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Gravitational Instability "Jeans Swindle" for Self-Gravitating Astrophysical Plasmas

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Introduction

We start from the following facts regarding a self-gravitating cloud:

1. A self-gravitating plasma cloud in **a static equilibrium** cannot be uniform on large scale: There is always the gravitational force present which makes the cloud intrinsically nonuniform.
2. The standard harmonic analysis of small perturbations in **a uniform cloud** is meaningful only if the wave-lengths of perturbations do not exceed the typical length of nonuniformity due to self-gravitational effects.
3. In the Cartesian geometry, a strict treatment of linear perturbations is possible only in **one-dimension**: Along the direction of gravitational force.
4. 2-D and 3-D analyses in Cartesian geometry are correct provided the wave-lengths perpendicular to the direction of gravity are sufficiently small as compared to the typical length of nonuniformity.

Basic Model in Cartesian Geometry

The self-gravitating plasma cloud is considered a perfect gas that is also:

- In a static equilibrium,
- isothermal ($T_0 = \text{const}$) with
- the gravitational force and the gradients of physical quantities along the $z-$axis, i.e. $\nabla = \vec{e}_z \partial/\partial z$,
- of uniform composition but non-uniform density $\rho(z)$,
- permeated by a nonuniform magnetic field $\vec{B}_0 = B_0(z)\hat{e}_x$ with
- nonuniform plasma flow $\vec{U}_0(z)$ along $\vec{B}_0(z)$.

The strength of the magnetic field $B_0(z)$ varies with z in such a way that the ratio of the gas pressure $p_{g0} = R_g T_0 \rho_0$ to the magnetic pressure $p_{m0} \equiv B_0^2/(2\mu_0)$ is constant, i.e. $\beta \equiv p_{g0}/p_{m0} = \text{const}$. Consequently, both the Alfvén speed v_A and the speed of sound v_s are constant

$$v_A^2 \equiv \frac{B_0^2}{\mu_0 \rho_0} = const, \quad v_s^2 \equiv \gamma R_g T_0 = const.$$

Here μ_0 is the magnetic permeability of free space, $R_g \equiv R/M$ is the gas constant for the cloud, R is the universal gas constant, M is the mean molar mass for the cloud, ρ_0 is the plasma density in the cloud and γ is the ratio of specific heats and hence $\gamma > 1$.

The constancy of v_A also means that the magnetic field is assumed stronger in the region with a higher plasma density ρ_0 and weaker where the density is lower. Such a distribution of the magnetic field in the cloud is rather realistic and may be found in highly conductive plasmas with frozen-in magnetic fields.

Magneto-hydrostatic Equilibrium

The initial unperturbed state of the isothermal self-gravitating plasma is determined by three equations:

- The equation of magneto-hydrostatic equilibrium:

$$-\nabla p_{g0} - \rho_0 \nabla \phi_0 + \frac{1}{\mu_0}(\nabla \times \vec{B}_0) \times \vec{B}_0 = 0,$$

  which reduces to

$$\left(1 + \frac{1}{\beta}\right) \frac{dp_{g0}}{dz} = -\rho_0 \frac{d\phi_0}{dz}. \qquad (1)$$

- The Poisson equation for the gravitational potential ϕ_0,

$$\frac{d^2\phi_0}{dz^2} = 4\pi G \rho_0, \qquad (2)$$

  where G is the gravitational constant.

- The perfect gas law

$$p_{g0} = R_g \rho_0 T_0, \qquad (3)$$

Profiles $\rho(z)$ and $B_0(z)$

The solution for the density profile of the cloud follows from Eqs (1)-(3) as

$$\rho_0 = \frac{\rho_{00}}{\cosh^2 Z}, \quad \text{where:} \quad Z \equiv \frac{z}{H}, \quad H^2 \equiv \frac{1+\beta}{\beta} \frac{v_s^2}{2\pi\gamma G \rho_{00}} \tag{4}$$

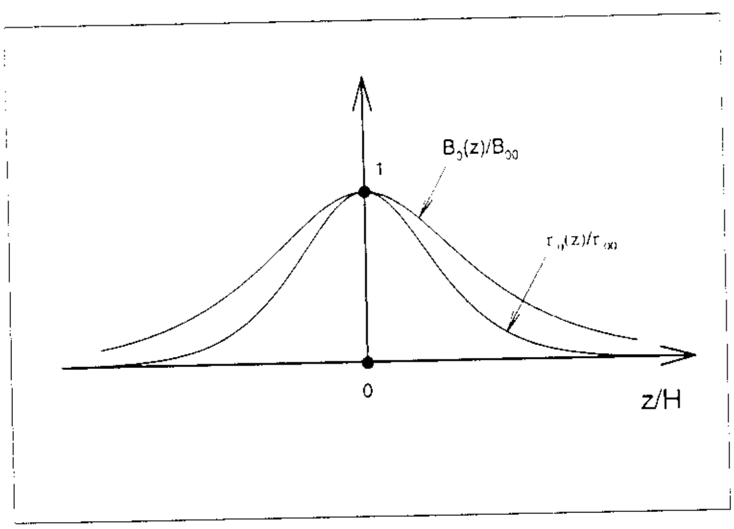
where, ρ_{00} is the density in the center of the cloud and it can be related to the total mass $\mathcal{M}$ per unit surface in the $xy-$plane:

$$\mathcal{M} \equiv \int_{-\infty}^{+\infty} \rho_0 dz = 2H\rho_{00}. \tag{5}$$

The magnetic field distribution follows from the initial assumption $\beta = const$ as

$$B_0 = \frac{B_{00}}{\cosh(z/H)}, \quad \text{where:} \quad B_{00} \equiv \left(\frac{2\mu_0}{\beta\gamma} v_s^2 \rho_{00} \right)^{1/2} \tag{6}$$

Here, B_{00} is the field strength at the center of the cloud and its value is prescribed by β and ρ_{00}.



Linearized Equations for Perturbations

We start from the following linearized set of equations for the perturbed quantities:

- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) + \nabla \cdot (\rho \vec{U}_0) = 0, \tag{7}$$

- Momentum equation:

$$\begin{aligned} &\rho_0 \frac{\partial \vec{v}}{\partial t} + \rho_0 \vec{v} \cdot \nabla \vec{U}_0 + \rho_0 \vec{U}_0 \cdot \nabla \vec{v} = -\nabla p - \rho \nabla \phi_0 - \rho_0 \nabla \phi \\ &+ \frac{1}{\mu_0} (\nabla \times \vec{B}_0) \times \vec{B} + \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B}_0, \end{aligned} \tag{8}$$

- Induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}_0) + \nabla \times (\vec{U}_0 \times \vec{B}), \tag{9}$$

- Poisson equation:

$$\frac{d^2 \phi}{dz^2} = 4\pi G \rho, \tag{10}$$

- Adiabaticity condition:

$$\frac{\partial p}{\partial t} + \vec{U}_0 \cdot \nabla p + \vec{v} \cdot \nabla p_0 = v_s^2 \left(\frac{\partial \rho}{\partial t} + \vec{U}_0 \cdot \nabla \rho + \vec{v} \cdot \nabla \rho_0 \right). \tag{11}$$

Fourier Transformed Equations

Taking $\nabla = \vec{e}_z \partial/\partial z$ in (7)–(11) and Fourier transforming them in time, one obtains the following set of scalar equations:

- Continuity equation

$$i\omega \frac{\rho}{\rho_0} = \frac{dw}{dz} + \frac{d\ln\rho_0}{dz} w,$$

- The x-, y- and z-component of the momentum equation:

$$\begin{aligned} i\omega u &= \frac{dU_0}{dz} w - v_A^2 \frac{d\ln B_0}{dz} b_z, \\ \omega v &= 0, \\ i\omega w &= \frac{1}{\rho_0}\frac{dP}{dz} + \frac{d\phi}{dz} + \frac{\rho}{\rho_0}\frac{d\phi_0}{dz}, \end{aligned}$$

- The x-, y- and z-component of the induction equation:

$$\begin{aligned} i\omega b_x &= \frac{d\ln B_0}{dz} w + \frac{dw}{dz} - \frac{dU_0}{dz} b_z, \\ \omega b_y &= \omega b_z = 0, \end{aligned}$$

- Poisson equation

$$\frac{d^2\phi}{dz^2} = 4\pi G\rho,$$

- Adiabacity of perturbations:

$$i\omega p - \frac{dp_0}{dz} w = v_s^2 \rho_0 \left(i\omega \frac{\rho}{\rho_0} - \frac{d\ln\rho_0}{dz} w \right),$$

Here:

- $P \equiv p + v_A^2 \rho_0 b_x$ is the total pressure perturbation,
- $\vec{v} = (u, v, w)$ and
- $\vec{b} = \vec{B}/B_0$.

Equations with New Variables

Introducing the Lagrangian displacement ξ in the $z-$direction and a new variable η by

$$w \equiv -i\omega\xi, \qquad \eta \equiv \rho_0\xi,$$

we obtain:

$$\begin{aligned}
& u + \eta\frac{1}{\rho_0}\frac{dU_0}{dz} = 0, \\
& \frac{d\phi_0}{dz}\frac{d\eta}{dz} + \omega^2\eta = \frac{dP}{dz} + \rho_0\frac{d\phi}{dz}, \\
& v_f^2\frac{d\eta}{dz} = \eta\left[\frac{d\ln\rho_0}{dz}\left(\frac{v_A^2}{2} + \frac{\gamma-1}{\gamma}v_s^2\right)\right] - P, \\
& \frac{d^2\phi}{dz^2} = -4\pi G\frac{d\eta}{dz},
\end{aligned} \tag{12}$$

where $v_f^2 \equiv v_A^2 + v_s^2$.

The non-uniform flow speed enters only in the first of the above equations and the only effects it causes are induced harmonic motions with speed u in the $x-$direction.

Final Equation for η

Integration of Poisson equation gives

$$\frac{d\phi}{dz} = -4\pi G\eta \quad \text{with the boundary condition} \quad \left.\frac{d\phi}{dz}\right|_{\eta=0} = 0.$$

Equations (12) reduce to a single one:

$$\frac{d^2}{dZ^2}(\eta \cosh\, Z) + \left(\frac{\omega^2 H^2}{v_f^2} + \frac{2}{\cosh^2 Z} - 1\right) \eta \cosh\, Z = 0, \tag{13}$$

where

$$Z \equiv \frac{z}{H}, \quad H^2 \equiv \frac{1+\beta}{\beta}\frac{v_s^2}{2\pi\gamma G\rho_{00}}.$$

Equation (13) **yields proper global solutions for small perturbations in a self-gravitating isothermal plasma cloud.**

Instead, if the standard procedure based on the assumption of a **uniform self-gravitating plasma** is performed, one obtains a significantly different equation

$$\frac{d^2\eta}{dZ^2} + \frac{H^2}{v_f^2}\left(\omega^2 + 4\pi G\rho_{00}\right)\eta = 0. \tag{14}$$

Local Solutions

1. Standard approach - starting form a uniform plasma and Equation 14:

$$\frac{d^2\eta}{dZ^2} + \frac{H^2}{v_f^2}\left(\omega^2 + 4\pi G\rho_{00}\right)\eta = 0.$$

with constant coefficients has harmonic solutions

$$\eta \sim \exp ikz/H$$

which yields the known **dispersion relation**

$$\omega^2 = v_f^2 k^2 - 4\pi G\rho_{00}$$

This relation allows for gravitational instability if

$$k \leq \frac{2}{v_f}\sqrt{\pi G\rho_{00}}, \qquad \omega^2 < 0$$

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2. Non-uniform plasma and equation 13:

$$\frac{d^2}{dZ^2}(\eta\cosh Z) + \left(\frac{\omega^2 H^2}{v_f^2} + \frac{2}{\cosh^2 Z} - 1\right)\eta\cosh Z = 0$$

can be treated locally provided the $Z-$ dependent term is negligible:

$$\frac{\omega^2 H^2}{v_f^2} - 1 \gg \frac{2}{\cosh^2 Z}$$

In this case, the **local dispersion relation** is

$$\omega^2 = v_f^2 k^2 + \frac{v_f^2}{H^2}$$

in which $\omega^2 > 0$ always. is satisfied.

Eigen-solutions

General properties of the eigen-problem

- The considered self-gravitating system can oscillate as a whole at certain allowed eigen-frequencies ω.
- Eigen-solutions for the displacement $\xi(z)$ have to be anti-symmetric with respect to the center $z = 0$ of the cloud:

$$\xi(z) = -\xi(-z)$$

- The energy of the system is finite and conserved.
- Consequently, the perturbations have to be localized, i.e. the energy density of the perturbations has to tend to zero at sufficiently large $|z|/H$.
- For example, the kinetic energy density E_k of perturbations has to satisfy the condition

$$\lim_{|z/H|\to\infty} \mathrm{E_k} \sim \lim_{|z/H|\to\infty} \xi^2(z)\rho_0(z) = 0$$

Condition for Anti-symmetric Solutions

Since the eigen-solution of Eq (13) for $\xi(z)$ is anti-symmetric everywhere in the $z-$space, we can consider it at the limit $|z/H| \to \infty$ where the local anti-symmetric analytical expressions are readily obtained as

$$\xi^{(-)} = -e^{\kappa z/H} \cosh(z/H), \quad \text{if } z < 0,$$
$$\xi^{(+)} = e^{-\kappa z/H} \cosh(z/H) \quad \text{if } z > 0. \tag{15}$$

where

$$\kappa \equiv \left(1 - \frac{\omega^2 H^2}{v_f^2}\right)^{1/2}$$

CONCLUSION

Frequencies of eigen-modes are real and they satisfy the condition for κ being a real quantity:

$$\omega \leq \frac{v_f}{H} \equiv \omega_0.$$

Perturbations having frequencies $\omega \geq \omega_0$ are propagating non-localized waves that carry energy away.

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