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SMR 1161/12

AUTUMN COLLEGE ON PLASMA PHYSICS

25 October - 19 November 1999

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S.M. Mahajan

Institute for Fusion Studies
The University of Texas at Austin
U.S.A.

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S.M. Mahajan
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

Z. Yoshida
Graduate School of Frontier Sciences, The University of Tokyo, Tokyo 113-8656, Japan
(March 16, 1999)

An appropriate extension of the recently derived double-Beltrami, self-organizing equilibria reproduces the essential observational features of the thin shear-layer associated with the H-mode tokamak discharges. Natural consequences of the theory are that the length scale of the shear-layer is of the order of the poloidal gyro-radius, and that the velocity achieves the poloidal Mach number of order unity.

52.55.Fa,52.30.-q,52.55.Dy,47.65.+a

Extending the framework of the recently developed magneto-fluid theory [1] (in which the velocity field is a fundamental determinant of the final state and is treated nonperturbatively) by embedding the charged fluid in a strong external field, we investigate the structure of a relatively thin self-organized shear-flow layer at the plasma edge. This layer represents a self-organized equilibrium state obtained by a simultaneous solution to the force balance and the induction equations and culminating in the form of coupled Beltrami conditions supplemented by corresponding Bernoulli relationships.

It is remarkable that such a self-organized states, in its essential aspects, resembles the shear-flow layer which develops when a tokamak plasma makes a transition to the high confinement mode (H-mode) [2–7]. The defining characteristics of an H-mode shear layer are: (1) There is a relatively large plasma flow, (2) The flow is highly sheared in that the characteristic length scale (the layer width) on which the velocity field builds up is rather short;it is found to be typically of the order of a poloidal gyro-radius and (3) There is a precipitous fall in the plasma density and pressure as we go across this thin layer.

It stands to reason, then, that an appropriate description of such a layer would require an equilibrium model in which the flows play an essential role, the flows and the magnetic fields produced by the currents in the layer are self-consistently determined, and in which the pressure drop in the layer is somehow related to the build-up of the velocity field. A subclass of structures investigated in [1], and reproduced below in a tokamak like setting, is precisely of this kind. The linking together of the velocity and the magnetic fields is brought about by the Hall term which also introduces a short characteristic length scale (λ_i, the ion skin depth) to the otherwise scale-less magnetohydrodynamics (MHD).

For simplicity, we consider a quasineutral plasma with singly charged ions. Neglecting the small electron inertia, the electron equation of motion is

$$E + V_e \times B + \frac{1}{en}\nabla p_e = 0, \tag{1}$$

where V_e and p_e are, respectively, the flow velocity and pressure of electrons, E and B are, respectively, the electric and magnetic fields, e is the electron charge, and n is the number density. The ion velocity V obeys

$$\frac{\partial}{\partial t}V + (V \cdot \nabla)V = \frac{e}{M}(E + V \times B) - \frac{1}{Mn}\nabla p_i, \tag{2}$$

where M is the ion mass ($M \gg$ electron mass), and p_i is the ion pressure.

Using $V_e = V - j/(en)$, $j = \mu_0^{-1}\nabla \times B$ (j is the electric current), and $E = -\partial A/\partial t - \nabla\phi$, where A (ϕ) is the vector (scalar) potential, and choosing the normalizations: coordinate $x = L_0\hat{x}$, $B = B_0\hat{B}$, $t = (L_0/V_A)\hat{t}$, $p = (B_0^2/\mu_0)\hat{p}$, and $V = V_A\hat{V}$, where L_0 and B_0 are arbitrary, and $V_A = B_0/\sqrt{\mu_0 M n}$ is the Alfvén speed, equations (1) and (2) transform to

$$\frac{\partial}{\partial \hat{t}}\hat{A} = \left(\hat{V} - \varepsilon\hat{\nabla} \times \hat{B}\right) \times \hat{B} - \hat{\nabla}\left(\hat{\phi} - \varepsilon\hat{p}_e\right), \tag{3}$$

$$\frac{\partial}{\partial \hat{t}}(\varepsilon\hat{V} + \hat{A}) = \hat{V} \times \left(\hat{B} + \varepsilon\hat{\nabla} \times \hat{V}\right) - \hat{\nabla}\left(\varepsilon V^2/2 + \varepsilon\hat{p}_i + \hat{\phi}\right), \tag{4}$$

where the scaling coefficient $\varepsilon = \lambda_i/L_0$ is a measure of the ion skin depth $\lambda_i = c/\omega_{pi} = V_A/\omega_{ci} = \sqrt{M/(\mu_0 n e^2)}$.

Here, for simplicity, we have assumed the density n to be constant. This is, of course, not true for the H-mode layers where the density falls sharply. It turns out that the effects of the variation of n (even when the variation is on the scale of the layer), though profound in determining the details of the fields in the layer, do not make a qualitative difference in the total change suffered by the observables as we go across the layer. Thus the essential features of the theory, which depend upon the jump conditions across the layer (separating the core plasma from the edge region), will be accessible within the constant n assumption. The details will be given in future work.

The Hall term $\varepsilon(\hat{\nabla} \times \hat{B}) \times \hat{B}$ of (3), which may be regarded as a singular perturbation to the conventional MHD equations, plays an essential role in determining the structure of a thin shear-flow layer which may appear at the edge of a high-temperature plasma. We now choose the length scale $L_0 = \lambda_i$ (and hence, $\varepsilon = 1$), and simplify the notation by dropping the $\hat{}$ on the normalized variables.

We shall consider the most basic stationary (or slowly evolving) structures of the electromagnetic fields and the associated flows which we can explore in the framework of the coupled Beltrami conditions [1]. To apply the theory to a tokamak plasma which is embedded in a strong external magnetic field, it is appropriate to decompose the magnetic field B into the sum of the self-field component B_s and the externally rooted component B_h. The B_s

is produced by the plasma current $\boldsymbol{j}$ in the region of our interest, while the $\boldsymbol{B}_h$ is current-free (curl-free, and thus "harmonic") in that region. From now on, the dynamical part of the field, $\boldsymbol{B}_s$, will be normalized by its representative value B_*. The velocities are, then, normalized by the corresponding Alfvén velocity V_{A*}.

We consider a one-dimensional system where the fields vary only in the "radial" direction, perpendicular to the magnetic surfaces implying $\boldsymbol{B}_h \cdot \nabla \equiv 0$. We also assume that $\boldsymbol{V}$ is incompressible ($\nabla \cdot \boldsymbol{V} = 0$). Then, we find $\nabla \times (\boldsymbol{V} \times \boldsymbol{B}_h) = 0$, which allows us to write

$$\boldsymbol{V} \times \boldsymbol{B}_h = \nabla P_i \tag{5}$$

with some potential field P_i. In a usual tokamak-like equilibrium, $\boldsymbol{V}$ is primarily the diamagnetic ion current $\boldsymbol{V} = -(\nabla p_i \times \boldsymbol{B})/B^2$. Plugging this expression in (5), we obtain $\nabla P_i = (\boldsymbol{B} \cdot \boldsymbol{B}_h / B^2)\nabla p_i$. We may, thus, estimate

$$p_i - P_i \approx \frac{\boldsymbol{B}_h \cdot \boldsymbol{B}_s + B_s^2}{B^2} p_i = \delta p_i, \quad \delta = O(B_*/B). \tag{6}$$

Similarly, we can write

$$\boldsymbol{V}_e \times \boldsymbol{B}_h = (\boldsymbol{V} - \nabla \times \boldsymbol{B}) \times \boldsymbol{B}_h = -\nabla P_e \tag{7}$$

with

$$P_e = -P_i - \boldsymbol{B}_h \cdot \boldsymbol{B}_s, \tag{8}$$

which primarily represents the zero-order diamagnetism. We may, now, rewrite the system (3) and (4) as

$$\frac{\partial}{\partial t}\boldsymbol{A}_s = (\boldsymbol{V} - \nabla \times \boldsymbol{B}_s) \times \boldsymbol{B}_s + \nabla \varphi_e, \tag{9}$$

$$\frac{\partial}{\partial t}(\boldsymbol{V} + \boldsymbol{A}_s) = \boldsymbol{V} \times (\boldsymbol{B}_s + \nabla \times \boldsymbol{V}) - \nabla \left(\varphi_i + V^2/2\right), \tag{10}$$

where $\nabla \times \boldsymbol{A}_s = \boldsymbol{B}_s$, and

$$\varphi_i = p_i - P_i + \phi, \quad \varphi_e = p_e - P_e - \phi, \tag{11}$$

represent the effective potential fields (the static electric potential and the "residual" pressures; see (6)) experienced, respectively, by the ion and electron fluids.

Taking the curl of (9) and (10), we can cast them in a system of vortex-dynamics equations

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}_j - \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\Omega}_j) = 0 \quad (j = 1, 2) \tag{12}$$

in terms of a pair of generalized vorticites $\boldsymbol{\Omega}_1 = \boldsymbol{B}_s$, $\boldsymbol{\Omega}_2 = \boldsymbol{B}_s + \nabla \times \boldsymbol{V}$, and the corresponding effective flows $\boldsymbol{U}_1 = \boldsymbol{V} - \nabla \times \boldsymbol{B}_s$, $\boldsymbol{U}_2 = \boldsymbol{V}$. The coupled Beltrami conditions [1,8] demand the alignment of the vorticites with the corresponding flows, i.e., $\boldsymbol{U}_j = \mu_j \boldsymbol{\Omega}_j$ with some scalar functions μ_j ($j = 1, 2$). Writing $a = 1/\mu_1$ and $b = 1/\mu_2$, and assuming that a and b are constants, the

Beltrami conditions translate to the simultaneous linear equations

$$B_s = a(V - \nabla \times B_s), \tag{13}$$
$$B_s + \nabla \times V = bV. \tag{14}$$

As a direct consequence of the Beltrami (13)-(14) and equilibrium conditions (9)-(10), we obtain a set of generalized "Bernoulli conditions"

$$\varphi_i + \frac{1}{2}V^2 = \text{constant}, \tag{15}$$
$$\varphi_e = \text{constant}. \tag{16}$$

We note that the constancy of the energy density (the sum of the potential and the kinetic energy) implied in (15)-(16) refers to the directions perpendicular, as well as parallel, to the streamlines of V. This is an essential difference from the conventional Bernoulli condition.

The Beltrami conditions represent the central paradigm of the present argument of "self-organization". The physical justification of this "homogeneity" (or equilibrium) of energy density in the transverse direction of the ambient streamlines relies on an implicit assumption of existing fluctuations which average out intensive variables. An interesting consequence that follows is the appearance of a non-trivial "structure" (with a characteristic length scale) in some physical quantities creating a thin boundary layer separating two regions, and enhancing the state of thermal non-equilibrium.

Combining (13) and (14) yields a second order partial differential equation

$$\nabla \times (\nabla \times V) + c_1 \nabla \times V + c_2 V = 0, \tag{17}$$

where $c_1 = (1/a) - b$ and $c_2 = 1 - b/a$. Denoting the curl derivative $\nabla\times$ by "curl", (17) becomes

$$(\text{curl} - \Lambda_+)(\text{curl} - \Lambda_-)V = 0, \tag{18}$$

where

$$\Lambda_\pm = \frac{1}{2}\left[-c_1 \mp (c_1^2 - 4c_2)^{1/2}\right]. \tag{19}$$

Since the operators $(\text{curl} - \Lambda_\pm)$ commute, the general solution to the "double curl Beltrami equation" (18) is given by the linear combination of two Beltrami fields $G_{\Lambda_\pm}$ satisfying $(\text{curl} - \Lambda_\pm)G_{\Lambda_\pm} = 0$, i.e.,

$$V = C_+ G_{\Lambda_+} + C_- G_{\Lambda_-}, \tag{20}$$

where $C_\pm$ are arbitrary constants. The corresponding magnetic field is given by

$$B_s = (b - \Lambda_+)\, C_+ G_{\Lambda_+} + (b - \Lambda_-)\, C_- G_{\Lambda_-}. \tag{21}$$

These interrelated flow and magnetic fields represent the structure of a "thin layer" which may be generated at the boundary of a plasma where a shock-like jump in the pressure emerges.

Let us work out an analytic solution in slab geometry (the coordinate x is radial, y is poloidal, and z is toroidal). We easily find that the sheared vector field

$$\mathcal{G}_\Lambda = {}^t(0,\ \sin(\Lambda x + \theta),\ \cos(\Lambda x + \theta)) \qquad (22)$$

is a Beltrami field solving $\nabla \times \boldsymbol{G}_\Lambda = \Lambda \boldsymbol{G}_\Lambda$ (θ is an arbitrary constant). A coupled Beltrami field is given by a linear combination of two of such Beltrami fields.

We consider a boundary layer $0 < x < \Delta$ in contact with a "core plasma" contained in $x < 0$. The exterior region $x > \Delta$ is scraped-off by a physical boundary. The layer thickness Δ is to be determined by the theory. We have the following boundary and "jump" conditions. In the core plasma, we assume $\boldsymbol{V} = 0$ (possibly in some inertial system). Since $\boldsymbol{B}_s$ is the magnetic field generated by the current in the layer, the poloidal (toroidal) field must vanish at the inner (outer) boundary [9]. We thus have

$$\begin{array}{ll} V_y(0) = 0, & B_{s,y}(0) = 0, \\ V_z(0) = 0, & B_{s,z}(\Delta) = 0. \end{array} \qquad (23)$$

The diamagnetic equation (8) yields a relation between the magnetic field and the pressure jumps across the layer. Denoting $[f] = f(\Delta) - f(0)$, we have

$$\begin{aligned} [p] &= [p_i + p_e] \\ &\approx [P_i + P_e] = -[\boldsymbol{B}_h \cdot \boldsymbol{B}_s] \approx -B[B_{s,z}]. \end{aligned} \qquad (24)$$

The ion Bernoulli condition (15) yields

$$2[\varphi_i] = -[V^2]. \qquad (25)$$

Using the boundary condition $B_{s,z}(\Delta) = 0$ and setting $p(\Delta) = 0$, we can write (24) as $B_{s,z}(0) = p(0)/B$. We use this value to normalize $\boldsymbol{B}_s$, i.e, we set $B_{s,z}(0) = 1$, and identify

$$B_* = \frac{p(0)}{B} = \frac{\beta}{2}B, \qquad (26)$$

where β is the conventional core plasma beta evaluated at the boundary $x = 0$. Since Δ is an unknown variable, we could treat $B_{s,z}(0) = 1$ as a boundary condition, and use $B_{s,z}(\Delta) = 0$ to determine Δ. It is appropriate to set $\varphi_i(\Delta) = 0$. Then, (25) yields $V^2(\Delta) = 2\varphi_i(0)$. For the energy density to have a monotone-decreasing profile in the layer, which may parallel the entropy condition in the shock problem, the peak of $V^2(x)$ must not appear inside the layer. The thickness of the layer is maximized when $V(x)^2$ achieve its first peak at $x = \Delta$. This condition, together with $V^2(\Delta) = 2\varphi_i(0)$, determines the required set of Beltrami parameters a and b (equivalently, $\Lambda_\pm$). The theory, then, will be parameter free; the known experimental quantities would be enough to describe the state.

For an explicit calculation, we start with $V = C_+\mathcal{G}_{\Lambda_+} + C_-\mathcal{G}_{\Lambda_-}$. Choosing $\theta = 0$ in (22) satisfies the boundary conditions $V_y(0) = B_{s,y}(0) = 0$. The other boundary conditions demand

$$C_+ + C_- = 0, \quad C_+ = \frac{1}{\Lambda_- - \Lambda_+}. \tag{27}$$

Using these relations, we obtain

$$V^2(x) = 4C_+^2 \sin^2[x(\Lambda_- - \Lambda_+)/2]. \tag{28}$$

Since $V^2(x)$ must be maximized at $x = \Delta$, we obtain

$$\Delta = \frac{\pi}{|\Lambda_- - \Lambda_+|} \tag{29}$$

This Δ must also satisfy $B_{s,z}(\Delta) = 0$ (see (23)) which is possible only if $b = 1/a$. Then, we have $\Lambda_+ = -\Lambda_- = \sqrt{b^2 - 1}$, $C_+ = 1/(2\sqrt{b^2-1})$, and $V^2(\Delta) = 1/(b^2 - 1)$. When the boundary value $V^2(\Delta)$ is given, all parameters are determined. Figure 1 shows the profiles of V and B_s in $x > 0$.

By relating the boundary value $V^2(\Delta) = 2\varphi_i(0)$ with the plasma pressure, we find a fascinating scaling for Δ and the flow velocities. Combining (27) and (29), we obtain $\Delta = \pi|C_+|$. Using (28), we observe $C_+^2 = V^2(\Delta)/4 = \varphi_i(0)/2$. Assuming $\phi(0) = 0$ [10], and using (26), we obtain $\varphi_i(0) = (B_*/B)p_i = (B/B_*)\beta_i/2 = \beta_i/\beta = T_i/T_i + T_e$ (β_i is the ion beta ratio). Thus the layer width $\check{\Delta}(= \Delta\lambda_i)$ in physical units is of the order λ_i. More conventionally,

$$\check{\Delta} = \pi\sqrt{\frac{\varphi_i(0)}{2}}\lambda_i = \frac{\pi\sqrt{\beta_i}V_A}{\sqrt{2\beta}\omega_{ci}} = \frac{\pi\rho_i}{\sqrt{\beta}} = \frac{\pi\rho_{i,p}}{\sqrt{\beta_p}} \tag{30}$$

where $\rho_{i(p)} = V_T/\omega_{ci(p)}$ is the ion gyro-radius (poloidal gyro-radius) and $V_T = \sqrt{\beta_i/2}V_A$ is the ion thermal speed. Since, typically, $\sqrt{\beta_p} = O(1)$, the layer width is also of the order of the poloidal gyro-radius.

The peak velocity $V_{max} = |V(\Delta)|$ ($\check{V}_{max}$ in physical unit) is given by (cf. (26))

$$\begin{aligned}\check{V}_{max} &= V_{max}V_{A*} = \sqrt{2\varphi_i(0)}V_{A*} = \sqrt{\frac{2B_*}{B}}V_T \\ &= \sqrt{\frac{2BB_*}{B_p^2}}\frac{B_p}{B}V_T = \sqrt{\beta_p}\frac{B_p}{B}V_T. \end{aligned} \tag{31}$$

In standard nomenclature, (31) implies that the peak velocity corresponds to the "poloidal Mach number" $(V/[V_T(B_p/B)])$ of order $\sqrt{\beta_p} = O(1)$.

In summary, we have derived a self-consistent model of a self-organized shear-flow layer which can be generated at the edge of a core plasma. The theory resembles a shock model because the jump conditions characterize the solution. The field distribution inside the thin layer (which is regarded as a jump in a coarse-grained model) is governed by the "collision-less" singular perturbation which stems from the Hall effect in the two-fluid MHD. The generalized Bernoulli conditions yield a self-consistent relation between the effective potential and the flow-kinetic energy. Across the shear-flow layer, the plasma pressure suddenly falls. The thickness of the layer and the flow velocity are uniquely determined by the magnetic field and the core plasma pressure. The predicted values are in good agreement with the experimental observations of the shear-flow boundary layers in the H-mode tokamak plasmas. A detailed comparisons of the present theory (including density gradients) with the H-mode phenomenologies will be discussed later.

SMM's work was supported by the U.S. Department of Energy Contact No. DE-FG03-96ER-54346. ZY's work was supported by Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture No. 09308011.

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[8] The Beltrami conditions can be deduced from various arguments. One of them is the "relaxation theory" which considers the minimization of the energy under the constraint on the "rugged" invariants (helicities); see L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997). We can apply the same argument for the helicities defined by the self-field component $\boldsymbol{B}_s$ as far as $\boldsymbol{B}_h \cdot \nabla \approx 0$ so that (5) and (7) are satisfied (which primarily means that the internal electric field is electrostatic).

[9] We consider a thin layer surrounding a torus. The currents in the layer produces both the poloidal ($B_{s,y}$) and toroidal ($B_{s,z}$) self-fields. By the Stokes theorem, $B_{s,y}$ must vanish at the inner boundary. The solenoidal current in the layer confines the toroidal field inside the outer boundary, and hence, $B_{s,z}$ vanishes there.

[10] The electric field is highlighted as a key characteristic of the H-mode [6]. In the present theory, $\phi(x)$ may be related with the electron pressure through (16). Biasing of the plasma will be discussed elsewhere.

FIG. 1. Profiles of (a) the shear flow and (b) the self-magnetic field ($V_z \equiv 0$). Here, we assume $a = 0.5$ and $b = 1/a$. The radial coordinate x is in the unit of the ion skin depth λ_i. The solution continues to oscillate as x increases. We cut off x at Δ (dotted line). Figure (c) shows the profile of the kinetic energy $V^2/2$.