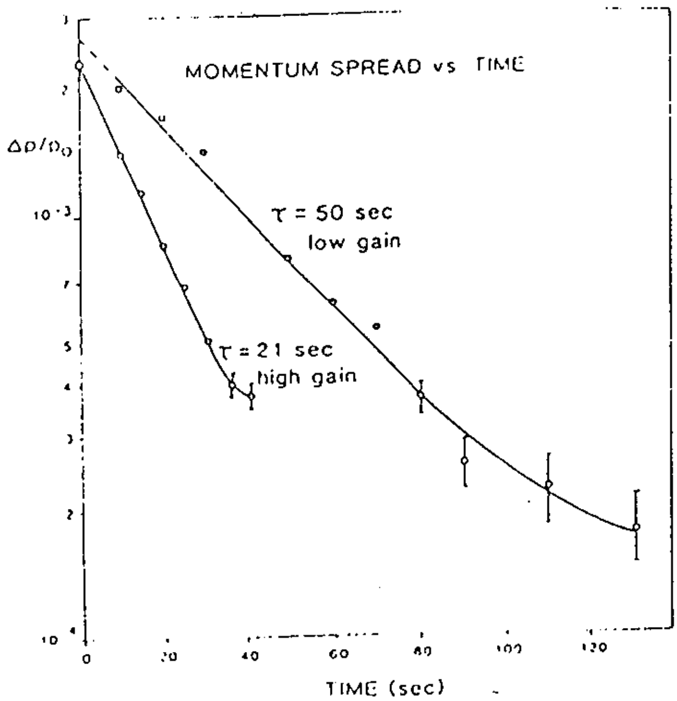


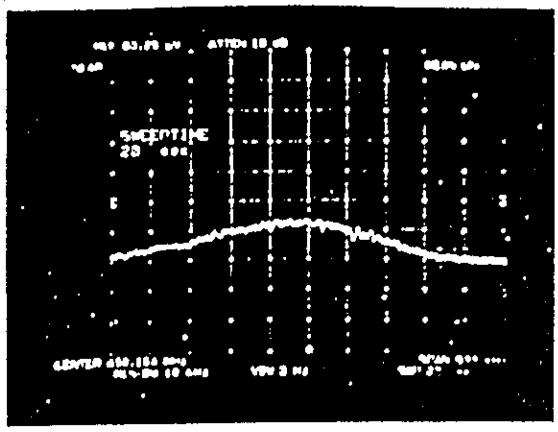


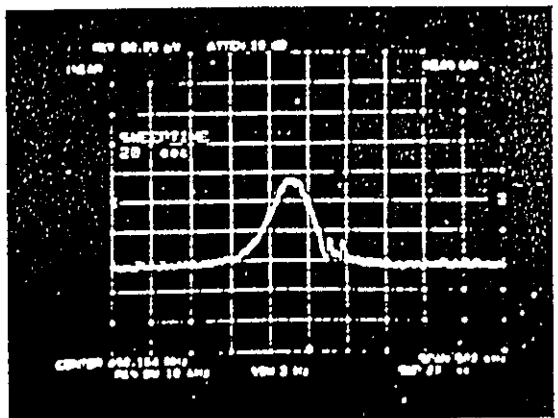
Signal/Noise ~ nV/μV

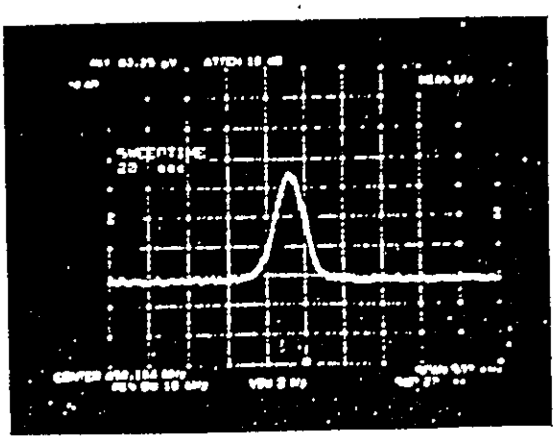
Experimental Demonstration of Longitudinal Stochastic Cooling
(CERN and FNAL, 1970's and 1980's)



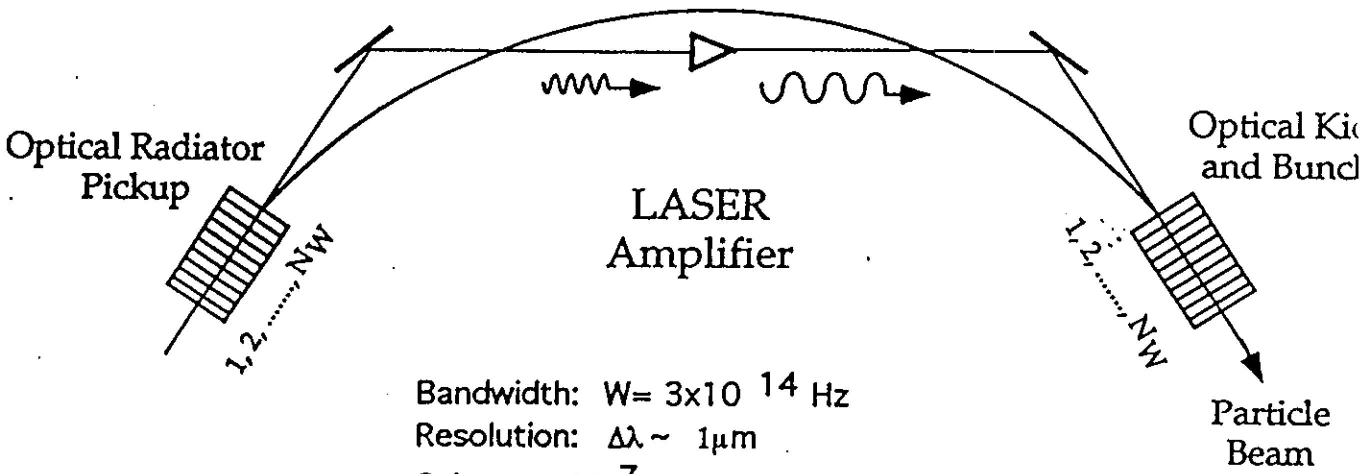






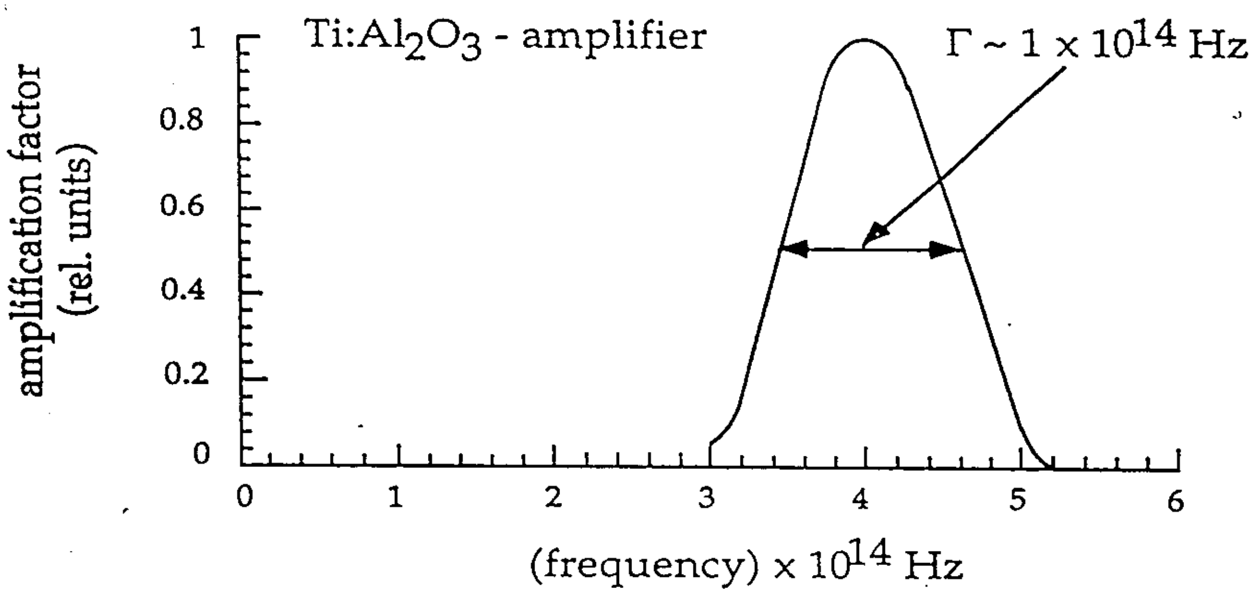


OPTICAL
STOCHASTIC
COOLING



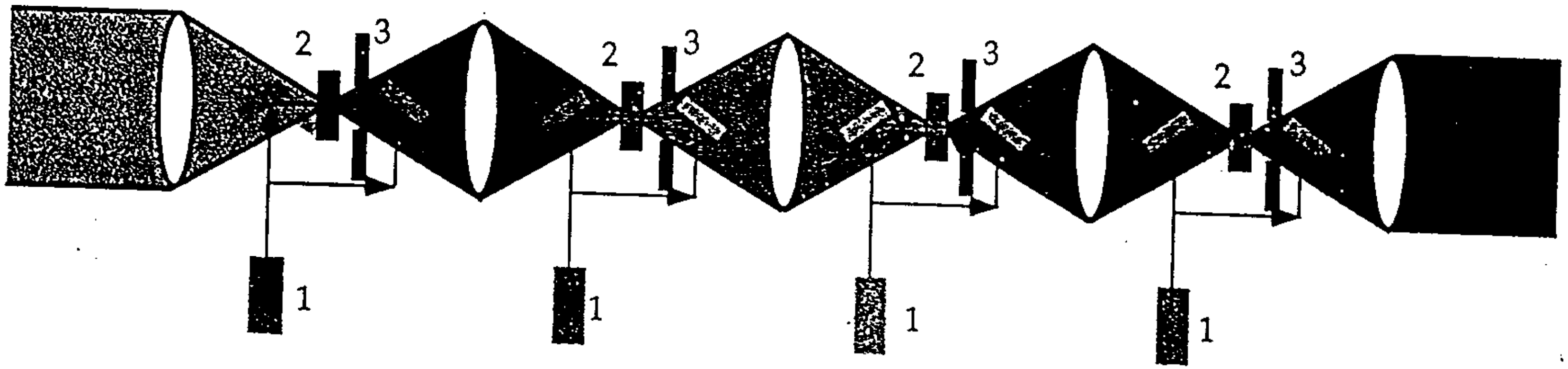
Bandwidth: W= 3×10^{14} Hz
Resolution: $\Delta\lambda \sim 1 \mu m$
Gain: g $\sim 10^{7}$
Photons/charged particle $\sim \alpha N_W K^2$

Broadband Optical Amplifiers ($Ti:Al_2O_3$, DYE)



A Schematic of Optical Amplifier

Average power	2 W
Bandwidth, FWHM	4×10^{13} Hz
Total gain	44 dB



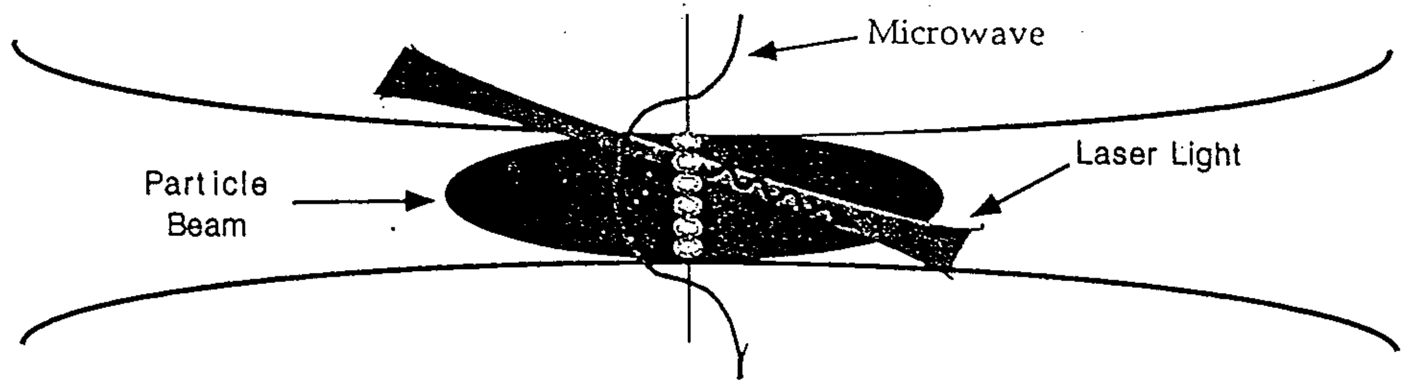
1 - argon-ion laser

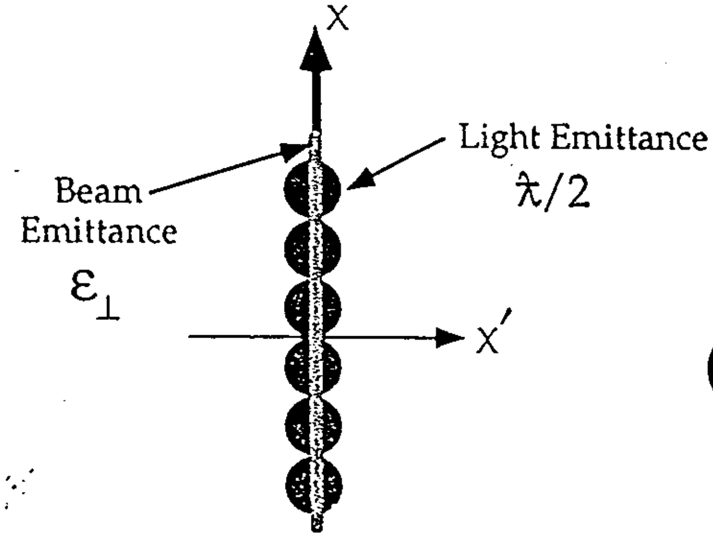
2 - Ti : sapphire crystal

3 - aperture

Optical Properties of Ti : sapphire at 300 k

Fluorescence peak	780 nm
Fluorescence lifetime	3.2 μ s
Saturation intensity	2.4×10^5 W/cm^2
Bandwidth, FWHM	10^{14} Hz
Refractive index, n	1.76
Temp. coefficient, $\frac{dn}{dT}$	1.3×10^{-5} K^{-1}





Particle

E

$(\Delta x_e, \Delta \phi_e)$

β

$\Delta x_e \bullet \Delta \phi_e = 2\,\pi \varepsilon_\perp$

Radiation

$\hbar\,\omega$

$(\Delta x_r, \Delta \phi_r)$

Z_r

$\Delta \phi_r \bullet \Delta x_r = \lambda\!\!\!^-/2$

Particle beam is fully resolved in space and time by light beam

Coherence Volume of Light < Beam Emittance

KINETIC THEORY

Consider 6N-dimensional Γ space in action-angle variable $\{\vec{I}_i, \vec{\psi}_i\}_{i=1,2,\ldots,N}$ with the density function

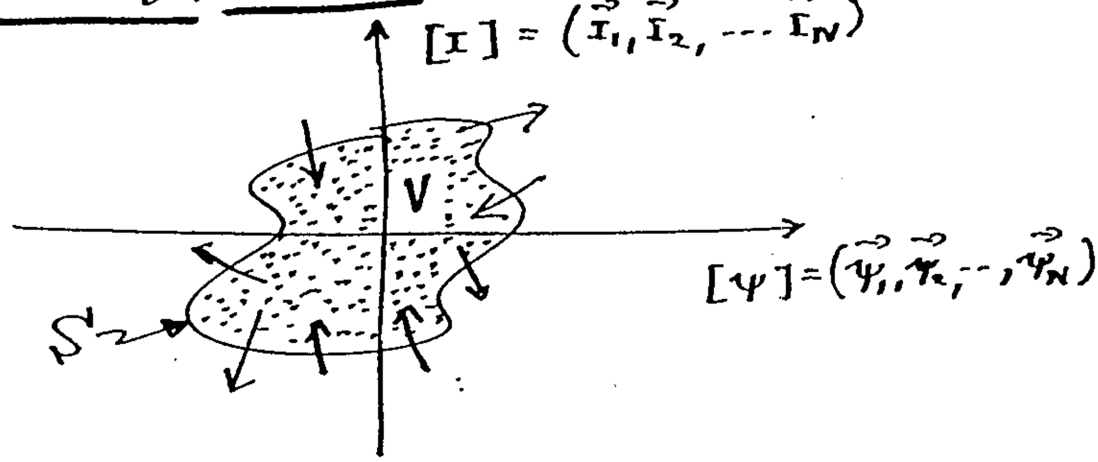
$$D\left(\vec{I}_1, \vec{\psi}_1; \vec{I}_2, \vec{\psi}_2; \cdots; \vec{I}_N, \vec{\psi}_N\right) \equiv D(\Gamma_N)$$

defined such that:

$$\int d\Gamma_N \, D(\Gamma_N) = 1$$

and we have the continuity equation expressing the conservation of probability:

$$\boxed{\frac{\partial D}{\partial t} + \sum_{i=1}^{N} \left[\frac{\partial}{\partial \vec{I}_i} \cdot \left(\dot{\vec{I}}_i D\right) + \frac{\partial}{\partial \vec{\psi}_i} \cdot \left(\dot{\vec{\psi}}_i D\right) \right] = 0}$$



Define "reduced distributions" by integrating over the variables we do not wish to care about.

REDUCED ONE-PARTICLE DISTRIBUTION:

$$f_1(1;t) \equiv f_1(\vec{I}_1, \vec{\psi}_1; t) = \int_{6(N-1)} d\Gamma_{N-1}\, D(\Gamma_N; t)$$

$$\equiv \int_{6(N-1)} (d\vec{I}_2 d\vec{\psi}_2) \cdots (d\vec{I}_N d\vec{\psi}_N)\, D(\Gamma_N; t)$$

REDUCED TWO-PARTICLE DISTRIBUTION:

$$f_2(1,2;t) \equiv f_2(\vec{I}_1, \vec{\psi}_1; \vec{I}_2, \vec{\psi}_2; t) = \int_{6(N-2)} d\Gamma_{N-2}\, D(\Gamma_N; t)$$

$$= \int_{6(N-2)} (d\vec{I}_3 d\vec{\psi}_3) \cdots (d\vec{I}_N d\vec{\psi}_N)\, D(\Gamma_N; t)$$

and so on.

To obtain equations for f_1, f_2; ... etc., we start integrating (103.2) over $6(N-1)$ and $6(N-2)$ and ---, etc. particle variables to get a chain of equations.

Thus the one-particle equation, after integrating over $(2, \cdots, N)$ variables, becomes:

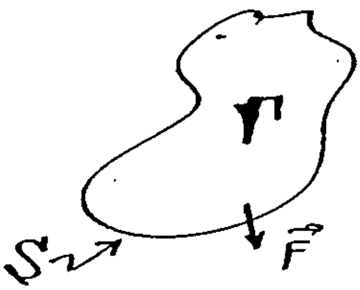
$$\frac{\partial f_1}{\partial t} = -\int_{6(N-1)} d\Gamma_{N-1} \left[\frac{\partial}{\partial \vec{I}_1} \cdot (\dot{\vec{I}}_1 D) + \frac{\partial}{\partial \vec{\psi}_1} \cdot (\dot{\vec{\psi}}_1 D) \right]$$

$$\underbrace{\left\{ - \int_{6(N-1)} d\Gamma_{N-1} \sum_{j=2}^{N} \left[\frac{\partial}{\partial \vec{I}_j} \cdot (\dot{\vec{I}}_j D) + \frac{\partial}{\partial \vec{\psi}_j} \cdot (\dot{\vec{\psi}}_j D) \right] \right\}}$$

Volume Integral : $\int d\Gamma_{N-1} \vec{\nabla}^{N-1} \cdot [\vec{u}^{N-1} D]$

$\Downarrow$ STOKE'S THEOREM

Surface Integral of $[\vec{u}^{N-1} D]$ on the $[6(N-1)-1]$-dimensional surface bounding the $6(N-1)$-dimensional volume Γ^{N-1}



$$\int_{\Gamma} d\Gamma (\vec{\nabla} \cdot \vec{F}) = \int_{S} d\vec{s} \cdot \vec{F}$$

This term vanishes because of boundary conditions on $[\vec{u}^{N-1} D]$ which must vanish at infinity.

$$\Rightarrow \boxed{\frac{\partial f_1}{\partial t} = - \int_{6(N-1)} (d\vec{I}_2 d\vec{\psi}_2) \cdots (d\vec{I}_N d\vec{\psi}_N) \left[\frac{\partial}{\partial \vec{I}_1} \cdot (\dot{\vec{I}}_1 D) + \frac{\partial}{\partial \vec{\psi}_1} \cdot (\dot{\vec{\psi}}_1 D) \right]}$$

COOLING INTERACTION

$$\boxed{\dot{\vec{I}}_i = \vec{G}(i,i) + \sum_{\substack{j=1\\ \neq i}}^{N} \vec{G}(i,j)}$$

$$\boxed{\dot{\vec{\psi}}_i = \vec{\omega}_i + \vec{H}(i,i) + \sum_{\substack{j=1\\ \neq i}}^{N} \vec{H}(i,j)}$$

Except for the non-conservative damping self-forces $\vec{G}(i,i)$ and $\vec{H}(i,i)$, the dynamics is governed by an underlying conservative Hamiltonian $\Rightarrow$ Hamiltonian Flow condition:

$$\boxed{\frac{\partial}{\partial \vec{I}_i} \cdot \left[\dot{\vec{I}}_i - \vec{G}(i,i)\right] = -\frac{\partial}{\partial \vec{\psi}_i} \cdot \left[\dot{\vec{\psi}}_i - \vec{H}(i,i)\right]}$$

The generalized forces are periodic in the 2π-periodic ange variables $\vec{\psi}_i, \vec{\psi}_j, \ldots$

$\Rightarrow$ Fourier decomposition:

$$\vec{G}\left[\vec{I}_i, \vec{\psi}_i ; \vec{I}_j, \vec{\psi}_j\right] = \sum_{\vec{n}_i} \sum_{\vec{n}_j} \vec{G}_{\vec{n}_i \vec{n}_j}(\vec{I}_i, \vec{I}_j) \cdot e^{i\left[\vec{n}_i \cdot \vec{\psi}_i + \vec{n}_j \cdot \vec{\psi}_j\right]}$$

$$\vec{H}\left[\vec{I}_i, \vec{\psi}_i ; \vec{I}_j, \vec{\psi}_j\right] = \sum_{\vec{n}_i} \sum_{\vec{n}_j} \vec{H}_{\vec{n}_i \vec{n}_j}(\vec{I}_i, \vec{I}_j) \cdot e^{\left[\vec{n}_i \cdot \vec{\psi}_i + \vec{n}_j \cdot \vec{\psi}_j\right]}$$

Using now the cooling interactions given by equations (103.6), (103.7) and the Hamiltonian Flow condition (103.8) and the symmetry of D under the interchange of particle indices, we obtain:

$$\frac{\partial f_1}{\partial t} + \vec{\omega}_1 \cdot \frac{\partial f_1}{\partial \vec{\varphi}_1} + (N-1) \int (d\vec{\varphi}_2 d\vec{I}_2) \left[\vec{G}(1,2) \cdot \frac{\partial f_2(1,2;t)}{\partial \vec{I}_1} + \vec{H}(1,2) \cdot \frac{\partial f_2(1,2;t)}{\partial \vec{\varphi}_1} \right]$$

$$= - \frac{\partial}{\partial \vec{I}_1} \cdot [\vec{G}(1,1) f_1] - \frac{\partial}{\partial \vec{\varphi}_1} \cdot [\vec{H}(1,1) f_1]$$

LHS $\equiv$ Liouvillian Conservative Flow

RHS $\equiv$ Dissipative Nonconservative Flow

$\Downarrow$

induces 'compression' or 'rarefaction' of phase space.

"LHS" includes integrals that expresses interaction with other beam particles

$$\left\{ \begin{array}{l} \text{Discrete 'pair' or collisions (1)} \\ + \\ \text{Correlations (2)} \\ + \\ \text{Self-consistent Average fields. (3)} \end{array} \right.$$

Similarly, integrating over variables $(3, \cdots, N)$, and using the same dynamics and symmetry assumption for D, one obtains equation for f_2 in terms of f_3:

$$\frac{\partial f_2}{\partial t} + \vec{\omega}_1 \cdot \frac{\partial f_2}{\partial \vec{\varphi}_1} + \vec{\omega}_2 \cdot \frac{\partial f_2}{\partial \vec{\varphi}_2} + (N-2)\int (d\vec{\varphi}_3\, d\vec{I}_3) \Big[\vec{G}(1,3) \cdot \frac{\partial f_3}{\partial \vec{I}_1} + H(1,3) \cdot \frac{\partial f_3}{\partial \vec{\varphi}_3} + (1 \rightleftarrows 2) \Big]$$

$$+ \vec{G}(1,2) \cdot \frac{\partial f_2}{\partial \vec{I}_1} + \vec{G}(2,1) \cdot \frac{\partial f_2}{\partial \vec{I}_2} + \vec{H}(1,2) \cdot \frac{\partial f_2}{\partial \vec{\varphi}_1} + \vec{H}(2,1) \cdot \frac{\partial f_2}{\partial \vec{\varphi}_2}$$

$$+ \left\{ \frac{\partial}{\partial \vec{I}_1} \cdot [\vec{G}(1,1) f_2] + \frac{\partial}{\partial \vec{\varphi}_1} \cdot [\vec{H}(1,1) f_2] + (1 \rightleftarrows 2) \right\}$$

$$= 0.$$

where $f_3 \equiv f_3(1,2,3;t)$.

Thus, an infinite hierarchy of relations between the reduced distributions is developing, which the terminates only at the flow equation for the full N-particle distribution, which is the grand-Continuity equation (103.2) itself.

$$\begin{cases} \dot{f}_1 \to f_1, f_2 \\ \dot{f}_2 \to f_2, f_3 \\ \quad \vdots \\ \dot{f}_{N-1} \to f_{N-1}, D \Longrightarrow \dot{D} \end{cases}$$

This is known as the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvonne) hierarchy. Some approximation is needed to terminate the sequence.

First we disentangle the totally and 'irreducibly' correlated i.e. "connected" parts of the distributions by the following "cluster decomposition":

$$f_1(1;t) = f(1;t)$$

$$f_2(1,2;t) = f(1;t)f(2;t) + g(1,2;t)$$

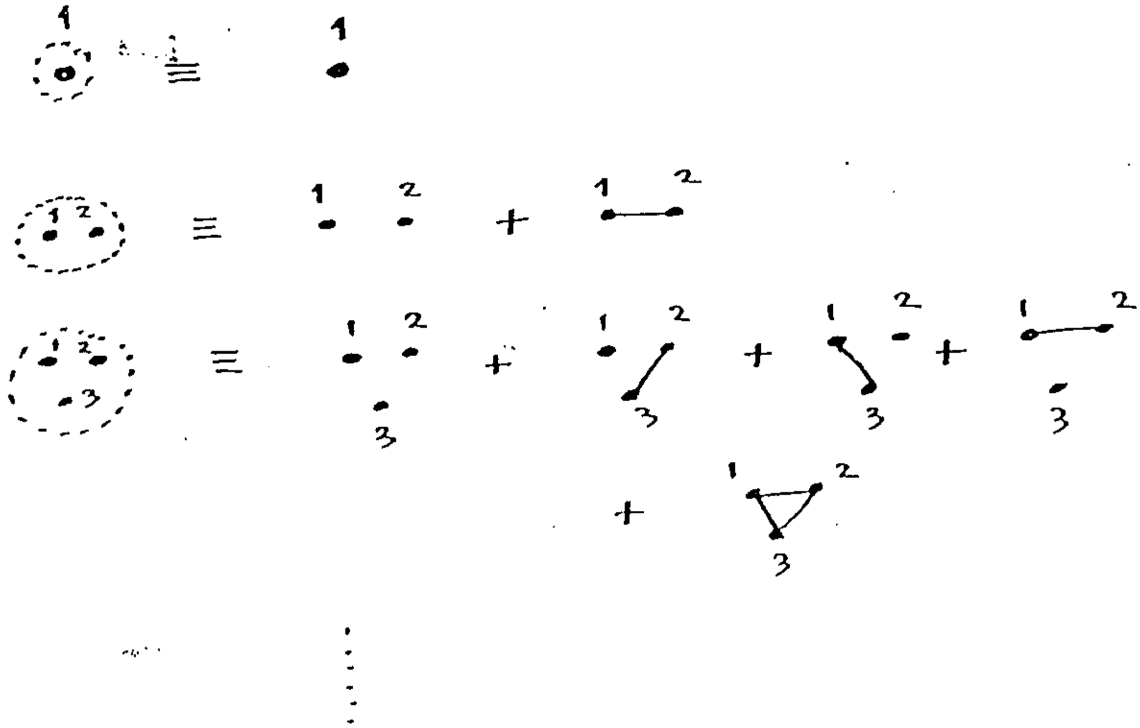
$$f_3(1,2,3;t) = f(1;t)f(2;t)f(3;t) + [f(1;t)\,g(2,3;t) + \text{cyclic permutation}] + h(1,2,3;t)$$

$$\vdots$$

etc.

We may then truncate the hierarchy beyond f_2 by setting $h(1,2,3;t) \approx 0$ i.e. neglecting three-body correlations with respect to two-body correlations which we retain as being non-negligible ($g(1,2;t) \neq 0$). For large N, we also assume $N \approx (N-1) \approx (N-2)$.

PICTORIAL CLUSTER DECOMPOSITION



Let the cooling interaction:

$$|\vec{G}|, |\vec{H}| \sim \varepsilon \ll 1$$

Then we assume a hierarchy of correlation strength scales:

$$\dots, \frac{h(1,2,3;t)}{g(1,2;t)} \sim \frac{g(1,2;t)}{f(1;t)f(2;t)} \sim |\vec{G}|, |\vec{H}| \sim \mathcal{O}(\varepsilon)$$

Then: $g \sim \mathcal{O}(\varepsilon)$ and $h \sim \mathcal{O}(\varepsilon^2)$

$(Gg), (Hg) \sim \mathcal{O}(\varepsilon^2)$

$N\cdot(Gg), N\cdot(Hg) \sim N\varepsilon^2, \quad N \gg 1$.

Therefore we $\begin{cases} \text{neglect } h, \; Gg, \; Hg, \ldots \text{etc.} \\ \text{retain } g, \; N(Gg), N(Hg), \ldots \text{etc.} \end{cases}$

The equations for one-body and two-body connected distributions, (f, g), take the form of the coupled closed equations:

$$\frac{\partial f(1;t)}{\partial t} + \vec{\omega}_1 \cdot \frac{\partial f(1;t)}{\partial \vec{\varphi}_1} + N \frac{\partial f(1;t)}{\partial \vec{I}_1} \cdot \int (d\vec{I}_2 d\vec{\varphi}_2) \, \vec{G}(1,2) f(2;t)$$

$$+ N \frac{\partial f(1;t)}{\partial \vec{\varphi}_1} \cdot \int (d\vec{I}_2 d\vec{\varphi}_2) \, \vec{H}(1,2) f(2;t)$$

$$= -\frac{\partial}{\partial \vec{I}_1} \cdot \left[\vec{G}(1,1) f(1;t) \right] - \frac{\partial}{\partial \vec{\varphi}_1} \cdot \left[\vec{H}(1,1) f(1;t) \right]$$

$$- N \int (d\vec{I}_2 d\vec{\varphi}_2) \left[\vec{G}(1,2) \frac{\partial g(1,2;t)}{\partial \vec{I}_1} + \vec{H}(1,2) \frac{\partial g(1,2;t)}{\partial \vec{\varphi}_1} \right]$$

LHS $\equiv$ Conservative un-correlated flow

RHS $\equiv$ Nonconservative "cooling" flow

$\oplus$ 2-body correlations.

Similarly:

$$\frac{\partial g(1,2;t)}{\partial t} + \vec{\omega}_1 \cdot \frac{\partial g(1,2;t)}{\partial \vec{\psi}_1} + \vec{\omega}_2 \cdot \frac{\partial g(1,2;t)}{\partial \vec{\psi}_2}$$

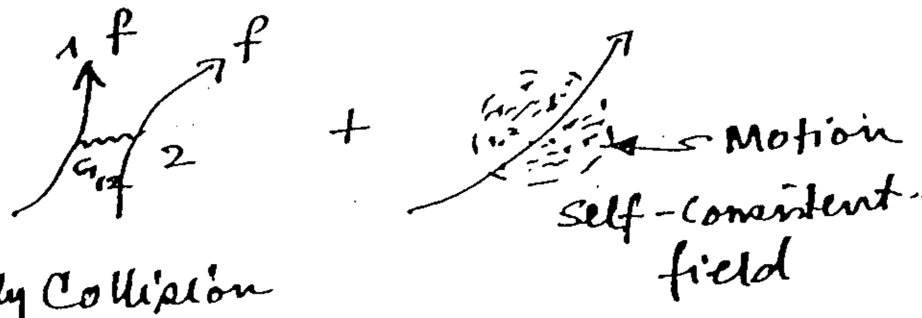
$$+ \left[\left\{ N \frac{\partial f(1;t)}{\partial \vec{I}_1} \cdot \int (d\vec{I}_3 d\vec{\psi}_3)\, \vec{G}(1,3)\, g(2,3;t) + (1 \leftrightarrow 2) \right\} + \left\{ \begin{matrix} I \to \psi \\ G \to H \end{matrix} \right\} \right]$$

$$= - \left[\left\{ N \frac{\partial g(1,2;t)}{\partial \vec{I}_1} \cdot \int (d\vec{I}_3 d\vec{\psi}_3)\, \vec{G}(1,3)\, f(3;t) + (1 \leftrightarrow 2) \right\} + \left\{ \begin{matrix} I \to \psi \\ G \to H \end{matrix} \right\} \right]$$

$$- \left[\left\{ \vec{G}(1,2) \cdot \frac{\partial f(1;t)}{\partial \vec{I}_1} f(2;t) + (1 \leftrightarrow 2) \right\} + \left\{ \begin{matrix} I \to \psi \\ G \to H \end{matrix} \right\} \right]$$

LHS $\Rightarrow$ Propagation of two body correlations

RHS $\Rightarrow$ Self-consistent field
+ collisions.



2-body Collision

Fourier - decompose in angles:

$$f(\vec{I},\vec{\psi};t) = \sum_{\vec{n}} f_{\vec{n}}(\vec{I};t)\, e^{i\vec{n}\cdot\vec{\psi}}$$

$$g(\vec{I}_1,\vec{\psi}_1;\vec{I}_2\vec{\psi}_2;t) = \sum_{\vec{n}_1}\sum_{\vec{n}_2} g_{\vec{n}_1,\vec{n}_2}(\vec{I}_1,\vec{I}_2;t)\, e^{i[\vec{n}_1\cdot\vec{\psi}_1 + \vec{n}_2\cdot\vec{\psi}_2]}$$

(103.15)

and using (103.9), we obtain for the angle- independent distribution

$$\Psi(\vec{I};t) \equiv f_0(\vec{I};t) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\vec{\psi}\, f(\vec{I},\vec{\psi};t) \quad (103.16)$$

the following:

$$\boxed{\frac{\partial\Psi(\vec{I};t)}{\partial t} + \frac{\partial}{\partial\vec{I}}\cdot\sum_{\vec{n}}\left[\vec{G}_{\vec{n},-\vec{n}}(\vec{I},\vec{I})\Psi(\vec{I};t)\right] = -\frac{\partial}{\partial\vec{I}}\cdot\left[\sum_{\vec{n}}\vec{R}^{*}_{\vec{n}\vec{n}}(\vec{I},\vec{I})\right]}$$

where

$$\vec{R}_{n_1n_2}(\vec{I}_1,\vec{I}_2) = N\sum_{\vec{n}_3}\int d\vec{I}_3\, \vec{G}^{*}_{\vec{n}_1\vec{n}_3}(\vec{I}_1,\vec{I}_3)\, g_{\vec{n}_2\vec{n}_3}(\vec{I}_2,\vec{I}_3;t)$$

$\vec{R}$ satisfies the integral Equation:

$$\vec{R}_{\vec{n}_2\vec{n}_1}(\vec{I}_2,\vec{I}_1) = -\pi N \sum_{\vec{n}_3} \int d\vec{I}_3 \; \delta_+[\vec{n}_3\cdot\vec{\omega}_3 - \vec{n}_1\cdot\vec{\omega}_1]\, \vec{G}^{*}_{\vec{n}_2\vec{n}_3}(\vec{I}_2,\vec{I}_3)$$

$$\Big\{ \cdot\Big[\vec{G}_{\vec{n}_1\vec{n}_3}(\vec{I}_1,\vec{I}_3)\cdot\frac{\partial f_0(1;t)}{\partial \vec{I}_1} f_0(3;t) - \vec{G}^{*}_{\vec{n}_3\vec{n}_1}(\vec{I}_3,\vec{I}_1)\cdot\frac{\partial f_0(3;t)}{\partial \vec{I}_3} f_0(1;t)\Big]$$

$$+ \Big[\frac{\partial f_0(1;t)}{\partial \vec{I}_1}\cdot\vec{R}^{*}_{\vec{n}_1\vec{n}_3}(\vec{I}_1,\vec{I}_3) - \frac{\partial f_0(3;t)}{\partial \vec{I}_3}\cdot\vec{R}_{\vec{n}_3\vec{n}_1}(\vec{I}_3,\vec{I}_1)\Big]\Big\}$$

(103.19)

and where

$$\pi\,\delta_+(x) = \pi\,\delta(x) - i\,\mathcal{P}V(1/x)$$

$$= \lim_{\eta\to 0^+} \frac{1}{[\eta + i x]}$$

(103.17) $\longrightarrow$ (103.19) $\Longrightarrow$ Coupled Integro-Differential Equations:

$$\begin{cases} \dot{\Psi} \longrightarrow f[\Psi, \vec{R}] \\ \vec{R} \longrightarrow g[\int \Psi \vec{R}] \end{cases}$$

TRANSPORT EQUATION : FRICTION AND DIFFUSION COEFFICIENTS

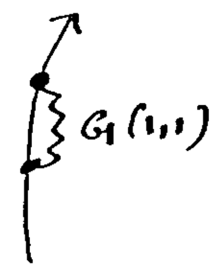
If we include only the nonconservati damping and cooling terms and the two-body correlations introduced by $G_1(i,j)$ but ignore that part of interaction of i with the averaged

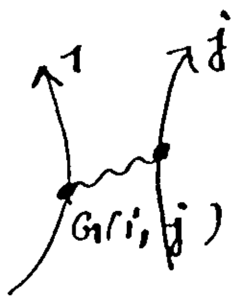
$$\langle G_1(i,j)\rangle_j \equiv \int G_1(x,x')\,dx' \equiv V(x),$$

then the solution is simple. Thus we keep:

$$G_1(1,1) \quad \text{and} \quad G_1(1, j \neq 1)$$

(A)

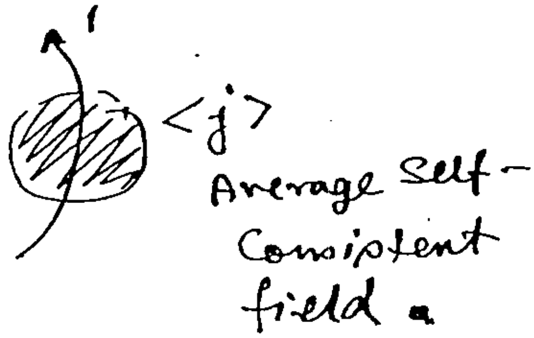




(B)

and neglect:

(C)



Then the transport equation takes the form of the Fokker-Planck Equation:

$$\boxed{\frac{\partial \psi(\vec{I};t)}{\partial t} = -\frac{\partial}{\partial \vec{I}} \cdot \left[\vec{F}(\vec{I}) \psi(\vec{I};t) \right] + \frac{1}{2} \frac{\partial}{\partial \vec{I}} \cdot \left[\underline{\underline{D}}(\vec{I}) \cdot \frac{\partial \psi(\vec{I};t)}{\partial \vec{I}} \right]}$$

where the Friction and Diffusion coefficients, $\vec{F}(\vec{I})$ and $\underline{\underline{D}}(\vec{I})$ are as follows:

$$\boxed{\vec{F}(\vec{I}) = \sum_{\vec{n}} G_{\vec{n},-\vec{n}}(\vec{I},\vec{I})}$$

$$\boxed{\underline{\underline{D}}(\vec{I}) = 2\pi N \sum_{\vec{n},\vec{n}'} \int d\vec{I}' \left| G_{-\vec{n},\vec{n}'}(\vec{I},\vec{I}') \right|^2 \psi_0(\vec{I}') \, \delta\left[\vec{n}' \cdot \vec{\omega}(\vec{I}') - \vec{n} \cdot \vec{\omega}(\vec{I}) \right]}$$

In general, $D(\vec{I}, \psi(\vec{I};t))$ changes slowly with time. For slow diffusion and cooling and F and D linear in I, one can solve the Fokker-Planck explicitly.

If the term ⓒ representing self-consistent field is non-negligible, one has:

$$\boxed{F(\vec{I}) = \sum_{\vec{n}} \frac{G_{\vec{n},-\vec{n}}(\vec{I},\vec{I})}{\epsilon_{\vec{n}}(\vec{I})}}$$

and

$$\boxed{D(\vec{I}) = 2\pi N \sum_{\vec{n}} \int d\vec{I}' \, \frac{|G_{-\vec{n},\vec{n}}(\vec{I},\vec{I}')|^2}{|\epsilon_{\vec{n}}(\vec{I})|^2} \, \psi_0(\vec{I}') \, \delta\left(\vec{n}\cdot[\vec{\omega}(\vec{I}') - \vec{\omega}(\vec{I})]\right)}$$

where

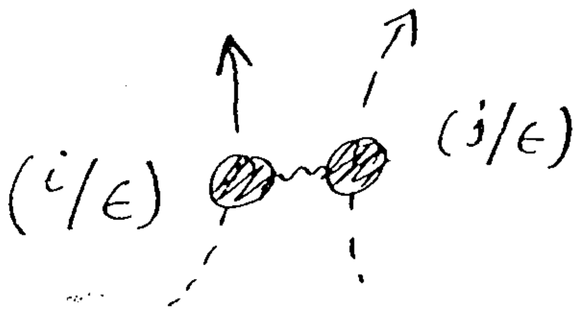
$$\epsilon_{\vec{n}}(\vec{I}) = \epsilon(\Omega)\Big|_{\Omega = \vec{n}\cdot\omega(\vec{I})}$$

and

$$\boxed{\epsilon(\Omega) = 1 - i N \lim_{\gamma \to 0^+} \int d\vec{I}' \, \frac{\left[G^*_{-\vec{n},\vec{n}}(\vec{I}',\vec{I}) \, \frac{\partial \psi_0(\vec{I}')}{\partial \vec{I}'}\right]}{\left[\Omega - \vec{n}\cdot\vec{\omega}(\vec{I}') + i\gamma\right]}}$$

Requires considerable gymnastics in the complex plane, involving Winer–Hopf techniques, etc. But the solution is complete. See e.g. S. Ichimaru, "Basic Principles of Plasma Physics" or any book on kinetic theory.

Particle appear "dressed" by the self-consistent averaged field so that both self-force and scattering get modified by the dressing function $\epsilon(r)$:



$$\Longrightarrow \frac{\langle ij \rangle}{|\epsilon|^2}$$

$$\Longrightarrow g\,\frac{i}{\epsilon}$$

EXPLICIT SOLUTION of FOKKER-PLANCK EQUATION FOR A SPECIAL CASE

The general Fokker-Planck equation can be written as:

$$\frac{\partial \psi(\vec{I};t)}{\partial t} = -\frac{\partial}{\partial \vec{I}} \cdot \left[\vec{F}(\vec{I})\, \psi(\vec{I};t) \right] + \frac{1}{2} \frac{\partial}{\partial \vec{I}} \cdot \left[\underline{\underline{D}}(\vec{I}) \cdot \frac{\partial \psi(\vec{I};t)}{\partial \vec{I}} \right]$$

Phase Space "Cooling"

Phase Space "Heating"

Let the dynamics of heating and cooling be operative only in one-dimension, $I_x \equiv I$, leaving the other dimension $I_Z = J$ intact and "unaffected", except that the coefficients of "friction" and "diffusion" will depend on the local phase space variable $\vec{I} \equiv (I, J)$ for a 2-D system with cooling in I. We consider the ideal special case where the coefficients F and G in the transport equation are linear in the "cooled" and "heated" variable I.

We thus consider:

$$\vec{F}(\vec{I}) = [F(I,J),\ 0,\ 0]$$

$$\underline{\underline{D}}(\vec{I}) = \begin{bmatrix} D(I,J) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$F(I,J) = \alpha(J) \cdot I$$
$$D(I,J) = \lambda(J) \cdot I.$$

The Fokker-Planck then reads:

$$\boxed{\frac{\partial \psi(I,J;t)}{\partial t} = -\frac{\partial}{\partial I}\left[\alpha(J) I\, \psi(I,J;t) - \frac{\lambda(J)}{2} I \frac{\partial}{\partial I} \psi(I,J;t)\right]}$$

The equilibrium distribution $\psi_{eq.}(I,J)$ such that $\partial \psi_{eq}/\partial t = 0$ is then given by:

$$\boxed{\psi_{eq.}(I,J) \equiv \exp\left[-\frac{I}{I_0(J)}\right]}$$

where $$I_0(J) = \frac{\lambda(J)}{2\alpha(J)}$$

$$\Rightarrow \quad \boxed{\psi_{eq.}(I,J) \equiv \exp.\left[-2\alpha(J)\frac{I}{\lambda(J)}\right]}$$

In this special case, one can find eigen-functions of the F-P equation. We assume:

$$\psi(I, J; t) = \psi(I, J) e^{-\delta t}$$

Then F-P reduces to:

$$\delta \psi - \frac{d}{dI}\left[\alpha I \psi - \frac{\lambda}{2} I \frac{d\psi}{dI}\right] = 0$$

Change variables:

$$I = -\frac{\lambda}{2\alpha} x \qquad \psi = h e^{x}$$

Then F-P reduces to the equation for Laguerre polynomials:

$$x h'' + (1-x) h' + k h = 0 \qquad (105\text{-}7)$$

where $k = \delta/(-\alpha) = n = 0, 1, 2, \ldots$ defines the eigensolutions, given by:

$$\boxed{\psi_n = L_n(x) e^{-x - n\alpha t}}$$

or $$\psi_n(I, J; t) = L_n\left(-\frac{2\alpha(J)}{\lambda(J)} I\right) \cdot \exp\left[\frac{2\alpha(J)}{\lambda(J)} I - n\alpha(J) t\right]$$

The general solution is given by:

$$\psi(I, J; t) = \int dI' \, G(I, I'; J | t) \, \psi(I', J; 0)$$

where $G(I, I'; J | t)$ is the Green's function given by:

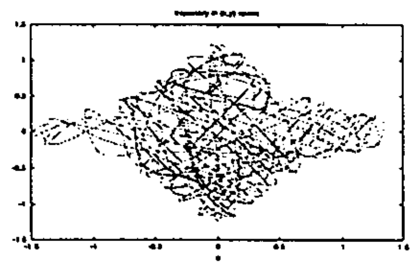
$$\boxed{\begin{aligned} G(I, I'; J | t) &= H(x, x' | t) \\ &= e^{-x} \sum_{n=0}^{\infty} L_n(x) \, L_n(x') \, e^{-n\alpha t} \\ &= \frac{\exp\left[-\frac{x - x' e^{-\alpha t}}{1 - e^{-\alpha t}}\right]}{\left[1 - e^{-\alpha t}\right]} \underbrace{I_0\left[\frac{\sqrt{x x'}}{\sinh\left(\frac{\alpha}{2} t\right)}\right]}_{\text{Bessel Function of the second kind with imaginary argument.}} \end{aligned}}$$

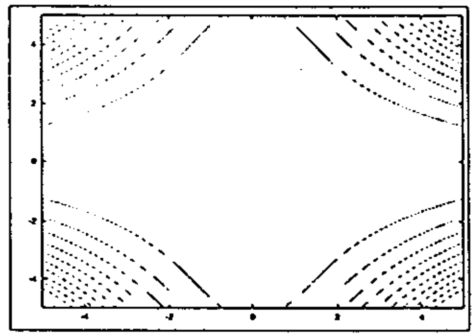
$$\left\{ \begin{aligned} H(x, x' | 0) &= \delta(x - x') \\ H(x, x' | \infty) &= e^{-x} \\ \int H(x, x' | t) \, dx' &= 1. \end{aligned} \right\}$$

Example: Parallel Langevin Simulation

- Interacting particles in a potential V(x,y) that admits chaos
- Damping & diffusion drive the beam to thermal equilibrium

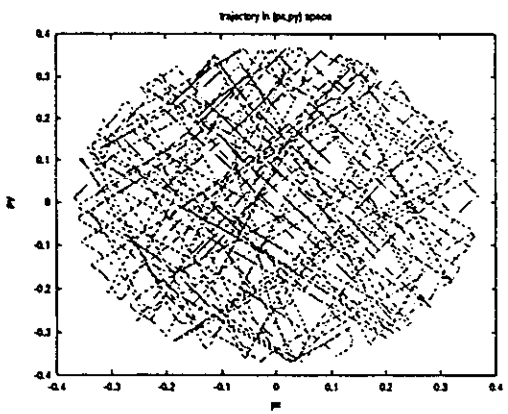
(x,y) trajectory

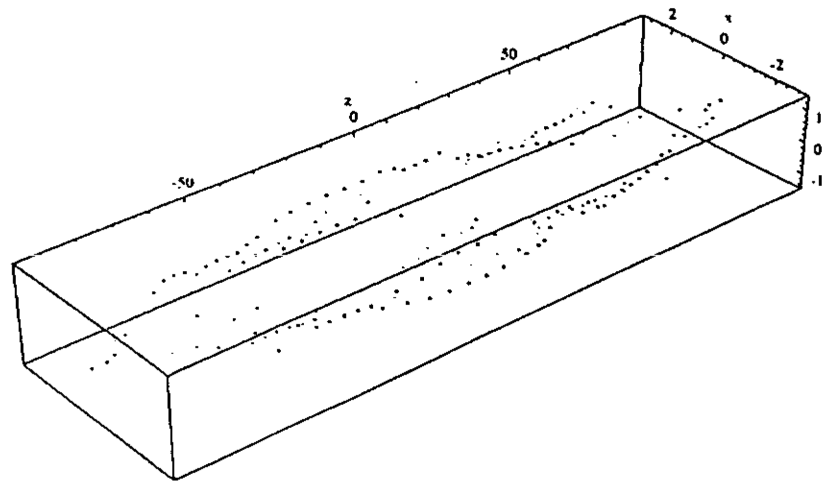




V(x,y)

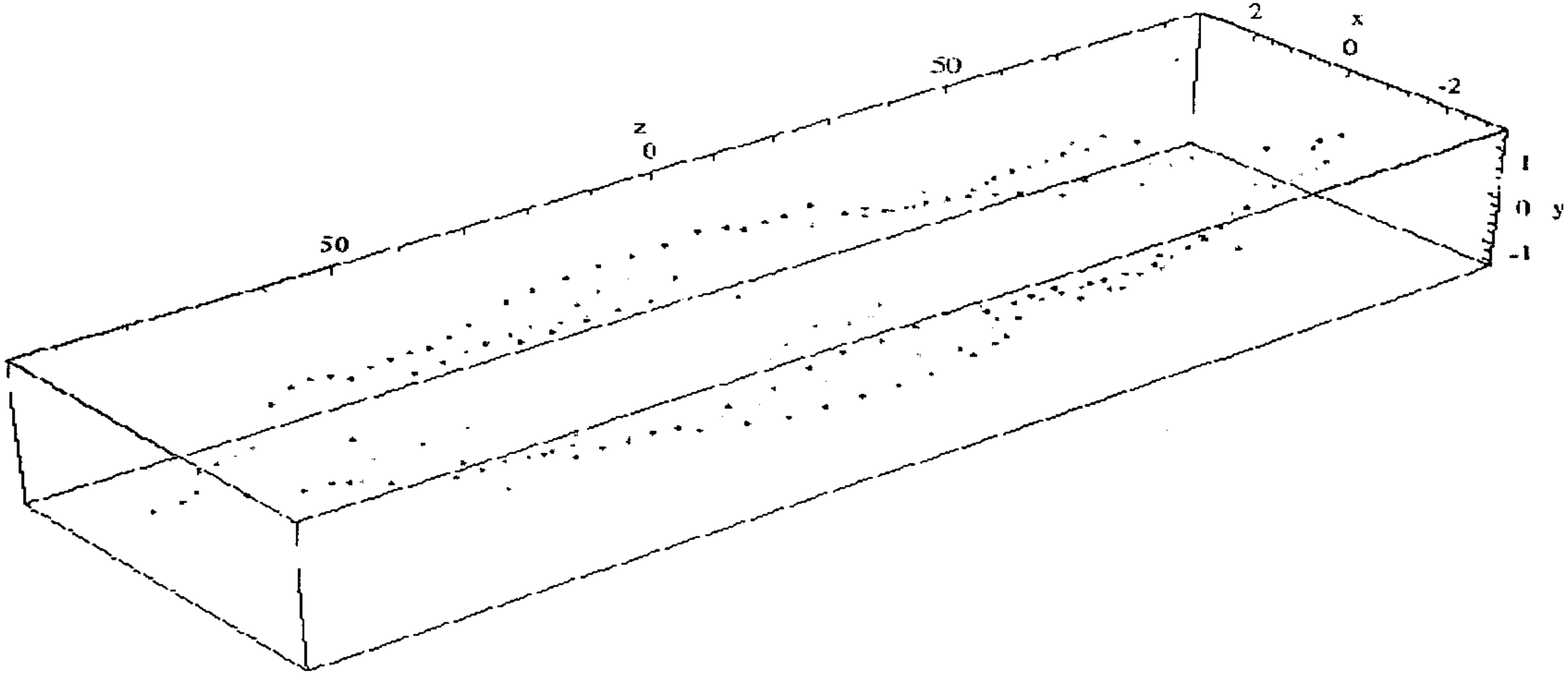
(p_x,p_y) trajectory





Courtesy of R. Ryne, LANL

Crystaline Beam, 200 Particles



Crystaline Beam, 400 Particles