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Applications of Beltrami Functions in Plasma Physics

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APPLICATIONS OF BELTRAMI FUNCTIONS IN PLASMA PHYSICS

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1 INTRODUCTION

The nonlinear magnetohydrodynamics (MHD) involves a variety of complex phenomena. It is well nigh impossible to construct physically nontrivial theory from a direct analysis of the basic equations. To elucidate a specific phenomenon, we must apply a reduction of the model with appealing to scale separations, singular perturbations, coarse-graining (averaging), etc.

In this paper, we discuss a slow motion (or a steady state) of a low-pressure magnetized plasma. In more specific terms, we consider the following singular limit. The general MHD equations read, in the standard normalized units,

$$\partial_t \boldsymbol{v} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \epsilon_A^{-2}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \beta\nabla p + \epsilon_R\Delta\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v}=0, \tag{1}$$

$$\partial_t \boldsymbol{B} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) - \epsilon_L\nabla\times(\nabla\times\boldsymbol{B}). \tag{2}$$

Unknown variables are the magnetic field $\boldsymbol{B}$, the flow velocity $\boldsymbol{v}$ and the pressure p. The Alfvén number ϵ_A, Lundquist number ϵ_L^{-1}, Reynolds number ϵ_R^{-1}, and the beta ratio β are nondimensional positive parameters. The incompressibility condition ($\nabla\cdot\boldsymbol{v}=0$) may be replaced by an evolution equation for the pressure p in a more sophisticated model.

This system of nonlinear parabolic equations (1)–(2) is a close cousin of the Navier-Stokes system describing neutral fluids (see [1, 2] and papers cited therein). The MHD system includes coupling between the magnetic field and the flow velocity through the nonlinear induction effect and its reciprocal Lorentz force, which adds a considerable complexity to the usual Navier-Stokes system. Surprisingly, however, we observe a more regular and ordered behavior in some MHD systems. Such phenomena are highlighted by a singular perturbation of $\epsilon_A^2 \to 0$, with fixing the time-scale, in the momentum-balance equation (1). This limit is amenable to slow motion of a strongly magnetized low β plasma. The determining equation becomes the force-free condition $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=0$, which is equivalent to the Beltrami condition

$$\nabla\times\boldsymbol{B} = \lambda\boldsymbol{B}. \tag{3}$$

Here λ is a scalar function. By the solenoidal condition ($\nabla \cdot \boldsymbol{B} = 0$) and identity $\nabla \cdot (\nabla \times \boldsymbol{B}) = 0$, taking the divergence of the both sides of (3) yields

$$\boldsymbol{B} \cdot \nabla \lambda = 0. \tag{4}$$

Since (4) means that the function λ should be constant along the streamline (field line) of $\boldsymbol{B}$, analysis of the system of equations (3)–(4) requires integration of the streamline equation

$$\frac{d}{ds}\boldsymbol{x} = \boldsymbol{B}(\boldsymbol{x}). \tag{5}$$

The solenoidal condition ($\nabla \cdot \boldsymbol{B} = 0$) parallels Liouville's theorem for the Hamiltonian flow, and hence one can formulate (5) in a canonical form [3]. For a general three-dimensional $\boldsymbol{B}$, the solution of (5) exhibits chaos. Hence, the general analysis of the system (3)–(4) includes an essential mathematical difficulty. Two special cases, however, can be studied rigorously. One is the case where $\boldsymbol{B}$ has an ignorable coordinate (two-dimensional). Then, (5) becomes integrable, and the system (3)–(4) reduces into a nonlinear elliptic equation [4, 5]. The three-dimensional problem involves the non-integrable streamline problem (5), however, it is decoupled from the Beltrami problem (3)–(4), if we assume a constant λ that make (4) trivial.

The plan of this paper is as follows. In Sec. 2, we give a concise review of the physical background of the Beltrami condition in plasma physics. The constant-λ Beltrami field is considered to be a "ground state" of a turbulent plasma. We define a mathematical problem that characterizes such an equilibrium, and discuss its implication in the "dynamo theory" of astrophysics. Section 3 is devoted to the mathematical analysis of the constant-λ Beltrami field. In Sec. 4, we develop a statistical mechanics of the MHD equilibrium that is amenable to the constant-λ Beltrami condition. Interactions among elements with inhomogeneous λ yield chaotic oscillations. In Sec. 5, we introduce a reduced finite-dimension model of such nonlinear dynamics, and present results of numerical analysis. The reduction from PDE into ODE uses a unique technique based on the singular perturbation, which differs from the usual mode truncation.

2 CONSTANT-LAMBDA BELTRAMI FIELD

The constant-λ condition for the Beltrami field is a strong ansatz based on the following physical reasons. The streamline equation (5) in a three-dimensional magnetic field is generally non-integrable, and hence, we may assume that streamlines (magnetic field-lines) are embedded densely in a volume. Since (4) demands that λ is constant along each field line, it is natural to assume a constant λ over such a volume. The theory of energy relaxation also derives the constant-λ condition. Woltjer [7] pointed out the importance of the magnetic helicity

$$K = \frac{1}{2}\int_{\Omega} \boldsymbol{A} \cdot \boldsymbol{B} d\boldsymbol{x}.$$

Here $\nabla \times \boldsymbol{A} = \boldsymbol{B}$, Ω is the entire volume of the plasma and $d\boldsymbol{x}$ is the volume element. The viscous dissipation does not change the helicity K, while the magnetic energy diminishes toward a "ground state". The magnetic field self-organized through this energy relaxation is characterized by a minimizer of the magnetic energy $W = \int_\Omega B^2 dx/2$ subject to a given helicity. This variational principle reads as $\delta(W - \lambda K) = 0$, where λ is the Lagrange multiplier. The formal Euler-Lagrange equation, under appropriate boundary conditions, is identical to (3). Taylor [8] formulated an equivalent variational principle, however, his model is based on a different hypothesis to justify the preferential conservation of the helicity. The energy dissipation proceeds faster than the change of the helicity, if the resistive dissipation is dominated by spatially concentrated fluctuation currents (see also Hasegawa [9]). Both effects, the viscous dissipation, resulting in ion heating, and the resistive dissipation, resulting in electron heating, were compared for a specific relaxation process [13].

There are many different observations suggesting the creation of constant-λ force-free fields in astrophysical, space and laboratory plasmas. Magnetic flux tubes (flux ropes), in which field lines are twisted, are produced through interactions between the magnetosphere and interplanetary magnetic fields [10]. In a laboratory plasma, detailed measurements of magnetic fields showed that the field produced after self-organization through turbulence is closely approximated by a solution of (3) [8]. Galactic jets are also considered to have similar configurations of magnetic fields [11].

The Beltrami field plays an essential role in the so-called "dynamo theory". To understand the rapid generation of magnetic fields in astrophysical systems, we have to invoke a "fast dynamo action" that has a growth rate of the magnetic energy independent of the resistivity (see [6] and papers cited therein). In a highly conductive plasma the evolution of the magnetic field $\boldsymbol{B}$ obeys Faraday's law (2) with $\epsilon_L \to 0$. A plasma flow $\boldsymbol{v}$ with chaotic streamlines (maps with positive Lyapunov exponents), which may have a large length-scale, bring about complex mixing of magnetic flux, and the length-scale of the inhomogeneity cascades toward a small scale, resulting in amplification of the magnetic field. If the length-scale reduces down to the dissipative range, and the resistive damping becomes comparable to the induction effect, then the magnetic field energy turns to diminish. In this classical picture of the kinematic dynamo, the magnetic field energy accumulates into small scale fluctuations, and the life-time of the amplified magnetic field is limited by the time-scale of the cascade process. To obtain a larger length-scale and a longer life-time of amplified magnetic fields, an appropriate limitation for the scale reduction should occur. The nonlinear effect of the amplified magnetic fields, that is the Lorentz back-reaction, plays an essential role in this "post-kinematic phase". Here we assume that the plasma achieves a quasi-steady state through the energy relaxation process. Then, the momentum balance equation reduces into (3), and the flow $\boldsymbol{v}$ must be chosen in such a way that $\boldsymbol{B}$ satisfies (3) implicitly. The parameter λ characterizes the length-scales of $\boldsymbol{B}$. Hence, the condition (3) imposes a bound for the length-scale of the field, if the magnitude of λ is restricted by some reason. This bound avoids scale reduction down to the resistive regime, and extends the life-time of the amplified magnetic field.

Through the kinematic dynamo process, the current ($\propto \nabla \times \boldsymbol{B}$) tends to concentrate in small volumes, which may be disconnected. When the sectional length-scale of such a volume becomes small enough, the Lorentz force dominates ($\epsilon_A^2 \ll 1$). Let Ω to be such a "clump" of the magnetic field. Its length-scale is denoted by ℓ_c. This Ω may have a complex topology. We want to find a constant-λ Beltrami field in Ω. If the parameter λ can be chosen such that $|\lambda| \leq \lambda_c = O(\ell_c^{-1})$, then equilibration of the clump into such a Beltrami field results in a lower bound for the length-scale. Here we solve the Beltrami condition (3) for a given helicity and an "external magnetic field". The external component of $\boldsymbol{B}$ is defined by decomposing $\boldsymbol{B} = \boldsymbol{B}_\Sigma + \boldsymbol{h}$, where $\nabla \times \boldsymbol{h} = 0$ and $\nabla \cdot \boldsymbol{h} = 0$. This $\boldsymbol{h}$, which represents the magnetic field rooted outside Ω, is assumed to be a given function. Its complement $\boldsymbol{B}_\Sigma$ is the unknown variable. We define the gauge-invariant helicity by

$$\mathcal{K} = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}_\Sigma dx \tag{6}$$

We prove the existence of a solution with $|\lambda| \leq \lambda_c = O(\ell_c^{-1})$ for every $\boldsymbol{h} \neq 0$ and $\mathcal{K}$ in the next section (Theorem3). The nonvanishing $\boldsymbol{h}$ plays the role of symmetry breaking.

3 EXISTENCE THEOREM AND COMPLETENESS THEOREM

The constant-λ Beltrami condition (3) is regarded as an eigenvalue problem with respect to the curl operator. Interestingly, the topology of the domain plays an essential role in this eigenvalue problem.

To study the spectrum the curl derivatives, we need the fundamental theory of vector function spaces. Let Ω ($\subset \boldsymbol{R}^3$) be a bounded domain with a smooth boundary $\partial\Omega = \cup_{i=1}^n \Gamma_i$ (Γ_i is a connected surface). We consider cuts of the domain Ω. Let $\Sigma_1, \cdots, \Sigma_m$ ($m \geq 0$) be cuts such that $\Sigma_i \cap \Sigma_j = \emptyset$ ($i \neq j$), and such that $\Omega \setminus (\cup_{i=1}^m \Sigma_i)$ becomes a simply connected domain. The number m of such cuts is the first Betti number of Ω. When $m > 0$, we define the flux through each cut by

$$\Phi_{\Sigma_i}(\boldsymbol{u}) = \int_{\Sigma_i} \boldsymbol{n} \cdot \boldsymbol{u} ds \qquad (i = 1, 2, \cdots, m),$$

where $\boldsymbol{n}$ is the unit normal vector on Σ_i with an appropriate orientation. By Gauss's formula, $\Phi_{\Sigma_i}(\boldsymbol{u})$ is independent of the place of the cut Σ_i, if $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ on $\partial\Omega$.

We denote $L^2(\Omega)$ the Lebesgue space of square-integrable (complex) vector fields in Ω, which is endowed with the standard innerproduct $(\boldsymbol{a}, \boldsymbol{b})$. We define the following subspaces of $L^2(\Omega)$;

$$\begin{aligned}
L^2_\Sigma(\Omega) &= \{\boldsymbol{w};\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{w} = 0 \text{ on } \partial\Omega,\ \Phi_{\Sigma_i}(\boldsymbol{w}) = 0\ (i = 1, \cdots, m)\}, \\
L^2_H(\Omega) &= \{\boldsymbol{h};\ \nabla \cdot \boldsymbol{h} = 0, \nabla \times \boldsymbol{h} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{h} = 0 \text{ on } \partial\Omega\}, \\
L^2_G(\Omega) &= \{\nabla\phi;\ \Delta\phi = 0 \text{ in } \Omega\}, \\
L^2_F(\Omega) &= \{\nabla\phi;\ \phi = c_i\ (\in \mathbf{C}) \text{ on } \Gamma_i\ (i = 1, \cdots, n)\}
\end{aligned}$$

We have an orthogonal decomposition [15]

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega) \oplus L^2_G(\Omega) \oplus L^2_F(\Omega).$$

The space of solenoidal vector fields with vanishing normal component on $\partial\Omega$ is

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega).$$

The subspace $L^2_H(\Omega)$ corresponds to the cohomology class, whose member is a harmonic vector field and $\dim L^2_H(\Omega) = m$ (the first Betti number of Ω). When Ω is simply connected, then $m = 0$ and $L^2_H(\Omega) = \emptyset$. We have the following expression

$$L^2_\Sigma(\Omega) = \{\nabla \times \boldsymbol{w};\ \boldsymbol{w} \in H^1(\Omega),\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega,\ \boldsymbol{n} \times \boldsymbol{w} = 0 \text{ on } \partial\Omega\}.$$

This implies that a member of $L^2_\Sigma(\Omega)$ can be expressed as the curl of a vector potential with the boundary condition $\boldsymbol{n} \times \boldsymbol{w} = 0$. We note that a member of $L^2_\sigma(\Omega)$ may not allow such an expression. We also note

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus \{\nabla\phi;\ \phi \in H^1(\Omega)\},$$

which implies that $L^2_\sigma(\Omega)$ is the orthogonal complement of the space of *potential flows*. This relation is called the Weyl decomposition. The gauge-invariance of the helicity $\mathcal{K}$ defined by (6) follows from this orthogonality.

We now study the spectra of the curl operator. A key step of the theory is finding a self-adjoint curl operator [14]. We have the following theorem [14].

Theorem 1 *Let $\Omega \subset R^3$ be a smoothly bounded domain. We define a curl operator $\mathcal{S}$ in the Hilbert space $L^2_\Sigma(\Omega)$ by*

$$\mathcal{S}\boldsymbol{u} = \nabla \times \boldsymbol{u}, \quad D(\mathcal{S}) = \{\boldsymbol{u} \in L^2_\Sigma(\Omega)\ ;\ \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)\}.$$

Then $\mathcal{S}$ is a self-adjoint operator. The spectrum of $\mathcal{S}$ consists of only point spectra $\sigma_p(\mathcal{S})$, which is a discrete set of real numbers.

If Ω is multiply connected ($m > 0$), it is interesting, both mathematically and physically, to extend the domain and range of the curl operator to the space $L^2_\sigma(\Omega)$ [14]. As an intermediate step, we consider another curl operator

$$\mathcal{T}\boldsymbol{u} = \nabla \times \boldsymbol{u}, \quad D(\mathcal{T}) = \{\boldsymbol{u} \in L^2_\Sigma(\Omega)\ ;\ \nabla \times \boldsymbol{u} \in L^2_\sigma(\Omega)\}.$$

Lemma 1 *For every $\lambda \in \mathbf{C} \setminus \sigma_p(\mathcal{S})$ and for every $\boldsymbol{f} \in L^2_\sigma(\Omega)$, the equation*

$$(\mathcal{T} - \lambda)\boldsymbol{u} = \boldsymbol{f} \tag{7}$$

has a solution.

(proof) First we show the existence of $\mathcal{T}^{-1}$, i.e., for $\boldsymbol{f} \in L^2_\sigma(\Omega)$ we solve $\mathcal{T}\boldsymbol{u} = \boldsymbol{f}$. Let $\tilde{\boldsymbol{f}}$ be the 0-extension of $\boldsymbol{f}$ in $\mathbf{R}^3$, i.e.

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$$

By $f \in L^2_\sigma(\Omega)$, one observes $\nabla \cdot \tilde{f} = 0$ in $\mathbf{R}^3$. We denote by $(-\Delta)^{-1}$ the vector Newtonian potential. We define $w_0 = \nabla \times [(-\Delta)^{-1}\tilde{f}]$ in Ω. We denote by P_Σ the orthogonal projection in $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$, and define $u_0 = P_\Sigma w_0$. Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(curl), we observe

$$\nabla \times u_0 = \nabla \times w_0 = \nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{f}]\}.$$

Since $\nabla \cdot [(-\Delta)^{-1}\tilde{f}] = 0$, we obtain $\nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{f}]\} = -\Delta[(-\Delta)^{-1}\tilde{f}] = \tilde{f}$. We thus have a solution $\mathcal{T}^{-1}f = \nabla \times u_0$.

Next we solve $(\mathcal{T} - \lambda)u = f$, for $f \in L^2_\sigma(\Omega)$. We decompose $f = g + h$ with $g = P_\Sigma f$ and $h \in L^2_H(\Omega)$. Let $u_0 = \mathcal{T}^{-1}h$ and $w = u - u_0$. Then (7) reads

$$(T - \lambda)w = g + \lambda u_0 \quad \in L^2_\Sigma(\Omega). \tag{8}$$

For $\lambda \notin \sigma_p(\mathcal{S})$, we may define $w = (\mathcal{S} - \lambda)^{-1}(g + \lambda u_0) \in D(\mathcal{S})$, which solves (8). In summary, we have a solution of (7)

$$u = \mathcal{T}^{-1}h + (\mathcal{S} - \lambda)^{-1}(P_\Sigma f + \lambda \mathcal{T}^{-1}h).$$

(Q.E.D.)

Theorem 2 *In $L^2_\sigma(\Omega)$ we define a curl operator $\tilde{\mathcal{S}}$ by*

$$\tilde{\mathcal{S}} = \nabla \times u, \quad D(\tilde{\mathcal{S}}) = \{u \in L^2_\sigma(\Omega)\ ;\ \nabla \times u \in L^2_\sigma(\Omega)\}.$$

(i) When $\dim L^2_H(\Omega) = 0$, i.e. if Ω is simply connected, then $\tilde{\mathcal{S}} \equiv \mathcal{S}$, and hence, the spectrum $\sigma(\tilde{\mathcal{S}}) = \sigma_p(\tilde{\mathcal{S}})$.

(ii) When $\dim L^2_H(\Omega) > 0$, i.e. if Ω is multiply connected, then $\tilde{\mathcal{S}}$ is an extension of $\mathcal{S}$. The spectrum $\sigma(\tilde{\mathcal{S}})$ consists of only spectra $\sigma_p(\tilde{\mathcal{S}})$, and $\sigma_p(\tilde{\mathcal{S}}) = \mathbf{C}$. Hence, for every $\lambda \in \mathbf{C}$,

$$(\tilde{\mathcal{S}} - \lambda)u = 0 \tag{9}$$

has a nontrivial solution.

(proof) The first part is straightforward. We prove the second part. For $\lambda \in \sigma_p(\mathcal{S})$, this has a solution as shown in Theorem 1. We assume $\lambda \notin \sigma_p(\mathcal{S})$. For a given $h \in L^2_H(\Omega)$, the equation

$$(\mathcal{T} - \lambda)v = \lambda h$$

has a solution (Lemma 1). Then, the function $u = v + h$ $[\in L^2_\sigma(\Omega) \cap H^1(\Omega)]$ solves (9).

(Q.E.D.)

Theorem 2 proves the general existence of the constant-λ Beltrami function for every $\lambda \in \mathbf{C}$, if Ω is multiply connected. In the next theorem, we solve the constant-λ Beltrami equation (3) for a given helicity $\mathcal{K}$ and harmonic field $\boldsymbol{h} \in L^2_H(\Omega)$. Now λ is an unknown variable. This problem is related with the magnetic clump discussed in Sec. 2.

We assume that Ω is multiply connected. Let $\{\varphi_j\}$ be the complete set of the eigenfunctions of the self-adjoint curl operator $\mathcal{S}$ (Theorem 1). The corresponding eigenvalues are numbered as

$$\cdots \le \mu_{-2} \le \mu_{-1} < 0 < \mu_1 \le \mu_2 \le \cdots. \tag{10}$$

For every $\boldsymbol{B} \in L^2_\sigma(\Omega)$, we have an orthogonal-sum expansion

$$\boldsymbol{B}(\boldsymbol{x},t) = \sum_j c_j(t)\varphi_j(\boldsymbol{x}) + \boldsymbol{h}(\boldsymbol{x},t), \tag{11}$$

where $\boldsymbol{h} \in L^2_H(\Omega)$. The harmonic field $\boldsymbol{h}$ is a given function, which plays an important role of "symmetry breaking" in the following discussion. The first summation in the right-hand side of (11) is denoted by $\boldsymbol{B}_\Sigma$. The energy of $\boldsymbol{B}$ is given by

$$W = \frac{1}{2}\sum_j c_j^2 + \frac{1}{2}\|\boldsymbol{h}\|^2. \tag{12}$$

There exists $\boldsymbol{g}$ such that $\boldsymbol{h} = \nabla \times \boldsymbol{g}$ (Lemma 1). The vector potential of $\boldsymbol{B}$ is given by

$$\boldsymbol{A} = \sum_j \frac{c_j}{\mu_j}\varphi_j + \boldsymbol{g}. \tag{13}$$

Denoting $D_j = (\varphi_j, \boldsymbol{g})$, the gauge invariant helicity (6) becomes

$$\mathcal{K} = \frac{1}{2}(\boldsymbol{A}, \boldsymbol{B}_\Sigma) = \frac{1}{2}\sum_j (\frac{c_j^2}{\mu_j} + D_j c_j). \tag{14}$$

For given $\mathcal{K}$ and $\boldsymbol{h}$, we can solve (3) by the variational principle $\delta(W - \lambda\mathcal{K}) = 0$, and obtain

$$c_j = \frac{\lambda\mu_j}{2(\mu_j - \lambda)} D_j \quad (\forall j). \tag{15}$$

The energy and the helicity become

$$W = \sum_j \frac{\lambda^2\mu_j^2}{8(\mu_j - \lambda)^2} D_j^2 + \frac{1}{2}\|\boldsymbol{h}\|^2, \qquad \mathcal{K} = \sum_j \frac{\lambda\mu_j(2\mu_j - \lambda)}{8(\mu_j - \lambda)^2} D_j^2. \tag{16}$$

We can show that $\mathcal{K}$ is a monotone function of λ in the range of $\mu_{-1} < \lambda < \mu_1$ (see definition (10)), if $D_j \neq 0$ $(\exists j)$, viz., if we have a "symmetry breaking" $\boldsymbol{h} \neq 0$. For every $\kappa \in \mathbf{R}$, the equation $\mathcal{K}(\lambda) = \kappa$ has a unique solution in this range of λ. Now we have the following theorem.

Theorem 3 *Let Ω ($\subset \mathbf{R}^3$) be a multiply connected bounded domain. Assume that $\boldsymbol{h}$ ($\in L^2_H(\Omega)$) is finite. For every κ ($\in \mathbf{R}$), the Beltrami condition (3) has a unique solution $\boldsymbol{B}$ such that its helicity $\mathcal{K} = \kappa$, and λ such that $\mu_{-1} < \lambda < \mu_1$.*

4 STATISTICAL EQUILIBRIUM

The complexity of the nonlinear dissipative dynamics of a plasma invokes a paradigm shift to a "coarse-grained" model. Using the phenomenological variational principle $\delta(W - \lambda\mathcal{K}) = 0$ (Sec. 2), we develop a statistical mechanical model that reproduces the constant-λ Beltrami field at the "zero *temperature* limit". A finite *temperature* (in the sense of MHD fluctuation) equilibrium includes fluctuations. The statistical theory predicts the spectra of macroscopic physical quantities such as the energy, helicity, etc.

A key step is to find an invariant measure of the temporal evolution equation. It corresponds to Liouville's theorem in the Hamiltonian dynamics. Montgomery *et al.* [16] used the "Chandrasekhar-Kendall functions", which are the eigenfunctions of the curl in a cylindrical geometry [17], to expand the solenoidal vector fields $\boldsymbol{B}$ and $\boldsymbol{v}$, and defined an infinite-dimensional phase space spanned by the expansion coefficients. The formal Lebesgue measure is shown to be invariant against the nonlinear ideal ($\epsilon_R, \epsilon_L \to \infty$) dynamics. The completeness theorem of the eigenfunctions (Theorem 1) gave a mathematical justification of the expansion, and generalized the Hilbert-space approach for an arbitrary geometry. An important development in recent work [18] is the treatment of the harmonic magnetic field, which brings about a symmetry breaking associated with a topological constraint. When we consider a multiply connected domain, the harmonic magnetic fields, which are rooted outside the domain, are represented by the cohomology class. If we impose the ideal conducting boundary conditions, these harmonic fields are invariant. The rest orthogonal complement spans the dynamical phase space. The invariant harmonic component plays the role of an externally applied symmetry breaking. Interestingly, this term yields "power-law spectra" of the energy, helicity and helicity fluctuation.

In this section, we give a brief sketch of the statistical mechanics of MHD.

Proposition 1 (Invariant Measure) *Let $\boldsymbol{v}(x,t)$ be a smooth vector field in Ω. Suppose that $\boldsymbol{B}(x,t)$ obeys*

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \quad \text{in } \Omega, \tag{17}$$

$$\boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \quad \text{on } \partial\Omega. \tag{18}$$

Using the eigenfunctions of the curl operator φ_j and the harmonic field $\boldsymbol{h}_\ell$, we write (cf. (11))

$$\boldsymbol{B}(x,t) = \sum_j c_j(t)\varphi_j(x) + \sum_{\ell=1}^{m} \check{c}_\ell(t)\boldsymbol{h}_\ell(x). \tag{19}$$

Then, $dC = d\check{c}_1 \cdots d\check{c}_m \prod_j dc_j$ is an invariant measure.

(proof) By the boundary condition (18), we observe $d\check{c}_\ell/dt = 0$ ($\forall \ell$). Using (17) and (18), we obtain

$$\begin{aligned} \frac{d}{dt}c_j &= (\nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \varphi_j) = (\boldsymbol{v} \times \boldsymbol{B}, \nabla \times \varphi_j) = \lambda_j(\boldsymbol{v} \times \boldsymbol{B}, \varphi_j) \\ &= \lambda_j \left[\sum_k c_k(\boldsymbol{v} \times \varphi_k, \varphi_j) + \sum_{\ell=1}^{m} \check{c}_\ell(\boldsymbol{v} \times \boldsymbol{h}_\ell, \varphi_j) \right]. \end{aligned} \tag{20}$$

Since $(\boldsymbol{v} \times \boldsymbol{\varphi}_j) \cdot \boldsymbol{\varphi}_j \equiv 0$, we find $\partial(dc_j/dt)/\partial c_j = 0 \quad (\forall j)$. Hence the measure dC is invariant.

(Q.E.D.)

The ansatz of the variational principle $\delta(W - \lambda\mathcal{K}) = 0$ suggests that two additive quantities W and $\mathcal{K}$ are the relevant state variables that characterize the statistical equilibrium. The possible ensemble consistent with this variational principle is the Boltzmann distribution

$$P(W, \mathcal{K}) \propto \exp[-\beta(W - \lambda\mathcal{K})] \tag{21}$$

where β is interpreted as an inverse temperature of the magnetic field. The helicity and the energy of each mode is $(c_j^2/\mu_j + D_j c_j)/2$ and $c_j^2/2$, respectively. The Boltzmann distribution for the amplitude c_j is

$$P_j \propto \exp\left[-\frac{\beta}{2}\left(c_j^2 - \frac{\lambda}{\mu_j}c_j^2 - \lambda D_j c_j\right)\right]. \tag{22}$$

The ensemble averages of W and $\mathcal{K}$ over the phase space become

$$\langle W \rangle = \sum_j \left[\frac{\mu_j}{4\beta(\mu_j - \lambda)} + \frac{\lambda^2 \mu_j^2}{8(\mu_j - \lambda)^2} D_j^2\right], \tag{23}$$

$$\langle \mathcal{K} \rangle = \sum_j \left[\frac{1}{4\beta(\mu_j - \lambda)} + \frac{\lambda \mu_j (2\mu_j - \lambda)}{8(\mu_j - \lambda)^2} D_j^2\right]. \tag{24}$$

These results are compared with (16). The first term of the right-hand side of (23) and that of (24) are the contributions of the fluctuations. In (23), the energy of the harmonic field, which is constant here, is omitted. This classical statistical model suffers from the Rayleigh-Jeans catastrophe, viz., when we pass the limit of the infinite summation over the all modes, the fluctuation terms diverge. To avoid this divergence, we can appeal to the Bose-Einstein statistics with second-quantizing the mode amplitude c_j and defining bosons MHD fluctuations [18].

5 REDUCED MODEL OF WEAK INTERACTIONS AND CHAOS

In this section, we consider nonlinear interactions among plasma elements with inhomogeneous λ. Each element satisfies the constant-λ Beltrami condition (3). Different elements are separated by a thin layer where the magnetic field lines are weakly chaotic. Hence, the connection lengths among different elements are considerably long. If we consider a small deviation from the Beltrami condition (3), and write

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B} + \boldsymbol{\alpha}, \tag{25}$$

then (4) receives a small correction and becomes

$$(\boldsymbol{B} \cdot \nabla)\lambda = -\nabla \cdot \boldsymbol{\alpha}. \tag{26}$$

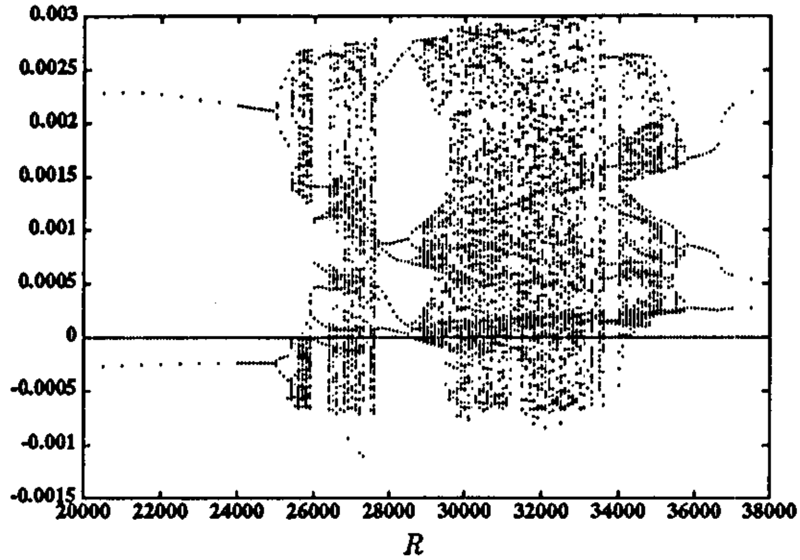

Figure 1: Feigenbaum diagram for R in the range of $2.0 \times 10^4 - 3.8 \times 10^4$.

Integrating (26) along a field line, we obtain a finite inhomogeneity in λ after a long distance. This allowas us to assume inhomogeneous λ in the following discussion. We consider a cylindrical plasma with radial inhomogeneity.

We introduce a reduced ODE system. The basic idea of the reduction is the use of the Beltrami condition ($\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$) to convert the spatial derivatives in the PDE system (1)–(2) into the multiplying of λ. We note that this procedure differs from the Fourier decomposition and truncation, which are usually used to derive reduced models in different problems. For each mode of Fourier decomposition, we can replace derivatives by multiplying of wavenumbers. In the present method, the conversion applies to the exact function, not to expansion modes. The λ is a dynamical variable to be determined by the evolutions equation. Here we invoke a quasilinear turbulence model of MHD fluctuations (see [19, 20, 22] and papers cited therein).

Magnetic field $\boldsymbol{B}$ is decomposed into the fluctuating component $\boldsymbol{b}$ and the ambient component $\boldsymbol{B}_0$. For the plasma velocity $\boldsymbol{v}$, we also assume two components; One the a uniform flow $\boldsymbol{V}$ and the other is the fluctuation $\tilde{\boldsymbol{v}}$ driven by the MHD instabilities. Assuming a quasilinear turbulence of resistive instabilities, we may write the ensemble average of the nonlinear term $\langle \tilde{\boldsymbol{v}} \times \boldsymbol{b} \rangle$ in terms of the growth rate of the instability, the energy of fluctuations, and some geometric factors. The parallel (with respect to the mean magnetic field $\boldsymbol{B}_0$) component of $\langle \tilde{\boldsymbol{v}} \times \boldsymbol{b} \rangle$ makes an essential contribution, which is denoted by $-E_{\parallel}^{(2)}$. The quasilinear turbulence theory [20] yields

$$E_{\parallel}^{(2)} = -\nabla \cdot (\eta^{(2)} \nabla j_{\parallel,0}),$$

where $\eta^{(2)}$ is the *hyper resistivity* given by

$$\frac{\eta^{(2)}}{\mu_0} = \frac{1}{2}\sum_k \frac{\gamma_k}{(\partial_r k_\parallel)^2}|b_{r,k}|^2 \ln\left(\frac{r_k^2}{(r-r_k)^2 + \gamma_k^2/(\partial_r k_\parallel)^2}\right). \tag{27}$$

Here the subscript k indicates a Fourier component of the fluctuation, γ_k is the growth rate and $k_\parallel$ is the parallel wavenumber with $k_\parallel(r_k) = 0$. The ensemble average of Faraday's law (2) becomes

$$\partial_t \boldsymbol{b} = -\epsilon_L \nabla\times(\nabla\times\boldsymbol{b}) + \nabla\times(\boldsymbol{V}\times\boldsymbol{b} - E_\parallel^{(2)}\nabla z). \tag{28}$$

We assume $\boldsymbol{B}_0 = \nabla z$. The Beltrami condition reads $\nabla\times\boldsymbol{b} = \lambda(\boldsymbol{b}+\boldsymbol{B}_0)$. We obtain

$$\begin{aligned}\nabla\times(\nabla\times\boldsymbol{b}) &= \lambda^2\boldsymbol{b} + \nabla\lambda\times(\boldsymbol{b}+\boldsymbol{B}_0),\\ \nabla\times(\boldsymbol{V}\times\boldsymbol{b}) &\approx -(\nabla\times\boldsymbol{b})\times\boldsymbol{V} = -\lambda\boldsymbol{b}\times\boldsymbol{V}.\end{aligned}$$

We consider a low pressure plasma, so that the parallel component of $\boldsymbol{b}$ is neglected. In the cylindrical coordinates such that $\nabla\varphi\times\nabla r = \nabla z$ (φ and r are the angle and radial coordinates, respectively), we write $p = b_r$ and $q = b_\varphi$. Inhomogeneity of the plasma is assumed in the direction of ∇r. We may write the φ and r components of (28) as

$$\partial_t p = -\epsilon_L\lambda^2 p - V\lambda q, \tag{29}$$

$$\partial_t q = -\epsilon_L\lambda^2 q + V\lambda p - \partial_r E_\parallel^{(2)} \tag{30}$$

The last term in the right hand side of equation (30) represents the effect of the MHD fluctuations. We can write

$$\frac{\partial E_\parallel^{(2)}}{\partial r} = -\frac{2}{\mu_0}\frac{\partial\eta^{(2)}}{\partial r}\frac{\partial^2\lambda}{\partial r^2}.$$

Two unknown variables p and q represent the amplitude of the magnetic perturbation $\boldsymbol{b}$. The third unknown variable λ characterizes the "curl" of $\boldsymbol{b}$. The equation obeyed by λ, which is also derived from (28), becomes [22]

$$\partial_t\lambda = C\langle p\rangle^2\lambda^2\frac{\partial^2\lambda}{\partial r^2} + E_h\lambda^2,$$

where E_h is the external driving electric field.

We note that the factor $\partial^2\lambda/\partial r^2$ represents the diffusion of the helicity induced by the MHD fluctuations. We discretize the radial coordinate into points where some different instabilities are resonant. Replacing the spatial derivatives of the helicity diffusion by difference quotients, we obtain the reduced model equations

$$\dot{p}_n = -\epsilon_L\lambda_n^2 p_n - V\lambda_n q_n, \tag{31}$$

$$\dot{q}_n = -\epsilon_L\lambda_n^2 q_n + V\lambda_n p_n + Q_n(p_{n-1}, p_{n+1}, \lambda_{n-1}, \lambda_n, \lambda_{n+1}), \tag{32}$$

$$\dot{\lambda}_n = \lambda_n^2\left(L_n(p_n, \lambda_{n-1}, \lambda_n, \lambda_{n+1}) + E_h\right), \tag{33}$$

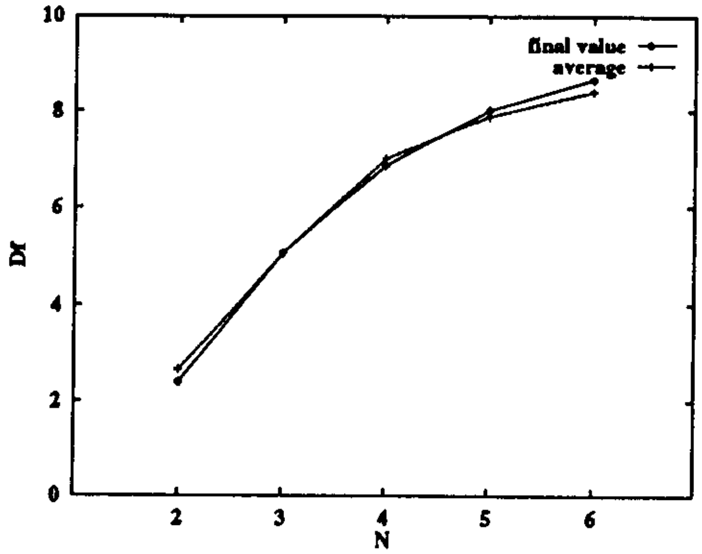

Figure 2: Dependence of the Ljapunov dimension on N.

where n $(= 1, 2, \cdots N)$ indicates the n^{th} radial position,

$$\begin{aligned} Q_n(p_{n-1}, p_{n+1}, \lambda_{n-1}, \lambda_n, \lambda_{n+1}) &= 2C\frac{(p_{n+1})^2 - (p_{n-1})^2}{\Delta}\frac{\lambda_{n+1} + \lambda_{n-1} - 2\lambda_n}{\Delta^2}, \\ L_n(p_n, \lambda_{n-1}, \lambda_n, \lambda_{n+1}) &= C(p_n)^2\frac{\lambda_{n+1} + \lambda_{n-1} - 2\lambda_n}{\Delta^2}, \end{aligned}$$

Δ is a radial distance between the radial grid points, and N is the number of interacting islands with different helicities.

The model equations (31)-(33) generate chaotic orbits of the solution. The typical parameters are

$$\begin{aligned} \lambda &= 10^{-1} \sim 1, \quad |p|,\ |q| = 10^{-4} \sim 10^{-2}, \quad \epsilon_L = 10^{-6} \sim 10^{-5}, \\ C &= 10^{-1} \sim 1, \quad \Delta \sim 10^{-1}, \quad V = 10^{-5} \sim 10^{-3}, \quad E_h = 10^{-8} \sim 10^{-7}. \end{aligned}$$

Changing the Lundquist number $R = \epsilon_L^{-1}$ (magnetic Reynolds number), we observe bifurcation and inverse cascade in the chaotic behavior of the solution. Figure 1 shows the Feigenbaum diagram, where we plot peak points of the time series with changing R. Here we fix other parameters as $N = 3$, $C = 1.0$, $\Delta = 0.1$, $V = 7.72 \times 10^{-5}$ and $E_h = 1.0 \times 10^{-8}$. We observe two branches bifurcate into chaos. For a larger R, the chaos quenches and periodic attractor appears.

The total number N of modes defines the freedom of the model equations, which is $3 \times N$. Figure 2 shows the dependence of Ljapunov dimension D_f on N.

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Part II
Simultaneous Beltrami Conditions in Coupled Vortex Dynamics

1 INTRODUCTION

The Beltrami condition, an expression of the alignment of a vorticity with its flow, describes the simplest and perhaps the most fundamental equilibrium state in a vortex dynamics system (Sec. II). The resulting Beltrami fields constitute a null set for the generator of the evolution equation describing the vortex dynamics. It is also believed that the Beltrami fields are accessible and robust in the sense that they emerge as the nonlinear dynamics of vortices tends to self-organize the system through a weakly dissipative process.

The simplest example of a Beltrami condition is provided by a three dimensional solenoidal field (flow) $\boldsymbol{u}$ obeying

$$\begin{cases} \nabla \times \boldsymbol{u} = \lambda \boldsymbol{u} & (\text{in } \Omega), \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 & (\text{on } \partial\Omega), \end{cases} \tag{1}$$

where λ is a real (or complex) constant number, Ω ($\subset \mathbf{R}^3$) is a bounded domain with a smooth boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit normal vector onto $\partial\Omega$. This system of linear equations is regarded as an eigenvalue problem with respect to the curl operator. The spectral theory of the curl operator reveals an interesting relation of this problem with the cohomology theory [1]. We have the following theorem.

(i) If Ω is simply connected, then (1) has a non-zero solution for special λ included in a set of discrete real numbers; these numbers represent the point spectrum of the self-adjoint part of the curl operator.

(ii) If Ω is multiply connected, then (1) has a non-zero solution for every $\lambda \in \mathbf{C}$ [2].

The aim of this paper is to generalize this theory for "coupled" (or higher-order) Beltrami conditions [3] that describe structures far richer than the ones contained in the single curl Beltrami equation (1). In an ideal plasma, the coupling between the magnetic field and the plasma flow yields the "double curl Beltrami equation"

$$\begin{cases} \nabla \times (\nabla \times \boldsymbol{u}) + \alpha \nabla \times \boldsymbol{u} + \beta \boldsymbol{u} = 0 & (\text{in } \Omega), \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{n} \cdot (\nabla \times \boldsymbol{u}) = 0 & (\text{on } \partial\Omega), \end{cases} \tag{2}$$

where $\boldsymbol{u}$ is either the magnetic field or the flow velocity of the plasma (Sec. III). Applying the spectral theory of the curl operator, we will show that (2) has a non-zero solution for arbitrary complex numbers α and β, if the domain Ω is multiply connected (Sec. IV). The method of present theory applies for general multi-curl Beltrami equations obtained from simultaneous Beltrami conditions in coupled systems.

2 VORTEX DYNAMICS AND BELTRAMI CONDITIOS

We start with reviewing the prototype equation for vortex dynamics. Let ω be a three-dimensional vector field representing a certain vorticity (contravariant vector field) in $\mathbf{R}^3$. We consider an incompressible flow $\boldsymbol{U}$ that transports ω. When the circulation associated with the vorticity is conserved everywhere, this ω obeys the equation

$$\frac{\partial}{\partial t}\omega - \nabla \times (\boldsymbol{U} \times \omega) = 0. \tag{3}$$

In $\mathbf{R}^2$, the vorticity becomes a pseudo-scalar field ω, and the vortex dynamics equation can be cast in the form of a Liouville equation

$$\frac{\partial}{\partial t}\omega + \{\phi, \omega\} = 0, \tag{4}$$

where ϕ is the Hamiltonian of an incompressible flow and $\{\ ,\ \}$ is the Poisson bracket, i.e.,

$$\boldsymbol{U} = \begin{pmatrix} \partial\phi/\partial y \\ -\partial\phi/\partial x \end{pmatrix}, \quad \{\phi, \omega\} = \frac{\partial\phi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\phi}{\partial x}\frac{\partial\omega}{\partial y}. \tag{5}$$

The Beltrami condition with respect to (3) is

$$\boldsymbol{U} = \mu\omega, \tag{6}$$

where μ is a certain scalar function. This condition assures the vanishing of the generator of the vortex dynamics equation (3). For (4), the Beltrami condition is simply

$$\phi = f(\omega), \tag{7}$$

which implies the commutation of the vorticity and the Hamiltonian of the flow.

The simplest example of the vortex dynamics equation is that of the Euler equation of incompressible ideal flows. Let $\boldsymbol{U}$ be an incompressible flow that obeys

$$\frac{\partial}{\partial t}\boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = -\nabla p, \tag{8}$$

where p is the pressure. Taking the curl of (8), we obtain the evolution equation for the vorticity $\omega = \nabla \times \boldsymbol{U}$, which reads, in $\mathbf{R}^3$, as (3), and in $\mathbf{R}^2$, as (4). In the Beltrami flow, ω parallels $\boldsymbol{U}$, i.e.,

$$\nabla \times \boldsymbol{U} = \mu\boldsymbol{U}. \tag{9}$$

We note that (9) is not Galilean invariant. We thus consider a bounded domain and impose a boundary condition (see (1)) to remove the freedom of the Galilei transform. Taking the divergence of (9), we find that the scalar function μ must satisfy

$$\boldsymbol{U} \cdot \nabla\mu = 0, \tag{10}$$

demanding that μ must remain constant along each streamline of the flow U. An analysis of the nonlinear system of elliptic-hyperbolic partial differential equations (9)-(10) involves extremely difficult mathematical issues. The characteristic curve of (10) is the streamline of the unknown flow U, which can be chaotic (nonintegrable) in general three-dimensional problems. If we assume, however, that μ is a constant number, the analysis reduces into a simple but nontrivial problem, i.e., the eigenvalue problem of the curl operator. In this paper, our analysis is restricted to this mathematically well-defined subclass of Beltrami fields.

We end this section by reviewing another example of vortex dynamics; the magnetohydrodynamic (MHD) description of a plasma. The two principal equations of the ideal (dissipation-less) conducting-fluid model are

$$\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0, \tag{11}$$

$$\frac{\partial}{\partial t}\boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = \frac{1}{\rho}(\boldsymbol{J} \times \boldsymbol{B} - \nabla p), \tag{12}$$

where $\boldsymbol{U}$, $\boldsymbol{J}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are, respectively, the flow velocity, the current density, the electric field and the magnetic field measured in a certain fixed coordinates, and ρ is the fluid mass density that is assumed to be constant. We may write

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi, \tag{13}$$

$$\boldsymbol{J} = \frac{1}{\mu_0}\nabla \times \boldsymbol{B} \tag{14}$$

in terms of a vector potential $\boldsymbol{A}$ (such that $\nabla \times \boldsymbol{A} = \boldsymbol{B}$) and a scalar potential ϕ. Using Faraday's law

$$\partial \boldsymbol{B}/\partial t = -\nabla \times \boldsymbol{E},$$

and taking the curl of (11) and (12), we obtain

$$\frac{\partial}{\partial t}\boldsymbol{B} - \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) = 0, \tag{15}$$

$$\frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla \times \left(\frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho} + \boldsymbol{U} \times \boldsymbol{\omega}\right) = 0, \tag{16}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{U}$. The Beltrami conditions for this system of vortex dynamics equations are

$$\boldsymbol{J} = \mu_1\boldsymbol{B} = \mu_2\boldsymbol{U} = \mu_3\boldsymbol{\omega}. \tag{17}$$

Using (14) in the first equality of (17), we get

$$\nabla \times \boldsymbol{B} = \mu\boldsymbol{B}, \tag{18}$$

which implies that $\boldsymbol{B}$ parallels its own vorticity (cf. (9)). This configuration, for which the magnetic stress $\boldsymbol{J} \times \boldsymbol{B}$ vanishes, is aptly called "force-free".

In order to characterize the stellar magnetic fields, solutions to (18) were intensively studied in 1950s [4, 5, 6]. For $\mu \neq 0$, the magnetic field $\boldsymbol{B}$ has a finite curl, and hence, the field lines are twisted. The current (proportional to $\nabla \times \boldsymbol{B}$), flowing parallel to the twisted field lines, creates what may be termed as "paramagnetic" structures. Such twisted magnetic field lines appear commonly in many different plasma systems such as the magnetic ropes created in solar and geo-magnetic systems [7], and galactic jets [8]. Some laboratory experiments have also shown that the "relaxed state" generated through turbulence is well described as solutions of the force-free equation [9, 10]

In the next section, we will show that a more adequate formulation of the plasma dynamics allows a much wider class of special equilibrium solutions. The set of new solutions contains field configurations which can be qualitatively different from the force-free magnetic fields.

3 DOUBLE CURL BELTRAMI FIELD

The two-fluid model for the macroscopic dynamics of a plasma differentiates between the electron and ion velocities. Denoting the electron (ion) flow velocity by $\boldsymbol{V}_e(\boldsymbol{V}_i)$, the macroscopic evolution equations become

$$\frac{\partial}{\partial t}\boldsymbol{V}_e + (\boldsymbol{V}_e \cdot \nabla)\boldsymbol{V}_e = \frac{-e}{m}(\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}) - \frac{1}{mn}\nabla p_e, \tag{19}$$

$$\frac{\partial}{\partial t}\boldsymbol{V}_i + (\boldsymbol{V}_i \cdot \nabla)\boldsymbol{V}_i = \frac{e}{M}(\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}) - \frac{1}{Mn}\nabla p_i, \tag{20}$$

where $\boldsymbol{E}$ is the electric field, p_e and p_i are, respectively, the electron and the ion pressures, e is the elementary charge, n is the number density of both electrons and ions (we consider a quasineutral plasma with singly charged ions), m and M are, respectively, the electron and the ion masses. In the electron equation, the inertial terms (left-hand side of (19)) can be safely neglected, because of their small mass ($m \ll M$) [11]. Therefore, (19) reduces to

$$\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B} + \frac{1}{en}\nabla p_e = 0. \tag{21}$$

When electron mass is neglected, $\boldsymbol{V}_i = \boldsymbol{V}$, the fluid velocity. We introduce the following set of dimensionless variables,

$$\begin{cases} \boldsymbol{x} = \lambda_i \hat{\boldsymbol{x}}, \quad \boldsymbol{B} = B_0 \hat{\boldsymbol{B}}, \\ t = (\lambda_i / V_A)\hat{t}, \quad p = (B_0^2/\mu_0)\hat{p}, \quad \boldsymbol{V} = V_A \hat{\boldsymbol{V}}, \\ \boldsymbol{A} = (\lambda_i B_0)\hat{\boldsymbol{A}}, \quad \phi = (V_A \lambda_i B_0)\hat{\phi}, \end{cases} \tag{22}$$

where the ion skin-depth

$$\lambda_i = \frac{c}{\omega_{pi}} = \frac{V_A}{\omega_{ci}} = \sqrt{\frac{M}{\mu_0 n e^2}}.$$

is a characteristic length scale of the system, and the Alfvén speed is given by $V_A = B_0/\sqrt{\mu_0 M n}$ (we assume n = constant, for simplicity) with B_0 as an appropriate measure of the magnetic field.

Writing $\hat{E} = -\partial\hat{A}/\partial\hat{t} - \hat{\nabla}\hat{\phi}$, the dimensionless version of (21) and (20) now read as

$$\frac{\partial}{\partial\hat{t}}\hat{A} = \left(\hat{V} - \hat{\nabla}\times\hat{B}\right)\times\hat{B} - \hat{\nabla}\left(\hat{\phi} + \hat{p}_e\right), \tag{23}$$

$$\frac{\partial}{\partial\hat{t}}(\hat{V} + \hat{A}) = \hat{V}\times\left(\hat{B} + \hat{\nabla}\times\hat{V}\right) - \hat{\nabla}\left(\hat{V}^2/2 + \hat{p}_i + \hat{\phi}\right). \tag{24}$$

In what follows, we shall drop the *hat* for a simpler notation. Taking the curl of (23) and (24), we can cast them in a revealing symmetric form

$$\frac{\partial}{\partial t}\omega_j - \nabla\times(U_j\times\omega_j) = 0 \quad (j = 1, 2) \tag{25}$$

in terms of a pair of generalized vorticities

$$\omega_1 = B, \quad \omega_2 = B + \nabla\times V,$$

and the effective flows

$$U_1 = V - \nabla\times B, \quad U_2 = V.$$

The simplest equilibrium solution to (25) is given by the "Beltrami conditions"

$$U_j = \mu_j\omega_j \quad (j = 1, 2), \tag{26}$$

which implies the alignment of the vorticities and the corresponding flows. Writing $a = 1/\mu_1$ and $b = 1/\mu_2$, and assuming that a and b are constants, the Beltrami conditions (26) read as a system of simultaneous linear equations in B and V

$$B = a(V - \nabla\times B), \tag{27}$$

$$B + \nabla\times V = bV. \tag{28}$$

These equations have a simple and significant connotation; the electron flow $(V - \nabla\times B)$ parallels the magnetic field B, while the ion flow V follows the "generalized magnetic field" $(B + \nabla\times V)$. This generalized magnetic field contains the Coriolis' force induced by the ion inertia effect on a circulating flow.

Combining (27) and (28) yields a second order partial differential equation

$$\nabla\times(\nabla\times B) + \alpha\nabla\times B + \beta B = 0, \tag{29}$$

where

$$\alpha = \frac{1}{a} - b, \quad \beta = 1 - \frac{b}{a}.$$

The double curl Beltrami equation (29) encompasses a wide class of steady-state equations of mathematical physics. The conventional force-free-field equation (18), which describes paramagnetic fields, is included in this system as a special case: $\alpha = 0$ and $\beta < 0$. On the other hand, when $\alpha = 0$ and $\beta > 0$, (29) resembles London's equation of super conductivity with its well-known fully diamagnetic solutions. We note that, in this version of the London equation, the characteristic shielding length for the magnetic field is the ion skin depth c/ω_{pi}, instead of the usual electron skin depth c/ω_{pe}, because it is the ion-dynamics that brings about the coupling of the magnetic field with the collective motion of the medium.

In the next section, we will study the mathematical structure of the double curl Beltrami equation with arbitrary complex α and β [12].

4 BELTRAMI FIELD AND HARMONIC FIELD

The single Beltrami condition (1) is known to have a non-zero solution for arbitrary complex number λ, if the domain Ω is multiply connected [1]. The harmonic field which represents the cohomology class of the differential forms in Ω plays an essential role to generate the Beltrami field. Similar relation holds in the double curl Beltrami equations (2). Here, we study the relation between the topology of the domain Ω and the degree of freedom in the solution of the double curl Beltrami fields.

It is convenient to denote the curl derivative $\nabla\times$ by "curl" to use it as an operator. Let us rewrite the differential equation of (2) in the form

$$(\mathrm{curl}-\lambda_+)(\mathrm{curl}-\lambda_-)\boldsymbol{u}=0, \tag{30}$$

where

$$\lambda_\pm=\frac{1}{2}\left[-\alpha\pm(\alpha^2-4\beta)^{1/2}\right]. \tag{31}$$

Because the two operators $(\mathrm{curl}-\lambda_\pm)$ commute, the general solution to (30) is given by a linear combination of two Beltrami fields. Let $\boldsymbol{G}_\pm$ be the Beltrami field such that

$$\begin{cases} (\mathrm{curl}-\lambda_\pm)\boldsymbol{G}_\pm=0 & (\text{in } \Omega), \\ \boldsymbol{n}\cdot\boldsymbol{G}_\pm=0 & (\text{on } \partial\Omega). \end{cases}$$

Then, for arbitrary constants $c_\pm$, the sum

$$\boldsymbol{u}=c_+\boldsymbol{G}_++c_-\boldsymbol{G}_- \tag{32}$$

solves (30). Since $\boldsymbol{n}\cdot(\nabla\times\boldsymbol{G}_\pm)=\lambda_\pm\boldsymbol{n}\cdot\boldsymbol{G}_\pm=0$ on $\partial\Omega$, $\boldsymbol{u}$ satisfies the boundary conditions given in (2). Therefore, the existence of a nontrivial solution to the double curl Beltrami equations (2) will be predicated on the existence of the appropriate pair of single Beltrami fields (cf. Appendix B). Let us briefly review the mathematical theory of single Beltrami fields [1].

Suppose that Ω ($\subset \boldsymbol{R}^3$) is a bounded domain with a smooth boundary $\partial\Omega=\cup_{i=1}^n\Gamma_i$. We consider cuts of the domain Ω. Let $\Sigma_1,\cdots,\Sigma_\nu$ ($\nu\geq 0$) be the cuts such that $\Sigma_i\cap\Sigma_j=\emptyset$ ($i\neq j$), and such that $\Omega\setminus(\cup_{j=1}^\nu\Sigma_j)$ becomes a simply connected domain. The number ν of such cuts is the first Betti number of Ω. When $\nu>0$, we define the flux through each cut by

$$\Phi_j(\boldsymbol{u})=\int_{\Sigma_j}\boldsymbol{n}\cdot\boldsymbol{u}\,ds \qquad (j=1,\cdots,\nu),$$

where $\boldsymbol{n}$ is the unit normal vector on Σ_j with an appropriate orientation. By Gauss' formula, $\Phi_j(\boldsymbol{u})$ is independent of the place of the cut Σ_j, if $\nabla\cdot\boldsymbol{u}=0$ in Ω and $\boldsymbol{n}\cdot\boldsymbol{u}=0$ on $\partial\Omega$.

Let $L^2(\Omega)$ the Lebesgue space of square-integrable (complex) vector fields in Ω, which is endowed with the standard innerproduct

$$(\boldsymbol{a},\boldsymbol{b})=\int_\Omega\boldsymbol{a}\cdot\overline{\boldsymbol{b}}\,dx.$$

We define the following subspaces of $L^2(\Omega)$:

$$
\begin{aligned}
L^2_\Sigma(\Omega) &= \{\boldsymbol{w};\ \nabla\cdot\boldsymbol{w}=0 \text{ in } \Omega,\ \boldsymbol{n}\cdot\boldsymbol{w}=0 \text{ on } \partial\Omega,\ \Phi_j(\boldsymbol{w})=0\ (j=1,\cdots,\nu)\},\\
L^2_H(\Omega) &= \{\boldsymbol{h};\ \nabla\cdot\boldsymbol{h}=0, \nabla\times\boldsymbol{h}=0 \text{ in } \Omega,\ \boldsymbol{n}\cdot\boldsymbol{h}=0 \text{ on } \partial\Omega\},\\
L^2_G(\Omega) &= \{\nabla\phi;\ \Delta\phi=0 \text{ in } \Omega\},\\
L^2_F(\Omega) &= \{\nabla\phi;\ \phi=c_i\ (\in \mathbf{C}) \text{ on } \Gamma_i\ (i=1,\cdots,n)\}
\end{aligned}
$$

in terms of which we have an orthogonal decomposition [13]

$$L^2(\Omega)=L^2_\Sigma(\Omega)\oplus L^2_H(\Omega)\oplus L^2_G(\Omega)\oplus L^2_F(\Omega).$$

The space of the solenoidal vector fields with vanishing normal components on $\partial\Omega$ is

$$L^2_\sigma(\Omega)=L^2_\Sigma(\Omega)\oplus L^2_H(\Omega).$$

The subspace $L^2_H(\Omega)$ corresponds to the cohomology class, whose member is a harmonic vector field and $\dim L^2_H(\Omega)=\nu$ (the first Betti number of Ω). When Ω is simply connected, then $\nu=0$ and $L^2_H(\Omega)=\emptyset$. We have the following expression

$$L^2_\Sigma(\Omega)=\{\nabla\times\boldsymbol{w};\ \boldsymbol{w}\in H^1(\Omega),\ \nabla\cdot\boldsymbol{w}=0 \text{ in } \Omega,\ \boldsymbol{n}\times\boldsymbol{w}=0 \text{ on } \partial\Omega\},$$

where $H^1(\Omega)$ is the Sobolev space of first order. This says that a member of $L^2_\Sigma(\Omega)$ can be expressed as the curl of a vector potential with the boundary condition $\boldsymbol{n}\times\boldsymbol{w}=0$.

The spectral theory of the curl operator provides the basic understanding of the mathematical structure of the Beltrami equations. We repeat Theorem 1 of Yoshida-Giga [1].

Theorem 1. *Suppose that Ω is a smoothly bounded domain in $\mathbf{R}^3$. We define a curl operator S in the Hilbert space $L^2_\Sigma(\Omega)$ by*

$$
\begin{aligned}
S\boldsymbol{u} &= \nabla\times\boldsymbol{u},\\
D(S) &= \{\boldsymbol{u}\in L^2_\Sigma(\Omega)\ ;\ \nabla\times\boldsymbol{u}\in L^2_\Sigma(\Omega)\}.
\end{aligned}
$$

The S is a self-adjoint operator. The spectrum of S consists of only point spectrum $\sigma_p(S)$ which is a discrete set of real numbers.

This theorem says that the Beltrami equation (1) together with the zero-flux condition (see the definition of the space $L^2_\Sigma(\Omega)$) has non-zero solution only for special discrete real numbers $\lambda\in\sigma_p(S)$. If Ω is simply connected ($\nu=0$), the topological flux $\Phi_j(\ )$ does not exist, so that $L^2_\Sigma(\Omega)=L^2_\sigma(\Omega)$. If Ω is multiply connected ($\nu\geq 1$), however, we can remove the zero-flux condition assumed in Theorem 1, and consider a wider set of functions to find solutions of (1). This is done by considering the curl operator defined in the space $L^2_\sigma(\Omega)$. Let us trace the method of Yoshida-Giga [1].

Lemma 1. *For every $\boldsymbol{f}\in L^2_\sigma(\Omega)$, the equation*

$$\nabla\times\boldsymbol{u}=\boldsymbol{f}\quad(\text{in } \Omega)\tag{33}$$

has a unique solution in $L^2_\Sigma(\Omega)$.

Proof. Let $\tilde{f}$ be the 0-extension of f over $\mathbf{R}^3$, i.e.

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Since $f \in L^2_\sigma(\Omega)$, we have $\nabla \cdot \tilde{f} = 0$ in $\mathbf{R}^3$. We denote by $(-\Delta)^{-1}$ the vector Newtonian potential. We define

$$w_0 = \nabla \times [(-\Delta)^{-1}\tilde{f}] \quad \text{in } \Omega.$$

We denote by $\mathcal{P}_\Sigma$ the orthogonal projection in $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$, and define $u_0 = \mathcal{P}_\Sigma w_0$. Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(curl), we observe

$$\nabla \times u_0 = \nabla \times w_0 = \nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{f}]\}.$$

Since $\nabla \cdot [(-\Delta)^{-1}\tilde{f}] = 0$, we obtain

$$\nabla \times \{\nabla \times [(-\Delta)^{-1}\tilde{f}]\} = -\Delta[(-\Delta)^{-1}\tilde{f}] = \tilde{f}.$$

We thus find that u_0 ($\in L^2_\Sigma(\Omega)$) is the solution of (33). Since $L^2_\Sigma(\Omega)$ is orthogonal to Ker(curl), this u_0 is the unique solution. □

This lemma shows that every solenoidal vector field (member of $L^2_\sigma(\Omega)$) has a unique vector potential in the space $L^2_\Sigma(\Omega)$. We apply this result to determine the vector potential of the harmonic field (member of $L^2_H(\Omega)$). Let ν (≥ 1) be the dimension of $L^2_H(\Omega)$ (first Betti number of Ω), and h_j ($j = 1, \cdots, \nu$) be the orthogonal basis of $L^2_H(\Omega)$ such that

$$\Phi_i(h_j) = \int_{\Sigma_i} n \cdot h_j \, ds = \delta_{i,j}. \tag{34}$$

By solving (33) for $f = h_j$, we obtain the corresponding vector potential which we denote by g_j, i.e.,

$$\nabla \times g_j = h_j \quad (\text{in } \Omega), \quad g_j \in L^2_\Sigma(\Omega) \quad (j = 1, \cdots, \nu).$$

Let us consider an arbitrary harmonic field and its vector potential, and write them as

$$h = \sum_{j=1}^{\nu} \xi_j h_j, \quad g = \sum_{j=1}^{\nu} \xi_j g_j. \tag{35}$$

For every $\lambda \notin \sigma_p(\mathcal{S})$, the resolvent operator $(\mathcal{S} - \lambda)^{-1}$ defines a unique continuous map on $L^2_\Sigma(\Omega)$. We consider

$$v = \lambda g + \lambda^2(\mathcal{S} - \lambda)^{-1} g.$$

This v is the unique solution (in $L^2_\Sigma(\Omega)$) of

$$(\text{curl} - \lambda)v = \lambda h \quad (\text{in } \Omega). \tag{36}$$

Now we find that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{h}$ solves

$$\begin{cases} (\text{curl } - \lambda)\boldsymbol{u} = 0 & (\text{in } \Omega), \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 & (\text{on } \partial\Omega), \end{cases}$$

Since $\boldsymbol{h}$ ($\in L^2_H(\Omega)$) and $\boldsymbol{v}$ ($\in L^2_\Sigma(\Omega)$) are orthogonal, $\boldsymbol{u} \not\equiv 0$.

We have shown that the single curl Beltrami equation (1) has a non-zero solution for every complex number λ, if the domain Ω is multiply connected. For $\lambda \notin \sigma_p(\mathcal{S})$, the solution is uniquely determined by the harmonic field $\boldsymbol{h}$. Although (1) appears as a homogeneous equation, the harmonic field (member of the kernel of curl) plays a role of a hidden inhomogeneous term; see (36). On the other hand, for $\lambda \in \sigma_p(\mathcal{S})$, the solution is given by the eigenfunction of the self-adjoint curl operator $\mathcal{S}$. Therefore, the solution is a zero-flux field, and $\boldsymbol{h}$ must be set to zero. The solution is not unique in the sense that any constant multiple of the eigenfunction is a solution.

Because of (32), it is now straightforward to generalize the theory for the double curl (and multi curl) Beltrami equations.

Theorem 2. *For a multiply connected smoothly bounded domain Ω, and for all complex numbers λ_1 and λ_2, the equation*

$$(\text{curl} - \lambda_1)(\text{curl} - \lambda_2)\boldsymbol{u} = 0 \tag{37}$$

has a non-zero solution.

Let us examine the relations among the solutions, the harmonic fields and the fluxes. If $\lambda_1, \lambda_2 \notin \sigma_p(\mathcal{S})$, then the solution is given by

$$\begin{aligned} \boldsymbol{u} &= c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2, \\ \boldsymbol{u}_j &= \boldsymbol{h} + \lambda_j\boldsymbol{g} + \lambda_j^2(\mathcal{S} - \lambda_j)^{-1}\boldsymbol{g} \quad (j = 1, 2), \end{aligned}$$

where $\boldsymbol{h} \in L^2_H(\Omega)$, $\nabla \times \boldsymbol{g} = \boldsymbol{h}$ and $\boldsymbol{g} \in L^2_\Sigma(\Omega)$. Let us decompose $\boldsymbol{h}$ in terms of the normalized bases as (35). The coefficients $c_1, c_2, \xi_1, \cdots, \xi_\nu$ are related to the fluxes of $\boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ by

$$\begin{cases} (c_1 + c_2)\xi_j = \Phi_j(\boldsymbol{u}), \\ (c_1\lambda_1 + c_2\lambda_2)\xi_j = \Phi_j(\nabla \times \boldsymbol{u}) \end{cases} \quad (j = 1, \cdots, \nu),$$

where $\Phi_j(\ )$ is the flux through the cut Σ_j. When $\nu = 1$ (as in the case of a simple toroid), we can give the fluxes of both $\boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$ independently to determine ξ_1 and c_1 with setting $c_2 = 1 - c_1$ (cf. Appendix B). For $\nu > 1$, the fluxes of $\nabla \times \boldsymbol{u}$ are not totally independent.

If $\lambda_1 \in \sigma_p(\mathcal{S})$ and $\lambda_2 \notin \sigma_p(\mathcal{S})$, we take $\boldsymbol{u}_1$ to be the eigenfunction corresponding to λ_1. Then, $\boldsymbol{u}_1$ is a zero-flux function, and hence, c_1 is an arbitrary constant. The other component $\boldsymbol{u}_1$ carries fluxes. Taking $c_2 = 1$, we can determine

$$\xi_j = \Phi_j(\boldsymbol{u}) \quad (j = 1, \cdots, \nu).$$

If $\lambda_1, \lambda_2 \in \sigma_p(\mathcal{S})$, then both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the corresponding eigenfunctions. Solution exists only for $\xi_j = 0$ $(j = 1, \cdots, \nu)$.

5 SUMMARY

The study of the solvability of the double curl equation is warranted both by physical as well as mathematical considerations. A more adequate modeling of plasma dynamics, containing a coupling of the magnetic and fluid aspects of a plasma, necessarily leads to a departure from the conventional single Beltrami equilibria (1) which are restricted to only force-free equilibria. This departure, then, leads to an immensely larger class of physically interesting equilibria which can be constructed by a superposition of several different Beltrami fields. In the example dealt with in this paper (where the coupling is introduced by the Hall term), a superposition of two Beltrami fields suffices. Notice that in the nonlinear vortex dynamics models such as (3) with coupled ω and U, a linear combination of Beltrami fields is no longer a Beltrami field. Hence, a finite pressure and coupled flows can exist in conjunction with the magnetic field, and the structures that are far richer than those of single Beltrami fields come within the scope of the theory [3].

The mathematical content of the paper may be summarized as follows: We have elucidated the general relation between the (double curl) Beltrami fields and the harmonic fields which, being members of Ker(curl), play the role of a hidden inhomogeneous term in the Beltrami equations. The existence of harmonic fields invokes the multiply-connectedness of the domain. For every $\lambda \in \mathbf{C}\setminus\sigma_p(\mathcal{S})$ (point spectrum of the self-adjoint curl operator), a harmonic field generates non-zero unique Beltrami field corresponding to λ. When $\lambda \in \sigma_p(\mathcal{S})$, the corresponding eigenfunction gives the Beltrami field. The linear combination of two Beltrami fields yields the double curl Beltrami field.

A Examples of Solutions

Some explicit forms of the Beltrami fields may help understanding of the structures of the solutions.

When we consider a cubic volume that has sides of length a and assume the periodic boundary condition, we have the so-called ABC flow. Let A, B and C be real (complex) constants and $\lambda = 2\pi n/a$ $(n \in \mathbf{N})$. In the cartesian coordinates, we define

$$\boldsymbol{u} = \begin{pmatrix} A\sin\lambda z + C\cos\lambda y \\ B\sin\lambda x + A\cos\lambda z \\ C\sin\lambda y + B\cos\lambda x \end{pmatrix}. \tag{38}$$

We easily verify that (38) gives an eigenfunction of the curl belonging to an eigenvalue λ. The linear combination of two ABC flows give the double curl Beltrami flow.

Solutions with the zero-normal boundary conditions are known for a cylindrical domain. In the (r,θ,z) cylindrical coordinates, the Chandrasekhar-Kendall function [6] is defined as

$$\boldsymbol{u} = \lambda(\nabla\psi\times\nabla z) + \nabla\times(\nabla\psi\times\nabla z) \tag{39}$$

with

$$\lambda = \pm(\mu^2 + k^2)^{1/2}, \tag{40}$$

$$\psi = J_m(\mu r)e^{i(m\varphi - kz)}, \quad k = 2\pi n/L, \ m, n \in \mathbf{N}, \tag{41}$$

where J_m is the ordinary Bessel function and L is the length of the periodic cylinder. We find that $\boldsymbol{u}$ is an eigenfunction of the curl corresponding to the eigenvalue λ ($\in \mathbf{R}$). The eigenvalue is determined by the boundary condition that the normal component of $\boldsymbol{u}$ vanishes on the surface of the cylindrical domain. This condition becomes trivial when $k = m = 0$. For these axisymmetric modes, we impose the "zero-flux condition"

$$\Phi(\boldsymbol{u}) = \int_\Sigma \boldsymbol{n} \cdot \boldsymbol{u} ds = 0, \tag{42}$$

where Σ is a cut of the cylinder (cf. Theorem 1).

When we do not impose the zero-flux condition, however, the eigenvalue μ can be an arbitrary real (and even complex) number for the $k = m = 0$ mode [2]. Therefore, we have non-zero Beltrami fields for arbitrary λ. For such a solution that has a finite flux $\Phi(\boldsymbol{u})$, the flux can be regarded as the variable of state. The double curl Beltrami field is a combination of two Beltrami fields, and hence, the degree of freedom is two and two fluxes $\Phi(\boldsymbol{u})$ and $\Phi(\nabla \times \boldsymbol{u})$ can be assigned.

In a two dimensional system, we can apply the Clebsch representation of solenoidal vector fields (cf. (4) and (5)). For example, let us assume that the fields are homogeneous in the direction of z in the cartesian coordinates x-y-z. We write $\boldsymbol{B}$ in a contravariant-covariant combination form

$$\boldsymbol{B} = \nabla\psi \times \boldsymbol{e} + \phi\, \boldsymbol{e}, \tag{43}$$

where $\boldsymbol{e} = \nabla z$. The ψ and ϕ are scalar functions of x and y. We have

$$\begin{aligned} \nabla \times \boldsymbol{B} &= \nabla\phi \times \boldsymbol{e} + (-\Delta\psi)\boldsymbol{e}, \\ \nabla \times (\nabla \times \boldsymbol{B}) &= \nabla(-\Delta\psi) \times \boldsymbol{e} + (-\Delta\phi)\boldsymbol{e}. \end{aligned}$$

Using these expressions in the double curl Beltrami equation (29), we obtain a system of coupled Helmholtz equations

$$\begin{cases} -\Delta\psi + \alpha\phi + \beta\psi = C, \\ -\Delta\phi - \alpha\Delta\psi + \beta\phi = 0, \end{cases} \tag{44}$$

where C is a constant. Biasing the potential ψ with $-C/\beta$, we can eliminate this constant. The system (44) can be casted into a symmetric form

$$\Delta \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ -\alpha\beta & \beta - \alpha^2 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix}. \tag{45}$$

Similar algebra applies for the case of axisymmetric (toroidal) systems, where we must take $\boldsymbol{e} = \nabla\theta$ in (43) and assume that ψ and ϕ are functions of r and z in the r-θ-z cylindrical coordinates. Then, the Laplacian Δ is replaces by the Grad-Shafranov operator

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}.$$

The coupled Grad-Shafranov equation of the type (44) was derived previously for the analysis of toroidal equilibrium in a plasma-beam system, where the inertia force of the beam particles brings about coupling of the magnetic field and the beam flow [21]

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