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Beltrami Fields

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PART - 1

Beltrami Fields

- Equilibrium (Kernel) of vortex dynamics.

  [May be robust.] $\left\{ \begin{array}{l} \text{mixing} \\ \text{stretching} \end{array} \right.$

- Beltrami / Bernoulli conditions.

  → Homogeneous Energy Density

- Chaos of 3D streamlines.

  → Relaxed, Homogenized, Mixed

- Twisted / spiral field.

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(1) Structures Represented by Beltrami Fields.

(2) Quantization by Beltrami Fields.

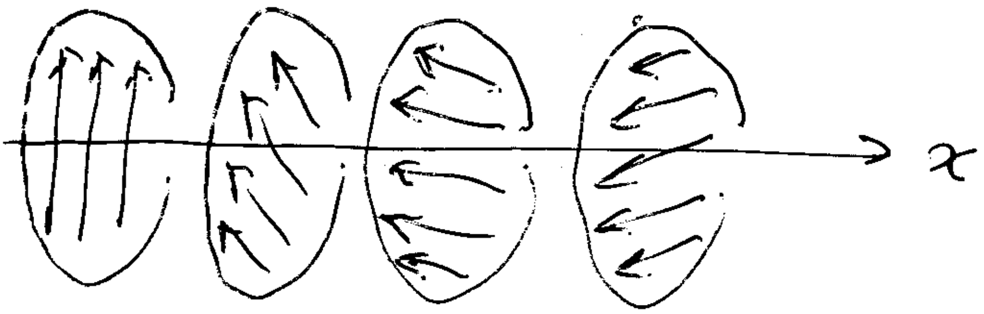
0. Examples

(1) The Beltrami Flow

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v} = \mu \boldsymbol{v}$$

1D solution

$$\boldsymbol{v} = A \begin{pmatrix} 0 \\ \sin f(x) \\ \cos f(x) \end{pmatrix} \quad \rightarrow \mu = f'(x)$$



Sheared flow, Circularly polarized

3D solution

$$\boldsymbol{v} = A \begin{pmatrix} 0 \\ \sin \mu x \\ \cos \mu x \end{pmatrix} + B \begin{pmatrix} \cos \mu y \\ 0 \\ \sin \mu y \end{pmatrix} + C \begin{pmatrix} \sin \mu z \\ \cos \mu z \\ 0 \end{pmatrix}$$

ABC Map.

(2) The Taylor Relaxed State

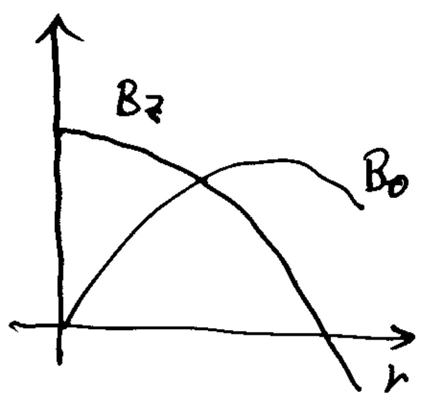
$$\mathbf{J} \equiv \nabla \times \mathbf{B} = \mu \mathbf{B}$$

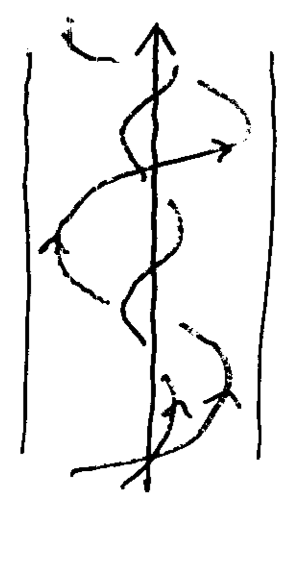
Force-Free Plasma Equilibrium

→ (Woltjer, Chandrasekhar)

<u>Cylindrical Solution</u>

$$\mathbf{B} = B \begin{pmatrix} 0 \\ J_1(\mu r) \\ J_0(\mu r) \end{pmatrix} \begin{array}{l} \leftarrow r \\ \leftarrow \theta \\ \leftarrow z \end{array}$$





Beltrami Fields

— application for H-mode boundary layer —

- Vortex dynamics
- Beltrami condition / Bernoulli condition
- Helicities
- Double Beltrami fields → diamagnetism
- Application for H-mode boundary layer

1. Vortex Dynamics

(3D) $$\frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla\times(\mathbb{U}\times\boldsymbol{\omega}) = 0$$

$\boldsymbol{\omega}$: vorticity (Kelvin's circulation

$\mathbb{U}$: flow

(2D)

$$\frac{\partial}{\partial t}\omega + \{H, \omega\} = 0$$

ω : vorticity

H : Hamiltonian $\quad \mathbb{U} = \begin{pmatrix} \partial_y H \\ -\partial_x H \end{pmatrix}$

$\boxed{\text{Steady state}}$

(3D) $$\mathbb{U}\times\boldsymbol{\omega} = \nabla\varphi$$

(2D) $$\{H, \omega\} = 0 \quad \text{, i.e., } \omega(H)$$

Mixing & Stretching

$$\frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) = 0$$

$\Updownarrow$ when $\nabla\cdot\boldsymbol{\omega}=0,\ \nabla\cdot\boldsymbol{u}=0$

$$\underline{\frac{\partial}{\partial t}\boldsymbol{\omega}} + \underline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}} - \underline{(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}} = 0$$

Lagrange derivative

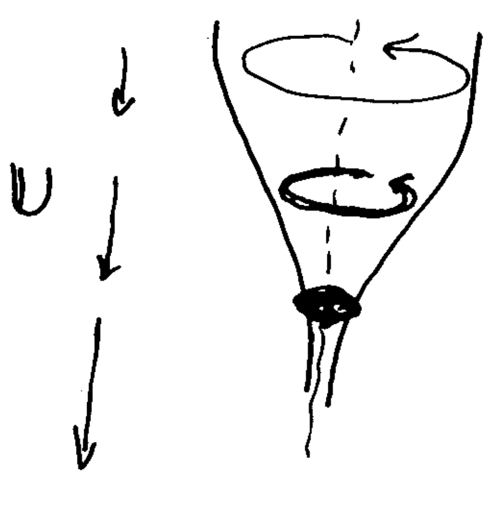
$\Downarrow$

$\boxed{\text{Mixing}}$

growth of $|\boldsymbol{\omega}|$ by $\nabla\boldsymbol{u}$

$\Downarrow$

$\boxed{\text{Stretching}}$ $\leftarrow$ 3D effect



Examples

(3D)	$\mathbb{W}$	$\mathbb{U}$
Euler flow	$\nabla \times \mathbb{V}$	$\mathbb{V}$
MHD (induction law)	$\mathbb{B}$	$\mathbb{V}$
Two-Fluid MHD (electrons)	$\mathbb{B}$	$\mathbb{V}_e = \mathbb{V} - \nabla \times \mathbb{B}$
(ions)	$\mathbb{B} + \nabla \times \mathbb{V}$	$\mathbb{V}$

(2D)	ω	H
Euler flow	$-\Delta \phi$	ϕ
Hasegawa-Mima	$-\Delta \phi + \phi$	ϕ
reduced MHD	$\begin{cases} -\Delta \phi \\ \psi \end{cases}$	$\begin{cases} \phi, \ \Delta \psi \\ \phi \end{cases}$

2. Beltrami Condition / Bernoulli Condition

- Beltrami Condition : $\boxed{\omega \parallel v}$
- homogeneous Beltrami condition : $\boxed{\begin{array}{l} \omega = \alpha v \\ \alpha = \text{constant} \end{array}}$

$$\underbrace{v \times \omega = 0}_{\text{Beltrami Condition}} \; = \nabla \varphi$$

$$0 = \nabla\varphi \quad \text{(Bernoulli condition)}$$

Examples

	Beltrami	Bernoulli
Euler flow	$\nabla \times V = a V$	$\frac{V^2}{2} + p = \text{const.}$
MHD	$\nabla \times B = a B$ (Taylor relaxed state)	$p = \text{const.}$ (force-free)
2F·MHD	$\begin{cases} B = a(V - \nabla \times B) \\ B + \nabla \times V = b V \end{cases}$	$\begin{cases} p_e - \phi = \text{const.} \\ p_i + \phi + \frac{V^2}{2} = \text{const.} \end{cases}$ $\left(p + \frac{V^2}{2} = \text{const.}\right)$

3. Helicities

$$\frac{\partial}{\partial t}\omega - \nabla\times(\mathbb{U}\times\omega) = 0$$

Helicity :
$$h = \int_\Omega (\mathrm{curl}^{-1}\omega)\cdot\omega \, dx / 2$$

Helicity conservation

When $\mathbb{U}\cdot n = 0$ on $\partial\Omega$,

then, h = constant.

gauge-invariant helicity

$$h^* = \int_\Omega (\mathrm{curl}^{-1}\omega)\cdot \underline{\omega_s} \, dx / 2$$

↑ solenoidal part

Example

magnetic helicity = $\int_\Omega \mathbb{A}\cdot\mathbb{B}\, dx/2$

(Woltjer)

Helicity is a measure of "linking".

Gauss' linking number

—— a <u>degree</u> theory ——

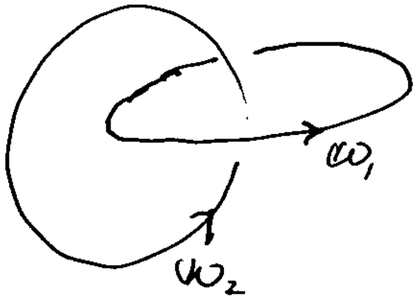
Consider a pair of line vortices with unit vorticity. Then,

$$h = \frac{1}{2}\int \operatorname{curl}^{-1}\omega \cdot \omega \, dx$$

is an integer representing the linking number.

$$\int \operatorname{curl}^{-1}\omega \cdot \underbrace{\omega\, dx}_{(\omega \cdot dS)\, d\vec{l}} = \sum_{\gamma} \oint_{\gamma} \operatorname{curl}^{-1}\omega \, d\vec{l}$$

$$= \sum \int \omega \cdot dS$$



heirarchy of invariants (bilinear forms)

MHD $\quad h_1 = \int \mathbb{A}\cdot\mathbb{B}\, dx/2$, $\quad h_\times = \int \mathbb{V}\cdot\mathbb{B}\, dx/2$, $\quad E = \int B^2\, dx/2$

2F-MHD $\quad h_1 = \int \mathbb{A}\cdot\mathbb{B}\, dx/2$, $\quad E = \int B^2 + V^2\, dx/2$, $\quad h_2 = \int (\mathbb{A}+\mathbb{V})\cdot(\mathbb{B}+\nabla\times\mathbb{V})\, dx/2$

2D-Euler $\quad E = \int V^2\, dx/2$, $\quad W = \int \omega^2\, dx/2$
(Enstrophy)

$\longrightarrow$ higher-order derivatives

$\longrightarrow$ strong topology

robust $\longleftrightarrow$ fragile

Variational Principle and Beltrami Condition

- $\delta(E - \lambda h_1) = 0$

$$\longrightarrow \nabla\times\mathbb{B} = \lambda\mathbb{B}$$

- $\delta(E - \lambda_1 h_1 - \lambda_2 h_x) = 0$

$$\longrightarrow \nabla\times\mathbb{B} = \lambda_1\mathbb{B}$$

$$\mathbb{V} = \lambda_2\mathbb{B}$$

- $\delta(h_2 - \lambda_1 h_1 - \lambda_2 E) = 0$

$$\longrightarrow \mathbb{B} = a_1(\mathbb{V} - \nabla\times\mathbb{B})$$

$$\mathbb{B} + \nabla\times\mathbb{V} = a_2\mathbb{V}$$

$$\longrightarrow \nabla\times(\nabla\times\mathbb{B}) - \alpha\,\nabla\times\mathbb{B} + \beta\,\mathbb{B} = 0$$

$$\left(\nabla\times(\nabla\times\mathbb{V}) - \alpha\,\nabla\times\mathbb{V} + \beta\,\mathbb{V} = 0\right)$$

4. Diamagnetism

- Taylor Relaxed State : $\nabla \times \mathbb{B} = \lambda \mathbb{B}$

  $\rightarrow$ force-free (0-beta)

  $\rightarrow$ paramagnetism

- Consider a linear combination of two Beltrami fields :

$$\nabla \times \mathbb{B}_1 = \lambda_1 \mathbb{B}_1 \qquad \nabla \times \mathbb{B}_2 = \lambda_2 \mathbb{B}_2$$

$$\mathbb{B} = C_1 \mathbb{B}_1 + C_2 \mathbb{B}_2$$

$$\rightarrow \quad (\nabla \times \mathbb{B}) \times \mathbb{B} = C_1 C_2 (\lambda_1 - \lambda_2)\, \mathbb{B}_1 \times \mathbb{B}_2 \neq 0$$

NOT Force-free

$\downarrow$

Diamagnetism

- $\mathbb{B} = C_1 \mathbb{B}_1 + C_2 \mathbb{B}_2$ solves

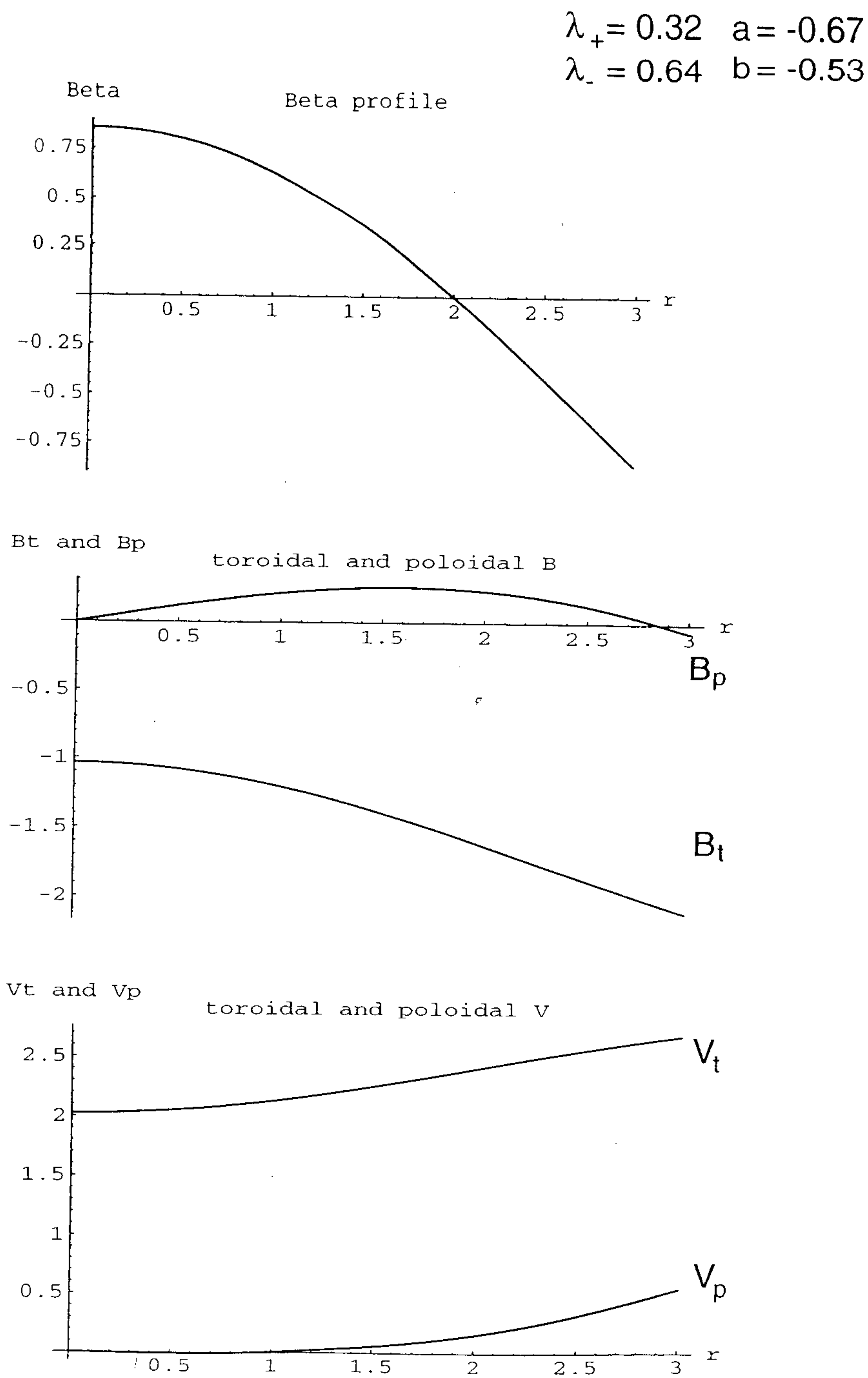
$$(\mathrm{curl} - \lambda_1)(\mathrm{curl} - \lambda_2)\mathbb{B} = 0$$

$$\Updownarrow$$

$$\nabla \times (\nabla \times \mathbb{B}) - \alpha \nabla \times \mathbb{B} + \beta \mathbb{B} = 0$$

Includes London eq. ($\alpha = 0$, $\beta < 0$)

$\rightarrow$ λ_1, λ_2 is imaginary



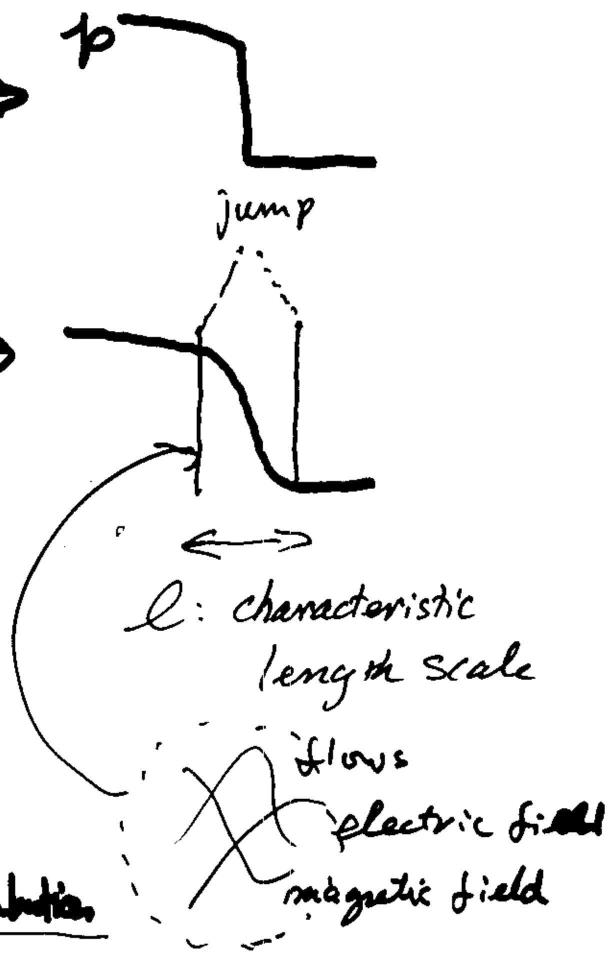
$\lambda_+ = 0.32$ $a = -0.67$
$\lambda_- = 0.64$ $b = -0.53$
Beta
Beta profile
r
Bt and Bp
toroidal and poloidal B
B_p
B_t
Vt and Vp
toroidal and poloidal V
V_t
V_p

5. H-mode boundary layer

" Equilibrium Theory "

Ideal-MHD view →

Singular perturbation →



2F · MHD provides a

<u>collision-less singular perturbation</u>

with $l = \lambda_i = c/\omega_{pi}$

For $\beta_p \simeq 1$; $l \sim \rho_{i.p}$

with $\begin{cases} \bullet \text{ flow } \sqrt{\beta_p} \cdot \frac{B_p}{B_0} V_T \text{ (poloidal Mach \# } \sim 1) \\ \bullet \ E_r < 0, \ E_r' < 0, \ e\phi \sim T_i \end{cases}$

Key Point

Separation of Self-Fields & External Field

$$\mathbb{B} = \underbrace{\mathbb{B}_h}_{\text{external}} + \underbrace{\mathbb{B}_S}_{\text{self-field}}$$

$\left(\begin{array}{c} \nabla \times \mathbb{B}_h = 0 \\ \text{in the layer} \end{array} \right)$

$\uparrow$ $O(B_0)$ $\qquad$ $\uparrow$ $O(B_*)$

Diamagnetic relation

$$\delta p = \delta \frac{B^2}{2\mu_0} \rightarrow \frac{B_*}{B_0} = \frac{\beta}{2}$$

2F. MHD

$$
\begin{cases}
\frac{\partial}{\partial t} \mathbb{A} = (\mathbb{V} - \nabla \times \mathbb{B}) \times \mathbb{B} - \nabla(\phi - p_e) \\
\frac{\partial}{\partial t}(\mathbb{V} + \mathbb{A}) = \mathbb{V} \times (\mathbb{B} + \nabla \times \mathbb{V}) - \nabla\left(\frac{V^2}{2} + p_i + \phi\right)
\end{cases}
$$

Separation of $\mathbb{B}$ into $\mathbb{B}_h + \mathbb{B}_s$

define
$$
\begin{cases}
\mathbb{V} \times \mathbb{B}_h = \left(\frac{\beta}{2}\right)^{-1} \nabla P_i \\
\mathbb{V}_e \times \mathbb{B}_h = -\left(\frac{\beta}{2}\right)^{-1} \nabla P_e
\end{cases}
$$

$$
\begin{cases}
\frac{\partial}{\partial t} A_s = (\mathbb{V} - \nabla \times \mathbb{B}_s) \times \mathbb{B}_s + \left(\frac{\beta}{2}\right)^{-1} \nabla(p_e - \underline{P_e} - \phi) \\
\frac{\partial}{\partial t}(\mathbb{V} + A_s) = \mathbb{V} \times (\mathbb{B}_s + \nabla \times \mathbb{V}) - \left(\frac{\beta}{2}\right)^{-1} \nabla\left(p_i - \underline{P_i} + \phi + \left(\frac{\beta}{2}\right)\frac{V^2}{2}\right)
\end{cases}
$$

Assume

Beltrami:
$$
\begin{cases}
\mathbb{B}_s = a(\mathbb{V} - \nabla \times \mathbb{B}_s) \\
\mathbb{B}_s + \nabla \times \mathbb{V} = b\,\mathbb{V}
\end{cases}
$$

Bernoulli:
$$
\begin{cases}
p_e - \underline{P_e} - \phi = c_e \\
p_i - \underline{P_i} + \phi + \left(\frac{\beta}{2}\right)\frac{V^2}{2} = c_i
\end{cases}
$$

general solution to the Beltrami eqs.

- $\mathbb{V} = C_+ \mathbb{G}_+ + C_- \mathbb{G}_-$
- $\mathbb{B}_S = (b - \lambda_+) C_+ \mathbb{G}_+ + (b - \lambda_-) C_- \mathbb{G}_-$

$$\mathbb{G}_\pm = \begin{pmatrix} 0 \\ \sin(\lambda_\pm x + \theta_\pm) \\ \cos(\lambda_\pm x + \theta_\pm) \end{pmatrix}$$

Parameters

λ_+ $\quad\lambda_-$ (equivalently a, b)

C_+ $\quad C_-$

θ_+ $\quad\theta_-$

Δ (layer width)

B.C.

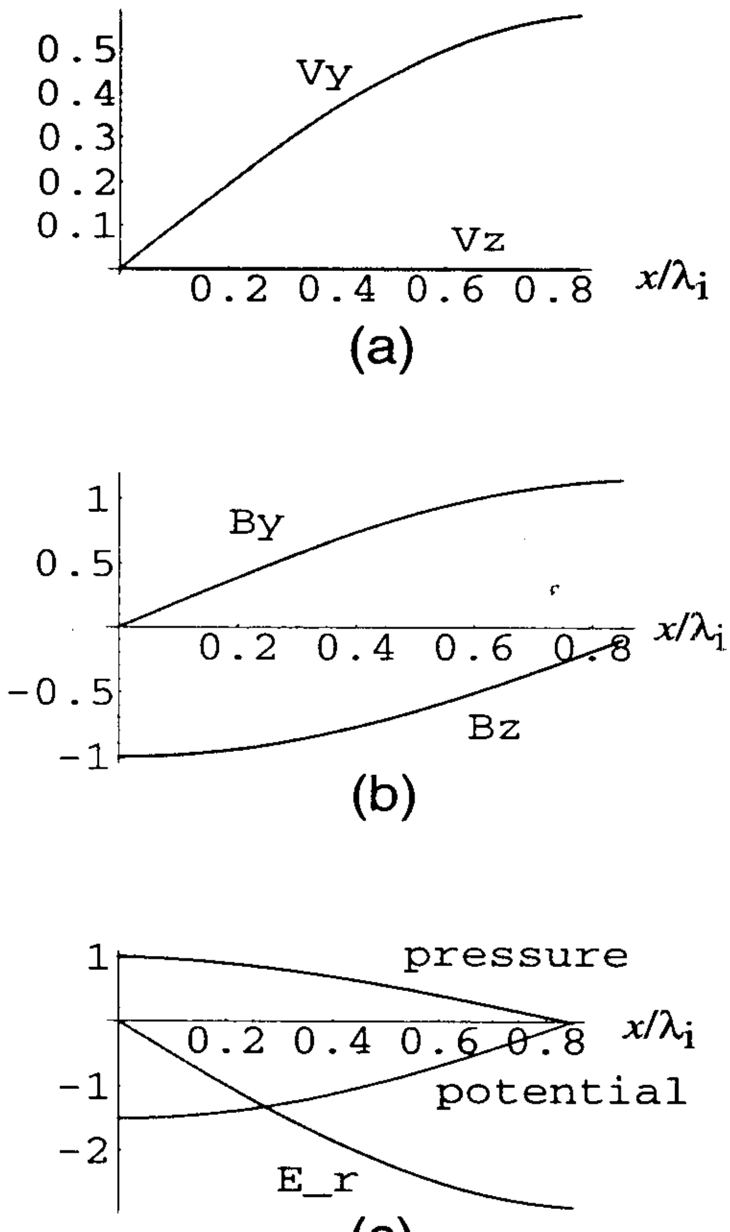
$V_y(0) = 0, \quad V_z(0) = 0$

$\boxed{B_{s,y}(0) = 0, \quad B_{s,z}(\Delta) = 0}$ Stokes thm.

$\boxed{B_{s,z}(0) = -1}$ diamagnetism.

- $V^2(x)$ achieves maximum at Δ

S.M. Mahajan and Z. Yoshida "A Collision-less Self-Organizing Model for the H-mode Boundary Layer"



Solution

$$\mathbb{V} = \begin{pmatrix} 0 \\ 2C \cdot \sin \lambda x \\ 0 \end{pmatrix}$$

$$\mathbb{B}_s = \begin{pmatrix} 0 \\ 2\lambda C \sin \lambda x \\ -2\lambda C \cos \lambda x \end{pmatrix}$$

$$E_x = -\frac{d\phi}{dx} = -V_y - \frac{d}{dx}\frac{B_{sz}}{2}$$

$$= -(2+\lambda^2) C \sin \lambda x \quad (<0)$$

$$(C, \lambda > 0)$$

$$\Delta = \pi/2\lambda$$

Summary

1. Beltrami condition characterizes robust structures in vortex-dynamics systems.

2. Coupled Beltrami fields of 2F·MHD are much different from the $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ state :
   - paramagnetic
   - $\mathbf{B} \Leftrightarrow \mathbf{V} \Leftrightarrow \phi$ coupling
   - characteristic length scale (λ_i)

3. H-mode ~~diamagnetic~~ boundary layer
   - $\Delta \sim \lambda_i = \rho_{ip} / \sqrt{\beta_p}$
   - $V \sim V_{A*} = \sqrt{\beta_p} \cdot \frac{B_p}{B_0} V_T \quad (M_p \sim 1$
   - E_r , $E_r' < 0$, $e\phi \sim T_i$

PART - 2

Quantization by Beltrami Fields

The homogeneous Beltrami equation:

$$\boxed{\mathrm{curl}\ \mathbf{u} = \lambda \mathbf{u}}$$

Does "curl" yield quantization?
(spectral resolution)

- Appropriately defined curl is self-adjoint.
  $\longrightarrow$ spectral resolution.

- "Appropriate definition" invokes the cohomology.

- The cohomology class yields a symmetry breakin[g]

$\boxed{\text{external field}}$

$\boxed{\text{fields}}$ (structurs)

intrinsic eigen state
spontaneous

response to external fields
(symmetry breaking)

$$\underset{}{A_\lambda u} = \underset{\text{external field}}{\underline{f}} \qquad (\text{typically } A_\lambda = A - \lambda I)$$

- If $\underline{A_\lambda u = 0}$ has a solution $u_\lambda\ (\neq 0)$,
  λ : eigen value, $\underline{u_\lambda \text{ : eigenfunction.}}$
  intrinsic structure

- If $A_\lambda u = 0$ does not have a solution,
  we can define

$$\underset{\text{response}}{\underline{u}} = A_\lambda^{-1} \underset{\text{external field}}{\underline{f}}$$

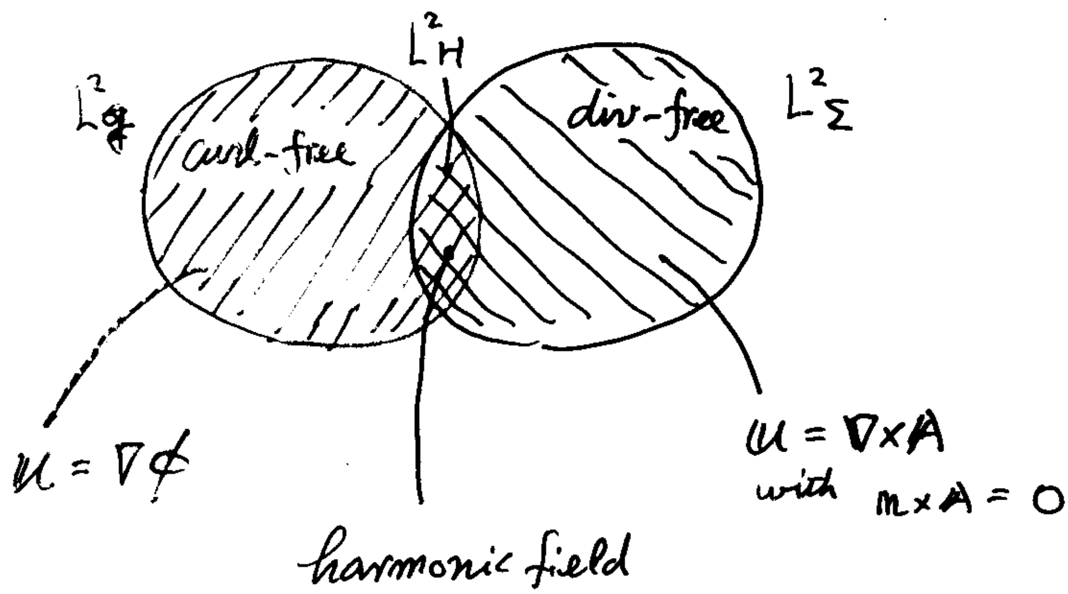
The Beltrami equation

$$(\mathrm{curl} - \lambda I)\, u = 0$$

includes a hidden external field.

function space of confined fields

$(\mathbf{n} \cdot \mathbf{u} = 0$ on $\partial\Omega)$



↑

This set is not void
when Ω is multiply connected.

⇓

defines the cohomology class.

Theorem

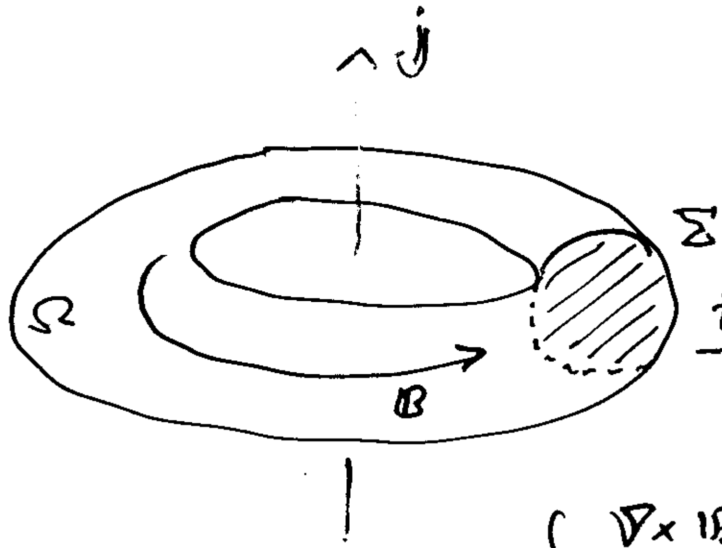
(1) curl defined in L^2_Σ is self-adjoint.
The spectrum consists of only point spectrum (eigen values)
The eigenfunctions are complete.

(2) curl defined in $L^2_\sigma = L^2_\Sigma \oplus L^2_H$ is not self-adjoint. The Beltrami eq.

$$\text{curl } \mathbf{u} = \lambda \mathbf{u}$$

has a nontrivial solution for $\forall \lambda \in \mathbb{C}$.

harmonic field and cohomology



toroidal flux.

$$\begin{cases} \nabla \times \mathbb{B} = 0 \\ \nabla \cdot \mathbb{B} = 0 \\ n \cdot \mathbb{B} = 0 \end{cases}$$

<u>NOTE</u> This $\mathbb{B}$ cannot be represented by $\nabla \phi$ with a single-value ϕ.

multi-value ϕ $\Leftrightarrow$ "angle"

<u>topological cut</u> $\Leftrightarrow$ Riemann cut

$\downarrow$

cohomology class

[How does the harmonic field work as an external field ?]

Let $\boldsymbol{h} \in L^2_H$, i.e. $\operatorname{curl} \boldsymbol{h} = 0$

For $\boldsymbol{u} = \boldsymbol{u}_\Sigma + \boldsymbol{h}$,

$$(\operatorname{curl} - \lambda I)\, \boldsymbol{u} = 0$$

reads

$$\operatorname{curl} \boldsymbol{u}_\Sigma + \cancel{\operatorname{curl} \boldsymbol{h}} - \lambda \boldsymbol{u}_\Sigma - \lambda \boldsymbol{h} = 0$$

$$(\operatorname{curl} - \lambda I)\, \boldsymbol{u}_\Sigma = \boxed{\lambda \boldsymbol{h}}$$

external field!

- If λ is <u>not</u> the eigenvalue of s.a. curl,

$$\boldsymbol{u}_\Sigma = (\operatorname{curl} - \lambda I)^{-1} \lambda \boldsymbol{h} \quad : \text{response}$$

- If λ is the eigenvalue of s.a. curl,

$$\boldsymbol{u}_\Sigma = \boldsymbol{u}_\lambda \quad ; \text{eigen function}$$

($\boldsymbol{h}$ must be 0)

Strange ?

Suppose that A is a matrix that has a 0 eigenvalue :

$$A \, \underline{u} = 0 \quad (\text{i.e. } \underline{u} \in \mathrm{Ker}(A))$$

Then $(A - \lambda I)\, \underline{u} = 0$ has a non-trivial solution for all λ ??

$\downarrow$

NO !

SUMMARY

Beltrami Fields

- Robust equilibrium in vortex dynamics.
  (kernel) $\left\{ \begin{array}{l} \text{mixing} \\ \text{stretching} \end{array} \right.$

- Beltrami / Bernoulli conditions.

- Twisted / spiral structurs
  characterized by helicities.

- Coupled Beltrami Fields
  A new type of equilibrium with coupled $\mathbb{B}$ and $\mathbb{V}$
  Stems from the Two-Fluid Effect
  $\uparrow$
  singular perturbation
  $+ \varepsilon \cdot$ [Hall effect] , $\varepsilon = \lambda_i / \ell$

- Beltrami Field : response / spontaneous
  eigen functions

  every solenoidal field $= \sum{}'$ spontaneous Beltrami fields
  $+$ harmonic field

PART - 3

Why "shear flow problems"?

① non-self-adjoint Hamiltonian generates dynam
(non-Hermitian)

linear theory, stability, transient phenomena
quantization (fillaments)

② mixing ⟷ structurs

continuous spectra, nonlinear theory
self-organization, patterns (spiral)

③ "Kernel"
(steady states)

topology, Beltrami/Bernoulli conditions
robust structure

1. Linear Theory

Example : shear-flow MHD (incompressible)

$$\begin{cases} \frac{\partial}{\partial t} \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla p, & \nabla\cdot\mathbf{u} = 0 \\ \frac{\partial}{\partial t}\mathbf{B} - \nabla\times(\mathbf{u}\times\mathbf{B}) = 0 \end{cases}$$

linearize for $\mathbf{u} = \underline{\underline{\tilde{u}}} + \mathbf{u}_0$, $\mathbf{B} = \underline{\underline{\tilde{b}}} + \mathbf{B}_0$

(1) When $\mathbf{u}_0 = 0$:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\mathbf{u}} = (\nabla\times\tilde{\mathbf{b}})\times\mathbf{B}_0 + (\nabla\times\mathbf{B}_0)\times\tilde{\mathbf{b}} - \nabla\tilde{p}, & \nabla\cdot\tilde{\mathbf{u}} = 0 \\ \frac{\partial}{\partial t}\tilde{\mathbf{b}} - \nabla\times(\tilde{\mathbf{u}}\times\mathbf{B}_0) = 0 \end{cases}$$

What is the "energy" of this system?

- Try the conventional energy.

$$H_0 = \frac{1}{2}\int(\tilde{u}^2 + \tilde{b}^2)\,dx$$

We find

$$\frac{d}{dt}H_0 = ((\nabla\times\mathbf{B}_0)\times\tilde{b}, \tilde{u}) = (\mathbf{J}_0\times\tilde{\mathbf{b}}, \tilde{\mathbf{u}}) \neq 0.$$

- Appropriate energy must have a <u>potential energy</u> associated with the force $\mathbf{J}_0\times\tilde{\mathbf{b}}$.

• For example, consider the case of $\mathbb{J}_0 = \lambda \mathbb{B}_0$.
(Beltrami)

We set

$$H = \frac{1}{2}\int\left[\dot{\xi}^2 + \tilde{b}^2 - \lambda(\mathbb{B}_0\times\xi)\cdot(\nabla\times(\mathbb{B}_0\times\xi))\right]dx$$

with $\tilde{u} = \dot{\xi}$

Then $$\frac{dH}{dt} = \underbrace{\frac{d}{dt}H_0}_{\substack{\shortparallel \\ (\mathbb{J}_0\times\tilde{\mathbb{b}},\, \tilde{u})}} - \lambda\underbrace{(\mathbb{B}\times\dot{\xi}, \nabla\times(\mathbb{B}\times\xi))}_{\substack{\shortparallel \\ -(\mathbb{J}_0\times\tilde{u},\, -\tilde{b})}} = 0$$

Hence, we have a Hamiltonian:

$$H = \frac{1}{2}\int(\dot{\xi}^2 + K\xi\cdot\xi)\,dx$$

The operator K generates the linear dynamics.
Hermitian

$$\ddot{\xi} = K\xi$$

$$\boxed{\text{Energy principle}}$$

(2) When $\mathbb{U}_0 \neq 0$, and $\mathbb{U}_0 \neq$ const. :

$$
\begin{cases}
\frac{\partial}{\partial t}\tilde{\mathbb{U}} = -\underbrace{(\nabla\times\mathbb{U}_0)\times\tilde{\mathbb{U}}} - \underbrace{(\nabla\times\tilde{\mathbb{U}})\times\mathbb{U}_0} - \underbrace{\nabla(\tilde{\mathbb{U}}\cdot\mathbb{U}_0)} \\
\qquad + (\nabla\times\tilde{\mathbb{b}})\times\mathbb{B}_0 + (\nabla\times\mathbb{B}_0)\times\tilde{\mathbb{b}} - \nabla p \quad , \quad \nabla\cdot\tilde{\mathbb{U}} = 0 \\
\frac{\partial}{\partial t}\tilde{\mathbb{b}} = -\nabla\times(\mathbb{B}_0\times\tilde{\mathbb{U}}) + \underbrace{\nabla\times(\mathbb{U}_0\times\mathbb{b})}
\end{cases}
$$

$$
\frac{dH_0}{dt} = (\underbrace{\tilde{\mathbb{U}}, \mathbb{U}_0\times(\nabla\times\tilde{\mathbb{U}})}) + (\mathbb{J}_0\times\tilde{\mathbb{b}}, \tilde{\mathbb{U}}) + (\underbrace{(\nabla\times\tilde{\mathbb{b}}), \mathbb{U}_0\times\mathbb{b}}) .
$$

These new actions do not allow a potential representation.

$\Downarrow$

We cannot apply the energy principle of

$\boxed{\text{kinetic energy}} + \boxed{\text{potential energy}} = \boxed{\text{const.}}$

Then what is the "stability"?

Dynamical Approach (linear theory)

We want to solve an initial value problem

$$\begin{cases} \frac{d}{dt} u = H u \\ u(0) = u_0 \end{cases}$$ (autonomous)

The solution may be given in the form of

$$u(t) = \underbrace{e^{tH}}_{} u_0$$

propagator or (semi) group operator

How can we define $\boxed{e^{tH}}$?

(1) If H is a number, $e^{tH} = \sum \frac{(tH)^n}{n!}$

: the exponential function.

or $= \frac{1}{2\pi i} \oint \frac{e^{t\lambda}}{\lambda - H} d\lambda$

((Cauchy's Thm)

(2) If H is a matrix,

⋮

(3) If H is a "bounded operator", $e^{tH} = \frac{1}{2\pi i} \oint e^{t\lambda} (\lambda -$

(spectral radius $< \infty$)

(4) If A is "dissipative", [Hille-Yosida]

$$e^{tH} = \lim_{n\to\infty} e^{tH(1-\frac{H}{n})^{-1}}$$

(5) If H is self-adjoint, [Von Neumann]

$$e^{tH} = \int e^{t\lambda}\, dE_\lambda$$ spectral Resolution

$$= \sum_i e^{t\lambda_i} (\cdot, \varphi_i)\varphi_i$$

Two Different Methods to Solve Wave eq.

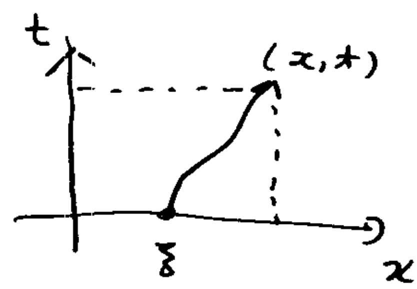
(1) Characteristics Method.

Wave $= \sum$ const.-phase points

$$\begin{cases} \left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right] u = 0 \\ u(0) = u_0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \frac{d}{dt} x = \mathbf{v} \\ x(0) = \xi \end{cases}$$



solution : $u(x,t) = u_0(\xi(x,t))$

$\begin{pmatrix} x \\ t \end{pmatrix} \to \begin{pmatrix} \xi \\ \tau \end{pmatrix}$ with $\frac{\partial}{\partial \tau} = 0$

(2) Mode Expansion

Wave $= \sum$ harmonic $\underbrace{\text{oscillators}}_{\text{modes}}$ $= \sum e^{j\omega t} \underbrace{\varphi_\omega(x)}_{\text{modes}}$

Combination of (1) & (2)

$$\underbrace{\left[\frac{\partial}{\partial t} + \mathbf{U}_0 \cdot \nabla\right]}_{\text{Lagrange derivative}} u = \underbrace{H u}_{\text{original s.a. part}}$$

Lagrange derivative $\Leftarrow$ non-self-adjointness

(1)

$$\frac{\partial}{\partial \tau} u = H u .$$

(2)

2. Nonlinear Theory

Inhomogeneous flow $\Rightarrow$ Mixing

small scale

complexity

simplicity $\Rightarrow$ large scale

"Structure"

Nonlinearity

- What is the "Kernel" of the nonlinear dynamics
- How it can be generated ?
  — relaxation , self-organization
- Variety , Robustness

Selective dissipation model

- Single fluid MHD.

$$\begin{cases} \frac{\partial}{\partial t} u + (u \cdot \nabla) u = (\nabla \times B) \times B - \nabla p , & \nabla \cdot u = 0 \\ \frac{\partial}{\partial t} B - \nabla \times (u \times B) = 0 \end{cases}$$

This system (ideal) conserves the enery and the helicity:

$$\begin{cases} E = \frac{1}{2} \int (u^2 + B^2)\, dx \\ h = \frac{1}{2} \int A \cdot B \, dx \end{cases}$$

Suppose that the magnetic part of the energy

$$E_M = \frac{1}{2} \int B^2 \, dx$$

is preferentially dissipated while $h \simeq$ const., and the minimum of E_M is achieved.

$$\Rightarrow \quad \delta (E_M - \lambda h) = 0$$

$$\Rightarrow \quad \nabla \times B = \lambda B \qquad \text{Beltrami Condition}$$

- How a flow can be gnerated ?
- What new effect may stem from a flow ?

3. Kernel (0-frequency eigenstates).

Vortex dynamics:

$$\frac{\partial}{\partial t}\Omega - \nabla\times(\mathbb{U}\times\Omega) = 0$$

$$\begin{cases} \Omega & : \text{vortex} \\ \mathbb{U} & : \text{flow} \end{cases}$$

The "steady state" (Kernel):

$$\mathbb{U}\times\Omega = \underline{\nabla\varphi}$$

$\leftrightarrow$ some physical potential field and gauge

The Beltrami / Bernoulli conditions:

$$\underbrace{\mathbb{U}\times\Omega = 0}_{\text{Beltrami}} \underbrace{= \nabla\varphi}_{\text{Bernoulli}}$$