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Velocity-Shear-Induced Wave Transformations

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Velocity-Shear-Induced Wave Transformations
in
Astrophysical Shear Flows

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"The Universe is transformation. Life is opinion." [1]
Marcus Aurelius ("Meditations")

Structure of the lecture:

- Definition of shear flow (SF);
- Appearance of SF;
- Puzzles of SF;
- Coupled oscillations;
- Example of wave transformations in SF;
- Astrophysical occurence (solar atmosphere, pulsar magnetosphere, galactic gaseous disks). What else?!

Definitions:

$\vec{U}_0 \neq 0$

- $U_0 = const$
  Uniform, stationary flows
- $U_0 = f(x, y, z)$
  Inhomogeneous, stationary flows
- $U_0 = f(x, y, z, t)$
  Inhomogeneous, nonstationary flows

Occurrence:

— Laboratory SF

— Terrestrial SF

— Astrophysical SF

Prinsipal question:

How SF affects modes of collective behaviour (waves, vortices, instabilities) in plasmas and fluids?

– Mathematical paradigms:

(a) Modal approach: SF makes operators non-self-adjoint;

(b) Nonmodal approach: evokes $\vec{K}(t)$ temporal "drift".
(Kelvin 1887, Marcus, Press 1977)

– Physical (nonmodal) shear-induced-phenomena:

(a) Creates new modes of collective behaviour [2-7]
("Kelvin modes" aka "shear vortices", shear beat waves)

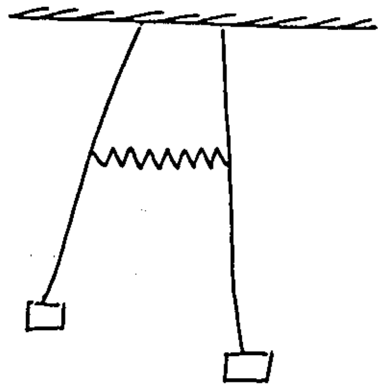
(b) Waves become able to extract energy from the SF; [4]

(c) Erases sharp margin between waves and vortices;
[4,6-8]

(d) Couples waves, induces their mutual transformations;
[9-10]

(e) Makes certain instabilities transient.

Coupled Oscillations:



$$\partial_t^2 F_1 + \omega_1^2 F_1 + C F_2 = 0$$

$$\partial_t^2 F_2 + \omega_2^2 F_2 + C F_1 = 0$$

ω_i - eigenfrequencies

C - coupling coefficient

Exact solution of the system:

$$F_1(t) = F_+ \cos(\Omega_+ t - \varphi_+) + F_- \cos(\Omega_- t - \varphi_-)$$

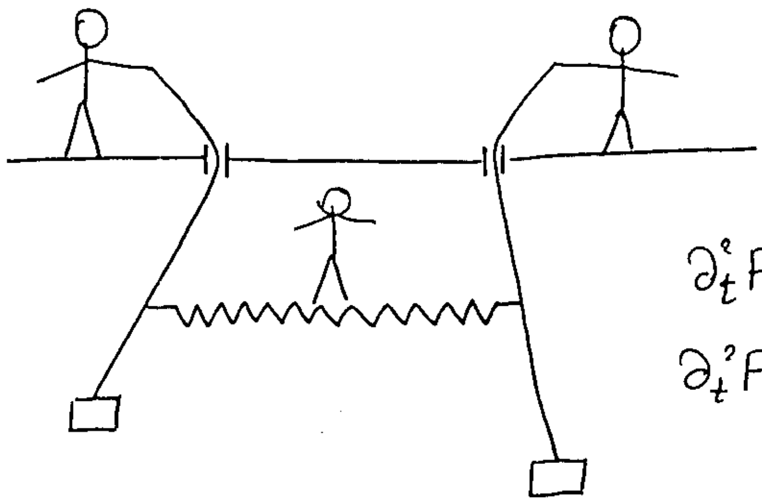
$$F_2(t) = \sigma_+ F_+ \cos(\Omega_+ t - \varphi_+) + \sigma_- F_- \cos(\Omega_- t - \varphi_-)$$

Fundamental (normal) frequencies defined by:

$$\Omega_\pm^2 \equiv \frac{1}{2}\left[(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4C^2}\right]$$

$$\sigma_\pm \equiv (\Omega_\pm^2 - \omega_1^2)/C = C/(\Omega_\pm^2 - \omega_2^2)$$

Decoupling of oscillations:



$$\partial_t^2 F_1 + \omega_1^2(t) F_1 + C(t) F_2 = 0$$
$$\partial_t^2 F_2 + \omega_2^2(t) F_2 + C(t) F_1 = 0$$

– Slow (adiabatic) variation:

$$|\partial_t \omega_i(t)| \ll 2\pi\, \omega_i^2(t) \;;\; |\partial_t C(t)| \ll 2\pi\, C(t)\,\omega_i(t)$$

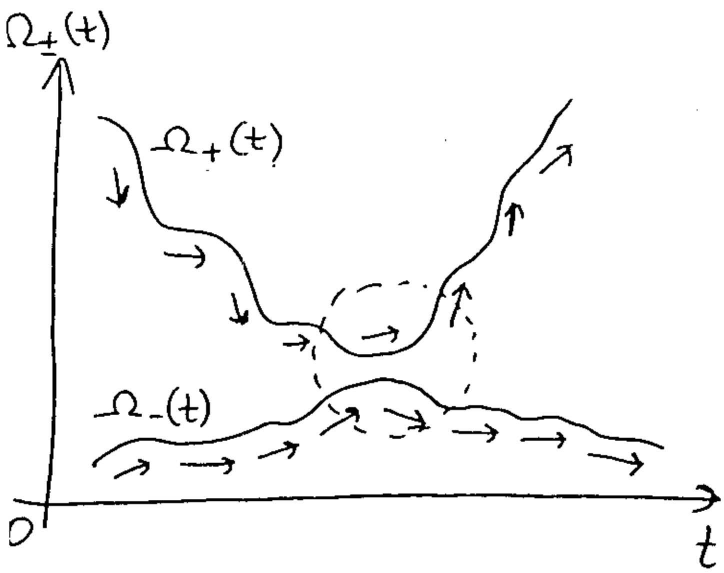
– Necessary conditions for effective mutual transformations:

(a) Existence of "degeneration region":

$$|\Omega_+^2(t) - \Omega_-^2(t)| \lesssim |C(t)|$$

(b) Slow "passing" condition:

$$|\partial_t \Omega_\pm(t)| \ll |C(t)|$$

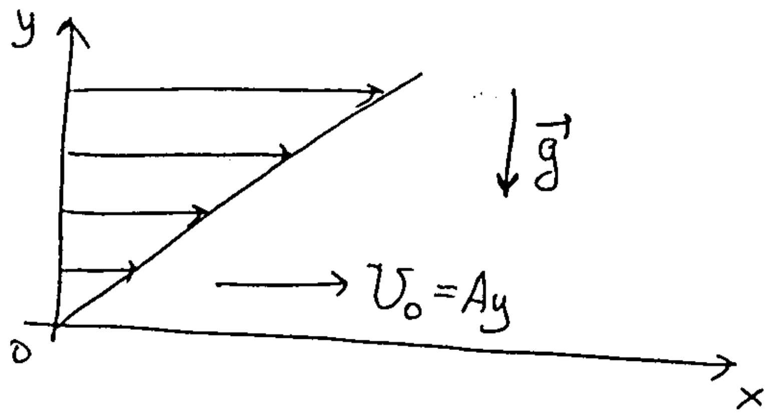


Conjecture:

$\boxed{\text{shear}} \rightarrow \boxed{\vec{k}(t)} \rightarrow \boxed{\omega_i(t)}$

Thus in $n>1$ multiwave case one may expect shear-induced wave transformations

Example: coupling of sound and internal gravity waves in plain Couette flow: [5] ⑥



$$\omega_0^2 \equiv -\left(\frac{g_0}{\rho_0}\right)\left(\frac{\partial P_0}{\partial s_0}\right)_{P_0} \partial_y s_0$$

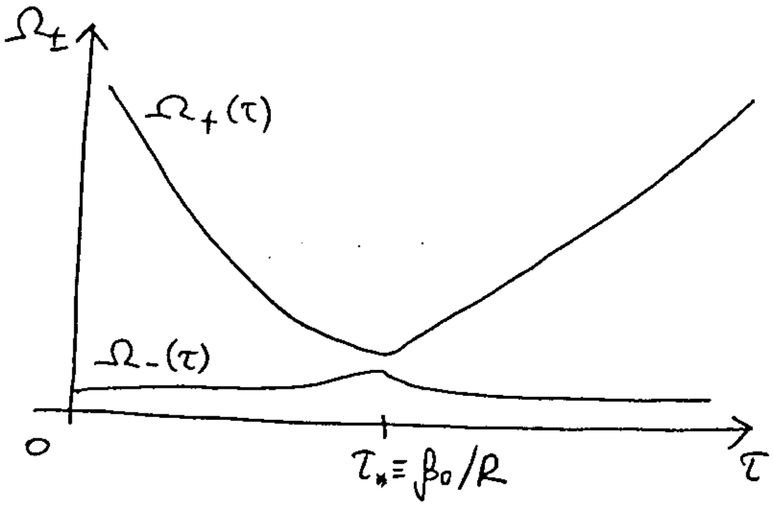
$$W^2 \equiv (\omega_0 / c_s k_x)^2$$

$$\tau \equiv c_s k_x t \,, \quad R \equiv A / c_s k_x$$

$\beta_0 \equiv k_y / k_x$, $\beta(\tau) = \beta_0 - R\tau$; $\psi(t)$, $e(t)$

$$\boxed{\begin{aligned} &\partial_\tau^2 \psi + \psi + \beta(\tau) e = 0 \\ &\partial_\tau^2 e + [W^2 + \beta^2(\tau)] e + \beta(\tau)\psi = 0 \end{aligned}}$$

$\omega_1^2 = 1$; $\omega_2^2 = W^2 + \beta^2(\tau)$; $C(\tau) = \beta(\tau)$

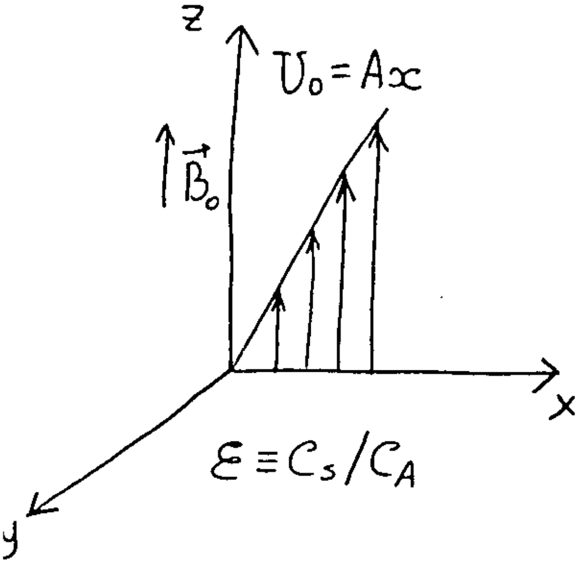


— Mutual transformations

— Beatings

$R \ll \beta_0 \ll 1$

MHD wave transformations: (Solar wind) [11]



$\varepsilon \equiv C_s / C_A$

$$\partial_t^2 \psi + \omega_1^2 \psi = C_1(t)\, b_x + C_2\, b_y$$

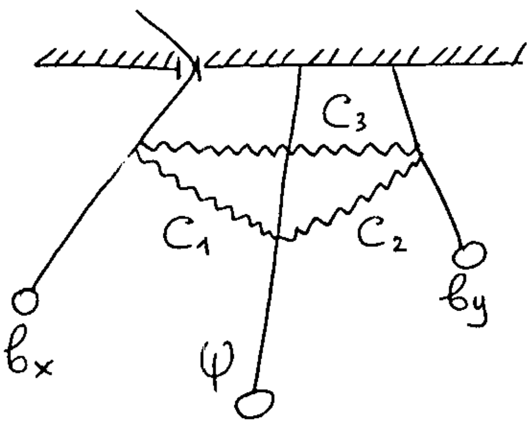
$$\partial_t^2 b_x + \omega_2^2(t)\, b_x = C_1(t)\,\psi + C_3(t)\, b_y$$

$$\partial_t^2 b_y + \omega_3^2\, b_y = C_2 \psi + C_3(t)\, b_x$$

$$\omega_1 = \varepsilon$$

$$\omega_2(\tau) = \sqrt{1 + (1+\varepsilon^2)\, k_x^2(\tau)}$$

$$\omega_3 = \sqrt{1 + (1+\varepsilon^2)\, k_y^2}$$



$$C_1(\tau) = \varepsilon^2 k_x(\tau)$$

$$C_2 = \varepsilon^2 k_y$$

$$C_3 = -(1+\varepsilon^2)\, k_y k_x(\tau)$$

(a) SMW $\leftrightarrow$ FMW transformations:

$$\partial_\tau^2 \psi + \varepsilon^2 \psi = \varepsilon^2 k_x(\tau)\, b_x$$

$$\partial_\tau^2 b_x + \left[1 + (1+\varepsilon^2)\, k_x^2(\tau)\right] b_x = \varepsilon^2 k_x(\tau)\, \psi$$

Favourable condition: $\varepsilon \simeq 1$

(b) AW $\leftrightarrow$ FMW transformations:

$$\partial_\tau^2 b_x + \left[1 + k_x^2(\tau)\right] b_x + k_y k_x(\tau)\, b_y = 0$$

$$\partial_\tau^2 b_y + \left[1 + k_y^2\right] b_y + k_y k_x(\tau)\, b_x = 0$$

Pulsar magnetosphere – radio emission: [13]

Escaping radio emission generation ?!

Arons and Smith conjecture (1979):

$$\boxed{\text{Stream instability generates Langmuir waves}} \xrightarrow{\text{Coupling ?}} \boxed{\text{Propagating electromagnetic modes}}$$

Mahajan, Machabeli, Rogava: Ap. J. Lett. (1997) **479**, 129.

$$\partial_\tau^2 D + W^2 D = -W^2 R \tau e^{i v_0 \tau} E$$

$$\partial_\tau^2 E + [1 + R^2 \tau^2] E = -R \tau e^{-i v_0 \tau} D$$

Velocity shear effectively converts nonescaping longitudinal Langmuir waves into propagating (escaping) electromagnetic waves.

Predicted by Blackmore et al. (1971)

"Smoke on the water" single...

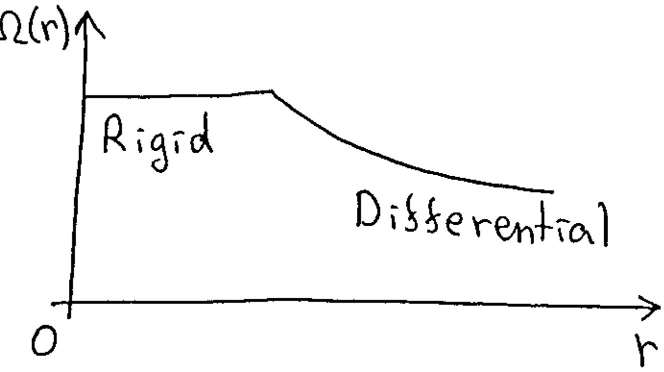
Coupling of galactic MHD waves: [14, 17]

Fan and Lou (1997) MHD waves in gas-rich spiral galaxies.

M51 – FMW

NGC 6946 – SMW

IC 342 – both waves?



Elmegreen's (1987) model: $D_t \equiv \partial_t + (\vec{V}\cdot\nabla)\vec{V}$

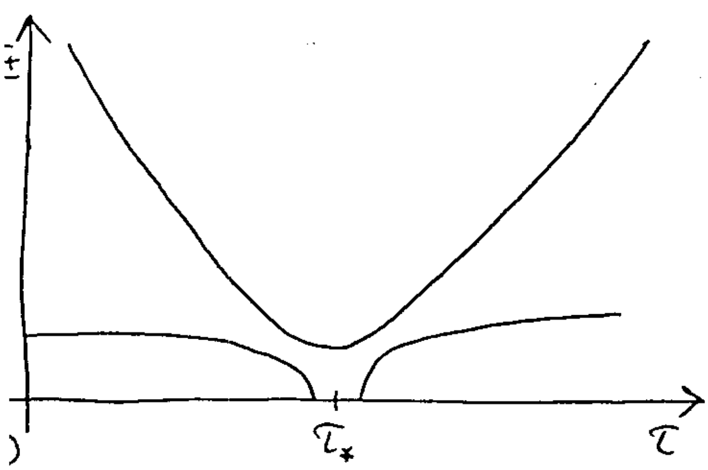
$$D_t \Sigma + \Sigma \nabla\cdot\vec{V} = 0$$

$$D_t \vec{V} = -\frac{1}{\Sigma}\nabla p + \frac{1}{4\pi\rho}(\nabla\times\vec{B})\times\vec{B} - 2\vec{\Omega}\times\vec{V} - \nabla\varphi$$

$$D_t \vec{B} = (\vec{B}\cdot\nabla)\vec{V} - \vec{B}(\nabla\cdot\vec{V})$$

$$\Delta^2\varphi = 4\pi G \sigma\, \delta(z)$$

$A \equiv (r/2)\partial_r \Omega$; $\tilde{B} \equiv \Omega + A$: first and second Oort's constants



$$\partial_\tau^2 \mathcal{L} + W_1^2 \mathcal{L} = 2\omega\,\partial_\tau \mathcal{B} + C\mathcal{B}$$

$$\partial_\tau^2 \mathcal{B} + W_2^2 \mathcal{B} = -2\omega\,\partial_\tau \mathcal{L} + C\mathcal{L}$$

$$W_1^2 = 1 + \alpha(\tau)$$

$$W_2^2 = \xi^2(1+\beta^2(\tau)) + (1+\alpha(\tau))\beta^2 + 4\omega R$$

$$C = (1+\alpha(\tau))\beta(\tau)$$

What else ?!

- Planetary atmospheres, rings;
- Stellar (pulsar) winds;
- Accretion columns;
- Disk - Star boundary layers;
- Jets in YSO, AGN...
- Accretion - ejection flows;

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See as the appendix to these lecture notes

Mon. Not. R. Astron. Soc. 307, L31–L36 (1999)

Are galactic magnetohydrodynamic waves coupled?

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ABSTRACT

Recently, Fan & Lou considered the excitation and time evolution of hydromagnetic density waves in a differentially rotating thin gaseous disc embedded in an azimuthal magnetic field. The authors found that both fast and slow hydromagnetic density waves are amplified while they 'swing' from leading to trailing configurations, and gave a detailed description of the phenomenon. Fan & Lou noticed that the results of their numerical study indicate the existence of a 'coupling' between slow and fast waves.

In this Letter we prove, in a simple and exact analytic way, that the coupling between slow and fast waves, presumed by Fan & Lou on the basis of their numerical study, indeed exists. We show that the coupling is induced exclusively by the presence of the velocity shear in the gaseous disc, and that it leads to the mutual transformations of the different density wave modes. We argue that the shear-induced wave transformations may play a significant role in the overall dynamics of galactic MHD density waves.

Key words: MHD – galaxies: magnetic fields – galaxies: spiral – galaxies: structure – radio continuum: galaxies.

1 INTRODUCTION

Currently it is well-acknowledged that galactic gaseous discs are magnetized, and hence the density waves sustained by them should be of magneto-acoustic origin. Therefore the theoretical study of hydromagnetic density waves in a thin magnetized gaseous disc is quite important in the context of general galactic dynamics and, in particular, in the context of its relevance to large-scale magnetic field structures in spiral galaxies. The pace of theoretical research in this area is driven by advances in synchrotron radio observations of the magnetic field structures in nearby spiral galaxies [for a recent review of the current state of the art in theoretical and observational domains see, for example, Lou & Fan (1998)].

Observations show that some galaxies (e.g. M51, M31 and NGC 2997) feature magnetic field spiral structures that are almost coincident with the optical spiral arms, while there are examples (e.g. NGC 6946) in which the magnetic field spiral arms lie between the optical spiral arms. This puzzling circumstance suggests that, in these two classes of galaxies, magnetic spiral patterns are related to density waves of a different origin. Galactic gaseous discs are essentially self-gravitating, differentially rotating magnetohydrodynamic (MHD) plasma shear flows, sustaining two modes of acoustic oscillations – slow magnetosonic waves (SMWs) and fast magnetosonic waves (FMWs). It is reasonable to suppose, therefore, that the above-specified classes of spiral structures are somehow associated with these two density wave modes.

Fan & Lou (1996, 1997) and Lou & Fan (1998) suggested this idea, and studied in this context fast and slow hydromagnetic (MHD) waves in a gaseous thin rotating disc embedded in an azimuthal magnetic field. Their study revealed that for fast MHD density waves the surface gas mass density perturbation is approximately in phase with the azimuthal magnetic field perturbation, so that, if the density waves in a galaxy are of a predominantly FMW origin, the optical and magnetic spiral structures should be roughly coincident. As regards slow MHD density waves, there is a substantial phase difference between the oscillation phases of these disturbances, which means, in turn, that the waves of SMW origin may account for the rough anticorrelation between optical and magnetic spiral arms. Thus these authors argued (Fan & Lou 1997, hereafter referred as FL97) that galaxies of M51 type appear to carry signatures of fast waves, while galaxies like NGC 6946 bear features of slow waves.

Fan & Lou also noted that their numerical results reveal the coupling between slow and fast MHD density waves in magnetized, self-gravitating, differentially rotating gaseous

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discs. This circumstance, if true, implies the coupling of the two kinds of MHD density waves in spiral galaxies. However, Fan & Lou did not consider and/or comment upon this intriguing issue in any more detail.

Are MHD waves in galaxies really coupled? In this Letter we give a simple analytic proof of the affirmative answer to this question. In particular, we show that the problem considered by FL97 is genuinely governed by the canonical system of ordinary differential equations describing a standard coupled oscillating system with two degrees of freedom. We show that the coupling is induced solely by the presence of the velocity shear in the system. The coupling leads to the mutual transformations of slow and fast density waves. The phenomenon is analogous to the one disclosed recently by Chagelishvili, Rogava & Tsiklauri (1996) for the parallel shear MHD flow. This kind of coupling is fairly universal, since it arises in many different kinds of shear flows (even without rotational and/or self-gravity effects) sustaining more than one mode of wave motion (Rogava, Mahajan & Berezhiani 1996; Rogava & Mahajan 1997; Mahajan, Machabeli & Rogava 1997; Chagelishvili, Rogava & Tsiklauri 1997; Poedts, Rogava & Mahajan 1998).

2 THEORY

It should be noted that the significance of shear-induced coupling between SMWs and FMWs in differentially rotating, magnetized gaseous discs is *not* connected exclusively with the idea that the magnetic spiral patterns in galaxies are associated with density waves of different origin. Essentially, what we prove below implies that, in various kinds of cylindrically symmetric magnetized astrophysical shear flows (dusty plasma rings around planets and stars, protostellar discs, accretion discs, etc.), velocity shear readily provokes mutual transformations of different MHD wave modes into one another. This circumstance may have profound consequences for a number of important physical issues associated with these flows.

One well-known example is the enigmatic problem of turbulence in accretion discs. Generally speaking, accretion discs provide kinematically complex, three-dimensional shear flows, which may be embedded in both toroidal and poloidal large-scale magnetic fields. All three kinds of MHD wave modes – Alfvén waves (AWs), SMWs and FMWs – may be excited in these complicated astrophysical plasma systems. However, even if one adopts a certain kind of 'minimalistic' approach, making a number of simplifying assumptions and admitting that the disc sustains a certain kind of 'internal' mechanism producing *only* a certain kind of MHD wave, the existence of shear-induced wave couplings ensures transformation of initial 'seed' waves into other kinds of MHD oscillations, establishing the regime of a perpetually oscillating 'sea' of mixed MHD waves. This implies that one should be quite careful when speaking about a particular kind of plasma turbulence (e.g. Alfvénic turbulence) in these systems. Rather, in the 'realm' of an accretion disc shear flow, the turbulence that may develop is likely to be of the 'mixed' type.

In FL97, the authors considered a thin, differentially rotating gaseous disc, embedded in a large-scale azimuthal magnetic field and sustained by its self-gravity. Below we give a brief account of their analysis. For the local analysis of the excitation and time evolution of compressible MHD perturbations, FL97 used the formalism of so-called 'sheared comoving coordinates' [see Goldreich & Lynden-Bell (1965), where the method was employed in an astrophysical context for the first time[1]]. The starting set of equations within this formalism is written in the locally small area of the disc, and perturbations are ordered to lie in the disc plane. The latter restriction excludes Alfvén waves from the dynamics of the system. All equations, except the Poisson equation, are written in terms of surface quantities (Elmegreen 1987) $[D_t \equiv \partial_t + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}]$:

$$D_t\Sigma + \Sigma\nabla\cdot\boldsymbol{V} = 0, \tag{1}$$

$$D_t\boldsymbol{V} = -\frac{1}{\Sigma}\nabla P + \frac{1}{4\pi\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - 2\boldsymbol{\Omega}\times\boldsymbol{V} - \nabla\Psi, \tag{2}$$

$$D_t\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{V} - \boldsymbol{B}(\nabla\cdot\boldsymbol{V}), \tag{3}$$

where ρ is the mass density, $\Sigma \equiv \rho h$ is the surface mass density; h is the disc thickness; and Ψ is the gravitational potential. Here, $\nabla \equiv \boldsymbol{e}_x\partial_x + \boldsymbol{e}_y\partial_y$ stands for the two-dimensional gradient operator.

The equilibrium state of the system is specified by the constant vectors of the angular rotation velocity $\boldsymbol{\Omega} \equiv (0, 0, \Omega < 0)$ and the large-scale background azimuthal field $\boldsymbol{B}_0 \equiv (0, B_0, 0)$, and by the linearized (locally plane-parallel) mean velocity field $\boldsymbol{V}_0 \equiv (0, 2\mathcal{A}x, 0)$, with $\mathcal{A} \equiv (r/2)\partial_r\Omega > 0$ being the first Oort constant.

The linear perturbation equations that come out of (1)–(3) are (FL97) $[\mathcal{D}_t \equiv \partial_t + 2\mathcal{A}x\partial_y]$

$$\mathcal{D}_t\sigma + \Sigma_0(\partial_x u + \partial_y v) = 0, \tag{4}$$

$$\mathcal{D}_t u - 2\Omega v = -\partial_x\chi + \frac{B_0}{4\pi\rho_c}(\partial_y b_x - \partial_x b_y), \tag{5a}$$

$$\mathcal{D}_t v + 2\mathcal{B}u = -\partial_y\chi, \tag{5b}$$

$$\mathcal{D}_t b_x = B_0\partial_y u, \tag{6}$$

$$\partial_x b_x + \partial_y b_y = 0, \tag{7}$$

where σ, u, b_x, v, b_y and ψ are the perturbations of surface mass density, velocity, magnetic field radial and azimuthal components and gravitational potential, respectively. The second Oort constant is $\mathcal{B} \equiv \Omega + \mathcal{A}$, and $\chi \equiv C_s^2\sigma/\Sigma_0 + \psi$, where C_s is the sound speed. These equations constitute a closed set together with the three-dimensional Poisson equation:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\psi = 4\pi G\sigma\delta(z), \tag{8}$$

where G is the gravitational constant and $\delta(z)$ is the Dirac delta function in terms of the vertical coordinate z.

The essence of the 'non-modal approach' is that the system sustains, together with conventional exponentially evolving disturbances, a class of solutions with non-exponential time evolution. In this simple case, when the background flow is plane-parallel and the spatial inhomogeneity in the equations is linear and one-dimensional, Kelvin's transformation of variables $x' = x$, $y' = y - 2\mathcal{A}xt$, $z' = z$, $t' = t$ is sufficient to recover that class of non-exponentially evolving perturbations, which is overlooked in the framework of the standard normal-mode approach. The switch to these variables transforms the initial spatial inhomogeneity of the system to a time inhomogeneity, because $\mathcal{D}_t \rightarrow \partial_{t'}$ and $\partial_x \rightarrow \partial_{x'} - 2\mathcal{A}t'\partial_{y'}$. This enables one to look for solutions in the form of spatial Fourier harmonics (SFHs) (Chagelishvili, Rogava & Segal 1994), bearing the form $\exp[\mathrm{i}(k_{x'}x' + k_{y'}y')]$. Employing this

[1] In the hydrodynamical literature, a similar approach was originally introduced by Lord Kelvin (1887) in the study of the linear stability of the plane Couette flow.

ansatz, one can effectively reduce the system to a set of first-order ordinary differential equations with time-dependent coefficients (FL97) for the perturbation amplitudes $(\hat{\sigma}, \hat{u}, \hat{b}_x, \hat{v}, \hat{b}_y, \hat{\chi})$ of the relevant Fourier components (the superscript $'$ is hereafter dropped):

$$d_\tau \hat{\sigma} + \frac{\mathrm{i}\Sigma_0 k_y}{2\mathcal{A}}(\hat{v} - \tau\hat{u}) = 0, \tag{9}$$

$$d_\tau \hat{u} - \frac{\Omega}{\mathcal{A}}\hat{v} = \frac{\mathrm{i}k_y}{2\mathcal{A}}\tau\hat{\chi} + \frac{\mathrm{i}k_y}{2\mathcal{A}}\frac{B_0}{4\pi\rho_0}(1 + \tau^2)\hat{b}_x, \tag{10}$$

$$d_\tau \hat{v} + \frac{B}{\mathcal{A}}\hat{u} = -\frac{\mathrm{i}k_y}{2\mathcal{A}}\hat{\chi}, \tag{11}$$

$$d_\tau \hat{b}_x = \frac{\mathrm{i}k_y B_0}{2\mathcal{A}}\hat{u}. \tag{12}$$

Here the Maxwell equation

$$\hat{b}_y = \tau\hat{b}_x \tag{13}$$

is already used to express $\hat{b}_y$ in terms of $\hat{b}_x$, and the dimensionless time variable of FL97, $\tau = 2\mathcal{A}t - k_x/k_y$, is introduced. The closure of the set is achieved by the solution of the Poisson equation which relates $\hat{\psi}$ to $\hat{\sigma}$ (FL97):

$$\hat{\psi} = -\frac{2\pi G\hat{\sigma}}{k_y(1 + \tau^2)^{1/2}}. \tag{14}$$

This is the set of equations derived by FL97. Further, they derived from these equations a complicated, fourth-order differential equation for the surface mass density Fourier amplitude $\hat{\sigma}$, and performed in the rest of their paper a numerical analysis of the system on the basis of that equation. Among other interesting qualities of the hydromagnetic waves, they noticed that MHD waves come into view as being coupled. Although noting this phenomenon, FL97 did not give a physical explanation, an explicit mathematical proof and/or a description of it. Does the coupling really exist? What is its physical nature? Does it lead to any new physical effects? The purpose of this Letter is to give answers to these questions.

First, let us simplify the notation and make our equations totally dimensionless. This is possible by introducing new dimensionless parameters: $R \equiv \mathcal{A}/C_s k_y$, $\beta_0 \equiv k_x/k_y$, $\xi \equiv C_A/C_s$ (with C_A the Alfvén speed), $\omega \equiv \Omega/C_s k_y$, $\delta \equiv -2\pi G\Sigma_0/C_s^2 k_y$; and new dimensionless variables $T \equiv tC_s k_y$, $\beta(T) \equiv \beta_0 - 2RT$, $\alpha(T) \equiv \delta/(1 + \beta^2)^{1/2}$, $\mathcal{S} \equiv \mathrm{i}\hat{\sigma}/\Sigma_0$, $\mathcal{U} \equiv \hat{u}/C_s$, $\mathcal{V} \equiv \hat{v}/C_s$, $b \equiv \mathrm{i}b_x/B_0$. Taking into account (13) and (14), we can rewrite (9)–(12) in the following totally dimensionless form:

$$\mathcal{S}^{(1)} = \beta\mathcal{U} + \mathcal{V}, \tag{15}$$

$$\mathcal{U}^{(1)} = -\beta(1 + \alpha)\mathcal{S} + 2\omega\mathcal{V} + \xi^2(1 + \beta^2)b, \tag{16}$$

$$\mathcal{V}^{(1)} = -(1 + \alpha)\mathcal{S} - 2(\omega + R)\mathcal{U}, \tag{17}$$

$$b^{(1)} = -\mathcal{U}, \tag{18}$$

where $F^{(n)} \equiv \partial_T^n$ denotes the nth-order time derivative of F. Note that the dimensionless parameters used here are in direct one-to-one correspondence to the parameters used by FL97. In particular, $\beta_0 = -\tau_0$, $\omega = 1/m\sqrt{\eta_1}$, $R = 1/2m\eta_0\sqrt{\eta_1}$, $\xi = \sqrt{\eta_2/\eta_1}$ and $\delta = -\eta_3/m\eta_1$. The dimensionless time variable τ in FL97 is related to our T as $T = m\eta_0\sqrt{\eta_1}(\tau - \tau_0)$.

The next important step in our proof is to introduce the new auxiliary variable

$$\mathcal{L} \equiv \mathcal{S} + \beta(T)b, \tag{19}$$

to find out that

$$\mathcal{L}^{(1)} = \mathcal{V} - 2Rb, \tag{20}$$

and taking one more derivative of $\mathcal{L}$ to reduce, eventually, the set (15)–(18) to the following pair of coupled, second-order equations:

$$\mathcal{L}^{(2)} + W_1^2\mathcal{L} = 2\omega b^{(1)} + Cb, \tag{21a}$$

$$b^{(2)} + W_2^2 b = -2\omega\mathcal{L}^{(1)} + C\mathcal{L}, \tag{21b}$$

with

$$W_1^2 = 1 + \alpha, \tag{22a}$$

$$W_2^2 \equiv \xi^2(1 + \beta^2) + (1 + \alpha)\beta^2 + 4\omega R, \tag{22b}$$

$$C \equiv (1 + \alpha)\beta. \tag{22c}$$

These equations already bear features of equations for coupled oscillatory systems. The coupling is exclusively evoked by the non-zero shear, i.e. by the differential character of the motion. This is obvious, since only the presence of the shear [$R \neq 0$, leading to $\alpha = \alpha(T)$ and $\beta = \beta(T)$] makes the coefficients of the system (21)–(22) time-dependent. In the absence of shear the system is decoupled in terms of effective normal coordinates (Morse 1981), and it possesses two uncoupled fundamental (normal) frequencies, the solutions of the following biquadratic equation:

$$\omega_\pm^4 - [4\omega^2 + (1 + \beta_0^2)(1 + \alpha_0 + \xi^2)]\omega_\pm^2 + (1 + \alpha_0)\xi^2(1 + \beta_0^2) = 0, \tag{23}$$

which correspond to fast and slow MHD density waves ($\omega_+ = \omega_f$ and $\omega_- = \omega_s$), respectively. It is relevant to note that, in the limit of non-rotating ($\omega = 0$) and non-self-gravitating ($\delta = \alpha = 0$) shear flow, the system reduces to the one considered by Chagelishvili et al. (1996), where the effect of the shear-induced wave coupling was originally disclosed.

The system (21)–(22) is not yet in the canonical form for the standard oscillating system with two degrees of freedom, because of the appearance of first-order derivatives in the coupling terms. However, the system may be formally reduced to the canonical form by employing the following rotational transformation:

$$\mathcal{L} = \mathcal{X}\cos\omega T + \mathcal{Y}\sin\omega T, \tag{24a}$$

$$b = -\mathcal{X}\sin\omega T + \mathcal{Y}\cos\omega T, \tag{24b}$$

which, after some laborious but straightforward algebra, leads to the following canonical system for coupled oscillations with two degrees of freedom:

$$\mathcal{X}^{(2)} + \Omega_1^2\mathcal{X} + \mathcal{C}\mathcal{Y} = 0, \tag{25a}$$

$$\mathcal{Y}^{(2)} + \Omega_2^2\mathcal{Y} + \mathcal{C}\mathcal{X} = 0. \tag{25b}$$

The eigenfrequencies (Ω_1 and Ω_2) and the coupling coefficient ($\mathcal{C}$) appearing in this system are expressed through W_1, W_2, and C in the following way:

$$\Omega_1^2 \equiv \omega^2 + W_1^2\cos^2\omega T + W_2^2\sin^2\omega T + C\sin 2\omega T, \tag{26a}$$

$$\Omega_2^2 \equiv \omega^2 + W_1^2\sin^2\omega T + W_2^2\cos^2\omega T - C\sin 2\omega T, \tag{26b}$$

$$\mathcal{C} \equiv \frac{1}{2}(W_1^2 - W_2^2)\sin 2\omega T - C\cos 2\omega T. \tag{26c}$$

Thus we have succeeded in reducing the initial system to the

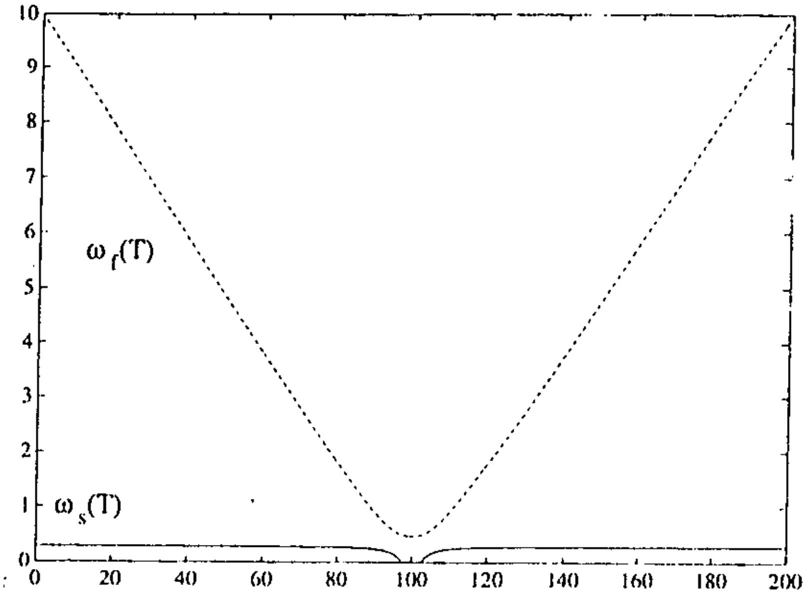

Figure 1. Dispersion curves $\omega_s(T)$ and $\omega_f(T)$ exhibiting adiabatic variation of SMW and FMW frequencies. The graph is drawn for $\beta_0 = 10$, $R = 0.05$, $\omega = -0.2$, $\xi = 0.3$ and $\delta = -1.04$.

above form, distinctive for coupled oscillations with two degrees of freedom. We have proved that the two wave modes sustained by the system – SMW and FMW – are actually coupled. The presence of shear in the disc flow ($R \neq 0$) ensures temporal variability of $\alpha(T)$ and $\beta(T)$ and makes the fundamental modes of oscillation *physically* coupled (Chagelishvili et al. 1996). Under certain circumstances the coupling leads to energy exchange between the waves, and to the transformation of waves into each other.

When the system parameters vary slowly (adiabatically), as they do when $2R/\beta_0 \ll 1$, the standard theory of coupled oscillations (Morse 1981) may still serve as a useful guide in understanding and interpreting the inherent physical processes. In particular, one can still operate with the roots of (23) as adiabatically varying characteristics of oscillations, replacing α_0s by $\alpha(T)$s and β_0s by $\beta(T)$s. Fig. 1 displays the dispersions of SMWs and FMWs, given by the functions $\omega_-(T) \equiv \omega_s(T)$ and $\omega_+(T) \equiv \omega_f(T)$. The graph is drawn for the case when the ratio of wavenumbers is $\beta_0 = 10$; the dimensionless first Oort constant and dimensionless angular rotation velocity are $R = 0.05$ and $\omega = -0.2$, respectively; and $\xi = 0.3$, $\delta = -1.04$. Note that in this particular case self-gravity causes, for SMW, the appearance of the limited instability interval around the moment of time $T_* \equiv \beta_0/2R$.

The usefulness of these phenomenological dispersion curves is associated with the general property of this kind of systems. The 'adiabatic behaviour' of the modes implies that they should normally follow dispersion curves of their own: the spectral energy density of either the FMW or SMW should be proportional to its corresponding normal frequency $E_\pm(T) \sim \omega_\pm(T)$ (Chagelishvili et al. 1996). For the problem under discussion, the total energy density of a perturbation, being the sum of its kinetic, magnetic and internal (compressional) energies, may be written as

$$E(T) \equiv (\mathcal{U}^2 + \mathcal{V}^2)/2 + \xi^2(1 + \beta^2)b^2/2 + (1 + \alpha)S^2/2, \qquad (27a)$$

and this is an exactly conserved quantity in the 'zero shear' ($R = 0$) case. This circumstance is apparent from the equation for $E^{(1)}$ which reads

$$E^{(1)} = R\left[-2\mathcal{U}\mathcal{V} + \frac{\alpha\beta}{1 + \beta^2}S^2 + \xi^2\beta b^2\right]. \qquad (27b)$$

When transformations are absent, the time evolution of $E(T)$ is indeed adiabatic with $E_\pm(T) \sim \omega_\pm(T)$. However, when a wave undergoes transformation its evolution is *not* adiabatic inside the so-called 'degeneracy region' (Chagelishvili et al. 1996), where efficient transformation of the initial wave into the other one takes place. If the degeneracy region overlaps with the region of transient instability, then the deviation from adiabaticity becomes more pronounced. Figs 2 and 3 vividly illustrate these remarkable circumstances. They are drawn for the same sample of parameters as Fig. 1, and the initial perturbation is chosen to be an SMW. Fig. 2 shows the temporal progress of the perturbation energy. Comparing the course of the evolution of the $E(T)$ with the dispersion curves given in Fig. 1, one can easily see that originally the energy is evolving via the $E(T) \sim \omega_-(T)$ law, remaining almost constant until it 'enters' the 'degeneracy region'. Here it becomes partially transformed into the wave mode with a significant FMW component. The remaining SMW component is boosted by the transient instability. The result is that the perturbation that leaves the 'degeneracy region' is a larger amplitude mixture of SMW and FMW modes, dominated by the FMW mode. The energy of the perturbation continues to evolve adiabatically, but this time with $E(T) \sim \omega_+(T)$: it follows the dispersion curve of the FMW, implying that at this stage of its evolution the perturbation steadily extracts energy from the background shear flow. Fig. 3 displays the temporal evolution of $S(T)$, and clearly shows how the initial, almost constant low-frequency SMW is transformed into the FMW.

Yet another interesting phenomenon associated with the shear-induced wave couplings is the appearance of shear-beat waves (Rogava & Mahajan 1997; Poedts et al. 1998). Our preliminary

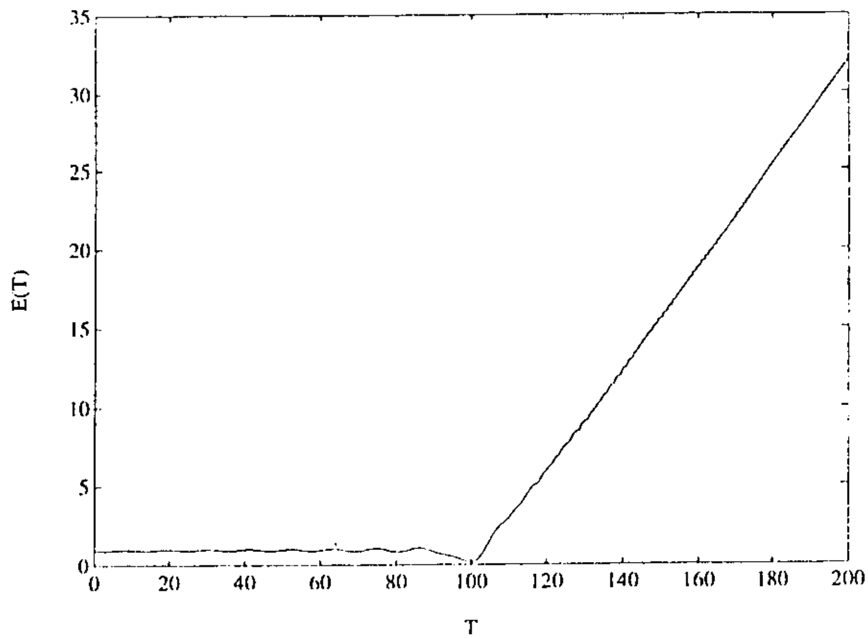

Figure 2. Temporal evolution of $E(T)$ corresponding to the example given by Fig. 1.

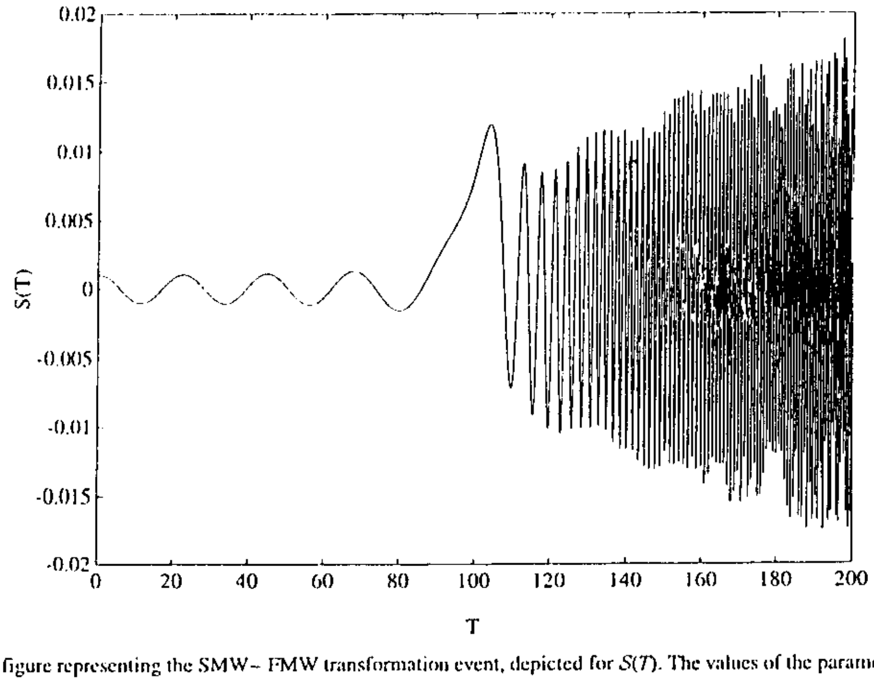

Figure 3. An example of a figure representing the SMW–FMW transformation event, depicted for $S(T)$. The values of the parameters are the same as in Figs 1 and 2.

numerical results (see Fig. 4) show that shear-beat waves are excited in this system too: in disc layers with low shear rates and for the perturbations with $R \ll \beta_0 \ll 1$.

These beat waves acquire a significant phase difference between surface density and magnetic field perturbations, and hence they may potentially account for the appearance of the magnetic field spiral structures anticorrelated with the optical spiral arms.

It is reasonable to assume that shear-induced wave couplings in the magnetized, self-gravitating, differentially rotating gaseous disc should have a considerable 'share' in the establishment of the overall physical dynamics of the MHD waves. The wave coupling naturally leads to mutual transformations of waves. In simple MHD systems the transformations are strongly pronounced when the Alfvén speed and the sound speed are approximately of the same order, i.e. when $\xi = 1$ (Chagelishvili et al. 1996). It is interesting to examine whether this criterion remains the same or whether it is somehow affected by the presence of the self-gravitation ($\delta \neq 0$) and Coriolis parameter ($\omega \neq 0$). These two physical factors (self-gravity and non-zero Coriolis parameter) lead to the appearance of the transient Jeans instability and 'epicyclic shaking', respectively. The interweaving of these physical effects with the shear-driven wave transformations demands careful, detailed examination and is beyond the scope of this Letter.

At the present stage of this study it seems clear that the

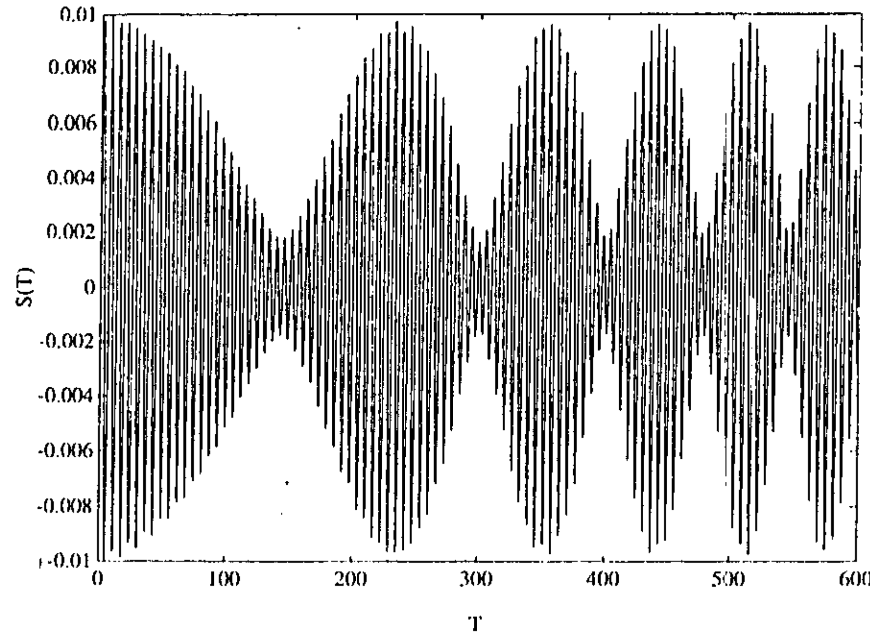

Figure 4. Shear-beat waves, shown for the density perturbation $S(T)$. The set of parameters is $\beta_0 = 10^{-2}$, $R = 10^{-4}$, $\omega = -10^{-2}$, $\xi = 1$ and $\delta = -10^{-2}$.

problem, considered by FL97, being of high theoretical and observational importance, needs to be revisited and considered in the light of this new knowledge about the velocity shear induced wave transformations. By saying this we have no desire to diminish the importance of the results obtained by Fan & Lou. On the contrary, we think that, by providing rigorous analytic evidence for the MHD density wave couplings in spiral galaxies, we emphasize the significance of the problem, and hopefully take a step towards the understanding of galactic magnetic spiral structures.

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