united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR 1161/43

AUTUMN COLLEGE ON PLASMA PHYSICS

25 October - 19 November 1999

Nonresonance Radiative Pierce Instability and its Saturation-Chaos

A.A. RUKHADZE

Russian Academy of Sciences
General Physics Institute
Moscow, Russia

These are preliminary lecture notes, intended only for distribution to participants.

Nonresonance Radiative Pierce Instability and its Saturation–Chaos

D.N. Klochkov, M.Yu. Pekar, A.A. Rukhadze

General Physics Institute, Russian Academy of Sciences
Vavilova 38, Moscow, Russia
e-mail: rukh @ fpl.gpi.ru

The review of theoretical investigations of a nonresonance stimulated Pierce radiation by the relativistic electron beam (REB) in a waveguide is presented. REB is supposed to be stabilized by an infinitely strong longitudinal magnetic field. Using the analytical methods of the linear theory the conditions of arising the radiative Pierce instabilities, its growth rate and the radiation spectrum are determined. By numerical modeling of the problem the efficiency of a beam energy transformation into electromagnetic field energy as a function of the beam current, its relativism and geometry are calculated. The physical nature of the instability is clarified and its saturation mechanisms are discussed. In particular, in the saturated state the chaotic motion of beam particles and chaos in the radiation spectrum are observed.

Typeset by $\mathcal{AMS}$-TEX

I. Introduction.

The mostly wide-spread device is ofthe Cherenkov microwave type using the slowing electrodynamical structures, such as: metallic waveguides with rippled walls or modulated guiding magnetic fields. It is just due to this reason in the vacuum microwave sources that the radiation breaks when the power exceeds 100 MW. The point is that the electric field in the vacuum microwave sources reaches its maximal value near the walls of waveguids and initiates surface discharge if the radiation power is sufficiently high. As a result the high power vacuum sources are working during no more then ≈ 10 ns.

In this sense the nonresonance microwave sources based on the stimulated Pierce radiation of REB in the smooth waveguids seem to be very perspective. Such type of sources using nonrelativistic electron beams were first theoretically proposed in 1940 in papers [1] and were called as monotrons. The first monotron oscillator with superconducting cavity has been realized in [2] having been used cylindrical waveguide with equal radius R and length L ($R = L =$ 3,8 cm). Microwave generation of three types has been reached: TM_{010} , TM_{011} and TM_{020} . The experimental results turned out to be in a good agreement with theory [1]. In particular the radiation condition, which is

$$\theta \equiv \frac{\omega L}{u} = \left(2n + \frac{1}{2}\right)\pi, \tag{1}$$

was confirmed. Here n is an integer, ω is a radiation frequency and u is an electron beam velocity. Under this condition in the system the positive feed-back arises, which leads to the instability, or radiation.

On the other hand in 1944 J. Pierce predicted quite another instability [3], which also was provided by the positive feed-back between the input and output electrodes in a diode. Later the another treatment of Pierce instability was given in [4, §50]. It was shown that when the beam current in a diode is higher than critical one (which was called Pierce current), in the beam the backward wave arises and provides the feedback in the system. So the existence of a new type of feed-back process, stimulated by the wave was indicated. It is quite natural to consider two apparently different instabilities as one with two regimes. The first, potential regime is possible when the beam current is higher than Pierce current, or $\omega_b \gamma^{-3/2}/k_\perp u > 1$; in this case the electric field is purely potential and increases aperiodic in time. The second, radiative (wave) regime occurs at frequency above the critical waveguide frequency, when the beam current is less than Pierce current, and when the electromagnetic waves propagating counter to the beam can exist in the system.

The aperiodic (potential) Pierce instability was studied in sufficiently detail [4–9]. Moreover some papers [10, 11] were devoted to investigating this instability and chaos in a Pierce diode. At the same time, the radiative Pierce instability, which provides monotron working, has been investigated relatively weakly up to day by several reasons.

Resonators in experimental devices were usually short ($L \approx R$) or very short ($L \ll R$) and nonrelativistic beams with very small current were used. So, in experiments [2] the beam current was of the order of $I = 1{,}25\,\mathrm{mA}$ and accelerating voltage was varied from 3,5 kV up to 15 kV. As a result the radiation power and efficiency turned out very small. The only advantage reached in [2] was high stability of radiation frequency, which was less than $2{,}2 \cdot 10^{-10}$ during the working time equal 10 s. But very complicate cryogen technology of the monotron oscillators with superconducting cavity makes them uncompetitive in comparison with the quartz oscillators.

The growing requirements in the high power and frequency wideband radiation sources stimulated the theoretical investigations of relativistic monotrons using the REB. As follows from formula (1), the efficiency of monotron increases with beam energy due to the broadening of radiation conditions. Besides, by increasing the beam current the radiation power and its frequency band also increase.

First attempts of developing the relativistic theory of monotron were carried out in papers [12,13]. However authors of these works neglected the beam waves as well as backward electromagnetic wave, that led to the incorrect description of the phenomenon. At the same time, in papers [14,15] taking into account the wave transformation on the unhomogenities in the nonequilibrium beam system, it was shown, that the new instabilities appeared. This phenomenon was treated as stimulated transit radiation of beam electrons at the entrance and outlet of the resonator. This treatment turned out to be incorrect for monotron, but nevertheless these papers may be accepted as the first consideration of stimulated radiative Pierce instability.

The complete theory of the phenomenon based on the solution of self-consistent problem was developed in the papers [16–19], where the role of boundary conditions was clarified as well as the beam waves were accounted. Bellow we will present results of these papers using general methods of plasma microwave electronics developed in [4].

The content of this paper is following: At first, using analytical methods we study the linear stage of the non-resonant, stimulated emission of electromagnetic radiation by a rectilinear relativistic electron beam in a smooth cavity. A detailed investigation of the cylindrically symmetric case, i.e., when only azimuthally symmetric modes are excited in a cavity with a circular cross section, is made. Then, in Sec. VIII, using numerical methods we study the nonlinear stage. Phase portraits of the beam electrons are obtained which can be analyzed to reveal the physical nature of saturation mechanisms for the instability. In the last part of the paper we examine the dynamic instability of the beam particles in the field of the excited waves.

II. Basic Assumptions and Equations.

Let us consider a smooth cylindrical waveguide with radius R, along the axis (Oz-axis) of which a straight-line REB propagates. The REB is injected into the resonator at the entrance, $z = 0$, where a thin metallic foil or a mesh, transparent for REB and reflective for electromagnetic waves, is located. From the outlet of the resonator, $z = L$, the incident electromagnetic waves partially or completely reflect, whereas the beam electrons pass over. In the region $z > L$ the radiation horn is located.

In unperturbed state the REB is homogeneous along the z-axis, but may be inhomogeneous in the radial direction. Moreover the azimuthal symmetry of the problem is supposed, i.e., only symmetrical modes of electromagnetic perturbations will be considered . The system is placed in a strong homogeneous longitudinal magnetic field $\vec{B}_0 \parallel Oz$, which completely prevents the transverse motion of beam electrons. Under these conditions beam electrons can interact only with field perturbations of E-type (TM-modes of waves), for which E_z component is nonzero.

In this case the Maxwell's equations for the nonzero E_z, E_r and B_φ field components can be reduced to the equations for the single Hertz polarization potential ψ [6]

$$\partial_z(\partial_z^2 + \Delta_\perp - \frac{1}{c^2}\partial_t^2)\psi = 4\pi\rho, \tag{2a}$$

$$\partial_t(\partial_z^2 + \Delta_\perp - \frac{1}{c^2}\partial_t^2)\psi = -4\pi j. \tag{2b}$$

Where $\Delta_\perp$ is the transverse part of Laplase operator, ρ and $j_i = j\delta_{iz}$ are densities of charge and current perturbations respectively (δ_{iz} is the Kronecker symbol). The field components can be expressed in terms of

ψ by the relations

$$E_z = (\partial_z^2 - \frac{1}{c^2}\partial_t^2)\psi, \quad E_r = \partial_z\partial_r\psi, \quad B_\varphi = -\frac{1}{c}\partial_t\partial_r\psi \tag{3}$$

Under the above accepted assumptions the following boundary conditions take place

$$E_z|_{r=R} = E_r|_{z=0} = 0. \tag{4}$$

Besides, for the monoenergy electron beam the perturbed charge density and the perturbed current density can be presented as (See Appendix A)

$$\rho = en_b(r)\left[\int \delta[z - z(t, z_0)]\, dz_0 - 1\right], \tag{5a}$$

$$j = en_b(r)\left[\int v_z\delta[z - z(t, z_0)]\, dz_0 - u\right]. \tag{5b}$$

Here $n_b(r) = n_b p_b(r)$ is the radial distribution of beam density, $p_b(r)$ is the profile of this distribution and $z(t, z_0)$ is the electrons trajectory, given by equations:

$$\frac{dz}{dt} = v_z; \quad \frac{dv_z}{dt} = \frac{e}{m}\gamma^{-3}(v_z)E_z, \tag{6}$$

with initial conditions $v_z(t = 0) = u$, $z(t = 0) = z_0$. Here $\gamma(v_z) = \left(1 - \frac{v_z^2}{c^2}\right)^{-1/2}$ is a relativistic factor.

The beam charge is assumed to be neutralized. When an electron enters the cavity its unperturbed velocity is u. The beam leaves the cavity without hindrance, carrying away the acquired perturbations.

The theory of monotron presented bellow is based on the system of Eqs. (2)–(6).

III. Field of TM-waves in the resonator with REB.

Let us begin with the analyses of the linear field structure in the resonator with beam. To derive the characteristic equation we represent polarization potential in the form

$$\psi(r, z, t) = \sum_\nu \tilde{\psi}_\nu(r)e^{-i\omega t + ik_\nu z} \tag{7}$$

and trajectory $z(t, z_0)$ and $v(t, z_0)$ in the form $z = z_0 + ut + \delta z$, $v_z = u + \delta v$. Here δz and δv are small perturbations of motion and therefore the Eq.(6) can be solved in the linear approximation. In this case the small perturbation of charge and current densities are

$$\rho = i\frac{\omega_b^2}{4\pi}p_b(r)\gamma^{-3}\sum_\nu \frac{\frac{\omega^2}{c^2} - k_\nu^2}{(\omega - k_\nu u)^2}k_\nu \tilde{\psi}_\nu(r)e^{-i\omega t + ik_\nu z}, \qquad (8a)$$

$$j = i\omega\frac{\omega_b^2}{4\pi}p_b(r)\gamma^{-3}\sum_\nu \frac{\frac{\omega^2}{c^2} - k_\nu^2}{(\omega - k_\nu u)^2}\tilde{\psi}_\nu(r)e^{-i\omega t + ik_\nu z}. \qquad (8b)$$

As a result we obtain the linear equation for transverse structure $\tilde{\psi}(r)$ of polarization potential [4]

$$\Delta_\perp\tilde{\psi}_\nu - \left(k_\parallel^2 - \frac{\omega^2}{c^2}\right)\left(1 - \frac{\omega_b^2\gamma^{-3}}{(\omega - k_\nu u)^2}p_b(\vec{r}_\perp)\right)\tilde{\psi}_\nu = 0, \qquad (9)$$

where $\omega_b = \left(\frac{4\pi e^2 n_b}{m}\right)^{1/2}$ is the Langmuir frequency of beam electrons, and $\gamma = (1 - u^2/c^2)^{-1/2}$ is an unperturbed Lorentz factor. Besides the first boundary Eq. (4) becomes $\tilde{\psi}_\nu(R) = 0$.

Now, for simplicity we consider the beam that is uniform over the cavity cross section, i.e. when $p_b(r) = 1$. Another geometry of beam is considered in Sec. VII. In this case we have $\tilde{\psi}(r) = A_\nu J_0(k_\perp r)$, and $\Delta_\perp\tilde{\psi} = -k_\perp^2\tilde{\psi}$. Then, from the Eq.(9) we obtain the characteristic equation

$$k_\perp^2 + \left(k_\nu^2 - \frac{\omega^2}{c^2}\right)\left[1 - \frac{\omega_b^2\gamma^{-3}}{(\omega - k_\nu u)^2}\right] = 0, \qquad (10)$$

in which $k_\perp = \frac{\mu_{s,0}}{R}$ and $\mu_{s,0}$ is the s–root of Bessel function ($J_0(\mu_{s,0}) = 0$).

The algebraic Eq.(10) determines four quantities $k_\nu(\omega)$, that allows us to represent $\psi(z, r, t)$ as

$$\psi = J_0(k_\perp r)e^{-i\omega t}\sum_{\nu=1}^{4} A_\nu e^{ik_\nu z}, \qquad (11)$$

where A_ν is a Hertz potential amplitude of wave with longitudinal wave number $k_\nu(\omega)$. In the limit of low density beam, when the Pierce parameter is small, i.e. when

$$\chi \equiv \frac{\omega_b^2\gamma^{-3}}{k_\perp^2 u^2} \ll 1, \qquad (12)$$

all $k_\nu(\omega)$ can be find out analytically. It must be noted that χ is the ratio of beam current to limiting Pierce current [4]. Introducing a notation

$$a^2 = \frac{\omega^2}{c^2} - k_\perp^2, \tag{13}$$

under the conditions (12) we have approximate solutions of Eq. (10)

$$k_{1,2} = \pm a \pm \frac{\beta_{1,2}}{2a}\omega_b^2, \quad \beta_{1,2} = -\frac{k_\perp^2 \gamma^{-3}}{(\omega \mp au)^2}; \tag{14a}$$

$$k_{3,4} = \frac{\omega}{u} \pm \alpha\omega_b, \quad \alpha = \frac{\omega}{u}\frac{\gamma^{-5/2}}{\sqrt{\omega^2 - a^2u^2}}. \tag{14b}$$

The first two solutions $k_{1,2}$ correspond to the electromagnetic waves ("1" — forward and "2" — backward), whereas the solutions $k_{3,4}$ are connected to the beam waves, slowing and fast respectively.

From the conditions that there be no perturbations in the charge densities and beam current at the entrance of the waveguide, $z = 0$, one can obtain two equations, coupling the wave amplitudes A_ν :

$$\sum_{\nu=1}^{4}(k_\nu^2 - a^2)A_\nu = 0, \quad \sum_{\nu=1}^{4} k_\nu(k_\nu^2 - a^2)A_\nu = 0. \tag{15}$$

The third equation following from the second boundary condition (4) is

$$\sum_{\nu=1}^{4} k_\nu A_\nu = 0. \tag{16}$$

These three equations allow us to express the forward waves amplitudes A_1, A_3 and A_4 in terms of backward wave amplitude A_2 :

$$\frac{A_1}{A_2} = 1, \quad \frac{A_{3,4}}{A_2} = \mp\omega_b \frac{k_\perp^2 u^2 \gamma^{-1/2}}{(\omega^2 - a^2u^2)^{3/2}}. \tag{17}$$

Note that the amplitudes of beam waves are small, but in spite of this they can't be neglected. It will be shown that neglecting leads to the significant mistake.

The phase velocity of forward electromagnetic wave in waveguide is higher then light speed c, i. e. $v_{ph} = c\sqrt{1 + (k_\perp/a)^2} > c$, and as consequence, $v_{ph} > u$. As a result, in this system the accompanying beam phase modulation by the forward wave is impossible and therefore the wave amplification is impossible too. Thus the stimulated Cherenkov radiation of REB is absent. Besides, as a consequence of the condition (12) aperiodic Pierce instability also is absent in the system.

At the same time in the system the radiative Pierce instability can develop when the positive feed-back process is realized by the backward wave A_2, which is possible only if the wave frequency is higher than the cut off frequency of waveguide, that is $\omega > k_{\perp}c$.

IV. Linear Theory of Radiative Pierce Instability.

In order to reveal the mechanism of transformation of the directed beam energy to the electromagnetic field energy, let us consider the work done by the longitudinal component E_z of the radiation field on an electron as it passes through the resonator,

$$W = e \int\limits_0^L E_z(t[z], z)\, dz. \tag{18}$$

The component E_z is expressed, with the aid of Eq.(3), in term of the potential ψ, which is given by the real part of expression (11).

If we average the work W of the field over the phases of the electrons or, equivalently, over the initial time t_0 of coming into the cavity electrons,

$$\langle W \rangle = \frac{1}{\tau} \int\limits_0^\tau W(t_0)\, dt_0, \quad \omega^{-1} \le \tau \le (\delta\omega)^{-1}, \tag{19}$$

then summing over the phases of the unperturbed electrons makes no contribution to the radiation. In order to obtain nonzero coherent radiation, it is necessary to account the radiation field influence on the motion of beam electrons. When the beam is modulated, the velocity and trajectory of an electron are slightly perturbed, so that

$$v = u + \tilde{v}, \quad t[z] = t_0 + \frac{z}{u} + \tilde{t}. \tag{20}$$

Here the $\tilde{v}$ and $\tilde{t}$ are the solutions of the linearized Eq. (6) of the characteristic system of Vlasov equations,

$$\frac{d\tilde{t}}{dz} = -\frac{\tilde{v}}{u^2}, \quad \frac{d\tilde{v}}{dz} = \frac{e\gamma^{-3}}{mu} E_z, \tag{21}$$

The solution of Eqs. (21) with the potential (11) has the form

$$\tilde{v} = \frac{e}{m}\gamma^{-3}\tilde{\psi}_s(r)\sum_{\nu=1}^{4}\frac{E_\nu}{\omega - k_\nu u}\sin\left[\omega t_0 + \left(\frac{\omega}{u} - k_\nu\right)z\right] \qquad (22a)$$

$$\tilde{t} = \frac{e}{mu}\gamma^{-3}\tilde{\psi}_s(r)\sum_{\nu=1}^{4}\frac{E_\nu}{(\omega - k_\nu u)^2}\cos\left[\omega t_0 + \left(\frac{\omega}{u} - k_\nu\right)z\right]. \qquad (22b)$$

Here $E_\nu = \left(\frac{\omega^2}{c^2} - k_\nu^2\right)A_\nu$ is amplitude of longitudinal component E_z of wave with wave number k_ν.

For a beam that is uniform over the cavity cross section, and for a cylindrical waveguide the transverse structure of potential is $\tilde{\psi}_s(r) = J_0(k_{\perp s}r)$. But if the geometry of beam is another $\tilde{\psi}_s(r)$ and correspondingly $k_{\perp s}$ must be taken as eigenfunction and eigenvalue of this geometry of system.

The work averaged over the phases of the electrons is

$$\langle W\rangle = \frac{e^2\omega}{2mu}\gamma^{-3}\tilde{\psi}_s^2(r)X\sum_{i<j}a_{ij}\left[\frac{1}{(\omega - k_j u)^2} - \frac{1}{(\omega - k_i u)^2}\right], \qquad (23)$$

where

$$X \equiv E_2^2, \quad a_{ij} = \frac{E_i E_j}{E_2^2}\frac{\cos(k_i - k_j)L - 1}{k_j - k_i}. \qquad (24)$$

The averaged work done on the beam electrons per unit time is

$$W_b = \int\limits_{S_\perp}\langle W\rangle n_b(r)u\,dS_\perp. \qquad (25)$$

The integral is taken over the transverse cross section of the cavity.

The concrete value of W_b depends on ratios A_i/A_2. Therefore, in order to calculate a specific value of the work W_b, it is necessary to use the boundary conditions. In the case of small beam current (condition (12)), the ratios A_i/A_2 are given by the relations (17), following from the left-hand boundary condition. But even in this case the quantity W_b turned out undetermined, since the parameter a, which appears in (25) after substituting Eqs. (23), (24) and Eqs. (14), (17), remains undetermined. In the absence of beam quantity a coincides with wave number k_z of the forward wave A_1. This means that this quantity as k_z must be determined from the left– and right–hands boundary conditions of cavity.

At first let us consider the high quality Pierce resonator supposing the electromagnetic field is completely closed in the cavity. Then using the relations (17) and condition of mirror reflection from the outlet of waveguide, which looks as

$$\sum_{\nu=1}^{4} k_\nu A_\nu e^{ik_\nu L} = 0, \tag{26}$$

we obtain (under the condition (12)) equation $\sin(aL) = 0$, which gives $a = \pi n/L$.

Now we can calculate the final expression for W_b :

$$W_b = (-1)^{n+1}\omega_b \frac{R^2}{2} J_1^2(\mu_{s,0}) \frac{\omega^2 u \gamma^{-1/2}}{(\omega^2 - a^2u^2)^{3/2}} \sin(\alpha\omega_b L) \sin\left(\frac{\omega L}{u}\right) X. \tag{27}$$

It is seen from (27) that the main contribution to the work W_b is given by just crossed electromagnetic-beam terms a_{ij} . Namely, when the beam is modulated by the electromagnetic waves of the radiation field, most of the work is done by the beam waves and, on the contrary, when the electron trajectories are perturbed by the beam waves the electromagnetic waves give the main contribution into the radiation. Thus, the two oscillatory systems are coupled : the beam and the electromagnetic field in the resonator. As a result of their coupling, or more correctly, their interaction, the frequency shift $\delta\omega$ arises, which represent the growth rate of the radiative Pierce instability. It is obvious that this instability has the collective character or, in other words, it is Raman type instability[4].

To calculate the growth rate we write down the balance equation for the energy of the cavity

$$\frac{d}{dt}\langle Q\rangle = -W_b, \tag{28}$$

where $\langle Q\rangle$ is the sum of the average kinetic energy of beam electrons and average electromagnetic field energy $\langle Q\rangle = \langle Q_f\rangle + \langle Q_b\rangle$. Under the condition (12), when the Pierce parameter is small, the varying part of energy is mainly determined by the electromagnetic field energy. Accounting this we can rewrite Eq. (28) as

$$\frac{dX}{dt} = 2(-1)^n \omega_b \frac{k_\perp^2 c^2 \gamma^{-1/2}}{(\omega^2 - a^2u^2)^{3/2}} \frac{u}{L} \sin(\alpha\omega_b L) \sin\left(\frac{\omega L}{u}\right) X. \tag{29}$$

X is quadratic in the field, so that

$$\frac{dX}{dt} = 2\delta\omega X, \qquad (30)$$

and therefore $\delta\omega$ representing the growth rate of the instability is

$$\delta\omega = (-1)^n \omega_b \frac{k_\perp^2 c^2 \gamma^{-1/2}}{(\omega^2 - a^2 u^2)^{3/2}} \frac{u}{L} \sin(\alpha\omega_b L) \sin\left(\frac{\omega L}{u}\right) \qquad (31)$$

at the frequency

$$\omega = \omega_{s,n} = c\sqrt{\left(\frac{\mu_{s,0}}{R}\right)^2 + \left(\frac{\pi n}{L}\right)^2}. \qquad (32)$$

The condition for the development of the instability is

$$(-1)^n \sin(\alpha\omega_b L) \sin\left(\frac{\omega L}{u}\right) > 0, \qquad (33)$$

which generalizes Eq. (1) and determines parameters of the beam-wave-guide systems for which the electromagnetic waves can be excited in the cavity. In accordance with Eq. (31) in general the lowest longitudinal and transverse modes have the maximal growth rate and they must be excited first of all. This conclusion agrees with experiment results [2].

Very important function is the dependence of growth rate (31) on the geometrical parameter $\xi = \dfrac{L}{R}$. In the sufficiently long systems when $\xi \gg 1$ we have $\omega \simeq k_\perp c$ and

$$\delta\omega = (-1)^n \omega_b \frac{u}{c} \frac{\gamma^{-1/2}}{k_\perp L} \sin\left(\frac{\omega_b L}{u} \gamma^{-5/2}\right) \sin\left(\frac{\omega L}{u}\right). \qquad (34)$$

The maximal value of $\delta\omega$ is of the order $\delta\omega \sim \omega_b \left(\dfrac{u}{c}\right) \gamma^{-1/2} \dfrac{R}{L}$. In the opposite case of short systems when $\xi \ll 1$, we obtain $\omega \approx ac = \pi nc/L$ and

$$\delta\omega \sim \omega_b^2 \left(\frac{k_\perp}{a}\right)^2 \approx \omega_b^2 \xi^2. \qquad (35)$$

A comparison of Eq.(34) with Eq.(35) permit to conclude that the optimal parameters for wave excitation correspond to $\xi \sim 1$ and $\gamma = \sqrt{3} \approx 1{,}7$.

The considered in this section radiative Pierce instability in high quality (ideal) resonator has no current threshold. But this statement lose its force as soon as the emission of radiation from the cavity will be account.

V. Dispersion equation.

In this section the dispersion equation of nonresonance radiative Pierce instability is derived in the most common form. Our consideration deals with only two geometries of electron beam: a beam that is uniform over the cavity cross section and a beam that is infinitely thin tubular with radius r_b. In first case, the amplitude A_ν is determined by Eq.(11), in the second case $A_\nu = \tilde{\psi}_\nu(r_b)$, where transverse structure of the field $\tilde{\psi}_\nu(r)$ is given by Eq.(7).

The conditions, that there be no perturbations in the charge densities and current of beam in the entrance of cavity $z = 0$, give two equations

$$\sum_{\nu=1}^{4} \frac{k_\nu^2 - \frac{\omega^2}{c^2}}{(\omega - k_\nu u)^2} k_\nu A_\nu = 0, \quad \sum_{\nu=1}^{4} \frac{k_\nu^2 - \frac{\omega^2}{c^2}}{(\omega - k_\nu u)^2} A_\nu = 0, \tag{36}$$

which are obtained from Eqs.(8). The boundary condition (4) leads to the third equation

$$\sum_{\nu=1}^{4} k_\nu \tilde{\psi}'_\nu(r) = 0, \tag{37}$$

which can be rewritten in the form

$$\sum_{\nu=1}^{4} k_\nu \left(\frac{\tilde{\psi}'_\nu}{\tilde{\psi}_\nu(r_b)} \cdot f \right) A_\nu \equiv \sum_{\nu=1}^{4} k_\nu \mu_\nu(f) A_\nu = 0. \tag{38}$$

Here $f = f(r)$ is an arbitrary integrable function. The scalar product is determined by

$$(f \cdot g) = \int_{S_\perp} f(r) g(r) dS_\perp. \tag{39}$$

The integral is taken over the transverse cross section of the cavity $S_\perp$.

The Eq.(36) and Eq.(38) determine transformation coefficients of waves on the left – hand (input) electrode. In case, when the beam current is small, the lows of waves dispersion in the cavity are defined by the following equations

$$k_{1,2} = \pm a \pm \delta k_{1,2} \tag{40a}$$

$$k_{3,4} = \frac{\omega}{u} \pm \delta k_3 . \tag{40b}$$

Here the following statement, that is $\delta k_{1,2} = O(\chi)$ and $\delta k_{1,2} = O(\sqrt{\chi})$, takes place. Therefore $\tilde{\psi}_1(r) = \psi_2(r) + O(\chi)$, and as a consequence $\mu_1(f) = \mu_2(f) + O(\chi)$. Under these conditions the coefficients of wave transformation on the input electrode are

$$R_1^L = \frac{A_1}{A_2} = 1 \tag{41a}$$

$$R_3^L = \frac{A_3}{A_2} = \frac{(\frac{\omega^2}{c^2} - a^2)u^2}{\omega^2 - a^2u^2}\gamma^2 \frac{u\delta k_3}{\omega} \tag{41b}$$

$$R_4^L = \frac{A_4}{A_2} = -R_3^L . \tag{41c}$$

Let us consider situation when the part of radiation leaves the cavity. In this case three forward waves A_1, A_2 and A_4 in the plane $z = L$ partially reflect transformating into the backward wave A_2 and partially emit transformating into the output radiation. In the most common form this process can be described by

$$A_2 e^{ik_2 L} = \sum_{\substack{\nu=1 \\ \nu\neq 2}}^{4} \varkappa_\nu A_\nu e^{ik_\nu L}, \tag{42}$$

where $\varkappa_\nu = \varkappa_{1,3,4}$ are the wave transformation coefficients in the boundary $z = L$, and what's more $\varkappa_3$ and $\varkappa_4$ are concerned with the beam waves transformations and $\varkappa_1$ represents the reflection coefficient for the forward electromagnetic wave. It is the most difficult to calculate coefficients $\varkappa_\nu$. Today it can be done only for cavity with low density beams, when condition (12) takes place. In this limit the quantity of $\varkappa_1$ coincides approximately with the value of reflection coefficient of electromagnetic wave in cavity without beam. Moreover in this limit the difference between $\varkappa_3$ and $\varkappa_4$ turns out to be of the order of ω_b and therefore in the first approximation we can accept $\varkappa_3 = \varkappa_4 = \varkappa_b$.

Having substituted Eqs.(41), in Eq.(42) we obtain the dispersion equation of the nonresonant stimulated Pierce instability

$$D(\omega) = D_0(\omega) + D_1(\omega) = 0. \tag{43}$$

Here

$$D_0(\omega) = \varkappa_1 e^{iaL} - e^{-iaL} \tag{44}$$

is unperturbed part of dispersion equation. Equation $D_0(\omega) = 0$ determines the spectrum of cavity frequencies, namely

$$a(\omega) = \frac{\pi n}{L} - \frac{1}{2L} \arg \varkappa_1 + \frac{i}{2L} \ln |\varkappa_1|, \tag{45}$$

where reflection coefficient $\varkappa_1$ can change within the limits $|\varkappa_1| \leq 1$, $-\pi \leq arg(\varkappa_1) \leq 0$.

The perturbed part of dispersion equation

$$D_1(\omega) = -2i\varkappa_b \frac{(\frac{\omega^2}{c^2} - a^2)u^2}{\omega^2 - a^2u^2} \gamma^2 \frac{u\delta k_3}{\omega} \sin(L\delta k_3) e^{i\frac{\omega L}{u}}, \tag{46}$$

determines the shift $\delta\omega$ in the frequency, that is

$$\delta\omega = -\frac{D_1(\omega)}{\frac{\partial D_0(\omega)}{\partial \omega}}. \tag{47}$$

Under condition $a\frac{d\varkappa_1}{da} \ll 1$ from Eq.(46) and Eq.(47) we obtain the finish result

$$\delta\omega = (-1)^n \frac{|\varkappa_b|}{\sqrt{|\varkappa_1|}} \frac{(\frac{\omega^2}{c^2} - a^2)u^2}{\omega^2 - a^2u^2} \gamma^2 \frac{d\omega}{da} \frac{u\delta k_3}{L\omega} \sin(L\delta k_3) e^{i\theta}. \tag{48}$$

Here

$$\theta = \frac{\omega L}{u} + arg(\varkappa_b) - \frac{1}{2} arg(\varkappa_1) \tag{49}$$

is a transformed electron drift angle.

Eq.(48) shows that development of instability is determined by shift δk_3 in the longitudinal wave number of slow beam wave.

VI. Linear Theory of Oscillator. Starting Current.

From the application point of view the most interesting case is the system with open resonator, or in other words, with a radiation horn at the outlet $z = L$. As the electromagnetic energy losses by the output emission is compensated by the development of the stimulated radiation instability, in the system the stationary state is established. A beam

Langmuir frequency corresponding to this state is called starting Langmuir frequency $\omega_{b,st}$ and corresponding beam current is called starting current I_{st} .

In this Sec. we consider the device with beam that is uniform over the cavity cross section. In this case the low of dispersion $k_\nu = k_\nu(\omega)$ is given by Eqs.(14). Then, a shift $\delta\omega$ in the frequency, whose imaginary part is nothing other than the growth rate of the instability, is

$$\delta\omega = (-1)^n \frac{|\varkappa_b|}{\sqrt{|\varkappa_1|}} \omega_b \frac{ac^2}{\omega L} \frac{k_\perp^2 u^2 \gamma^{-1/2}}{(\omega^2 - a^2u^2)^{3/2}} \sin(\alpha\omega_b L) e^{i\theta}, \tag{50}$$

A frequency of emission is

$$\omega = c\sqrt{\left(\frac{\mu_{s,0}}{R}\right)^2 + (\mathrm{Re}\, a)^2}. \tag{51}$$

From Eqs.(45) under condition $Re(a) \gg Im(a)$ one can obtain decrease rate due to emission from cavity

$$\delta = \frac{1}{2L} \frac{\partial\omega}{\partial k_z} \ln \frac{1}{|\varkappa_1|} \tag{52}$$

The condition for the development of the instability $Im\delta\omega > \delta$ looks as

$$(-1)^n \sin(\alpha\omega_b L) \sin\theta > \frac{\omega_{b\,st}}{\omega_b}, \tag{53}$$

and determines the current threshold. Here starting beam Lengmuir frequency is

$$\omega_{b\,st} = k_\perp c \frac{\gamma^{1/2}}{2} \left(\frac{c}{u}\right)^2 \left(1 + \frac{a^2}{k_\perp^2 \gamma^2}\right)^{3/2} \frac{\sqrt{|\varkappa_1|}}{|\varkappa_b|} \ln \frac{1}{|\varkappa_1|}. \tag{54}$$

Corresponding starting beam current is

$$I_{st} = \frac{mc^3}{e} \frac{\mu_{s,0}^2}{4} \frac{u}{c} \left(\frac{S_b}{S_\perp}\right) \left(\frac{\omega_{b\,st}}{k_\perp c}\right)^2 \tag{55}$$

Note that, the quantity $\frac{mc^3}{e}$ has the units of current and equals to 17,03 kA.

The Eq.(54) generalizes Eq.(33) and gives the half-width of electron drift angle

$$\Delta = \arccos\left(\frac{\omega_{b\,st}}{\omega_b|\sin(\alpha\omega_b L)|}\right). \tag{56}$$

within limits of which the instability develops. Here we supposed, that $\omega_{b\,st} \leq \omega_b|sin(\alpha\omega_b L)|$.

From the linear theory one can estimate the radiation efficiency of the oscillator. Giving energy to the electromagnetic field the beam electron decelerate, and the condition (53) is violated. It is this mechanism for the instability saturation that is typical of the Raman regime. The maximum change in the velocity for which the wave amplification no longer occurs is

$$\delta u = \frac{u^2}{\omega L}\Delta, \tag{57}$$

In this case, the power pumped out into the wave energy flux is

$$P = n_b umc^2\delta\gamma = n_b u^2 m\gamma^3\frac{u}{\omega L}\Delta, \tag{58}$$

From Eq.(58) we obtain an upper bound for the radiation efficiency η

$$\eta = \frac{\delta\gamma}{\gamma-1} \approx \left(\frac{u}{c}\right)^2\frac{\gamma^3}{\gamma-1}\frac{u}{\omega L}\Delta. \tag{59}$$

For $\gamma \gg 1$ we have $\eta \sim \gamma^2\Delta/\xi$. We can see that relativistic – electron lasers are more efficient than nonrelativistic ones. In addition, from above presented estimation one can conclude, that the beam current ought to exceed the starting current more than two times to give effective generation. However this statement is only approximate. The exact calculations of optimal beam current can be done only by numerical simulation of nonlinear problem of wave excitation in considered system.

If the cavity is closed, that is the mirror reflection in the plane $z = L$ takes place, then $\varkappa_1 = 1$ and $\varkappa_b = \omega/au > 1$.

The simplest model of the open resonator with radiation horn is a half–infinite cylindrical waveguide fulfilled in space $z \geq L$ by an uniform dielectric material with permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + \varepsilon''(\omega)$. By choosing $\varepsilon(\omega)$ one can simulate any radiation horn. Supposing that the dielectric is transparent for REB one can easily obtain

$$\varkappa_1 = \frac{\varepsilon a - a_\varepsilon}{\varepsilon a + a_\varepsilon} \tag{60a}$$

$$\varkappa_b = \frac{\varepsilon - 1}{\varepsilon}\frac{\omega}{au}\frac{a_\varepsilon + \frac{\omega}{u}\gamma^{-2}}{\left(1 + \frac{a_\varepsilon}{\varepsilon a}\right)\left(a_\varepsilon + \frac{\omega}{u}\right)}, \tag{60b}$$

where $a_\varepsilon^2 = \varepsilon(\omega)\frac{\omega^2}{c^2} - k_\perp^2$. For high quality resonator, when $\varepsilon(\omega) \gg 1$,

$$\varkappa_1 \approx 1 - 2\frac{\omega}{ac}\frac{1}{\sqrt{\varepsilon}}, \quad \varkappa_b \approx \frac{\omega}{au}. \tag{61}$$

In this limit the starting current is equal

$$I_{st} = \frac{mc^3}{e}\frac{\mu_{s,0}^2}{4}\frac{\gamma}{|\varepsilon|}. \tag{62}$$

VII. Inhomogeneous beam system.

Finally, we consider thin annular beams that are most widely used in real experiment [22] to obtain high efficiency in exploitation of beams. Let us supposed that the waveguide is only partially fulfilled by the beam in the region $r_1 \le r \le r_2 < R$. The thickness of the beam is $\Delta = r_2 - r_1$, the average radius of beam is $r_b = (r_1 + r_2)/2$. If the condition for thin annular beam $\Delta_b \ll r_b$ takes place, then we can take the profile of the beam as $p_b(r) = \Delta_b\delta(r - r_b)$. In this case Eq.(9) leads to the boundary conditions

$$\tilde{\psi}_\nu'(r_b + 0) - \tilde{\psi}_\nu'(r_b - 0) = -\omega_b^2\gamma^{-3}\frac{k_\parallel^2 - \frac{\omega^2}{c^2}}{(\omega - k_\parallel u)^2}\Delta_b\tilde{\psi}_\nu(r_b). \tag{63}$$

Here $\tilde{\psi}_\nu(r)$ is continuous. Below we consider electromagnetic and beam waves separately.

The electromagnetic waves ($\nu = 1, 2$) are volumetric and therefore the solution of the Eq.(9) can be represent as

$$\tilde{\psi}_\nu(r) = \begin{cases} J_0(k_{\perp\nu}r), & r < r_b \\ \dfrac{J_0(k_{\perp\nu}r_b)}{\mathcal{G}_0(k_{\perp\nu}r_b)}\mathcal{G}_0(k_{\perp\nu}r), & r > r_b \end{cases} \tag{64}$$

where

$$\mathcal{G}_n(k_{\perp\nu}r) = J_n(k_{\perp\nu}r) - \frac{J_0(k_{\perp\nu}R)}{Y_0(k_{\perp\nu}R)}Y_n(k_{\perp\nu}r). \tag{65}$$

Substituting Eq.(64) into the Eq.(63) we obtain the equation for wave number $k_{\perp\nu}$ to define

$$\frac{J_0(k_{\perp\nu}R)}{Y_0(k_{\perp\nu}R)} = \frac{\pi}{2}\omega_b^2\gamma^{-3}\frac{k_{\parallel\nu}^2 - \frac{\omega^2}{c^2}}{(\omega - k_{\parallel\nu}u)^2}\Delta_b r_b J_0(k_{\perp\nu}r_b)\mathcal{G}_0(k_{\perp\nu}r_b). \tag{66}$$

This equation can be approximately solved under following condition

$$\frac{\omega_b^2 \Delta_b r_b}{u^2 \gamma^3} \ll 1, \tag{67}$$

which is equivalent to Eq.(12). In this limit one can represent $k_{\perp\nu} = k_{\perp s} + \delta k_{\perp\nu}$, where $k_{\perp s} = \mu_{s,0}/R$ is the solution of Eq.(66) in the absence of the beam, and $\delta k_{\perp\nu}$ is a small correction being equal

$$\delta k_{\perp\nu} = \omega_b^2 \gamma^{-3} \frac{\frac{\omega^2}{c^2} - k_{\|\nu}^2}{(\omega - k_{\|\nu} u)^2} \cdot \frac{G_s}{2k_{\perp s}}. \tag{68}$$

Here

$$G_s = 2\frac{r_b \Delta_b}{R^2} \frac{J_0^2(k_{\perp s} r_b)}{J_1^2(\mu_{s,0})} = \frac{S_b}{S_\perp} \cdot \frac{J_0^2(k_{\perp s} r_b)}{J_1^2(\mu_{s,0})}, \tag{69}$$

Taking into account Eq.(68), from Eq.(9) we obtain the characteristic relation

$$k_{\perp s}^2 + \left(k_{\|\nu}^2 - \frac{\omega^2}{c^2}\right)\left[1 - \frac{\omega_b^2 \gamma^{-3}}{(\omega - k_{\|\nu} u)^2} G_s\right] = 0, \tag{70}$$

The solution of this relation under conditions (67) is

$$k_{\|1,2} = \pm a \pm \frac{\beta_{1,2}}{2a} G_s \omega_b^2, \tag{71}$$

The quantities a and $\beta_{1,2}$ are determined by Eqs.(13),(14).

Quite another situation takes place for beam waves ($\nu = 3,4$), which turn out to be surface, and therefore their transverse structure is

$$\tilde{\psi}_\nu(r) = \begin{cases} I_0(k_{\perp\nu} r), & 0 \leqslant r \leqslant r_b \\ \dfrac{I_0(k_{\perp\nu} r_b)}{H_0(k_{\perp\nu} r_b)} H_0(k_{\perp\nu} r), & r_b \leqslant r \leqslant R \end{cases} \tag{72}$$

Here

$$H_n(k_{\perp\nu} r) = I_n(k_{\perp\nu} r) K_0(k_{\perp\nu} R) - I_0(k_{\perp\nu} R) K_n(k_{\perp\nu} r). \tag{73}$$

Substitution Eq.(72) into Eq.(63) leads to the characteristic equation

$$\left(k_{\|\nu} - \frac{\omega}{u}\right)^2 = \frac{\omega_b^2 \gamma^{-3}}{\varkappa_\perp^2 u^2}\left(k_{\|\nu}^2 - \frac{\omega^2}{c^2}\right), \tag{74}$$

determinating the $k_{\|\nu}$. Here parameter $\varkappa_\perp^2$ is given by the expression

$$\varkappa_\perp^2 = \left[r_b \Delta_b \left(\frac{K_0(k_{\perp\nu} r_b)}{I_0(k_{\perp\nu} r_b)} - \frac{K_0(k_{\perp\nu} R)}{I_0(k_{\perp\nu} R)} \right) I_0^2(k_{\perp\nu} r_b) \right]^{-1} \tag{75}$$

The solutions of Eq.(74) under the condition (67) are

$$k_{\|\nu} = \frac{\omega}{u} \pm \tilde{\alpha}\omega_b, \quad \tilde{\alpha} = \frac{\omega}{u} \frac{\gamma^{-5/2}}{\varkappa_\perp u}. \tag{76}$$

Using the dispersion equation for cable waves in vacuum we find quantities $k_{\perp\nu} = \frac{\omega}{u}\gamma^{-1} \pm \tilde{\alpha}\gamma\omega_b$. Now we can calculate the parameter $\varkappa_\perp$, which turns out to be approximately equal for both fast and slow beam waves.

The growth rate of instability is equal $Im\delta\omega$, where the shift in frequency is

$$\delta\omega = (-1)^n \frac{|\varkappa_b|}{\sqrt{|\varkappa_1|}} \omega_b \frac{\gamma^{-1/2}}{\varkappa_\perp L} au \frac{k_{\perp s}^2 c^2}{\omega^2 - a^2 u^2} \sin(\tilde{\alpha}\omega_b L) e^{i\theta} \tag{77}$$

The frequency is given by Eq.(51).

The condition for development of the instability $Im\delta\omega > \delta$ can be rewritten in form

$$(-1)^n \sin(\tilde{\alpha}\omega_b L) \sin\theta > \frac{\omega_{b\,st}}{\omega_b}, \tag{78}$$

where starting Langmuir frequency is

$$\omega_{b\,st} = \frac{1}{2} \frac{\sqrt{|\varkappa_1|}}{|\varkappa_b|} \ln \frac{1}{|\varkappa_1|} \cdot \varkappa_\perp c \frac{\gamma^{3/2}}{(\gamma^2 - 1)^{1/2}} \left(1 + \frac{a^2\gamma^{-2}}{k_{\perp s}^2} \right). \tag{79}$$

The corresponding starting beam current is given by Eq.(55).

In conclusion let us consider some numerical parameters for microwave oscillator based on the radiative Pierce instability. We will take beam parameters very close to the experimental ones: $I_b = 1\,\mathrm{kA}$, $r_b = 2{,}0\,\mathrm{cm}$, $\Delta_b = 0{,}2\,\mathrm{cm}$ and $V = 450\,\mathrm{kV}$.

Then $\omega = 1{,}8 \cdot 10^{10}\,s^{-1}$ and $\gamma = 1{,}9$. If $R = 4{,}0$ cm then the radiation frequency is $\omega_b = 1{,}76 \cdot 10^{10}\,s^{-1}$. For this parameters the inequality (78) looks as

$$(-1)^n \sin(0{,}016L) \sin(0{,}71L + \varphi) > 6{,}0\Lambda, \tag{80}$$

where $\Lambda = -\sqrt{|\varkappa_1|}|\varkappa_b|^{-1}\ln|\varkappa_1|$; L is given in centimeter (cm). Starting Langmuir frequency is equal $\omega_{b\,st} = 1{,}06 \cdot 10^{11}\Lambda(s^{-1})$. The calculation using Eq.(80) shows that excitation of wave must be only for $\Lambda \leqslant 0{,}05$. Thus the instability develops in cavity with high Q–factor. For $L = 33 \div 38$ cm and $L = 42 \div 47$ cm the even longitudinal modes may excited, whereas for $L = 31 \div 33$ cm and $L = 38 \div 43$ cm the odd longitudinal mode may excited.

VIII. Nonlinear theory of instability.

The nonlinear system of Maxwell – Vlasov equations can be studied only by numerical simulation of the instability. According to the linear theory, several modes with similar frequencies are excited at once in the cavity. The absence of a distinct frequency, as well as wavelength, makes it impossible to separate the field into slowly and rapidly varying components with subsequent averaging over time or position. Thus, we found it necessary to use the methods described in Ref. 21. The Maxwell's equations are solved directly using by finite – different elements. The beam electrons were simulated by a Particle In Cell (PIC) method [21]. In our case, the Maxwell's equations are solved under ideal boundary condition $E_\tau|_s = 0$, when the radiation is closed in the cavity. The initial conditions for particle of beam are $v_z(z=0) = 0$, $t_z(z=0) = t_0$. The following unitless quantities were used in the numerical simulation:

$$\tau = \frac{u}{L}t, \quad p = \frac{p_z}{mc}, \quad \mathcal{E} = \frac{eL}{mc^2\gamma^3}E_z. \tag{81}$$

Here $p_z = m\gamma(v_z)v_z$, $\gamma = (1 - u^2/c^2)^{-1/2}$ is a nonperturbed relativistic factor of beam electrons.

In the numerical calculation, we considered an infinitely thin tubular beam and used a soft regime for its entry into the cavity (the beam front was very smooth). The Pierce parameter χ was defined as the ratio of the working current I to the limiting Pierce current I_0 , which is

$$I_0 = \frac{mc^3}{2\,e}\left(\frac{u}{c}\right)^3 \frac{\gamma^3}{\ln\left(\frac{R}{r_b}\right)} \tag{82}$$

for considered configuration. In all the calculations we chose $r_b/R = 0{,}4$. Here the instability saturated after 10 — 10000 transit times τ , depending on the beam current and the cavity geometric parameter $\xi = L/R$. In most cases the first transverse mode s=1 was exited. For very short systems $\xi \sim 1$ and for some values of ξ in long systems, higher transverse modes were exited. For short systems, with $\xi \leq 12$, the instability regime was mainly single-mode; for certain values of ξ two longitudinal harmonics corresponding to the selection rule (33) were exited. In long systems ($\xi > 12$), a multimode regime sets in. In fact, for large ξ ($\xi \gg 1$), the relation

$$\xi = \frac{1}{2}\frac{c}{u}\frac{\pi}{\mu_{s,0}}\frac{n_2^2 - n_1^2}{\Delta\theta} \tag{83}$$

holds, where n_i is the longitudinal mode number. Two modes with the same growth rates can be exited if the difference between their drift angles obeys $\Delta\theta < \pi$. Since Eq.(33) implies that either even or odd harmonics can be exited simultaneously in long systems, we find that for $\gamma = 2$, when $\xi \geq 3$, a two-mode regime exists and for $\xi \geq 12$, a three-mode regime.

We have distinguished two pictures of saturation of the instability. The first scenario occurs in short ($\xi < 12$) systems with single-mode regimes. In this case, the nonlinear shift of longitudinal wave number $k_\nu = k_\nu(A_\mu)$ takes place. In addition, the parametric instability develops [23]. In particular, confluence-decay process

$$\omega_{1,n} + \omega_{1,n} \rightarrow \omega_{1,n-1} + \omega_{1,n+1}, \tag{84}$$

which results in pumping of energy from the wave back into the beam in accordance with the condition (33), can occurs in the system.

Figure 1 shows the results of a calculation for $\xi = 6$, $\gamma = 2$ and $\chi = 0{,}05$ (which roughly corresponds to a current $I \approx 2{,}4$ kA). Regular oscillations in the field amplitude with a modulation frequency on the order of $\delta\omega$ in a steady-state saturation regime can be seen clearly. The position of the beam electrons in the phase plane is represented by Fig. 1b. For time $\tau = 45$, when the instability is still linear, the modulation of the beam is purely harmonic. As the instability is develops, nonlinear distortions appear, which cause breaking; the electrons begin to overtake each other and the beam stratifies in velocity. Breaking occurs when the instability passes into a stationary regime. As the instability becomes saturated, changes in the electron density acquire the character of a deep modulation. This corresponds to an increase in the ratio of the amplitudes of the beam waves to the electromagnetic wave and to a shift in the longitudinal wave numbers k_ν ($\nu = 3, 4$) toward larger values. The beam remains cold, despite the rather long time after the radiation amplitude reaches its stationary level.

In long systems ($\xi \geq 12$) when the instability is multimode from the outset the mechanism responsible for saturation is the randomization of the beam particles in the field of many waves. As a result of the stochastic motion of the beam electrons, the modulation of the beam becomes uniform, i.e. the phases of the electromagnetic field relative to the electrons are distributed completely random in the phase interval $[0, 2\pi]$. Thus the contribution to the stimulated emission goes to zero. As the numerical studies showed, saturation of the instability sets in simultaneously with the chaos of the beam particles. Fig.2 shows the calculations for $\xi = 18$, $\gamma = 2$ and $\chi = 0{,}05$. The phase plane of beam electrons is shown for the time of instability saturates, $\tau = 40$. By the middle of the cavity the beam is completely randomized. Although the beam is highly chaotic it is still modulated at the initial level. For this operating regime of the generator the field in the cavity has a broad spectrum of longitudinal harmonics. Beam bunching was absent for both instability saturation mechanisms.

In order to estimate the degree of chaos in the motion of electrons at the time when the instability began to saturate two test particles separated by a rather small distance in the phase plane with velocities roughly equal to the beam electron velocity u were launched into the cavity. The maximum distance by which particles could separate in the phase plane as they pass through the cavity was chosen as a measure of the chaos of the beam electron motion. Fig.3 is a plot of the maximum phase separation between the probe particles, $D = \sqrt{(z/L)^2 + p^2}$, as a function of the controlling parameter which is the geometric factor ξ for $\gamma = 2$ and $\chi = 0{,}05$. This graph shows clearly the existence of two saturation pictures for the instability which appear in regions consistent with the above estimates.

As the beam current is raised and the Pierce parameter χ approaches unity (or greater) Pierce potential aperiodic instability develops in the system. Fig.4 shows a phase pattern of the beam at the time the instability saturates when $\tau = 10$ holds for $\xi = 18$, $\chi = 0{,}95$ and $\gamma = 2$. A virtual cathode is observed to form at the cavity entrance and the beam electrons are partially reflected from it. In the meantime the chaotization of the beam corresponds to the presence in the cavity of a radiative instability whose development is somewhat suppressed by the potential instability. As χ is increased further the growth rate of the aperiodic Pierce instability increases more rapidly than that of the radiative instability. As a result the potential branch suppresses the radiative branch. Thus the aperiodic and radiative Pierce instability can be regarded as two regimes of a single instability which are realized for different values of χ.

At last let us consider the question of the generator efficiency. Since the radiation is trapped inside the resonant cavity we take the conversion efficiency of the beam electron energy to electromagnetic radiation energy be the ratio of the radiation flux of forward electromagnetic wave to the incident beam energy flux,

$$\eta = \frac{\langle|\vec{S}|\rangle}{mc^2 nu\gamma}, \tag{85}$$

where $\langle|\vec{S}|\rangle$ is the magnitude of the Pointing vector of the forward electromagnetic wave near the right boundary of the waveguide averaged over a long time interval $t \gg 2\pi/\omega$. Fig.5 shows that the function $\eta = \eta(\xi)$ attains its maximum in relatively long systems with $2 < \xi < 8$. When the cavity length is increased further the conversion efficiency falls off rapidly in agreement with the linear theory. Therefore ξ of order 5–6 is the optimum. The dependence of η on the relativistic factor γ of the beam is consistent with the linear theory. As γ increases the energy conversion efficiency η initially increases quadratically with γ but then saturates for $\gamma > 5$ (Fig.6). This is similar for that of the Cherenkov instability [24].

IX. Dynamic instability of the beam particle motion.

After the instability reaches the nonlinear stage the field amplitude continues to rise slowly. The regular amplitude modulations in the field are replaced by random oscillations. The alternation in the system takes place (Fig.7). This is because as the instability develops further, the frequency spectrum of the oscillations broadens (Fig.8) owing to nonlinear many-wave processes which cause a redistribution of the energy in the radiation spectrum

$$\omega_{1,n} + \omega_{1,n} \to \omega_{1,n-m} + \omega_{1,n+m} \tag{86}$$

in terms of the integer $m < n$.

Even under ideal boundary conditions a Pierce oscillator is an open system since it exchanges energy with the surroundings by means of an electron beam. The development of an instability assumes the existence of strong positive feedback. Naturally, in such systems there is a dynamic instability of motion — an exponential spread in the particle trajectories [25] which was observed in the nonlinear stage of the numerical calculations for long systems with $\xi \geq 8$. It is development of dynamic chaos that causes broadening of the spectrum of the oscillations.

Since the time of residence of particles in cavity is limited by value having magnitude in the order of L/u to make use of k-entropy as a feature of dynamic chaos is difficult. Therefore the following alternately approach is made use of. Under conditions being defined by external parameters as χ, ξ, u the evolution of test particles motions is studied in phase and coordinate spaces. In small vicinity of initial point (t_0, u) the ensemble of $50 \div 60$ points corresponding initial conditions of launched particles is taken.

Since the initial conditions for trajectories of electrons are almost similar the test particles can be considered as ensemble. And therefore we can obtain the ensemble — averaged characteristics such as

$$\hat{R}(t) = \langle k_\perp |\delta z(t_0, t)| \rangle \tag{87a}$$

$$\hat{D}(t) = \left\langle \sqrt{(k_\perp \delta z)^2 + \left(\frac{\delta p}{mc}\right)^2} \right\rangle, \tag{87b}$$

where brackets $\langle \ldots \rangle$ mean taking an ensemble average; δz and δp are phase coordinates being defined by

$$\delta z = z(t_0, t) - u(t - t_0), \quad \delta p = p_z - m\gamma u \tag{88}$$

Using numerical calculation we find (Fig.9) that the exponential scattering of particles is replaced by oscillations of $\hat{R}(t)$ and $\hat{D}(t)$. The oscillations of $\hat{D}(t)$ means that in system the electrons of beam are confused and this process is similar to the "baker's transformation".

During the numerical calculations for some values of ξ an intermittence between the ordered and chaotic regimes was observed in time. Besides the regime instability stratification of the beam into two components was observed: cold, in which the particle motion was ordered, and hot, in which the particles were subject to a dynamic instability. Here the transit time through the cavity for the cold particles was considerably shorter than the residence time in the cavity for the hot particles.

X. Conclusions.

The nonresonance radiative Pierce instability can develop in the systems with sufficiently strong positive feed-back effect. It is this feature that makes Pierce instability to be universal unlike resonance instabilities (Cherenkov or cyclotron ones). The same feature discards the creation of amplifier working with this effect but allows to create the wide class of oscillators.

Some advantages of radiative Pierce instability ought to be noted.

1. The wave slowing structures are not needed for the device to operate which is very important for realization of high power microwave oscillator.

2. Existence of two different instability regimes allows to realize both narrow and wide band microwave oscillators. The fact that the regimes depend only on the geometric parameter ξ of the cavity makes an attempt to construct a tunable generator tempting.

3. Numerical simulation showed that for optimum generation the device efficiency can be on the order of 20–30%.

4. The Pierce oscillator with beam is the open self-oscillatory spreading system. Naturally, in this system the dynamic chaos develops. To understand this phenomenon is very important from the fundamental science of view and therefore this instability ought to be studied very carefully as soon as possible.

Appendix A.

Let us consider one–dimensional Vlasov equation for collisionless plasmas

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{e}{m}\gamma^{-3}E_z\frac{\partial f}{\partial v} = 0 \qquad \text{(A1)}$$

The solution of Eq.(A1) can be represented in form

$$f(t,z,v) = \int\int f_0(v_0)\delta[z - z(t,z_0,v_0)]\delta[v - v(t,z_0,v_0)]\,dz_0dv_0. \qquad \text{(A2)}$$

Here $f_0(v_0)$ is an initial distribution function for beam electrons; $z(t,z_0,v_0)$ and $v(t,z_0,v_0)$ are solutions of the characteristic system (6) for Vlasov equation; $z_0 = z(t=0)$, $v_0 = v(t=0)$.

For monoenergy electron beam the initial distribution function is

$$f_0(v_0) = n_b p_b(r)\delta(v_0 - u). \qquad \text{(A3)}$$

In this case it is easy to obtain the perturbed parts for charge and current densities, namely Eqs.(5).

REFERENCES

1. Muller J.J., Rostas E., "Helvet. Phys. Acta" **13** (1940), 435–350.
2. Biquard F., Grivet P., Septier A., "IEEE. Trans." **4** (1968), 354.
3. Pierce J.R., "J. Appl. Phys." **15** (1944), 721.
4. Kuzelev M. V., Rukhadze A. A., *"Plazma Free Electron Lazers"*, Paris, Edit. Frontier, 1995.
5. Vladimirov V. V., Mosiyuk A. N., Mukhtarov M. A.,, Sov. Journ. "Plazma Physics" **9** (1983), 992.
6. Mosiyuk A. N., Mukhtarov M. A., Sov. Journ. "Plazma Physics" **10** (1984), 878.
7. Mosiyuk A. N., Sov. Journ. "Plazma Physics" **12** (1986), 1493.
8. Mosiyuk A. N., Sov. Journ. "Plazma Physics" **13** (1987), 42.
9. Kuhn S., "Contrib. Plasma Phys." **34** (1994), 495.
10. Friedel H., Grauer R., Spatschek K.H., "Physics of Plasmas" **5** (1998), 3187.
11. Hörhager M., Kuhn S., "Phys. Fluids B" **2** (1990), 2741.
12. Yulpatov V. K., Sov. Journ. "Radiophysics" **13** (1970), 1784.
13. Smorgonsky A. V., Sov. Journ. "Radiophysics" **16** (1973), 150.
14. Kalmikova S. S., Sov. Journ. "Doklady" **208** (1973), 1062.
15. Kalmikova S. S., Sov. Journ. "Doklady" **215** (1974), 814.
16. Klochkov D. N., Rukhadze A. A., "Plasma Physics Reports" **23** (1997), 598.
17. Klochkov D. N., Pekar M. Yu., Rukhadze A. A., Sov. Journ "Radophysics and Electronics" **44** (1999), 379.
18. Klochkov D. N., Pekar M. Yu., Rukhadze A. A., "Plazma Physics Reports" **25**, (1999), 54.
19. Klochkov D. N., Pekar M. Yu., Rukhadze A. A., JETP **116** (1999), 1115.
20. Kuzelev M. V., Loza O. T., Ponomaryov A. V., Rukhadze A. A., Strelkov P. S., Ulyanov D. K., Shkvarunetz A. G., JETP **109** (1996), 2048.
21. Berezin Yu. A., Vshivkov V. A., *"Particle in Cells in the Dynamic of Rare Plasma"*, Nauka, Novosibirsk, 1980, 190 p.
22. Berezin Yu. A., Fedoruk M. P., *"Modeling of the Nonstationar Plasma Processes"*, Nauka, Novosibirsk, 1993, 340 p.
23. Sagdeev R. Z., Zaslavsky G. H., *"Introduction into nonlinear physics"*, Nauka, Moscow, 1988, 368 p.
24. Kuzelev M. V., Rukhadze A. A., Plasma Phys. Reports **24** (1998), 486.
25. Klimontovich Yu. L., *"Turbulent Motions and Chaos"*, Nauka, Moscow, 1990, 320 p.

FIG 1. (a) Wave amplitude dynamics in a short resonator for $\xi = 6$, $\gamma = 2$ and $\chi = 0{,}05$.

(b) Phase plane of beam electrons in a cavity at different times (1) $\tau = 40$; (2) $\tau = 45$; (3) $\tau = 60$; (4) $\tau = 65$.

FIG 2. Phase plane of beam electrons in a long cavity at the time saturation sets in $\xi = 18$, $\gamma = 2$ and $\chi = 0{,}05$.

FIG 3. The maximum distance between two probe particles in the phase plane as a function of parameter ξ at the time moment the the instability enters the nonlinear stage.

FIG 4. Phase plane of the beam electrons at the time of saturation for $\xi = 25$, $\chi = 0{,}95$ and $\gamma = 2$.

FIG 5. Energy conversion coefficient η as a function of the geometry parameter ξ of resonator for $\gamma = 2$ and $\chi = 0{,}05$.

FIG 6. Energy conversion coefficients as a function of the relativism of electrons γ for $\chi = 0{,}05$: (1) $\xi = 4$; (2) $\xi = 5$; (3) $\xi = 6$.

FIG 7. Wave amplitude dynamics in time.

FIG 8. Fourier spectrum of the electromagnetic oscillations at different tims: (a) at the time the instability saturates, (b) in the nonlinear stage and (c) advanced nonlinear stage.

FIG 9. The average radius of probe particles ensemble as a function of time (1) in coordinate plane; (2) in phase plane for $\xi = 14$, $\gamma = 2$ and $\chi = 0{,}05$.

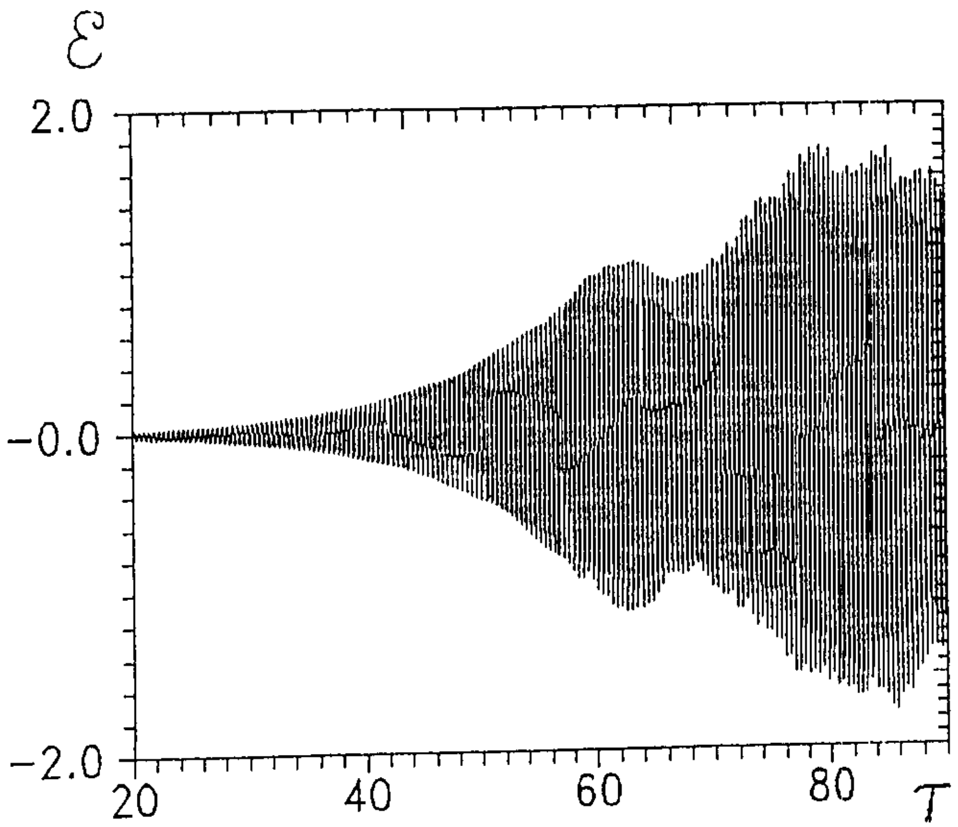

FiG 1. (a)

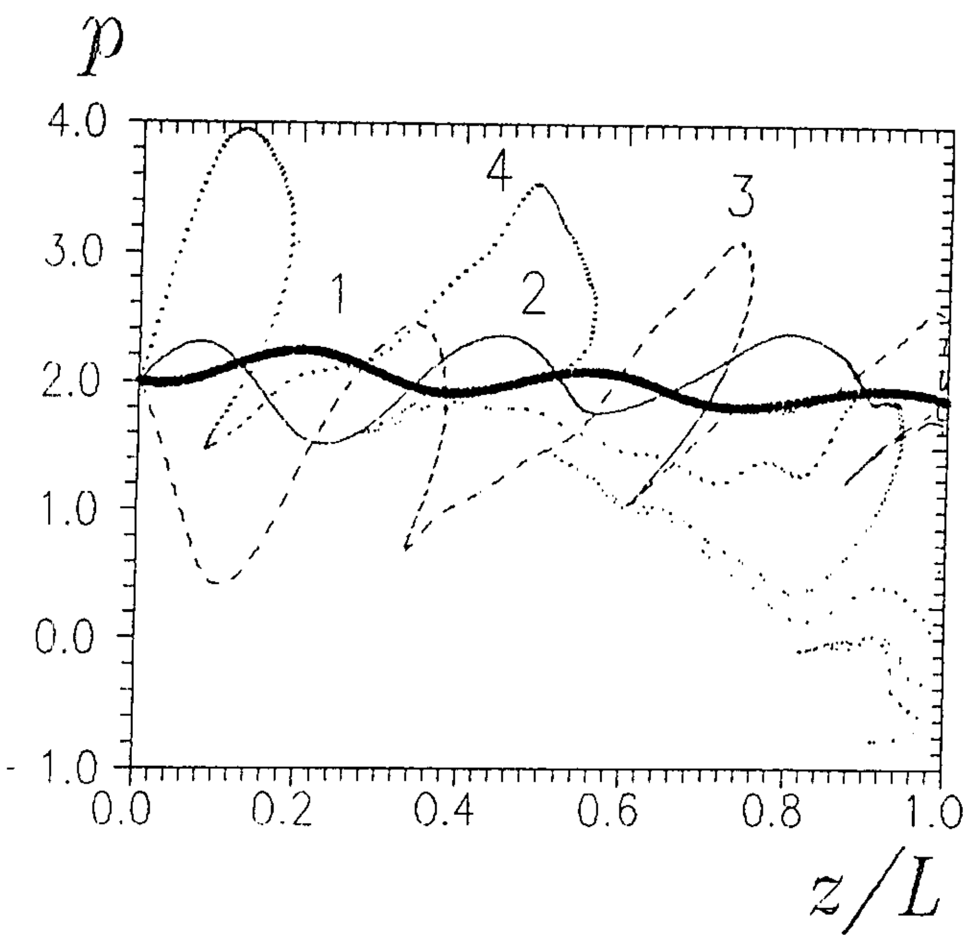

Fig 1 (b)

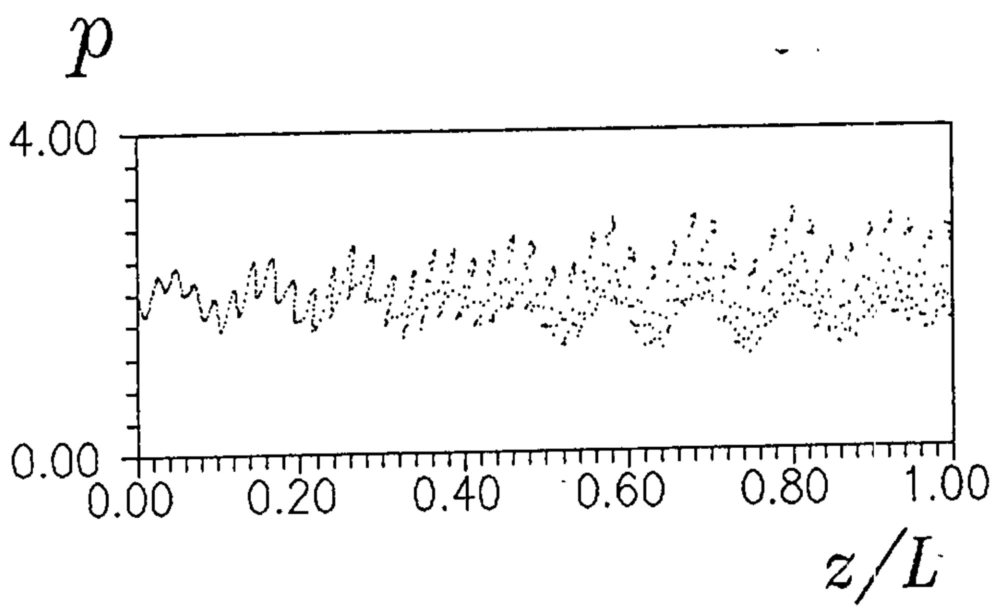

FIG 2.

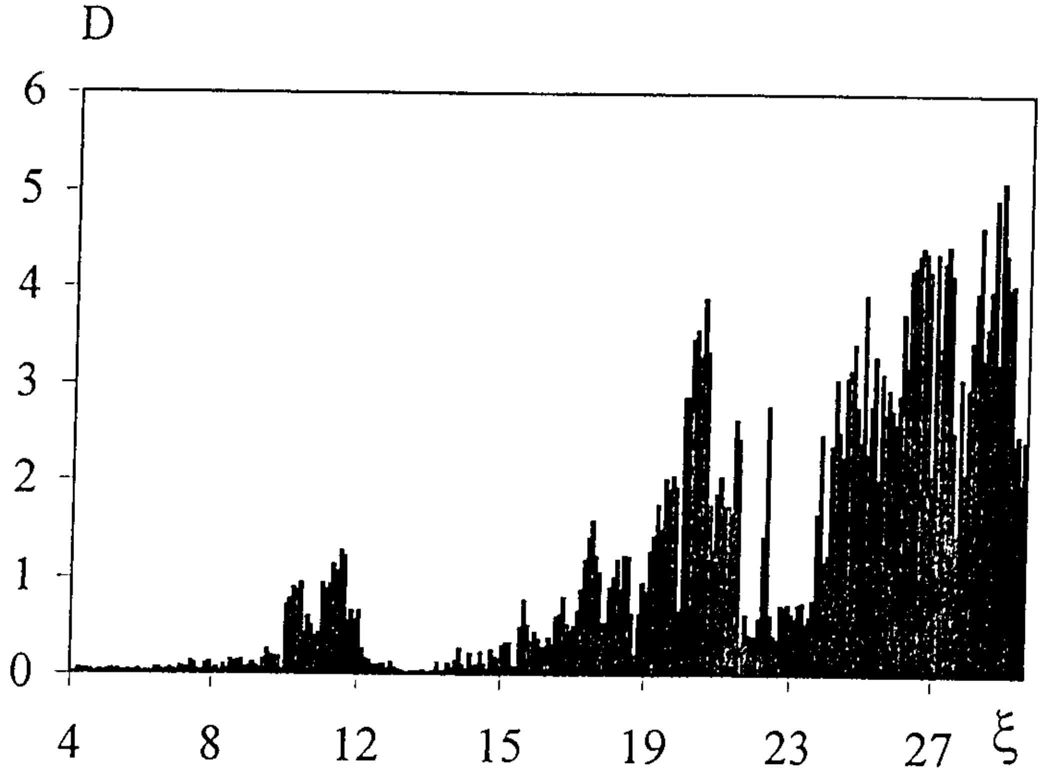

FIG 3.

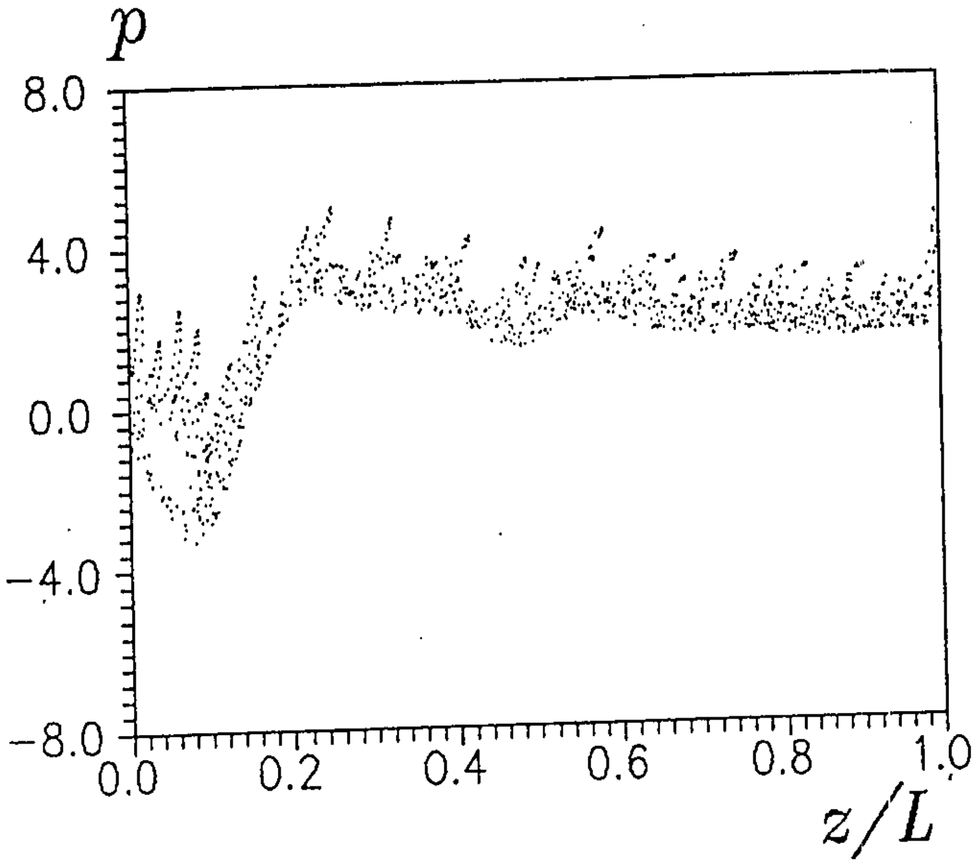

FiG. 4

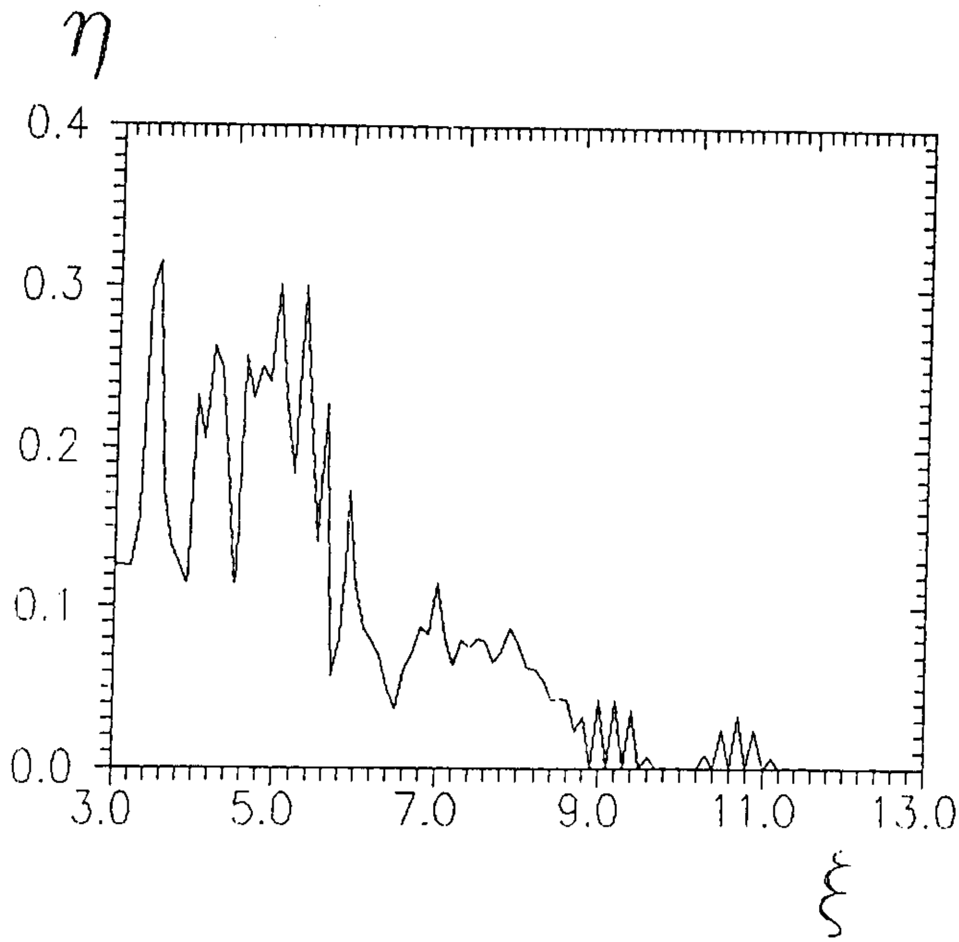

FIG. 5

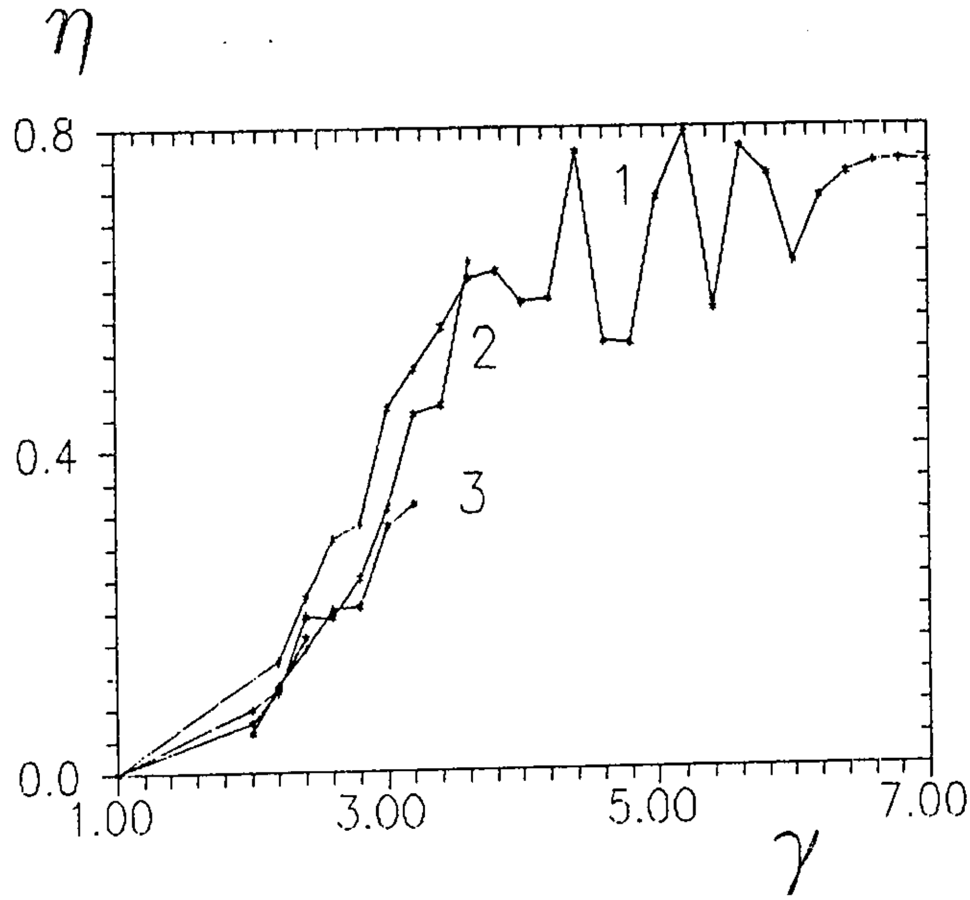

FiG. 6

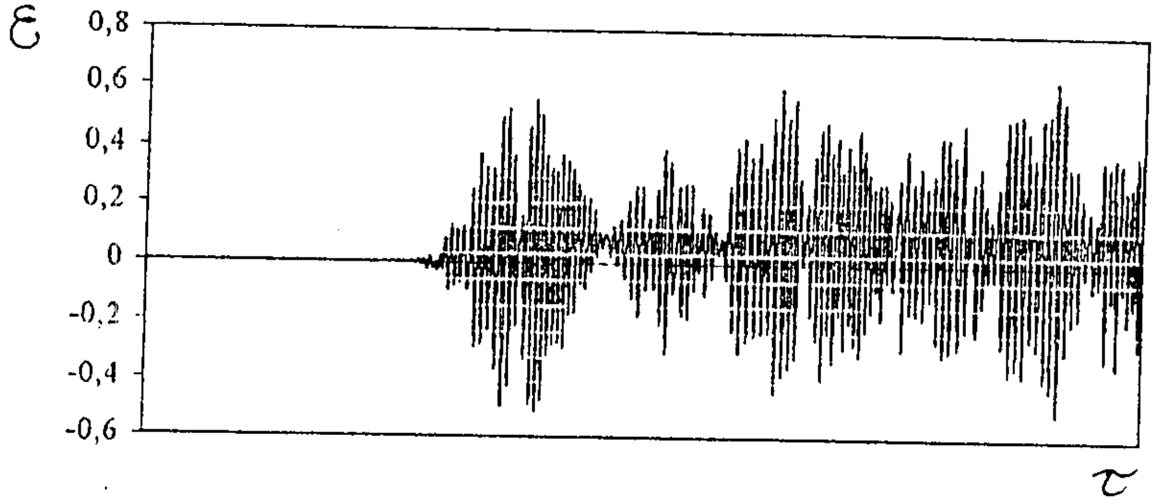

FiG. 7

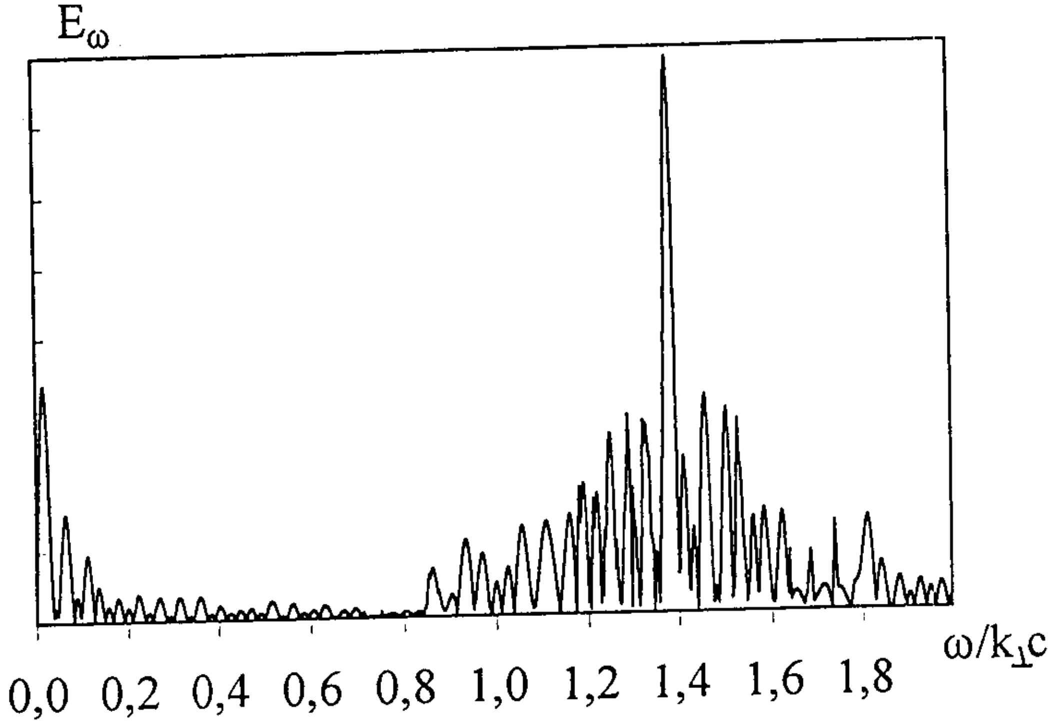

FiG. 8 (a)

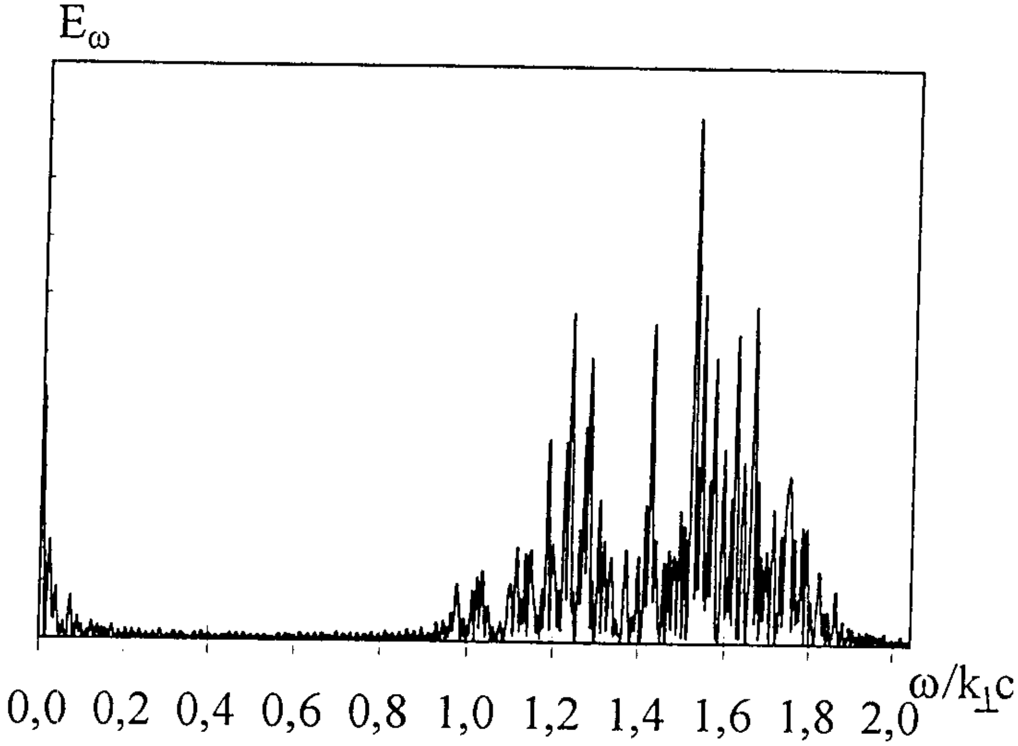

FiG 8 (b)

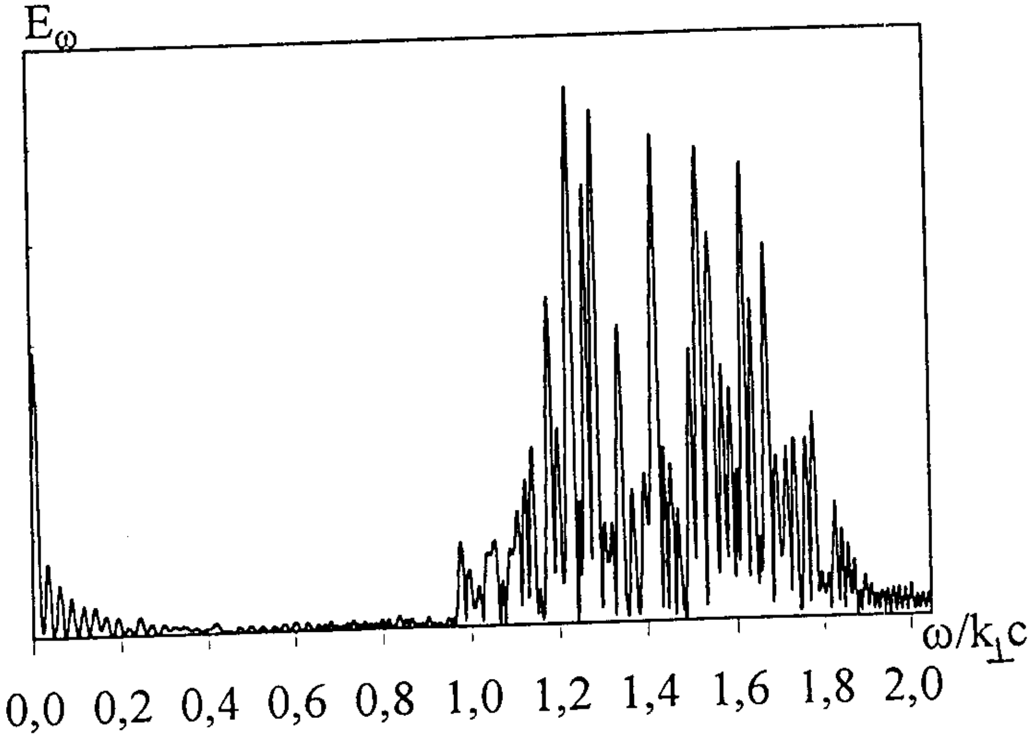

FIG. 8 (c)

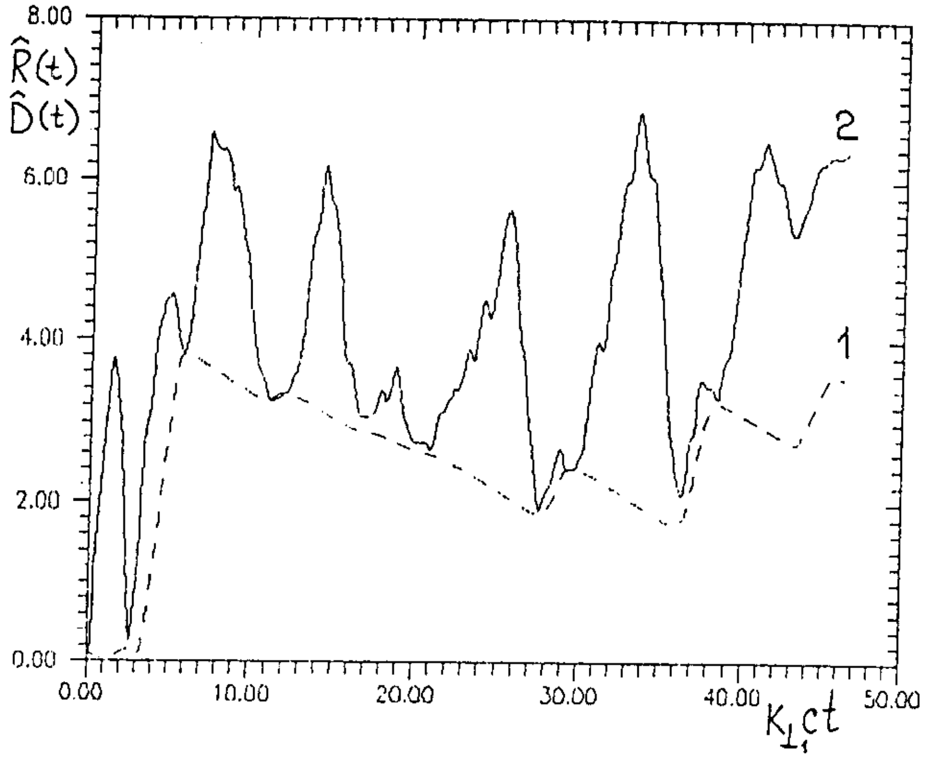

FiG. 9.

©1999 Uspekhi Fizicheskikh Nauk, Russian Academy of Sciences

METHODOLOGICAL NOTES

PACS numbers: 52.25.-b, 52.25.Dg, 52.90.+z

On the quantum description of the linear kinetics of a collisionless plasma

M V Kuzelev, A A Rukhadze

Abstract. It is demonstrated that the linear kinetics of a collisionless quantum plasma can be described in a simple and effective way by means of a self-consistent-field scheme in which the quantum hydrodynamic equations are derived directly from the Schrödinger equation.

1. We show that the known system of equations of cold hydrodynamics in the Eulerian form [1] [1]

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{V}) = 0,$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \times \nabla)\mathbf{V} = \frac{e}{m}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{V} \times \mathbf{B}]\right\} \qquad (1)$$

can also be used with profit, at least in the linear approximation, for describing the kinetic properties of a plasma with a thermal scatter in the particle velocities (a Vlasov plasma).

Let some group of particles with number density n in a homogeneous isotropic plasma (without an external magnetic field $\mathbf{B}_0$) possess a velocity $\mathbf{V}$. A small perturbation of this state by a weak electromagnetic field $\mathbf{E}$, $\mathbf{B}$ will give rise to perturbations of density δn and velocity $\delta\mathbf{V}$, which are found from the linearized system (1). Since n and $\mathbf{V}$ are constant, the quantities δn and $\delta\mathbf{V}$ can be sought as $\exp(-i\omega t + i\mathbf{kr})$. On determining δn, $\delta\mathbf{V}$ and then the current density

$$j_i = en\delta V_i + e\delta n V_i = \sigma_{ij}(\omega, \mathbf{k})E_j, \qquad (2)$$

we shall find the conductivity σ_{ij} and the dielectric constant of the particle group under consideration:

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + \frac{4\pi i}{\omega}\sigma_{ij}(\omega, \mathbf{k})$$
$$= \delta_{ij} - \frac{4\pi e^2 n}{m\omega^2}\left[\delta_{ij} + \frac{k^2 V_i V_j}{(\omega - \mathbf{kV})^2} + \frac{k_i V_j + V_i k_j}{\omega - \mathbf{k}V}\right]. \qquad (3)$$

[1] For brevity of the following presentation, we consider only one plasma component, for instance, the electron component.

M V Kuzelev Moscow State University of Publishing, ul. Pryanishnikova 2a, 127550 Moscow, Russian Federation
Tel. (7-095) 976 40 88. Fax (7-095) 976 06 35
E-mail: mgup@mail.cnt.ru
A A Rukhadze Institute of General Physics, Russian Academy of Sciences, ul. Vavilova 38, 117942 Moscow, Russian Federation
Tel. (7-095) 135 02 47. Fax (7-095) 135 80 11
E-mail: rukh@fpl.gpi.ru

Received 5 March 1999, revised 16 March 1999
Uspekhi Fizicheskikh Nauk 169 (6) 687 – 689 (1999)
Translated by E K Ragozin; edited by A Radzig

Now we can go over from a group of particles to the entire plasma by averaging over the momentum distribution function $f_0(p)$ with the substitution $n \to f_0(p)\,\mathrm{d}p$ and subsequent integration

$$n \to \int \mathrm{d}\mathbf{p} f_0(p)[\ldots]. \qquad (4)$$

The last bracketed factor of the integrand stands for the factor in expression (3) enclosed in square brackets. In consequence we find the known expression for the permittivity tensor for an isotropic plasma, which is usually obtained by solving the kinetic Vlasov equation [2]:

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\mathrm{tr}}(\omega, \mathbf{k}) + \frac{k_i k_j}{k^2}\varepsilon^{\mathrm{l}}(\omega, \mathbf{k}),$$

where

$$\varepsilon^{\mathrm{l}}(\omega, \mathbf{k}) = 1 - \frac{4\pi e^2}{k^2}\int \frac{\mathbf{k}\,\partial f_0/\partial \mathbf{p}}{\omega - \mathbf{kV}}\,\mathrm{d}\mathbf{p}, \qquad (5)$$

$$\varepsilon^{\mathrm{tr}}(\omega, \mathbf{k}) = 1 - \frac{4\pi e^2}{m\omega^2}\int \mathrm{d}\mathbf{p}\left[f_0(p) + \frac{mV_\perp^2\,\mathbf{k}\,\partial f_0/\partial \mathbf{p}}{2(\omega - \mathbf{kV})}\right].$$

Naturally, the outlined method is applicable not only for calculating the dielectric constant of an isotropic plasma. The substitution (4) is appropriate whenever the plasma with a thermal velocity scatter can be treated as a collection of groups of particles described by Eqns (1). In this case, the square brackets under the integral in (4) should enclose all expressions dependent on the hydrodynamic characteristics of each group of particles.

2. We shall generalize the outlined method to the case of a quantum plasma. In doing this, we proceed from the Schrödinger equation for the electrons without a spin, following Ref. [3] in the derivation:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = \left[-\frac{\hbar^2}{2m}\Delta + i\hbar\frac{e}{mc}\mathbf{A}\nabla + \frac{e^2}{2mc^2}\mathbf{A}^2 + e\varphi\right]\psi. \qquad (6)$$

Here $\mathbf{A}$ and φ are the vector and scalar potentials of the fields $\mathbf{E}$ and $\mathbf{B}$, with

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi, \quad \mathbf{B} = [\nabla \times \mathbf{A}], \quad (\nabla\mathbf{A}) = 0. \qquad (7)$$

We represent the wave function as

$$\psi = a(\mathbf{r}, t)\exp\left[\frac{i}{\hbar}S(\mathbf{r}, t)\right] \qquad (8)$$

and draw on the definitions of charge and current densities

$$\rho = en = e|\psi|^2 = ea^2\,,$$

$$\mathbf{j} = en\mathbf{V} = \frac{ie\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{e^2}{mc}\mathbf{A}\psi\psi^*$$

$$= \frac{ea^2}{m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right), \tag{9}$$

to obtain the system of equations

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{V}) = 0\,,$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\times\nabla)\mathbf{V} = \frac{e}{m}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{V}\times\mathbf{B}]\right\}$$

$$+ \frac{\hbar^2}{4m^2}\nabla\left\{\frac{1}{n}\left[\Delta n - \frac{1}{2n}(\nabla n)^2\right]\right\} \tag{10}$$

from the Schrödinger equation (6).

The first of these equations coincides with the equation of continuity, and the second with the Euler equation of system (1). Therefore, by analogy with system (1), system (10) will be referred to as the quantum equations of cold plasma hydrodynamics.

Eqns (10) differ from Eqns (1) in that the Euler equation includes the quantum force resulting from the Heisenberg uncertainty principle. This is easily verified by considering small perturbations of the uniform state with $n = \text{const}$ and $\mathbf{V} = 0$. In the limit $n \to 0$ when the self-consistent fields $\mathbf{E}$ and $\mathbf{B}$ can be neglected, for solutions of the type $\exp(-i\omega t + i\mathbf{kr})$ the linearized system (10) yields the dispersion relation

$$\omega = \frac{\hbar k^2}{2m} \equiv \omega_q\,, \tag{11}$$

which describes the oscillations of a single electron. This expression relates the temporal (proportional to $1/\omega$) and spatial (proportional to $1/k$) domains of localization of a free electron, or the energy $\hbar\omega$ and the momentum $\hbar\mathbf{k}$. The quantity (11) is the frequency of the quantum oscillations of a free electron.

Following the outlined procedure, we can now derive the dielectric constant of a quantum isotropic plasma with a thermal scatter in electron velocities. First, for any group of plasma particles we obtain the corresponding quantum dielectric constant, i.e. the quantum analog of tensor (3). Assuming the perturbed quantities to be of the form $\exp(-i\omega t + i\mathbf{kr})$, from Eqn (10) it follows that

$$\varepsilon^q_{ij}(\omega,\mathbf{k}) = \varepsilon^{cl}_{ij}(\omega,\mathbf{k}) - \frac{\omega_q^2}{\omega_{L_e}^2}\,\delta\varepsilon^{cl}_{i\mu}(\omega,\mathbf{k})\frac{k_\mu k_\nu}{k^2}\,\delta\varepsilon^{cl}_{i\nu}(\omega,\mathbf{k})$$

$$\times\left[1 + \frac{\omega_q^2}{\omega_{L_e}^2}\frac{k_\mu k_\nu}{k^2}\,\delta\varepsilon^{cl}_{\mu\nu}(\omega,\mathbf{k})\right]^{-1}, \tag{12}$$

where $\omega_{L_e} = \sqrt{4\pi e^2 n/m}$ is the electron Langmuir frequency, and $\varepsilon^{cl}_{ij} = \delta_{ij} + \delta\varepsilon^{cl}_{ij}$ is the classical dielectric constant tensor defined by expression (3). In the derivation of expression (12), we drew on the obvious substitution

$$\mathbf{E}^q = \mathbf{E}^{cl} - i\mathbf{k}\frac{\hbar k^2}{4m}\frac{\delta n}{n}\,, \tag{13}$$

which follows in the linear approximation from the Euler equation (10).

We next substitute expression (3) into (12) and pass on to the kinetic description with the help of change (4) to obtain by straightforward calculations the known expressions for the quantum longitudinal and transverse dielectric constants of an isotropic plasma [2]

$$\varepsilon^l(\omega,\mathbf{k}) = 1 + \frac{4\pi e^2}{\hbar k^2}\int\frac{d\mathbf{p}}{\omega - \mathbf{kV}}\left[f_0\left(\mathbf{p} + \frac{\hbar\mathbf{k}}{2}\right) - f_0\left(\mathbf{p} - \frac{\hbar\mathbf{k}}{2}\right)\right],$$

$$\varepsilon^{tr}(\omega,\mathbf{k}) = 1 - \frac{\omega_{L_e}^2}{\omega^2} + \frac{2\pi e^2}{\hbar\omega^2}$$

$$\times\int\frac{d\mathbf{p}}{\omega - \mathbf{kV}}V_\perp^2\left[f_0\left(\mathbf{p} + \frac{\hbar\mathbf{k}}{2}\right) - f_0\left(\mathbf{p} - \frac{\hbar\mathbf{k}}{2}\right)\right]. \tag{14}$$

Notice that in Ref. [2] expressions (14) were derived by solving the Wigner quantum kinetic equation, which involved tedious calculations. In the limit $\hbar \to 0$, formulas (14) obviously transform to formulas (5).

3. Now consider a homogeneous magnetoactive plasma. Let the external magnetic field $\mathbf{B}_0$ be aligned with the OZ-axis. For simplicity, we shall restrict our consideration to the case of a potential field $\mathbf{E} = -\nabla\varphi$, $\mathbf{A} = 0$. As above, we consider a group of particles with number density n, which possess longitudinal velocity V_z and rotate about the magnetic lines of force with the Larmor frequency $\Omega = eB_0/mc$ and the Larmor radius $R_L = V_\perp/\Omega$. The longitudinal dielectric constant of this classical cold plasma (group of particles) is easy to obtain from the general formula given in Ref. [1]. It is of the form

$$\varepsilon(\omega,\mathbf{k}) = \frac{k_i k_j}{k^2}\varepsilon_{ij}(\omega,\mathbf{k}) = 1 - \frac{\omega_{L_e}^2}{k^2}$$

$$\times\sum_s\left[\frac{k_z^2 J_s^2(z)}{(\omega - k_z V_z - s\Omega)^2} + \frac{2sk_\perp^2 J_s(z)J_s'(z)}{z\Omega(\omega - k_z V_z - s\Omega)}\right], \tag{15}$$

where $J_s(z)$ is the Bessel function of the real argument $z = k_\perp R_L$.

We average expression (15) over the distribution function $f_0(\mathbf{p})$ according to the above recipe (4) to obtain the known expression for the longitudinal dielectric constant of a classical magnetoactive plasma [1]

$$\varepsilon(\omega,\mathbf{k}) = 1 + \frac{4\pi e^2}{mk^2}\int d\mathbf{p}\sum_s\frac{J_s^2(z)}{\omega - k_z V_z - s\Omega}$$

$$\times\left(k_z\frac{\partial f_0}{\partial V_z} + \frac{s\Omega}{V_\perp}\frac{\partial f_0}{\partial V_\perp}\right). \tag{16}$$

It is also an easy matter to write out the longitudinal dielectric constant of a quantum magnetoactive plasma. To accomplish this, it should be recognized that the total force in the right-hand part of the Euler equation (10) does not depend on the type of plasma at all. Consequently, relation (13) is universal in character too, and with it formula (12). Hence, the longitudinal dielectric constant is given by

$$\varepsilon^q(\omega,\mathbf{k}) = \varepsilon^{cl}(\omega,\mathbf{k}) - \frac{\omega_q^2}{\omega_{L_e}^2}\frac{\delta\varepsilon^{cl}(\omega,\mathbf{k})\delta\varepsilon^{cl}(\omega,\mathbf{k})}{1 + (\omega_q^2/\omega_{L_e}^2)\delta\varepsilon^{cl}(\omega,\mathbf{k})}\,, \tag{17}$$

where $\varepsilon^{cl} = 1 + \delta\varepsilon^{cl}$ is defined by expression (15).

Therefore, expression (17) refers to the longitudinal dielectric constant of a cold quantum plasma. As above, the passage to the kinetic description is accomplished by averaging expression (17) over the distribution function $f_0(\mathbf{p})$ with the help of substitution (4). Substitution of expression (15) into (17) with subsequent averaging results in cumbersome expressions, which we omit here.

It is more expedient to address the question of the $f_0(\mathbf{p})$ distribution itself over which the averaging is performed. The point is that, in general, account must be taken of the energy of quantization of the transverse electron motion in a magnetoactive plasma. This has no effect on the magnitude of ω_q but substantially affects the shape of the distribution function $f_0(\mathbf{p})$.

In the case of Maxwellian statistics (nondegenerate electrons) [4], one obtains

$$f_0(\mathbf{p}) = \frac{n}{(2\pi m)^{3/2} T^{1/2} E_\perp} \exp\left(-\frac{p_z^2}{2mT} - \frac{p_\perp^2}{2mE_\perp}\right), \qquad (18)$$

where

$$E_\perp = \frac{\hbar\Omega}{2} \coth\frac{\hbar\Omega}{2T} \approx \begin{cases} T, & \hbar\Omega \ll 2T, \\ \hbar\Omega/2, & \hbar\Omega \gg 2T \end{cases} \qquad (19)$$

is the average energy of the transverse electron motion. The condition for nondegeneracy is written as

$$E_F \ll T^{1/3} E_\perp^{2/3}, \qquad (20)$$

where $E_F = (3\pi^2)^{2/3} \hbar^2 n^{2/3}$ is the Fermi energy for $B_0 = 0$.

When inequality (20) is violated, the degeneracy should be taken into consideration and the function $f_0(\mathbf{p})$ becomes more complicated. Nevertheless, in the Hartree approximation it has the simple form [5]

$$f_0(\mathbf{p}) = \frac{2}{(2\pi\hbar)^3} \sum_s (-1)^s$$

$$\times \frac{L_s(p_\perp^2/m\hbar\Omega)\exp(-p_\perp^2/m\hbar\Omega)}{1 + \exp\left\{T^{-1}\left[p_z^2/2m + \hbar\Omega(s+1/2) - \zeta\right]\right\}}, \qquad (21)$$

where $L_s(x)$ is the Laguerre function, and ζ is the chemical potential, which coincides with the Fermi energy E_F for free electrons. The summation in expression (21) is extended over all the Landau levels s.

Notice that the extension of the results derived in the foregoing to a multicomponent plasma medium is apparent and reduces to a simple summation over the components in formulas (3), (5), (12), (14)–(17). It is significant that the plasma dielectric constant in a quantized magnetic field can equally be derived through the direct solution of the Wigner equation with the distributions (18) or (21). However, this procedure is found to be very complicated owing to the arduous mathematical treatment [6, 7]. The application of formulas (16) and (17) may prove to be preferable.

Thus, with the appropriate averaging over the distribution function, the simple cold hydrodynamic model describes the kinetic properties of a quantum plasma as fully as of a classical one. This was demonstrated above in the linear approximation. But nonlinear processes call for special consideration.

References

1. Aleksandrov A F, Bogdankevich L S, Rukhadze A A *Osnovy Elektrodinamiki Plazmy* (Foundations of Electrodynamics of Plasma) 2nd ed. (Moscow: Vysshaya Shkola, 1988) [Translated into English (Berlin: Springer, 1984)]
2. Silin V P, Rukhadze A A *Elektromagnitnye Svoĭstva Plazmy i Plazmopodobnykh Sred* (Electromagnetic Properties of Plasmas and Plasma-like Media) (Moscow: Gosatomizdat, 1961)
3. Blokhintsev D I *Osnovy Kvantovoĭ Mekhaniki* (Principles of Quantum Mechanics) (Moscow: Vysshaya Shkola, 1963) [Translated into English as *Quantum Mechanics* (Dordrecht: Reidel, 1964)]
4. Rukhadze A A, Shafer V Yu *Kratk. Soobshch. Fiz.* (10) 41 (1980)
5. Zil'berman P S *Fiz. Tverd. Tela* (Leningrad) 12 1697 (1970)
6. Zyryanov P S, Kalashnikov V P *Zh. Eksp. Teor. Fiz.* 40 1119 (1961) [*Sov. Phys. JETP* 13 894 (1961)]
7. Eleonskiĭ V M, Zyryanov P S, Silin V P *Zh. Eksp. Teor. Fiz.* 42 896 (1962) [*Sov. Phys. JETP* 15 804 (1962)]