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Complex MHD Shock Interactions

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Complex MHD shock interactions

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Time evolution schemes

- explicit / implicit / numerical stability / upwind schemes

Shock capturing schemes

- Finite Volume Method
- TVD schemes / Riemann (Godunov) solvers

Complex MHD shock interactions

- recent 2D and 3D results on shocks in switch-on regime
- intermediate shocks and compound shocks in 2D and 3D!

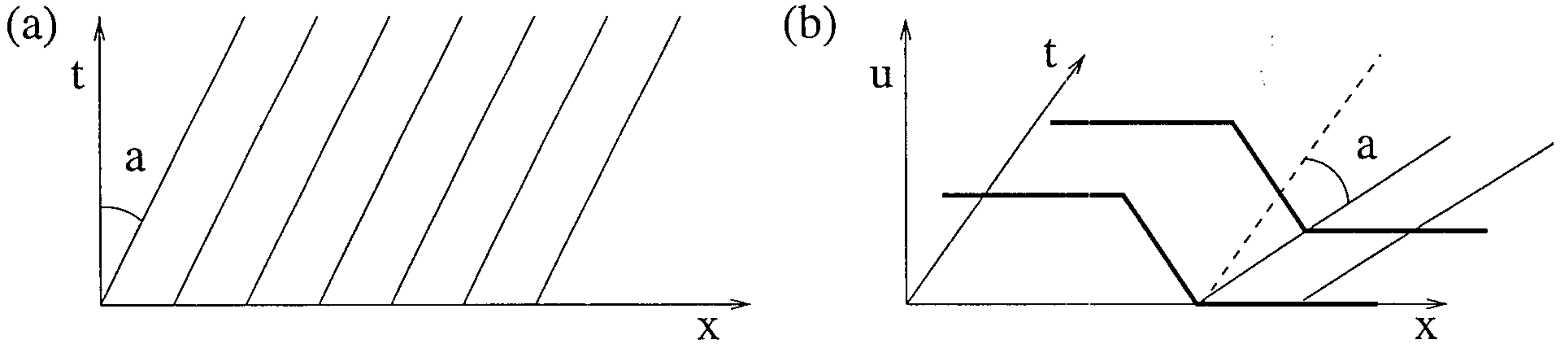
Model problem

- consider $u(x,t)$: $\boxed{\dfrac{\partial u}{\partial t} = -v\,\dfrac{\partial u}{\partial x}}$ or $\boxed{\dfrac{\partial u}{\partial t} + \dfrac{\partial vu}{\partial x} = 0}$ with v constant

$\Rightarrow$ solution: $u = u(x - vt)$, *i.e.* wave propagating in x-direction

$\Rightarrow$ dispersion relation: assume $u \sim e^{i(kx-\omega t)}$ $\Rightarrow$ $\boxed{\frac{\omega}{k} = v}$ $\Rightarrow$ no dispersion!

$\Rightarrow$ characteristics: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = v$ $\Rightarrow$ straight, parallel lines



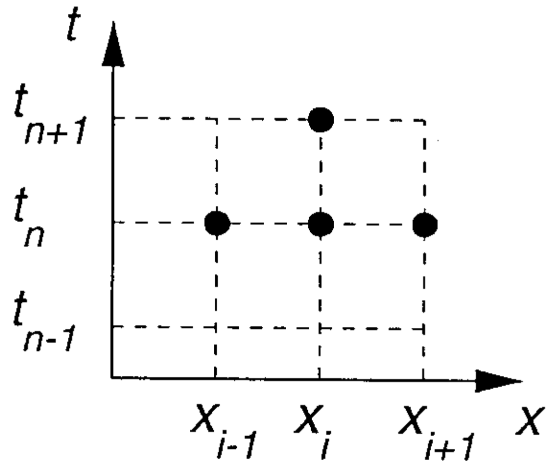
Explicit schemes

- notation: $u_i^n \equiv u(x_i, t_n) \quad \Rightarrow \quad \underbrace{\frac{u_i^{n+1} - u_i^n}{\Delta t}}_{\text{1st-order forward FD}} = -v \underbrace{\frac{u_{i+1}^n - u_{i-1}^n}{2\,\Delta x}}_{\text{2nd-order central FD}}$

$$\Rightarrow \boxed{u_i^{n+1} = u_i^n - v\,\frac{\Delta t}{\Delta x}\,\frac{u_{i+1}^n - u_{i-1}^n}{2}}$$

'FTCS Euler scheme'

$\Rightarrow O(\Delta x^2, \Delta t)$ with stencil:



- 'explicit': u_i^{n+1} determined only by u^n's of previous time step
- 'consistent': truncation errors vanish for $\Delta x \to 0, \Delta t \to 0$

Numerical stability

- stable scheme $\Leftrightarrow$ round-off errors shrink or do not grow

  $\Rightarrow$ evaluated by (local) *'von Neumann' method* $\Rightarrow$ $\epsilon(x,t) = \hat{\epsilon}_k e^{\lambda t} e^{ikx}$

  $\Rightarrow$ stable scheme $\Leftrightarrow$ $\left|\dfrac{\epsilon_k^{n+1}}{\epsilon_k^n}\right| = \left|e^{\lambda \Delta t}\right| \leq 1$ for every k

- substitute $\underbrace{\hat{u}}_{\text{num. sol.}} = \underbrace{E}_{\text{exact sol.}} + \underbrace{\epsilon}_{\text{error}}$ in backward Euler scheme:

  $$\Rightarrow e^{\lambda \Delta t} = 1 - \frac{v\Delta t}{\Delta x}\,\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2} = 1 - i\,\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

  $\Rightarrow \left|e^{\lambda \Delta t}\right| > 1$ for every k! $\Rightarrow$ *unconditionally unstable!*

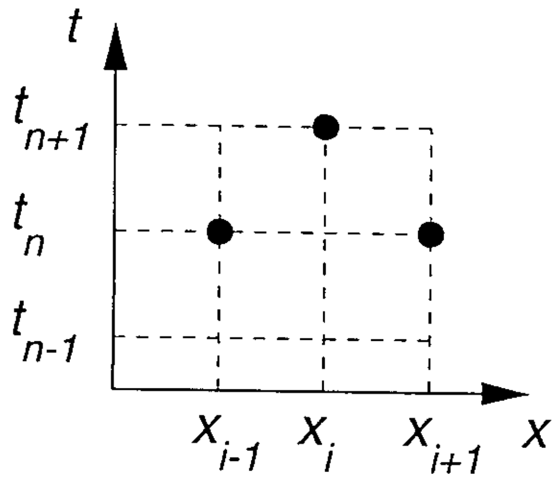
$\Rightarrow$ $\exists$ 3 types of solutions:
- addition of 'numerical diffusion'
- use of same space-time symmetry as the original PDE
- use of an implicit scheme

- replacing u_i^n in FTCS Euler scheme by an average value we get:

$$\boxed{u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - v\,\frac{\Delta t}{\Delta x}\,\frac{u_{i+1}^n - u_{i-1}^n}{2}}$$

'Lax-Friedrichs'

$\Rightarrow O(\Delta x^2, \Delta t)$ with stencil:



- rewriting: $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} = -v\,\dfrac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \dfrac{(\Delta x)^2}{2\Delta t}\,\dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$

$\Rightarrow$ discretization of: $\dfrac{\partial u}{\partial t} = -v\,\dfrac{\partial u}{\partial x} + \underbrace{\dfrac{(\Delta x)^2}{2\Delta t}\,\dfrac{\partial^2 u}{\partial x^2}}_{\text{diffusion term}}$

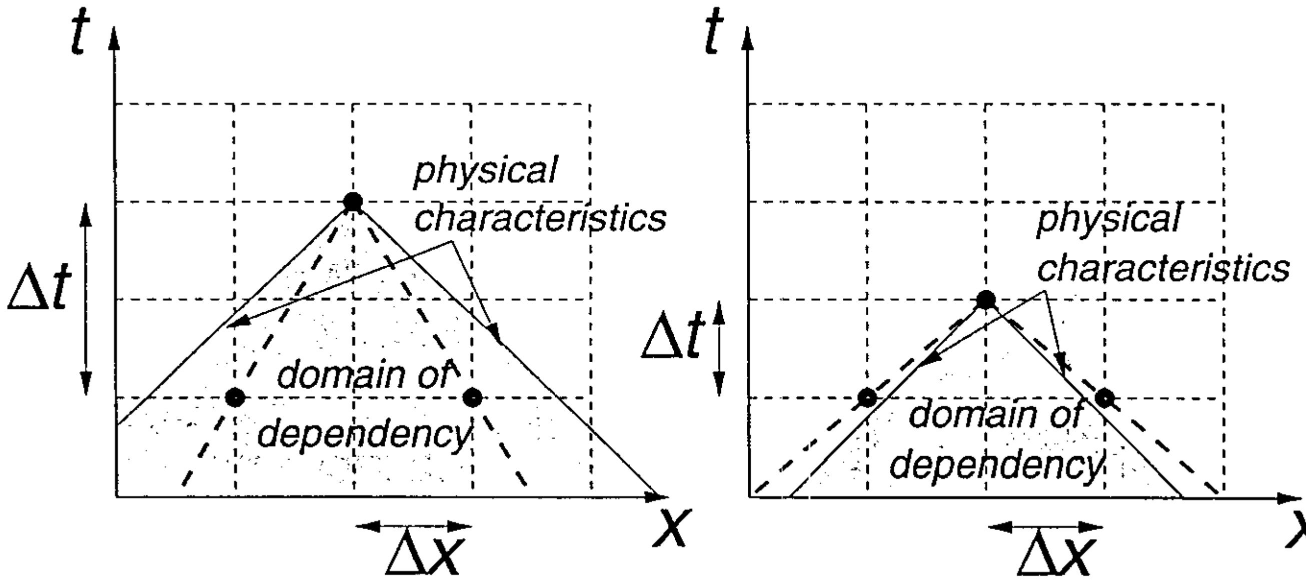
$\Rightarrow$ *'numerical dissipation'* or *'viscozity'* added!

- von Neumann method on Lax-Friedrichs scheme yields:

$$e^{\lambda\Delta t} = \cos(k\Delta x) - i\,\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

$$\Rightarrow \left|e^{\lambda\Delta t}\right| = \cos^2(k\Delta x) + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)$$

$\Rightarrow$ condition for stability: $\boxed{C \equiv \frac{|v|\Delta t}{\Delta x} \leq 1}$ = 'CFL condition



- *physical meaning:*

- cf. chars. $x = vt$: information propagates only in positive x-direction (for $v > 0$)
- Lax-Friedrichs: information propagates in two directions ($\pm x$)

$\Rightarrow$ update upwind nodes:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \begin{cases} -\dfrac{F_i^n - F_{i-1}^n}{\Delta x} & \text{for } v > 0 \\ -\dfrac{F_{i+1}^n - F_i^n}{\Delta x} & \text{for } v < 0 \end{cases}$$

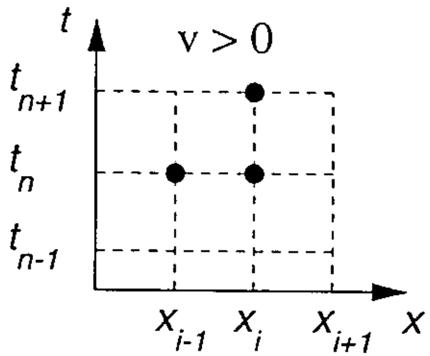
'upwind difference scheme'

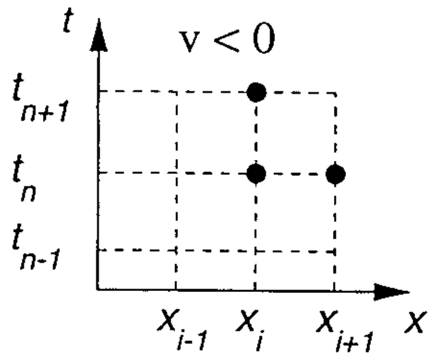
- *'conservative' form*

(model eq.: flux function $F = vu$)

$\Rightarrow O(\Delta x, \Delta t)$

$\Rightarrow$ with stencil:





- widely used in shock-capturing schemes for nonlinear problems
- produces 'oscillation-free' solutions: *'monotone' scheme*

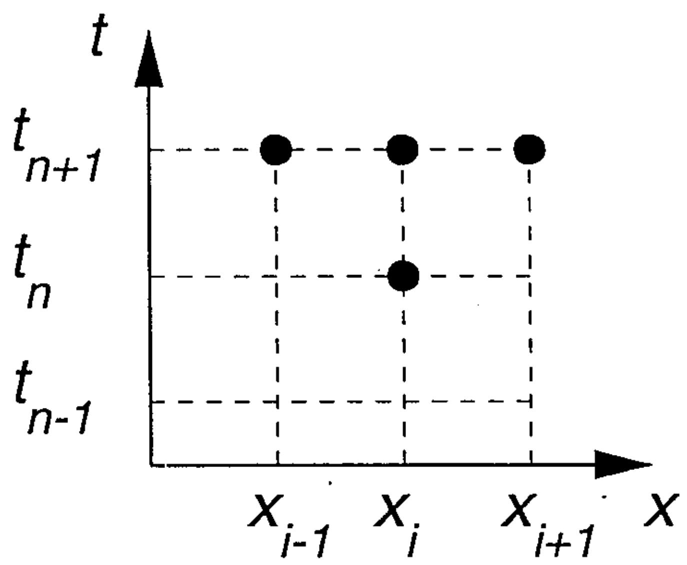
Implicit schemes

- evaluate spatial derivative in FTCS Euler scheme in t^{n+1}

$$\Rightarrow \boxed{u_i^{n+1} = u_i^n - v\,\frac{\Delta t}{\Delta x}\,\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2}}$$

'BTCS Euler scheme'

$\Rightarrow O(\Delta x^2, \Delta t)$ with stencil:



- *'implicit'*: u_i^{n+1} *not* expressed in u^n's of previous time step

$\Rightarrow$ tridiagonal system in each time step

- stability: $e^{\lambda \Delta t} = 1 - iC \sin(k\Delta x)$

$$\Rightarrow \left|\frac{\epsilon_k^{n+1}}{\epsilon_k^n}\right| < 1\,,$$ *for every k*: *unconditionally stable!*

Total Variation Diminishing schemes

- total variation of u on $[0,1] \equiv \int_0^1 \left|\frac{\partial u}{\partial x}\right| \mathrm{d}x \overset{\text{discr.}}{\Longrightarrow} TV(u^n) = \sum_{i=0}^{N} |u_{i+1}^n - u_i^n|$

- scheme is TVD in time $\Leftrightarrow \quad TV(u^{n+1}) \leq TV(u^n) \quad \forall n$

  $\Rightarrow$ *'TV-stable' schemes always converge!*

  e.g. all monotone schemes are TVD schemes

- 'general' explicit method can be written in form:

$$u_{i+1}^n = u_i^n + A_{i+1/2} \underbrace{(u_{i+1}^n - u_i^n)}_{\Delta u_{i+1/2}^n} - B_{i-1/2} \underbrace{(u_i^n - u_{i-1}^n)}_{\Delta u_{i-1/2}^n}$$

$$\Rightarrow \text{if} \left.\begin{array}{c} A_{i+1/2} \geq 0 \quad \forall i \\ B_{i-1/2} \geq 0 \quad \forall i \\ 0 \leq A_{i+1/2} + B_{i+1/2} \leq 1 \quad \forall i \end{array}\right\} \Rightarrow \quad \underline{\text{scheme is TVD}}$$

- example: 1st-order TVD scheme for Burger's equation, with $F = \frac{u^2}{2}$:

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0} \quad \Rightarrow \quad u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \begin{cases} F_{i+1}^n - F_i^n & \text{for} \quad \alpha_{i+1/2} < 0 \\ F_i^n - F_{i-1}^n & \text{for} \quad \alpha_{i+1/2} > 0 \end{cases}$$

where $\alpha_{i+1/2} \equiv \begin{cases} u_i^n & \text{if} \quad u_i^{n+1} - u_i^n = 0 \\ \frac{F_{i+1}^n - F_i^n}{u_i^{n+1} - u_i^n} & \text{if} \quad u_i^{n+1} - u_i^n \neq 0 \end{cases}$

$\Rightarrow$ can be rewritten in different forms, *e.g.*

$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[\alpha_{i+1/2}\Delta u_{i+1/2}^n - \left|\alpha_{i+1/2}\right| \Delta u_{i+1/2}^n + \alpha_{i-1/2}\Delta u_{i-1/2}^n + \left|\alpha_{i-1/2}\right| \Delta u_{i-1/2}^n\right]$$

$$\Rightarrow \begin{cases} A_{i+1/2} = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\left|\alpha_{i+1/2}\right| - \alpha_{i+1/2}\right) & \geq 0 \; \forall i \\ B_{i-1/2} = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\left|\alpha_{i-1/2}\right| + \alpha_{i-1/2}\right) & \geq 0 \; \forall i \\ 0 \leq \left|\alpha_{i-1/2}\right| \frac{\Delta t}{\Delta x} \leq 1 \quad \Rightarrow \quad \textit{CFL type condition!} \end{cases}$$

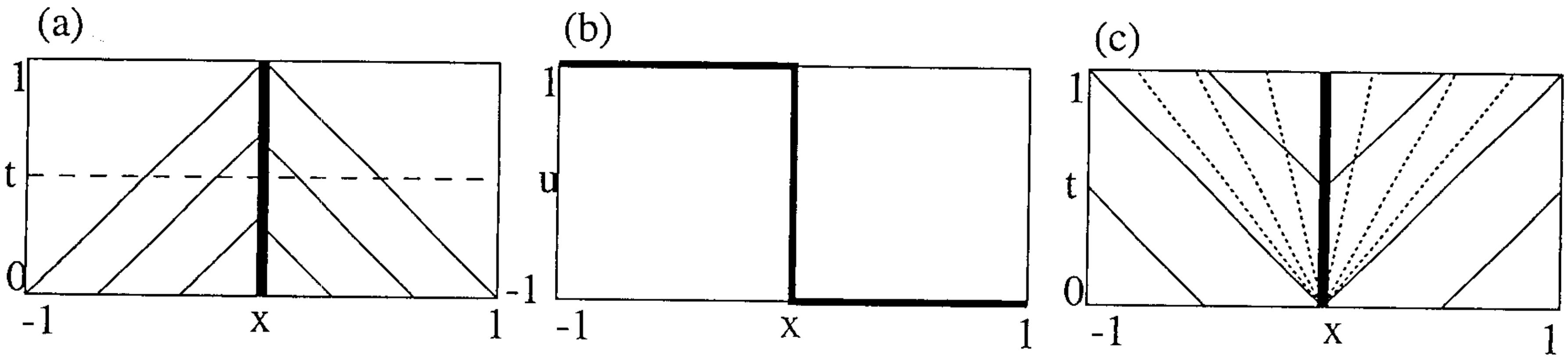
The Riemann problem

= *conservation law* + *piecewise constant* initial data with a *single discontinuity*

$e.g.$ Burger's eq: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ with $u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$

- case a) $u_l > u_r$: $\exists$ unique weak solution:

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases} \text{ with } s = \frac{1}{2}(u_l + u_r) = \text{'shock speed'}$$



a) case $u_l = 1, u_r = -1$, *b) shock profile,* *c) case* $u_l = -1, u_r = 1$

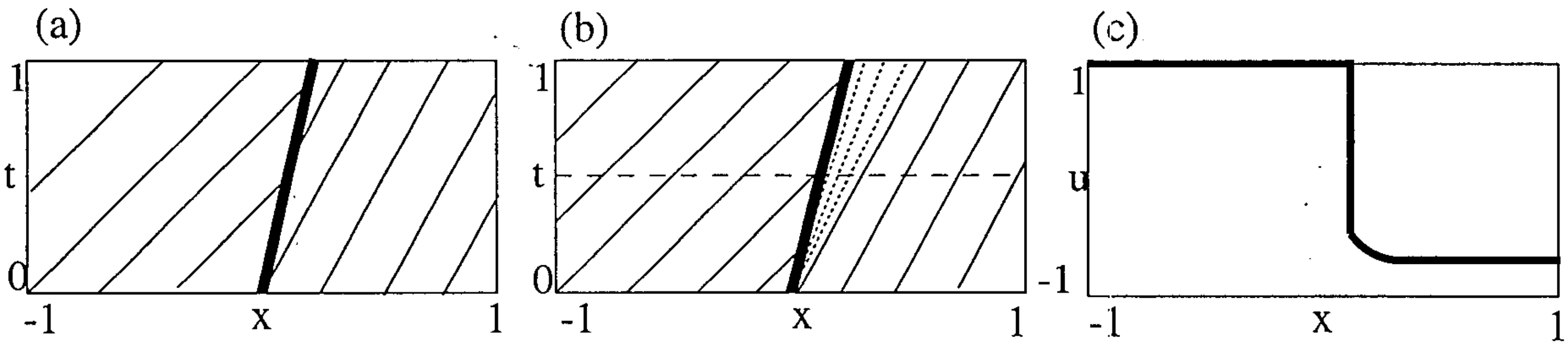
- case b) $u_l < u_r$: $\exists\, \infty$ weak solutions

  $e.g.$ rarefaction wave: $u(x,t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t \leq x \leq u_r t \\ u_r & x > u_r t \end{cases}$

- for non-convex flux function, $e.g.$ $F(u) = \frac{u^3}{3}$

  case $u_l < u_r$: no admissible shock

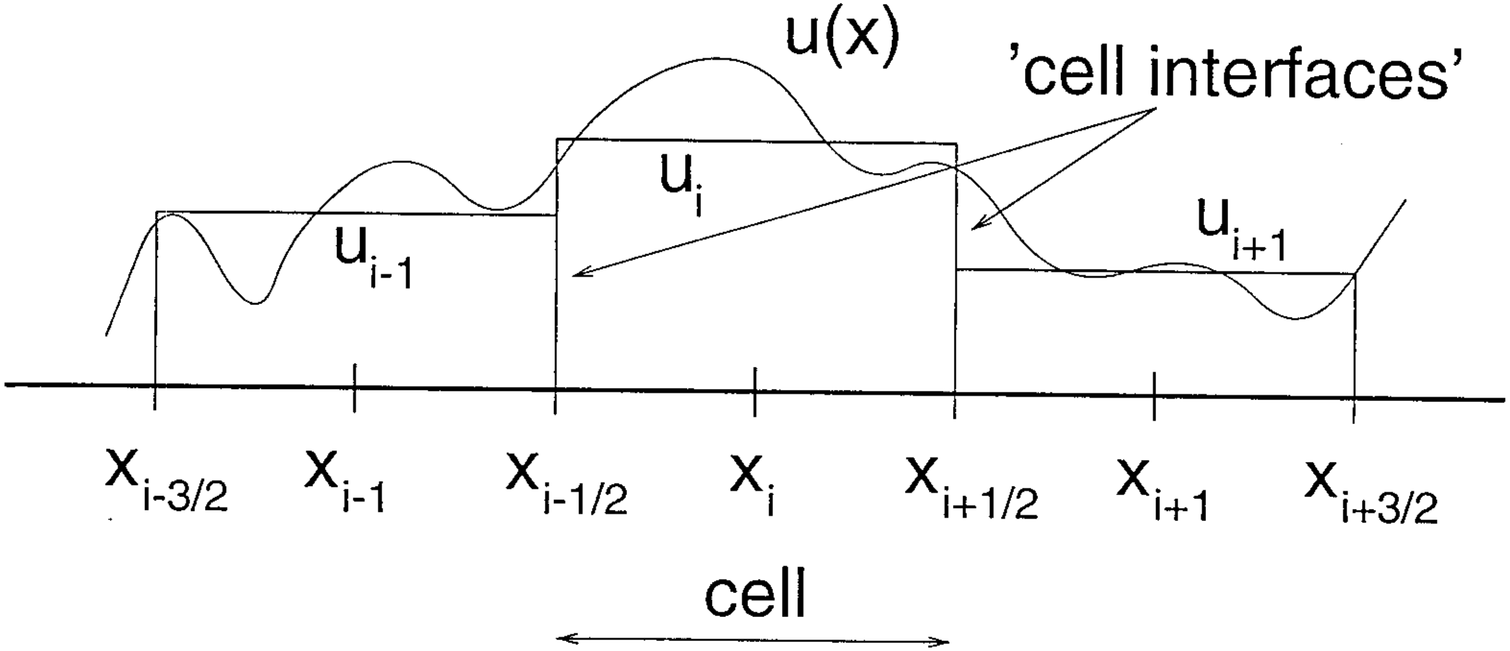
  $\Rightarrow$ compound shock: sonic shock followed by attached rarefaction:



a) case $u_l = 1, u_r = -3/4$, b) compound shock, c) shock profile at $t = 1/2$

Finite volume methods (FVM)

⇒ obtained by interpreting u_i as the *average value of* $u(x)$ *in* $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$:



$$\Rightarrow u_i(t) \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)\, \mathrm{d}x$$

- domain subdivided in a set of *cells* (covering whole domain)

⇒ advantage: *discretized equation itself* can be seen as an *integral law*

⇒ 'most physical' discretization of conservation law

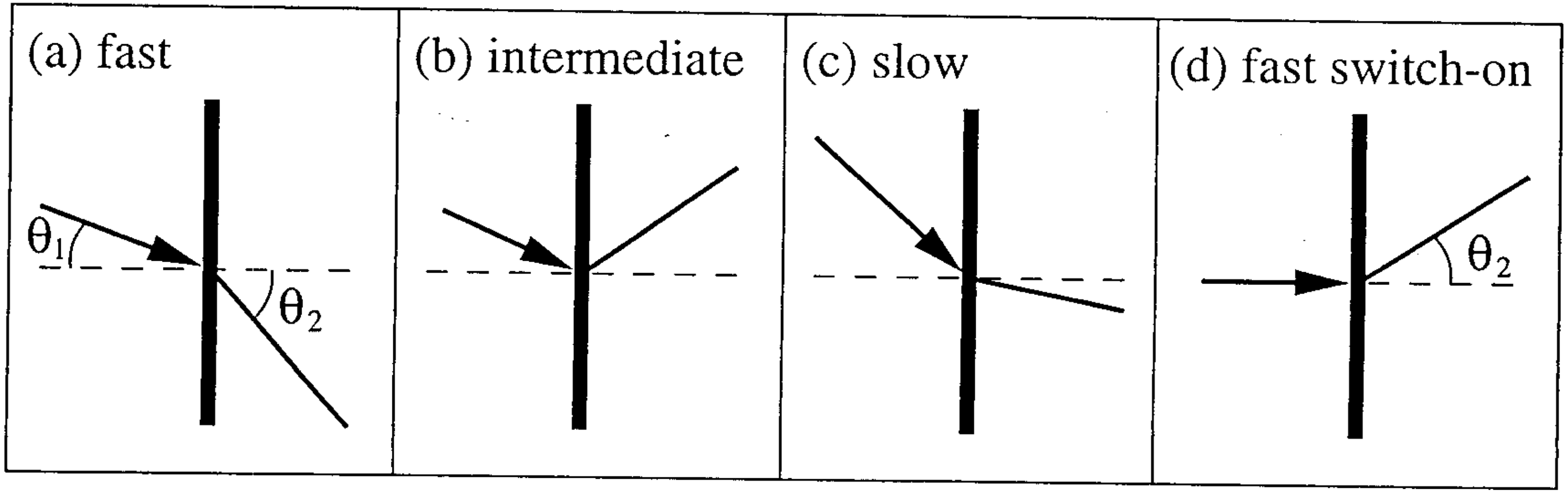
- methods based on this idea = *'finite volume'* schemes
  - almost universally used in CFD for shock-capturing codes
  - recently also in MHD!
- conservation laws applied on each cell to determine the time evolution of the dependent variables (= conserved quantities themselves!) in some discrete points of the cells (=*'nodes'*)
- generalization to 2D ('volumes' (cells) are surfaces) and 3D (cells = volumes) is obvious
- *very flexible*, *e.g.* also on unstructured triangular grids
  - cells can overlap
  - choice of cells and nodes decoupled ⇒ many different combinations!

Complex MHD shock interactions

- supersonic flow around object ⇒ bow shock
- *anisotropy* MHD waves ⇒ complex MHD bow shocks
- 4 possible positions of flow speed in direction x:

$$\boxed{\boxed{1} \geq c_{fx} \geq \boxed{2} \geq c_{Ax} \geq \boxed{3} \geq c_{sx} \geq \boxed{4}}$$

⇒ three types of MHD shocks:



- three wave speeds depend on direction

$\Rightarrow$ three Mach numbers too:

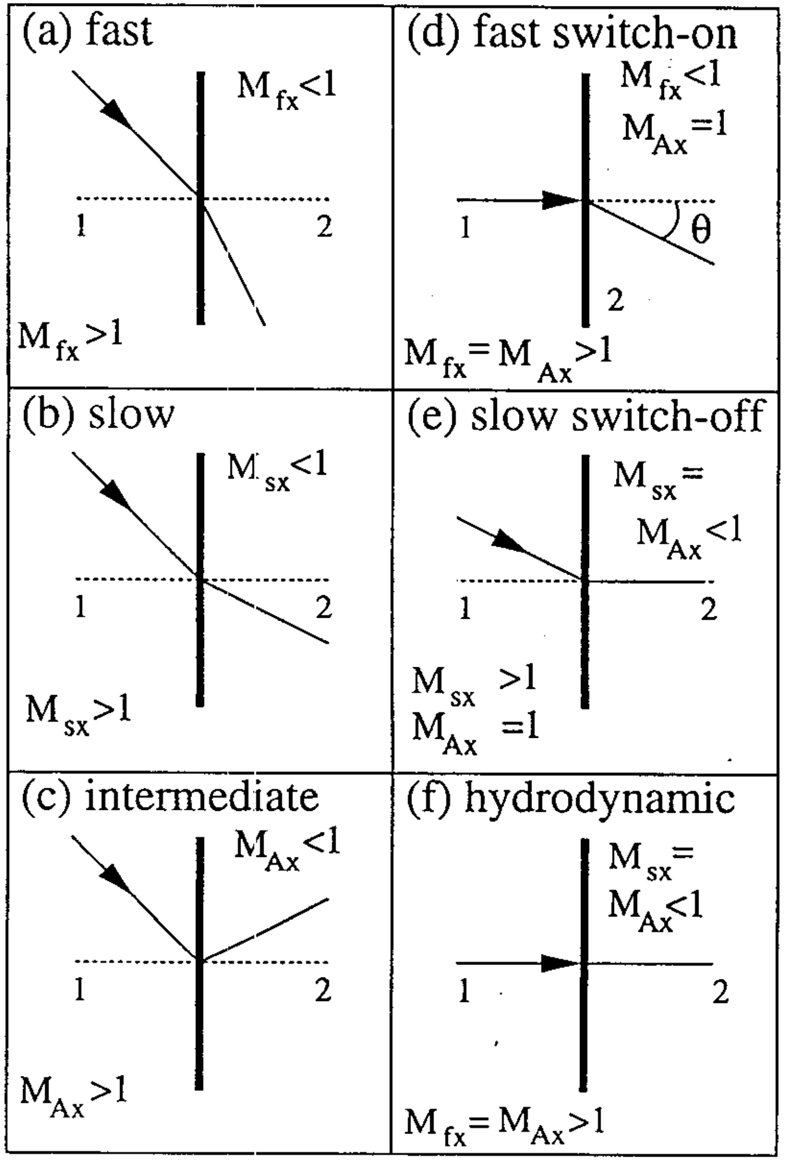
  - $M_{fx} = \dfrac{|v_x|}{c_{fx}}$
  - $M_{Ax} = \dfrac{|v_x|}{c_{Ax}}$
  - $M_{sx} = \dfrac{|v_x|}{c_{sx}}$

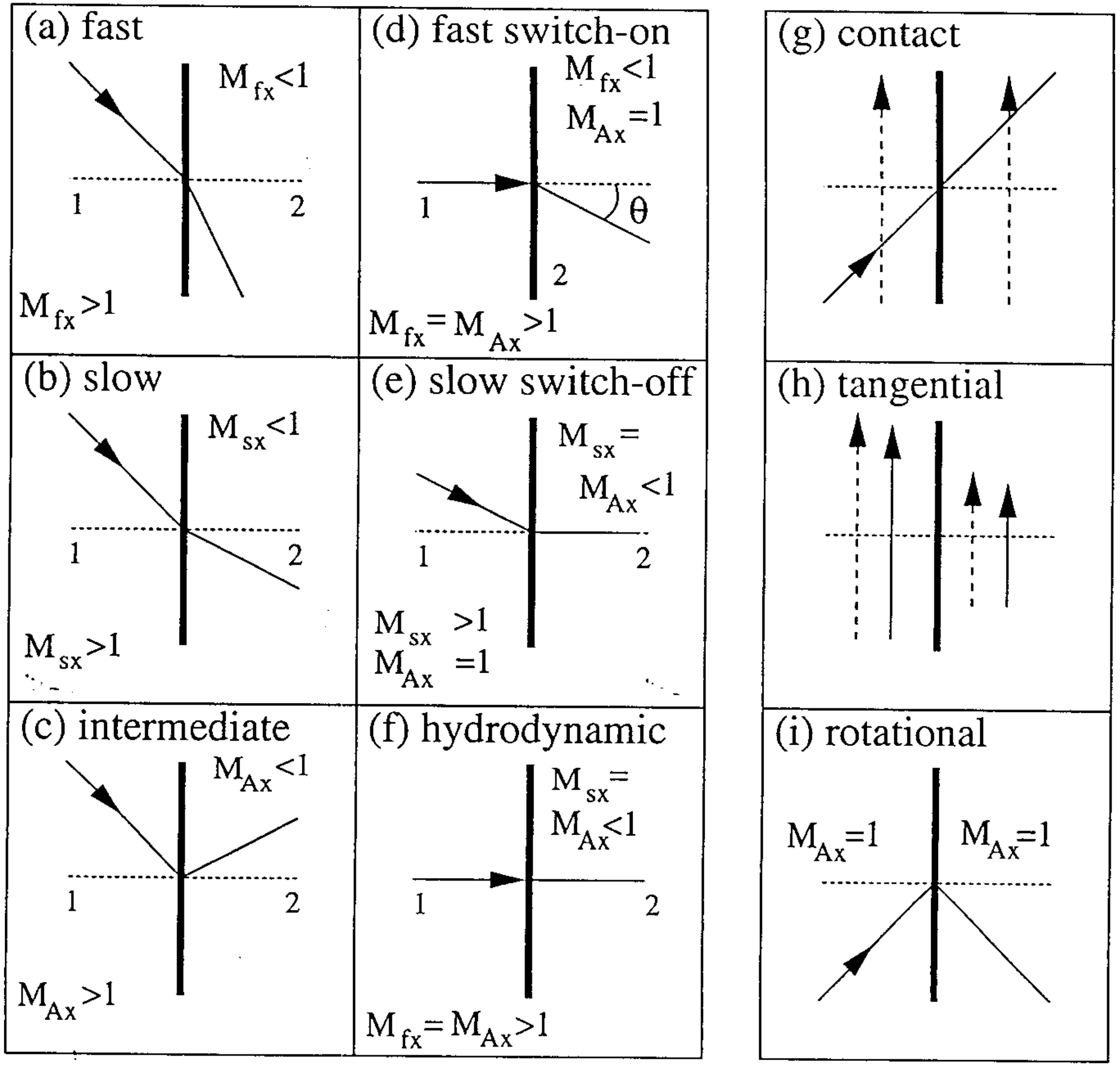
- fast switch-on shocks = intrinsic magnetic effect
- only occur when:

  1) $B_1^2 > \gamma p_1$

  2) $B_1^2 > \rho_1 v_{x,1}^2 \dfrac{\gamma - 1}{\gamma(1-\beta_1)+1}$

$\Rightarrow$ upstream flow is *magnetically dominated*





<u>other discontinuities:</u>

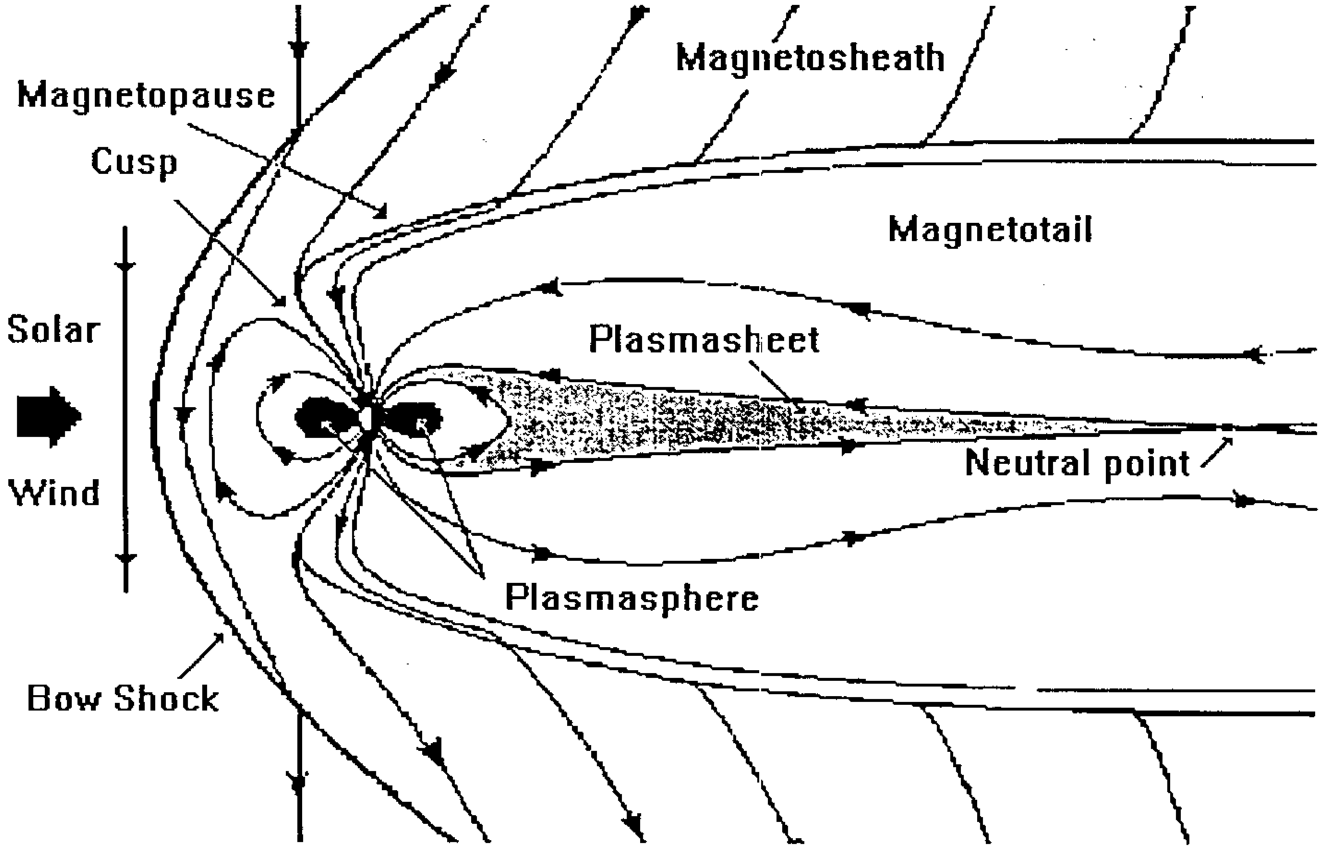
g) $v_x = 0,\ B_x \neq 0$, only jump in ρ and s

h) $v_x = B_x = 0$, jump in ρ, p, and $v_{\|}$ and $B_{\|}$

i) magnetic field rotated $180°$ around normal, no jump in entropy

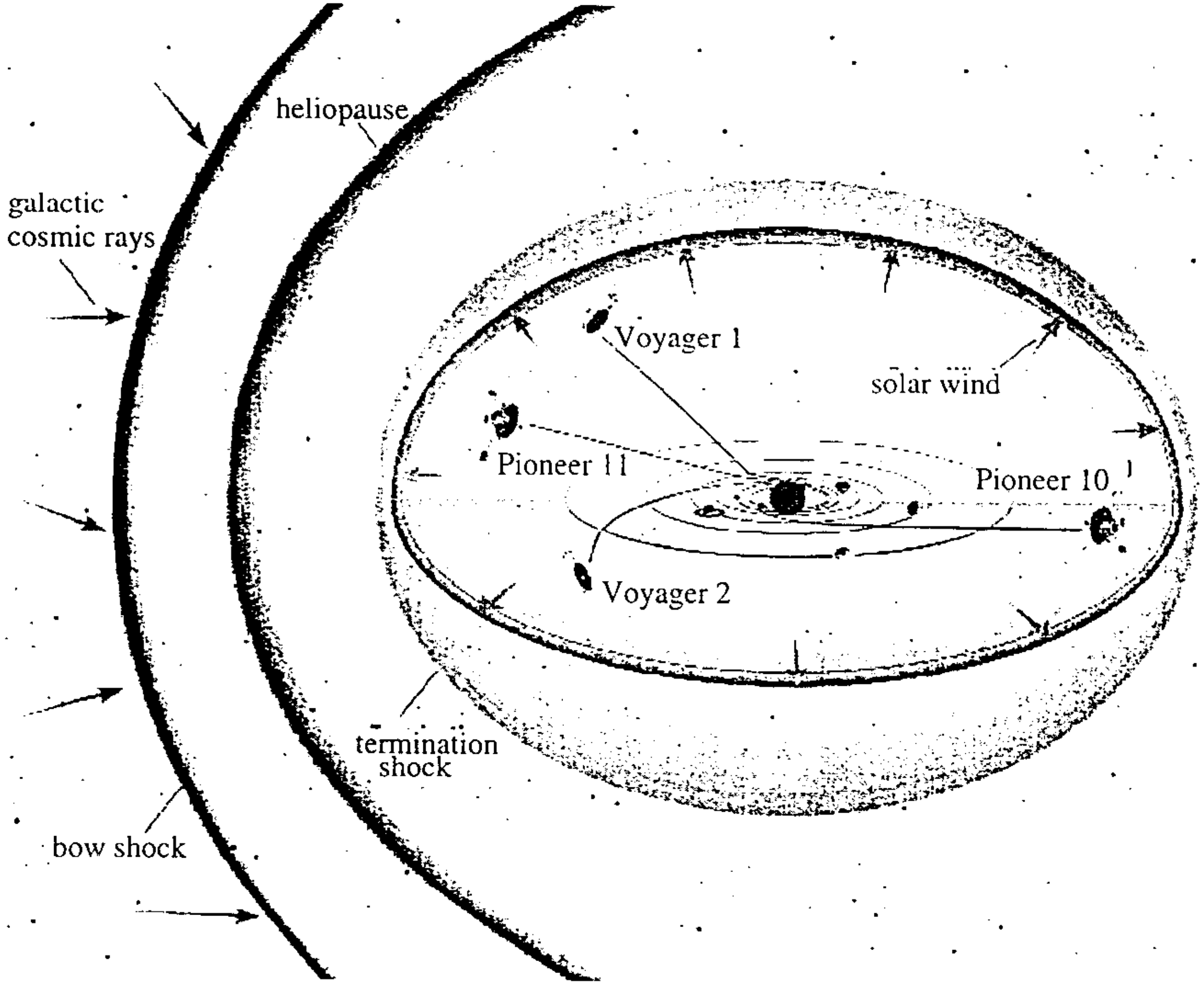
Bow shocks in the solar system

⇒ where superfast solar wind encounters 'obstacles'

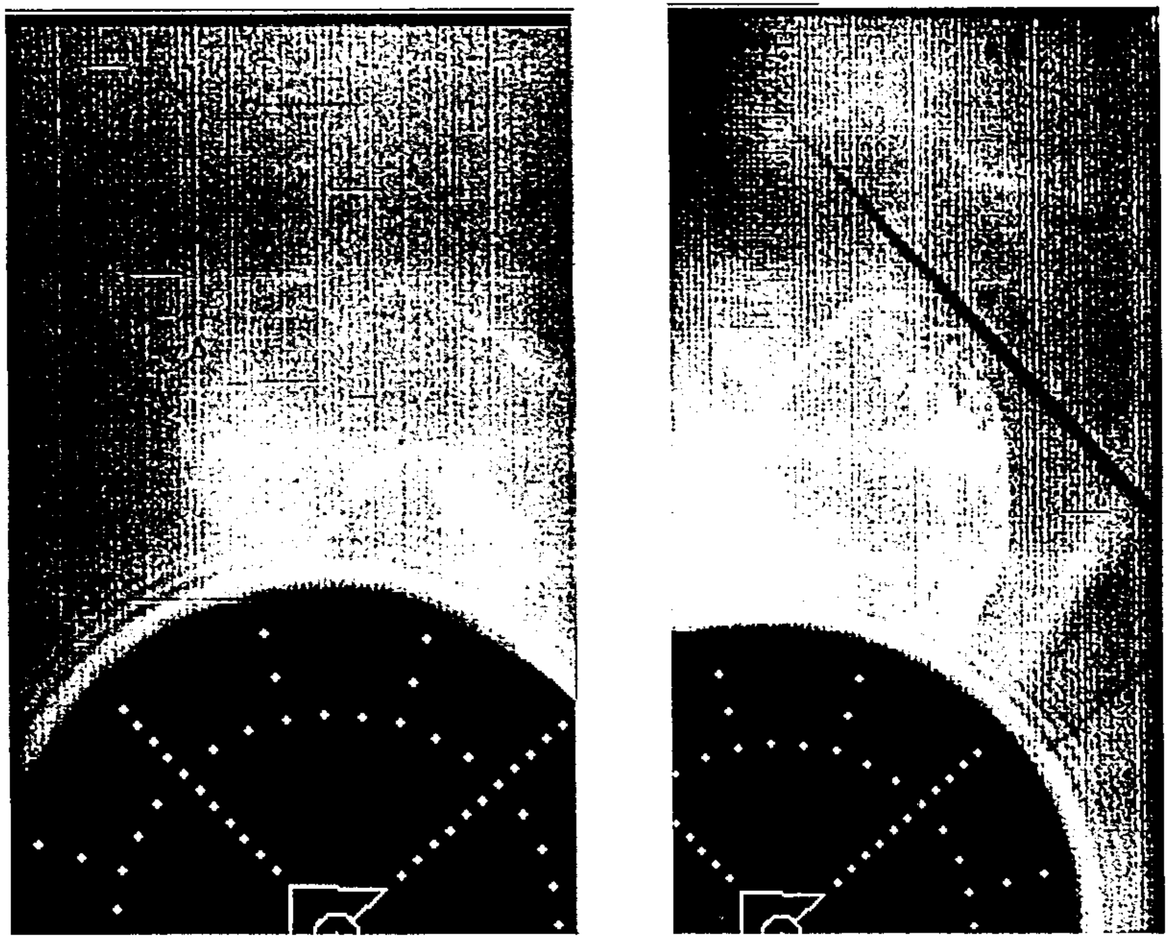


Examples:

- Earth's bow shock
- all other planets
- comets



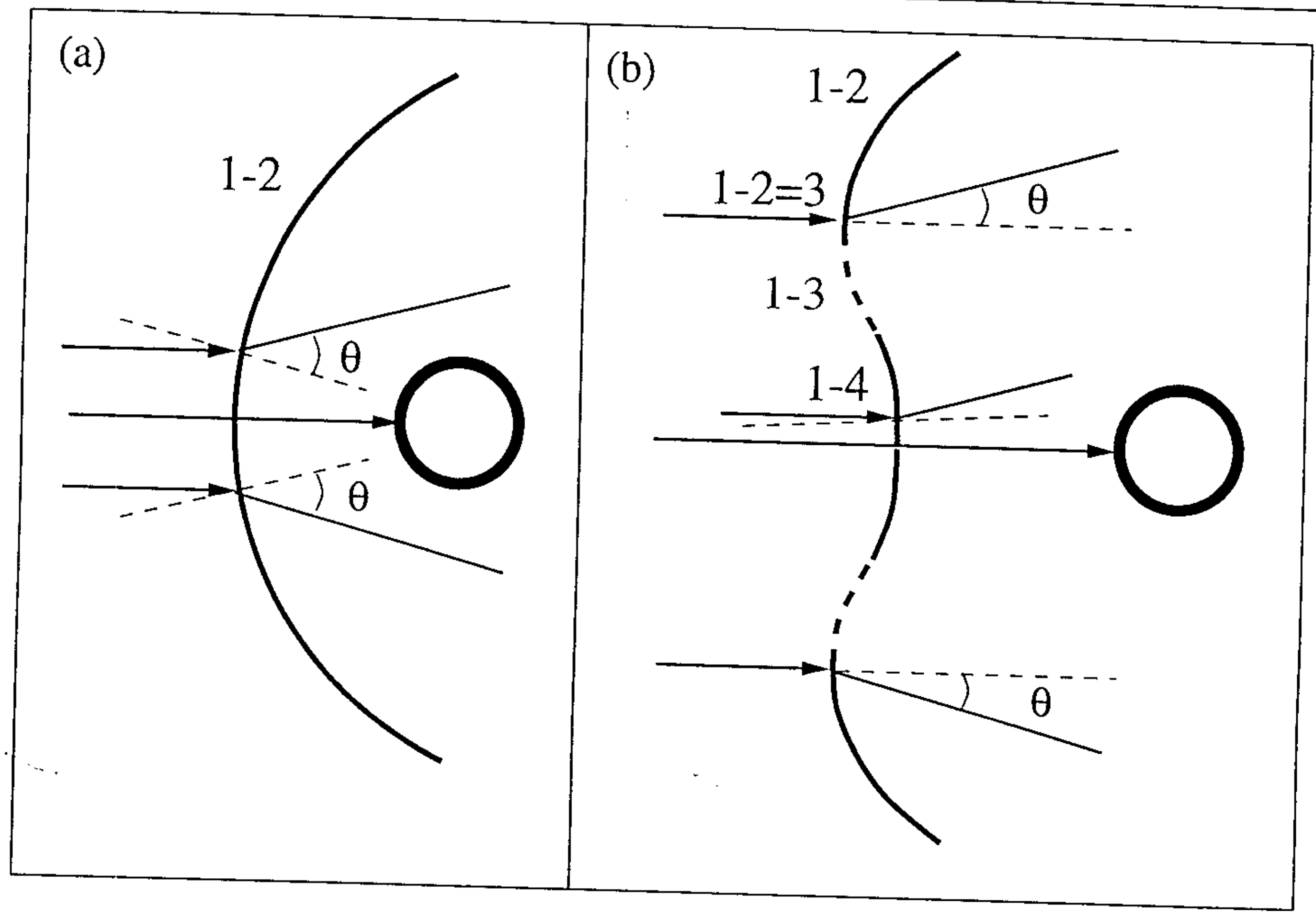
Heliosphere: magnetic environment of the solar system



SMM coronagraph images of a CME with 17 min interval ($R_{disk} = 1.6 \cdot R_{\odot}$)

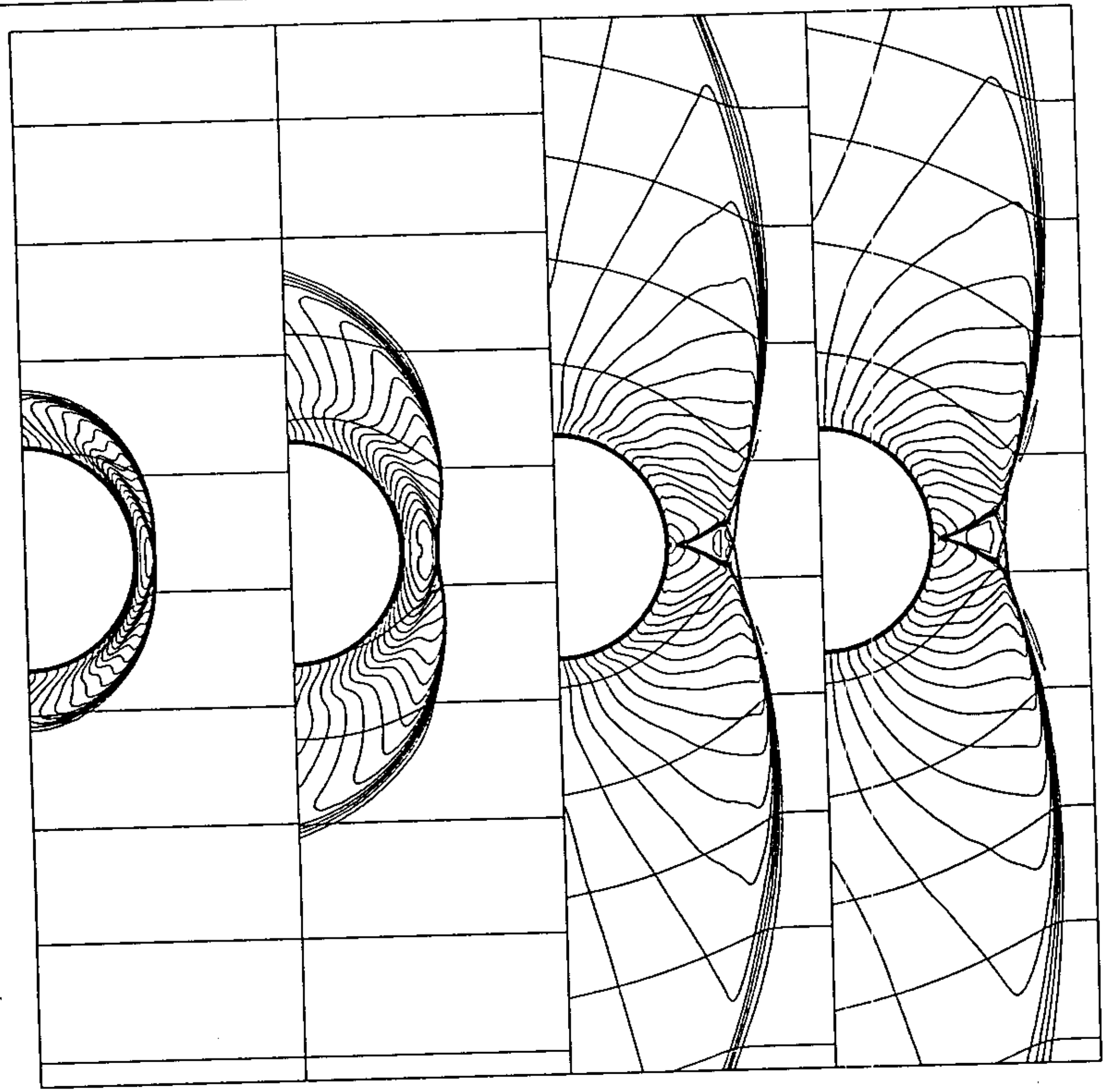
$\Rightarrow$ speed ≈ 1000 km/s, dimpled shock front

- *Steinolfson & Hundhausen ('90)*: fast switch-on shocks occur in CMEs

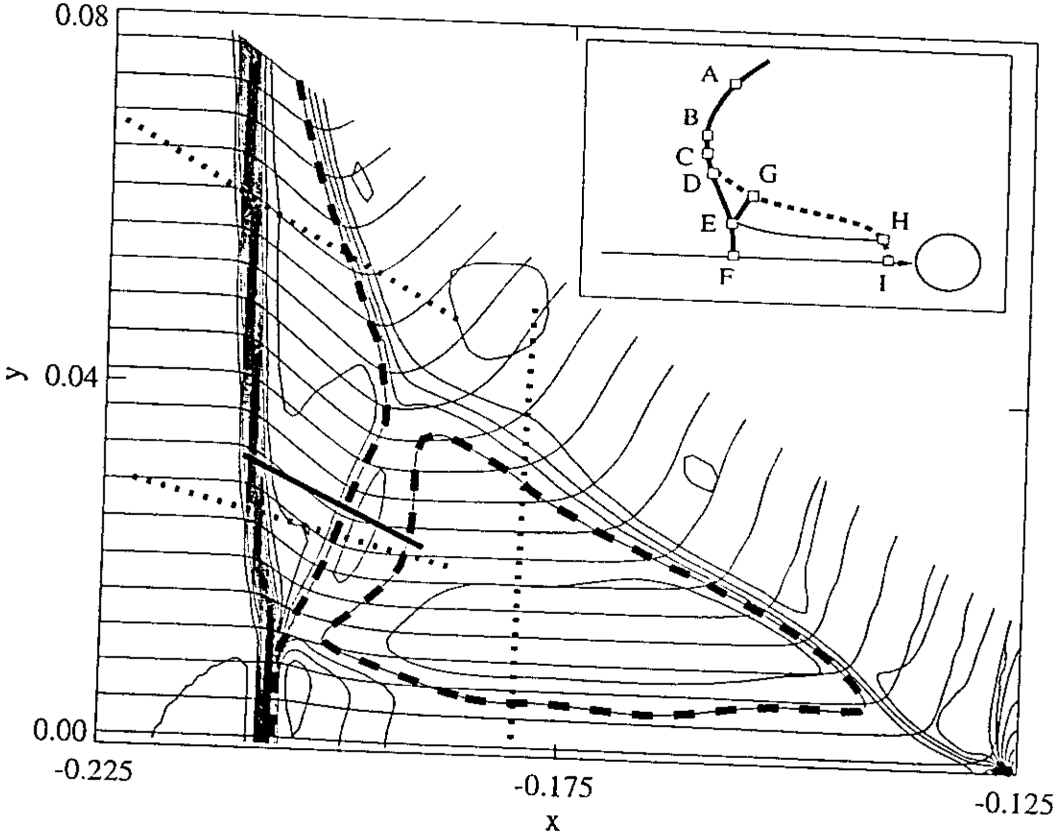


a) standard fast shock, b) topology proposed by Steinolfson & Hundhausen

- *Steinolfson & Hundhausen ('90)*: dimpled fast switch-on shocks occur in CMEs
- *De Sterck et al. ('99)*: all possible shocks occur!

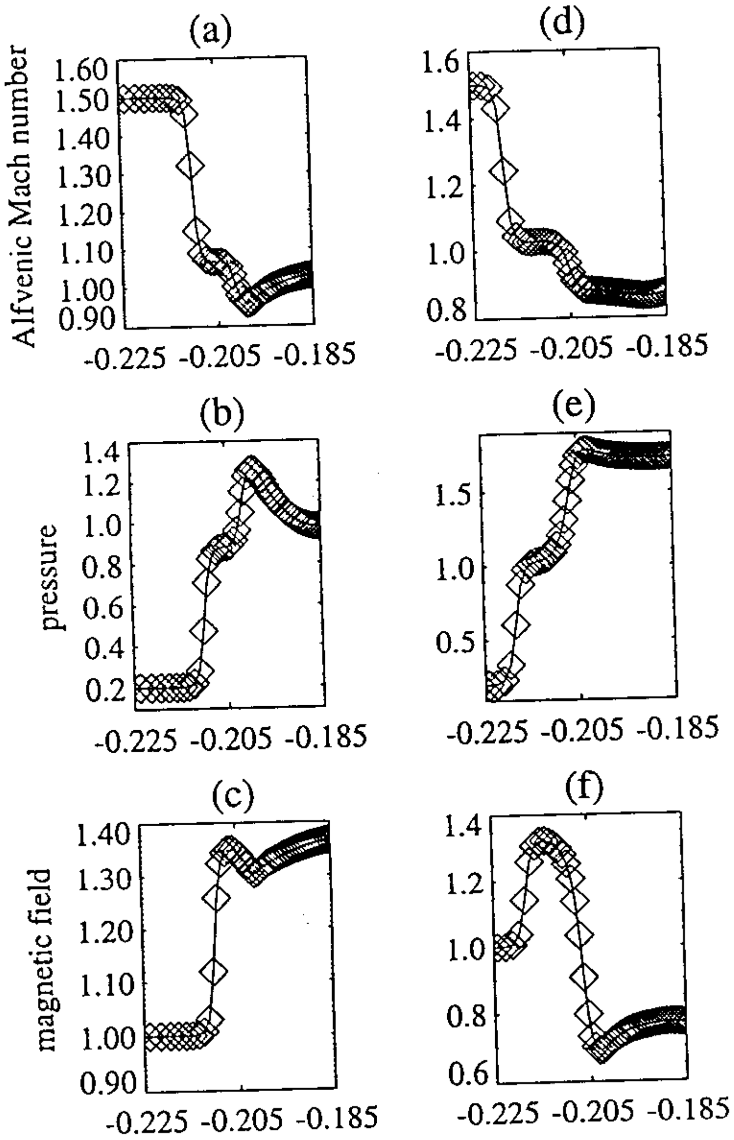


Symmetric 2D result (Hans De Sterck).



Magnetic field lines and M_A contour lines (from De Sterck et al. '98)

D-E	fast shock (almost switch-on)	E-H	tangential discontinuity
E-F	[1]-[4] intermediate shock	E-G	intermediate shock
at F	hydrodynamic shock	D-G-H-I	[2]≈[3]-[4] intermediate shock

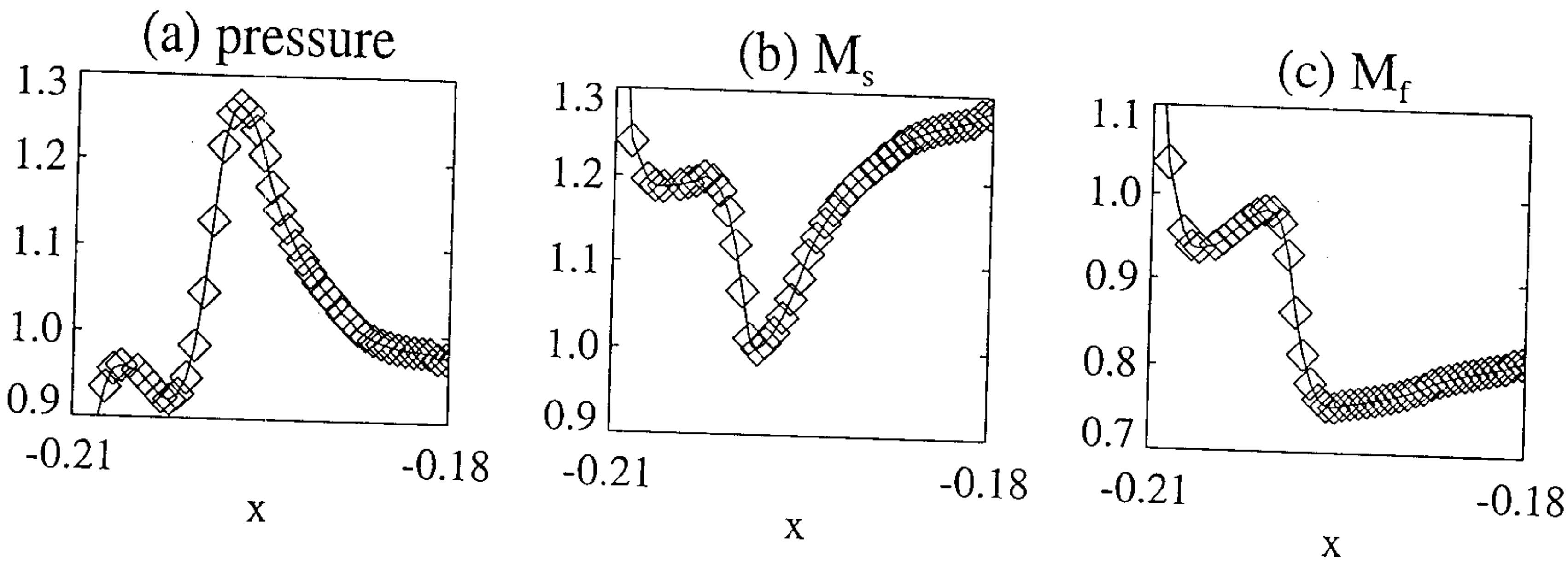


- (a)-(c): cut along lower dotted line in previous fig

⇒ fast shock and intermediate shock with rarefaction

- (d)-(f): cut along upper dotted line in previous fig

⇒ fast shock and intermediate shock

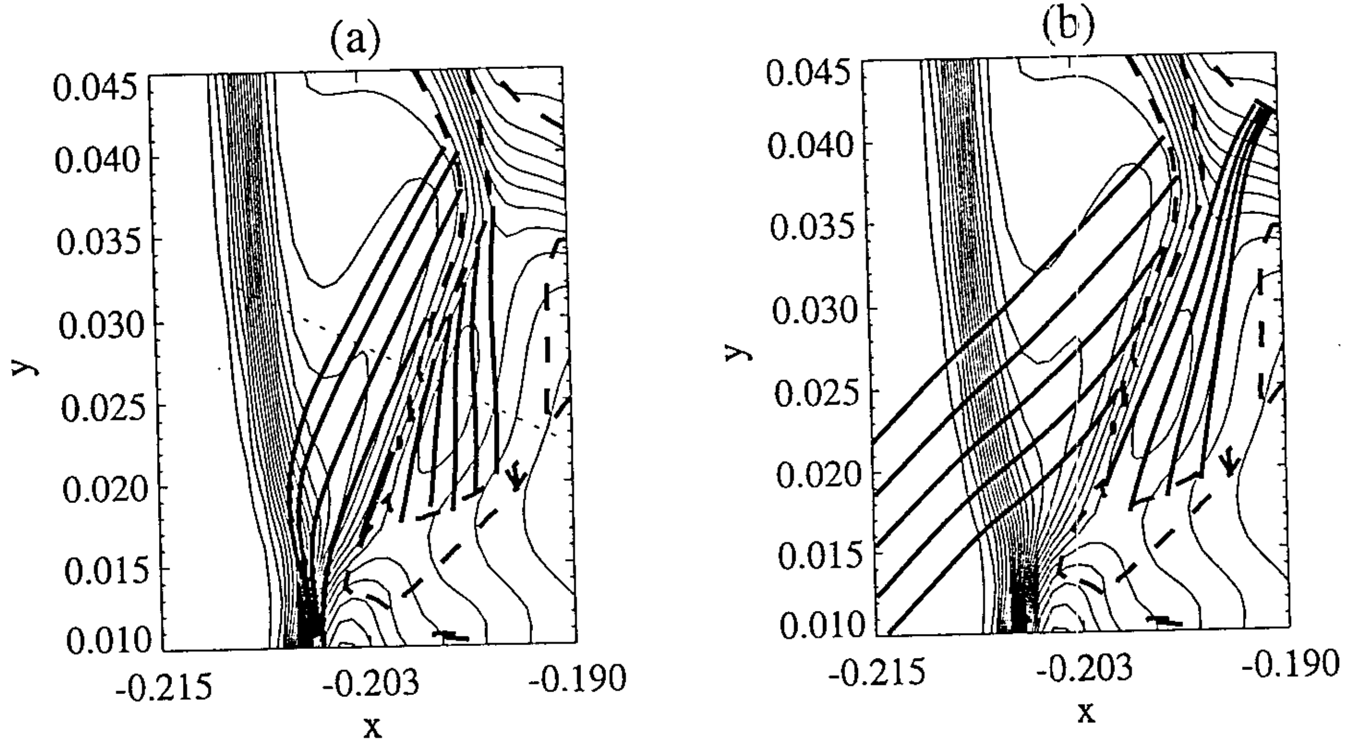


Cut along solid line (from De Sterck et al. '98)

- E-G shock is preceded and followed by rarefaction regions
- $M_f = 1$ where upstream (left) rarefaction is attached to shock
- $M_s = 1$ where downstream (right) rarefaction is attached to shock

$\Rightarrow$ E-G: $\boxed{1}$=$\boxed{2}$ - $\boxed{3}$=$\boxed{4}$ shock

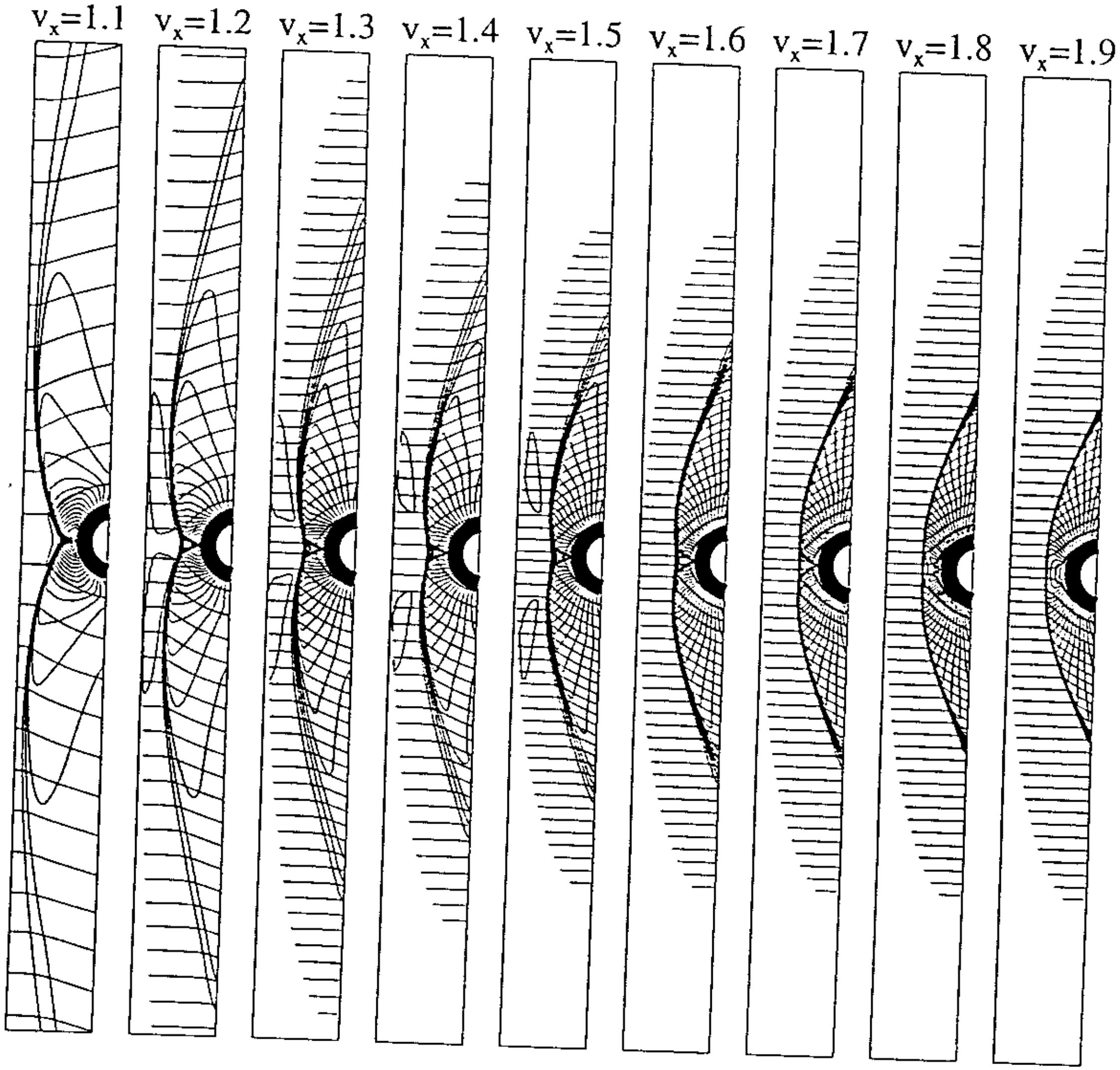
$\Rightarrow$ stationary double compound shock!



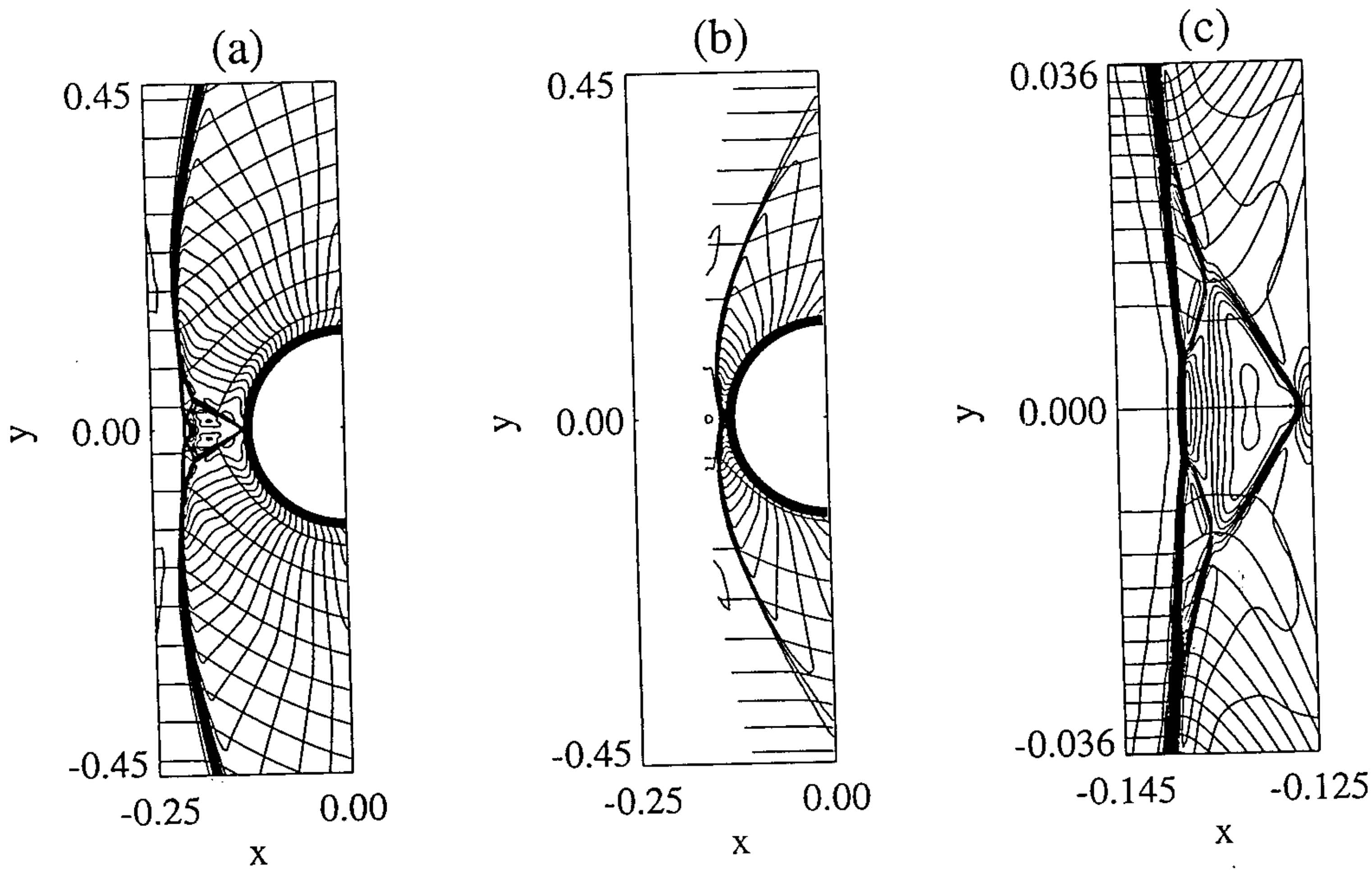
Two families of characteristics near E-G shock (from De Sterck et al. '98)

$\Rightarrow$ characteristic analysis $\Rightarrow$ stationary double compound shock!

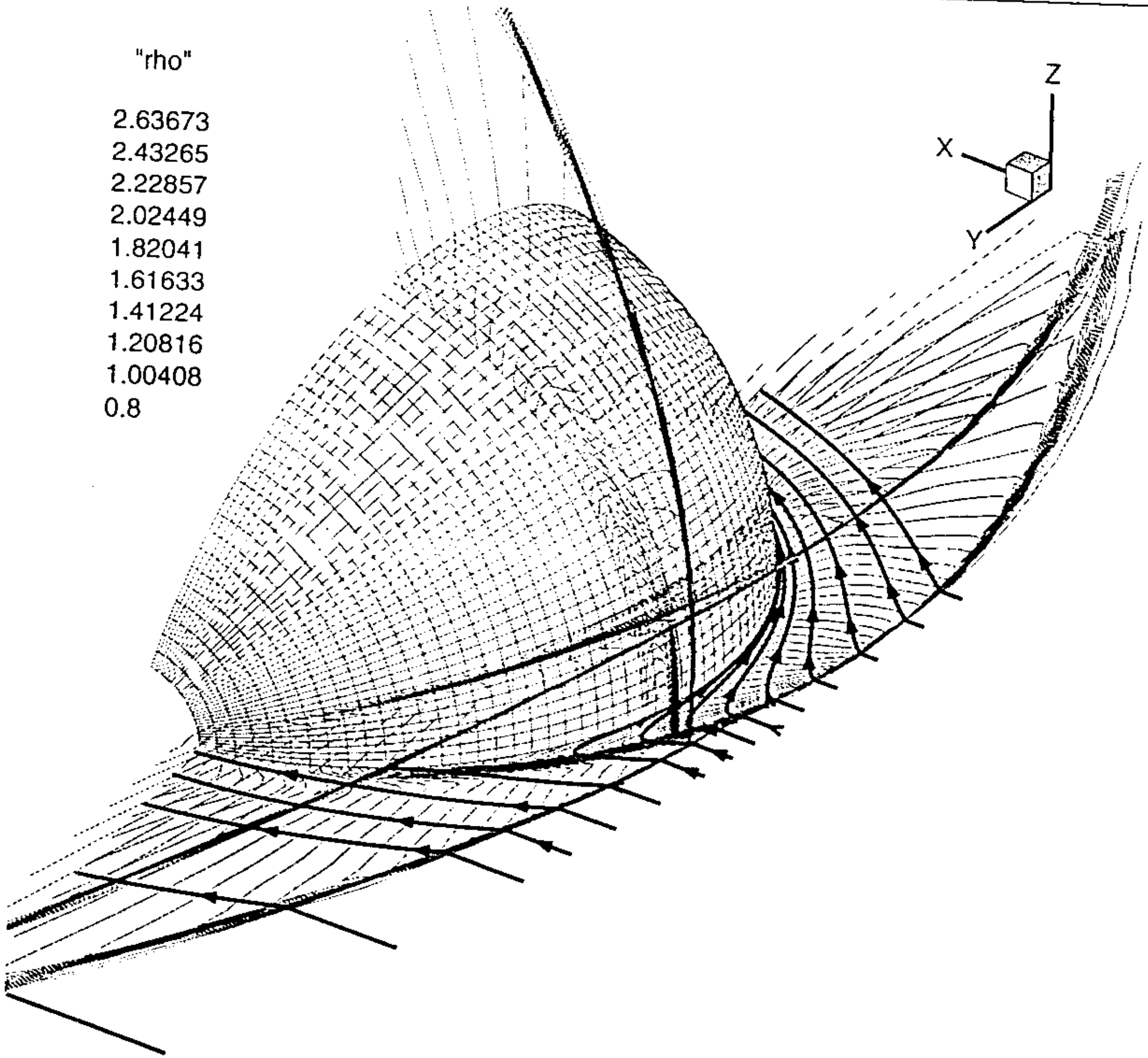
$\Rightarrow$ mathematical equivalent of xt double compound shocks of Myong and Roe ('97)



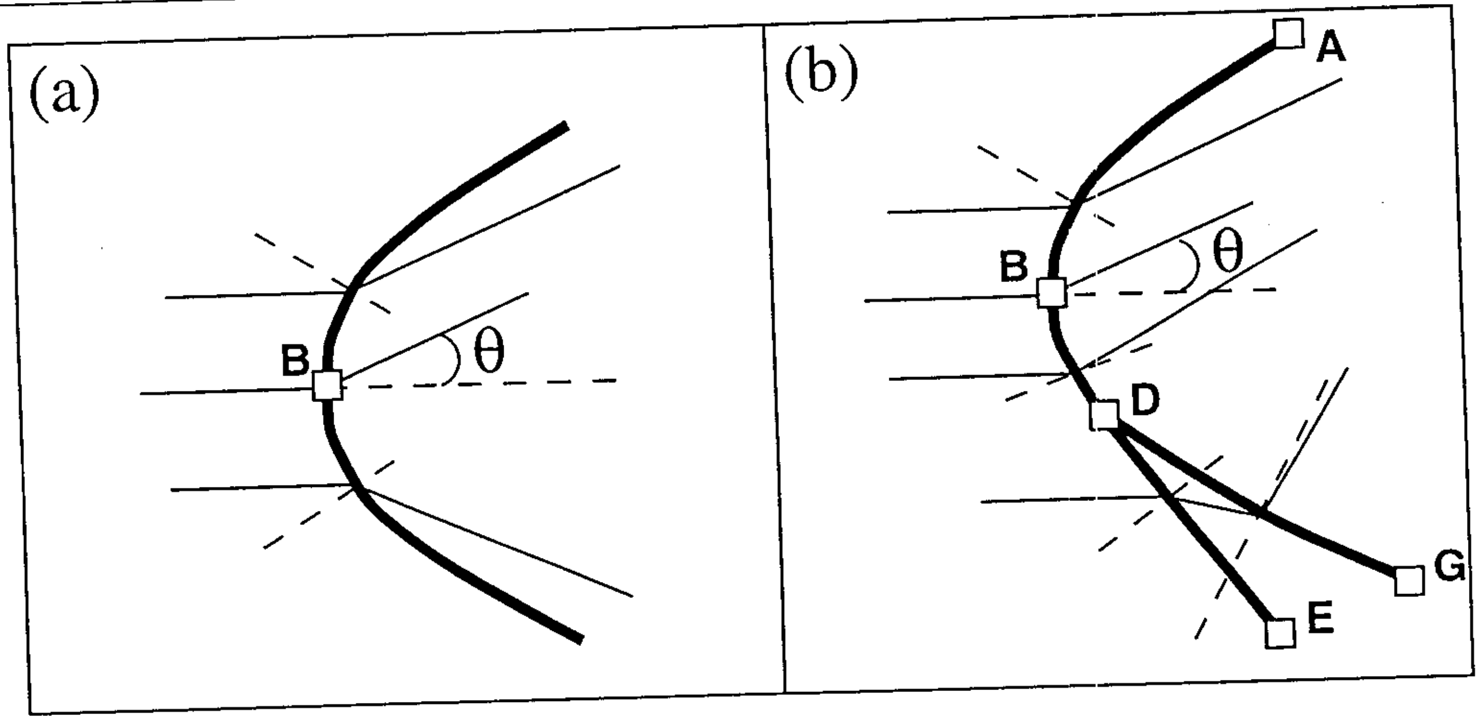
Sweep over parameter domain (from De Sterck et al. '99)



Comparison of 2D flow over cylinder and 3D axisymmetric flow over a sphere (from De Sterck et al. '99)



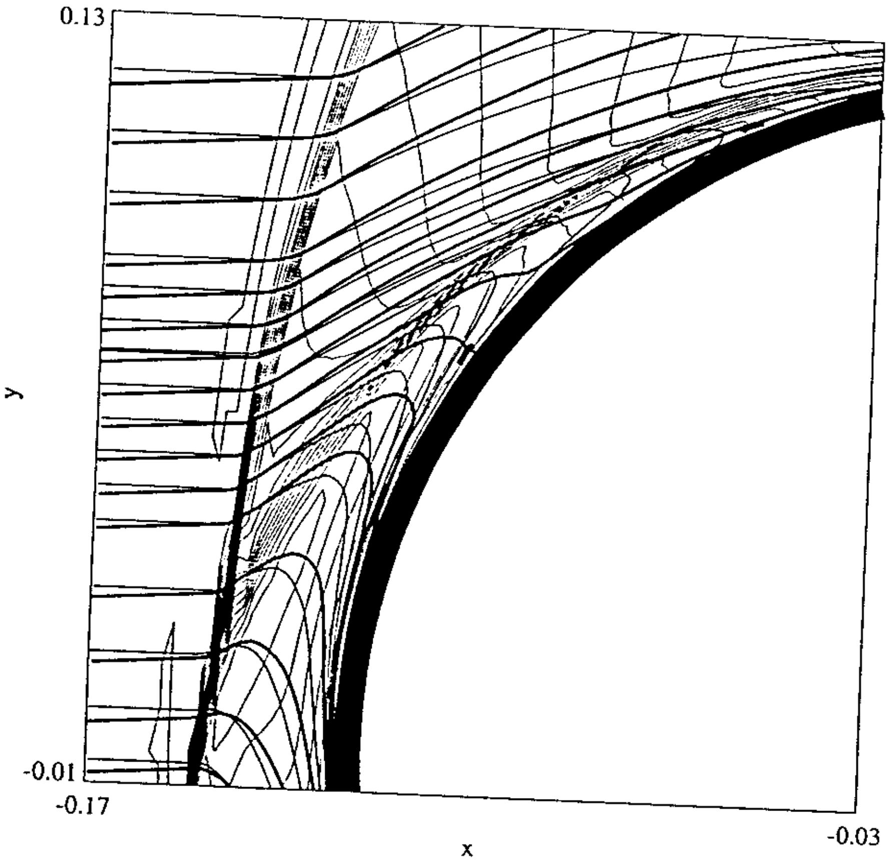
3D flow over a sphere ($\theta_{vB} = 5^\circ$) (from De Sterck et al. '99)



Topology of 3D flow over a sphere (from De Sterck et al. '99)

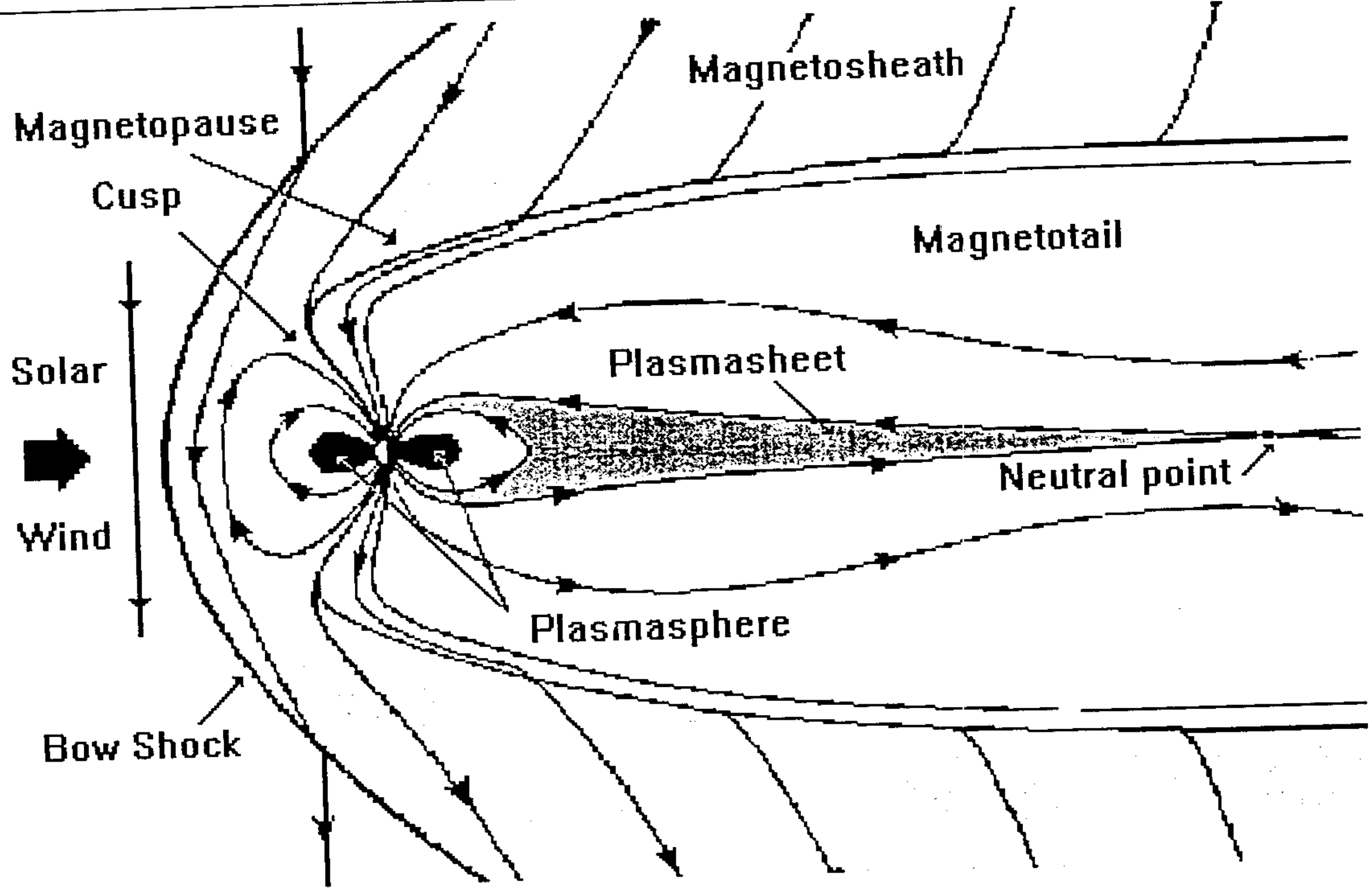
a) shock front can not be entirely of the [1]-[2] fast type near a perpendicular pointwith a [1]-[2=3] switch-on shock

b) complex shock front needed, in case $\theta_{vB} \neq 0$:
A-B is [1]-[2] fast shock, B-D is [1]-[3] intermediate, D-E is [1]-[2]fast, and D-G is [2]-[4] intermediate evolving into [2=3]-[4] switch-off and [3]-[4] slow along the shock front



Detail of 3D flow over a sphere (from De Sterck et al. '99)

⇒ density contours (thin solid) with streamlines (thick solid) and magnetic field lines (thin solid)



Earth's magnetosphere (from De Sterck et al. '98)

⇒ related to SPACE WEATHER!