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Secondary Instabilities in Temperature Gradient Driven Turbulence

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Secondary Instabilities in E/ITG Turbulence

- Gyrokinetic equation (GKE)
- Instability from electron temperature gradient
- Relationship between GKE and Hasegawa-Mima (HM)
- Main secondary instability: Kelvin-Helmholtz
  - Physical picture
  - Standard KH
  - KH for HM (ETG)
  - KH for corrected HM (ITG)
- Numerical simulations
- Implications

Gyrokinetic Equation

- Gyrokinetic equation appropriate for small amplitude fluctuations with

$$\frac{\omega}{\Omega_c} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\delta f}{f} \sim \frac{e\delta\Phi}{T} \sim \frac{\delta B}{B} \sim \frac{\rho}{L} \ll 1$$

- Gyrokinetic equation describes evolution of perturbed distribution function h. For $F_0 = F_0(\epsilon, \Psi)$:

$$\left(\frac{d}{dt} + v_\parallel \hat{\mathbf{b}} \cdot \nabla + i\omega_d + C\right) h = i\omega_*^T \chi - \frac{q}{m} \frac{\partial F_0}{\partial \epsilon} \frac{\partial \chi}{\partial t}.$$

- The total derivative is

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{c}{B} \left\{\chi, h\right\}.$$

- The drift frequency $i\omega_*^T = n_0 c\, \partial F_0 / \partial \Psi$, where n_0 is the toroidal mode number of the perturbation and Ψ is the equilibrium poloidal magnetic flux enclosed by the magnetic surface of interest.

- The perpendicular drifts (curvature, grad-B) are

$$\omega_d = \mathbf{k}_\perp \cdot \mathbf{B}_0 \times \left(m v_\parallel^2 \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} + \mu \nabla B_0\right) / (m B_0 \Omega_c),$$

- The fields are represented by

$$\chi = J_0(\gamma) \left(\Phi - \frac{v_\parallel}{c} A_\parallel\right) + J_1(\gamma) \frac{v_\perp}{c} \frac{\delta B_\parallel}{k_\perp}.$$

Here, $\gamma = k_\perp v_\perp / \Omega_c$.

Gyrokinetic Maxwell Equations

- Fields determined by Maxwell equations, neglecting displacement current.

- Poisson's equation (neglecting Debye term for now):

$$\sum_s \int d^3v\, q \left[q\Phi \frac{\partial F_0}{\partial \epsilon} + h \exp(iL) \right] = 0,$$

  where $L = (\mathbf{v} \times \hat{\mathbf{b}} \cdot k_\perp)/\Omega_c$ accounts for the gyrophase dependence.

- Preferred velocity space coordinates are (ϵ, μ, ξ), so that

$$\int d^3v = \frac{B}{m^2} \int \frac{d\epsilon\, d\mu\, d\xi}{|v_\parallel|}$$

- Integrate over the gyrophase to find

$$\sum_s \frac{2\pi B}{m^2} \int \frac{d\epsilon\, d\mu}{|v_\parallel|} q \left[q\Phi \frac{\partial F_0}{\partial \epsilon} + J_0(\gamma) h \right] = 0.$$

- Similarly, Ampere's equation provides the two components of the perturbed magnetic field:

$$k_\perp^2 A_\parallel = \frac{4\pi}{c} \sum_s \frac{2\pi B}{m^2} \int \frac{d\epsilon\, d\mu}{|v_\parallel|} q v_\parallel J_0(\gamma) h$$

$$k_\perp^2 \delta B_\parallel = -4\pi \sum_s \frac{2\pi B}{m^2} \int \frac{d\epsilon\, d\mu}{|v_\parallel|} m v_\perp^2 \frac{J_1(\gamma)}{\gamma} h$$

ETG Instability

- Analogue of ITG instability. Roles of electrons and ions reversed.
- Typical ETG instability has $k_\perp \rho_e \sim 1, \omega \sim k_\theta \rho_e v_{te}/L \ll \Omega_{ci}$.
- Ions respond to perturbations adiabatically because $k_\perp \rho_i \gg 1$.
- Principle differences from ITG: magnetic well physics, magnetic flutter physics, details of adiabatic response, zonal flow physics.
- Nonlinear simulations dramatically different. It is sufficient to consider the electrostatic limit.

Hasegawa-Mima Equation

- Nonlinear Hasegawa-Mima equation is starting point for many studies of plasma turbulence.
- Derived in the $T_i/T_e \ll 1$ limit
- Ion continuity equation, ignoring parallel ion inertia, is
$$\frac{\partial n}{\partial t} + \nabla_\perp \cdot \left[n_0 \left(\mathbf{v}_E + \mathbf{v}_p \right) \right] = 0.$$
- Polarization drift is
$$\mathbf{v}_p = \frac{1}{\Omega_{ci} B} \left[-\frac{\partial}{\partial t} \nabla_\perp \Phi - \left(\mathbf{v}_E \cdot \nabla \right) \nabla_\perp \Phi \right].$$
- **E**×**B** drift is
$$\mathbf{v}_E = \frac{c}{B} \hat{\mathbf{b}} \times \nabla \Phi$$
- Electrons assigned adiabatic response, $n = |e|\Phi/T_e$.
- Quasineutrality then gives
$$\left(1 - \rho_s^2 \nabla_\perp^2\right) \frac{\partial \Phi}{\partial t} + i\omega_* \Phi - \frac{c}{B} \left\{ \Phi, \nabla_\perp^2 \Phi \right\} = 0$$

From Gyrokinetic Equation to Hasegawa-Mima

- Frieman and Chen derived HW from electrostatic, nonlinear gyrokinetic equation (1982).

$$\left(\frac{d}{dt} + v_{\parallel}\hat{\mathbf{b}}\cdot\nabla + i\omega_d\right) h = i\omega_*^T J_0\Phi + \frac{q}{T}\frac{\partial J_0\Phi}{\partial t}.$$

- It is convenient to work with the non-adiabatic part of the ion distribution function, $f \equiv h - f_M|e|J_0\Phi/T_i$.
- Integrate over velocity to find density evolution; cold-ion limit allows neglect of most FLR terms. Neglecting the ion-sound term:

$$\frac{dn}{dt} - i\omega_{de}\Phi + i\omega_{*e}\Phi = 0.$$

- Electrons assigned adiabatic response, $n_e = |e|\Phi/T_e$.
- Find density from Poisson's equation:

$$\int d^3v\, J_0 f + (\Gamma_0 - 1)\frac{|e|\Phi}{T_i} = \frac{|e|\Phi}{T_e}$$

For $T_i/T_e \ll 1$, $\Gamma_0 \sim 1 - k_\perp^2\rho_i^2$ and $\int d^3v\, J_0 f = n$, so that

$$n = \left(1 - \rho_s^2\nabla_\perp^2\right)\frac{|e|\Phi}{T_e}$$

- In summary:

$$\left(1 - \rho_s^2\nabla_\perp^2\right)\frac{\partial\Phi}{\partial t} - i\omega_{de}\Phi + i\omega_*\Phi - \frac{c}{B}\left\{\Phi, \nabla_\perp^2\Phi\right\} = 0$$

Gyrokinetic Hasegawa-Mima

- Note that the guiding center ion density in the gyrokinetic theory may be naturally identified with the ion vorticity:

$$n = \left(1 - \rho_s^2 \nabla_\perp^2\right) \frac{|e|\Phi}{T_e}$$

- Curvature term identified, $\propto i\omega_{de}\Phi$.

- However, most importantly, both Hasegawa and Mima and Frieman and Chen mistreated the electron response! Adiabatic electron response is not correct for disturbances with $k_\parallel = 0$.

- Electron adiabaticity can be found from Ohm's Law (which can be obtained from $\int d^3v\, v_\parallel$ moment of GKE):

$$\frac{\partial A_\parallel}{\partial t} + \hat{\mathbf{b}} \cdot \nabla \left(|e|\Phi - p_e\right) = 0.$$

- In isothermal, electrostatic limit, second term implies

$$|e|\Phi = n_e T_0 + f(\Psi)$$

- The integration constant $f(\Psi)$ is missing from the HM equation, but is very important for the secondary Kelvin-Helmholtz instability, which determines the turbulence saturation level.

Flux-Surface Averaged Electron Response

- Integration constant $f(\Psi)$ is free function related to initial equilibrium conditions.
- If the initial equilibrium has no radial electric field, then the appropriate choice for $f(\Psi)$ is

$$f(\Psi) = |e| < \Phi >$$

  where $< \cdots >$ represents the flux-surface average (which annhilates $\hat{\mathbf{b}} \cdot \nabla$). Thus, the perturbed flux-surface averaged electron response is initially zero and remains zero, which is appropriate, since adiabatic electrons correspond to no radial transport of electrons.

- In general, one may take other values of $f(\Psi)$, allowing, for example, for a sheared radial electric field. However, such a field must also be included in the GKE.

Ion Response for the ETG Mode

- For the ETG mode, the ion response is adiabatic, because $k_\perp \rho_i \gg 1$.
- The gyrokinetic form of Poisson's equation allows one to examine this statement closely.

$$n_i^{\text{tot}} = \bar{n}_i - n_{i0}\left(1 - \Gamma_0(b_i)\right)\frac{|e|Z_i\Phi}{T_i}.$$

  Here, $b_i = (k_\perp \rho_i)^2$. The function $\Gamma_0(b) = I_0(b)e^{-b}$, in which $I_n(b) = i^{-n}J_n(ib)$ is the modified Bessel function.

- The gyrophase independent component of the ion density is $\bar{n}_i = \int d^3v\, J_0(\gamma) f_i \simeq 0$, which is negligible for perturbations with $k_\perp \rho_i \gg 1$.
- Similarly, $\Gamma_0(b_i) \ll 1$ for $b_i \gg 1$.
- Upon summing over species (and assuming the different ion components have equal temperatures) one finds

$$\sum_i Z_i n_i^{\text{tot}} = -\tau(|e|\Phi/T_e),$$

  where $\tau = Z_{\text{eff}} T_e/T_i$.

- A stronger adiabatic ion response reduces the ETG growth rate. Thus, higher Z_{eff} is stabilizing, and higher T_i/T_e is destabilizing.

Secondary Instabilities

- Linear toroidal (E)ITG eigenmodes typically have $\gamma(k_x)$ peaked at zero. Corresponds to "ballooning" structure.

- At the outboard midplane, the perturbed quantities are basically sinusoidal, with poloidal wavelength satisfying $k_\theta \rho_{e,i} \sim 0.4$.

- There are therefore exponentially growing poloidal gradients of temperature, density, potential, *etc.*

- Conjecture: Most potential secondary instabilities are not able to grow because of strong $\mathbf{E}\times\mathbf{B}$ shear from poloidal gradients of Φ.

- However, sheared flow itself may be unstable to Kelvin-Helmholtz type mode.

- Question: Does the difference in adiabatic response affect the stability properties of the secondary instabilities?

- Answer: Yes. The Kelvin-Helmholtz stability is particularly affected. The difference is qualitatively consistent with the simulation results.

Basic Secondary Kelvin-Helmholtz Instability

- Consider the model equation:

$$\frac{\partial n}{\partial t} + \{\Phi, n\} = 0.$$

- To recover the standard K-H instability, use

$$n = -\nabla_{\perp}^2 \Phi$$

- "Linearize" around the exponentially growing eigenmode. For a quantity f, this implies

$$n = n^{(0)}(y) + \delta n(x, y)$$

- Note: we do not assume that the perturbed quantity is slowly varying in time. (Different from Diamond, *et al.*)

- The "linearized" equation becomes

$$\frac{\partial \delta n}{\partial t} + \{\Phi, n\} = \gamma \delta n + ik_x \left(n_y^{(0)} \delta\Phi - \Phi_y^{(0)} \delta n \right)$$

- Define $\bar{\gamma} = \gamma - ik_x \Phi_y^{(0)}$. Then,

$$\frac{\partial \delta n}{\partial t} + \{\Phi, n\} = \bar{\gamma} \delta n + ik_x n_y^{(0)} \delta\Phi$$

Secondary Kelvin-Helmholtz Continued

- One may get an analytic answer in the long wavelength limit. Consider the case $\partial^2/\partial y^2 \gg k_x^2$. To zeroth order, the LHS must vanish, so that

$$G_0 = \text{const}$$

- The eigenvalue can be found at next order by averaging over one wavelength in the y direction:

$$k_x^2 < \bar{\gamma}^2 >= 0; \qquad \gamma^2 = \frac{1}{2} k_x^2 V_0^2$$

where $V_0 = |\Phi_y^{(0)}|$.

- The growth rate of the secondary instability is

$$\gamma = k_x k_y \Phi^{(0)}.$$

- When this growth rate is comparable to the linear instability's growth rate, expect nonlinear saturation. This implies

$$\Phi_{\text{sat}} \sim \frac{\gamma_\ell}{k_\perp^2},$$

which is an unsurprising result.

Secondary Kelvin-Helmholtz for ETG

- The calculation proceeds along exactly the same lines, except that now

$$n = -\left(\tau - \lambda^2 \nabla_\perp^2\right) \Phi$$

- The presence of the "τ" term changes the stability properties of the KH mode.
- The predicted saturation level increases significantly:

$$\Phi_{\mathsf{sat}} \sim \frac{\gamma_\ell}{k_\perp^4}$$

- NOTE: This calculation is only qualitative, and cannot really predict the nonlinear state. However, since the peak of the nonlinear fluctuation spectrum occurs for small $k_\perp$, this calculation suggests that the ETG/HM/FC system saturates at a large level.
- It is interesting to explore the modified Hasegawa-Mima/Frieman-Chen equation, to see if the long wavelength KH instability remains suppressed.

Secondary Kelvin-Helmholtz for ITG

- The modified Hasegawa-Mima/Frieman-Chen nonlinear equation is the appropriate equation for the consideration of ITG modes.
- Question: Does the inclusion of the flux-surface averaged component of the electrostatic potential change the stability of the KH mode?
- Answer: Yes! The stability properties revert to the classic KH case in the long-wavelength limit.
- Why? The equation reduces to:

$$\frac{\partial}{\partial y}\left[\bar{\gamma}^2 \frac{\partial G}{\partial y}\right] = \bar{\gamma}^2 k_x^2 G + (\gamma\bar{\gamma}G - \bar{\gamma} < \delta\Phi >)$$

- Surprisingly, setting G = const to make the LHS vanish also causes the new term on the RHS to vanish, since $<\gamma>=<\bar{\gamma}>$ and $<G>=G$.
- The eigenvalue is that of the classic Kelvin-Helmholtz instability.
- Finally, the predicted saturation level is

$$\Phi_{\mathrm{sat}} \sim \frac{\gamma_\ell}{k_\perp^2}$$

which is (to us) a surprising result.

Simplified ETG Physics

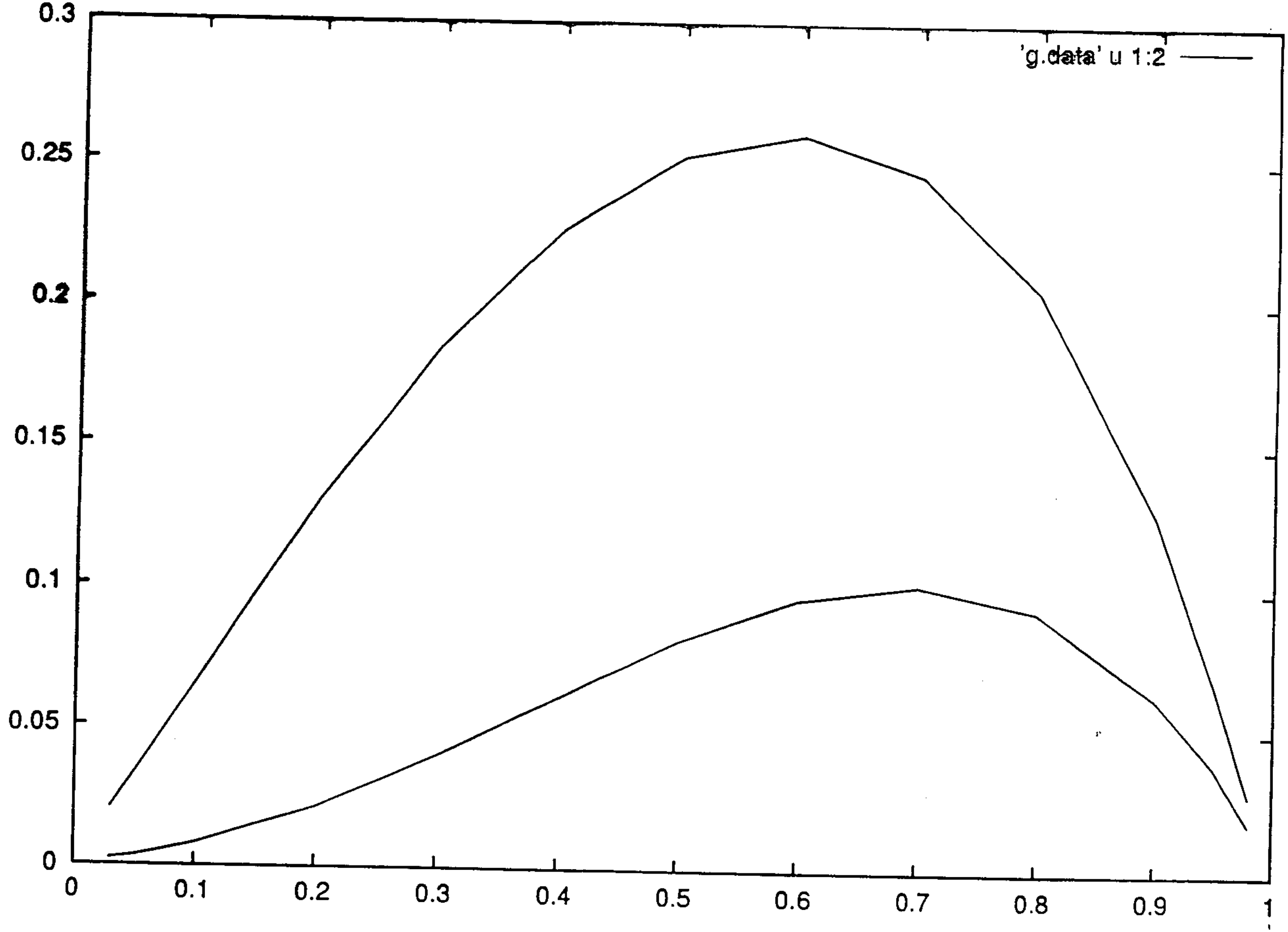
- Unlike ITG modes or other long-wavelength drift waves, the simplest nonlinear ETG-like equation does correspond to the Frieman-Chen generalization of the Hasegawa-Mima equation.

- Consider limit of $\rho_i \gg \lambda_D \gg \rho_e$. Then, one must retain the Debye shielding term in Poisson's equation:
$$-\nabla_\perp^2 \Phi = 4\pi\rho$$

- Since the ion response is adiabatic, this equation yields:
$$n_e = -(\tau - \lambda_D^2 \nabla_\perp^2)\Phi$$

- This has the same mathematical form as the ion vorticity. Only the normalizations are different. Note that there is no complicating flux-surface averaged component.

- The electron GKE has the same form as the ion GKE. Thus, one finds
$$\left(\tau - \lambda^2 \nabla_\perp^2\right) \frac{\partial \Phi}{\partial t} - i\omega_{de}\Phi + i\omega_* \Phi - \lambda^2 \left\{\Phi, \nabla_\perp^2 \Phi\right\} = 0.$$
where $\lambda = \lambda_D/\rho_e$, lengths are normalized by ρ_e, and time is normalized by L_n/v_{te}.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Summary

- ITG and ETG systems are virtually identical mathematically. The only difference is in the details of the response of the adiabatic species.
- Long wavelength, secondary Kelvin-Helmholtz instabilities are suppressed in the ETG system, but not in the ITG system.
- One expects from this observation that the ITG mode should saturate at a lower level than the ETG mode. This is found in gyrofluid and gyrokinetic simulations.
- The Hasegawa-Mima equation is missing important physics in the limit in which it was derived. The Frieman-Chen generalization is likewise deficient.
- The Frieman-Chen generalization is, however, mathematically equivalent to a simplified description of ETG physics if one renormalizes all the terms of the equation to reflect the short wavelength nature of ETG modes.