united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

SMR 1161/6

AUTUMN COLLEGE ON PLASMA PHYSICS

25 October - 19 November 1999

Magnetic Confinement and Pressure-driven Instabilities

M. WAKATANI

Kyoto University, Japan

These are preliminary lecture notes, intended only for distribution to participants.

Magnetic Confinement and Pressure-driven Instabilities

M. Wakatani
Kyoto University
Gokasho, Uji. Japan
611-0011

Contents

- Reduced MHD equations
- Ideal and resistive interchange modes
- Nonlinear evolution of interchange mode
- Comparison to experiments

Rayleigh - Taylor Instability

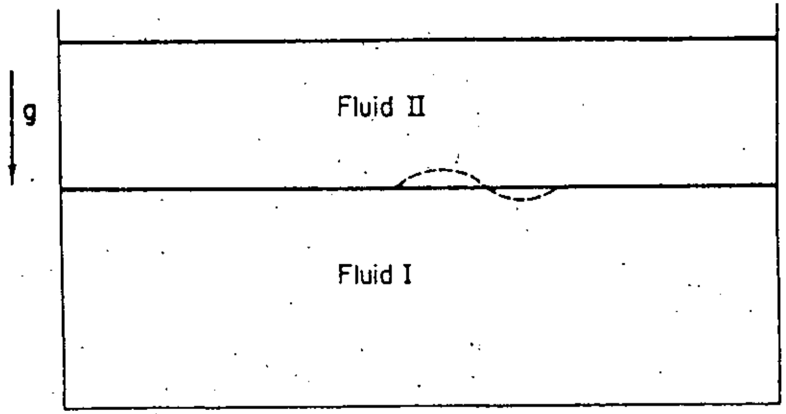
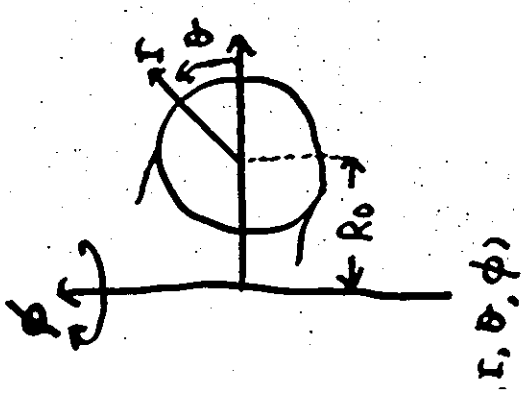


FIG. 5-3. Hydrostatic equilibrium of two fluids in a gravitational field.

$g \longleftrightarrow$ "curvature of magnetic field line"



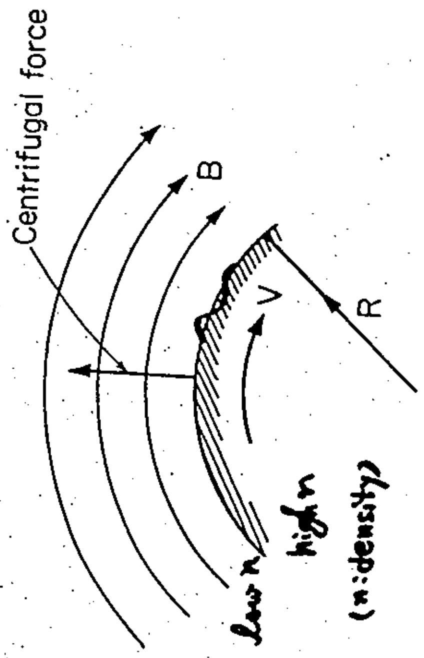


Fig. 10-8

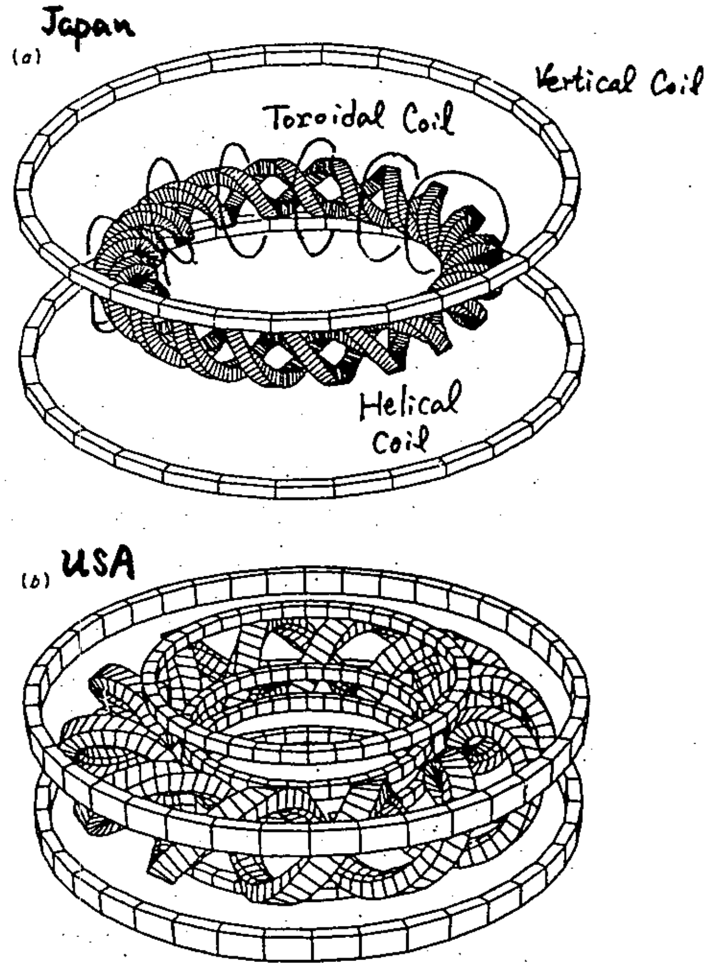


FIG. 2.6. Coil sets for heliotron/torsatron configurations: (a) Heliotron E, (b) ATF.

H-E

safety factor

$q = \frac{1}{\iota}$

$\iota = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} \Delta\theta_i}{N} \cdot \frac{1}{2\pi}$

(rotational transform)

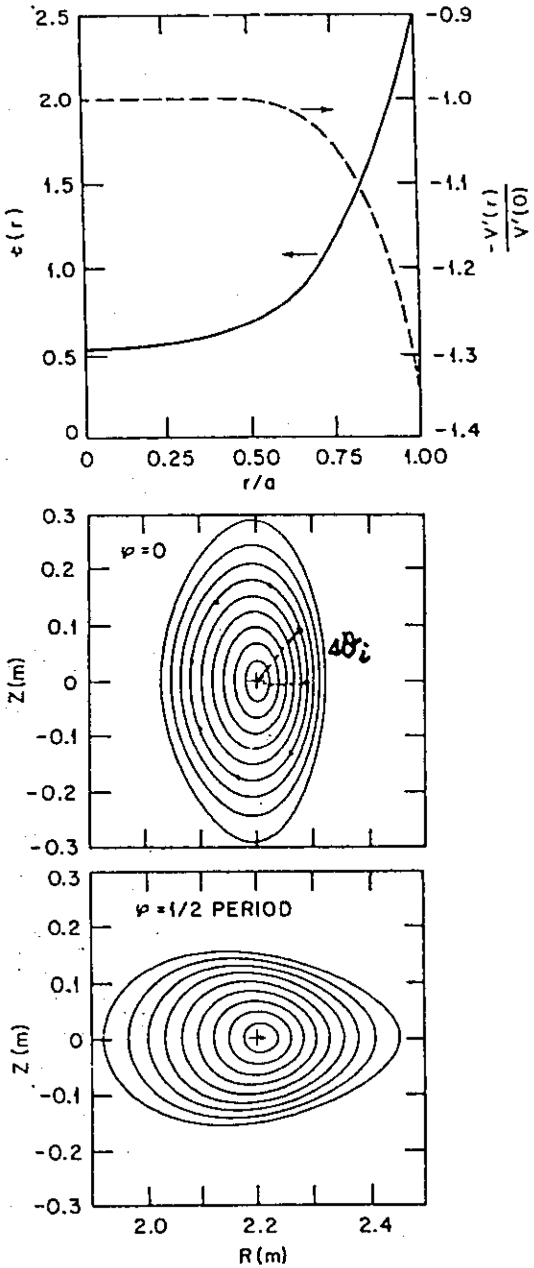


FIG. 2.1. Vacuum magnetic surfaces and profiles of ι and V' for a representative high-transform, high-shear stabilized configuration (Heliotron E). In this and the following diagrams, −V' is plotted, so that an upward-sloping curve corresponds to a magnetic well and a downward-sloping curve

Magnetic Field in Stellarator/Heliotron

$$\mathbf{B} = B_0\hat{\xi} + \delta\nabla\Phi_h + \delta^2\nabla\psi_J \times \hat{\xi} + \delta^2 B_2\hat{\xi}$$

$B_0\hat{\xi}$: longitudinal field

$\nabla\Phi_h$: stellarator field

$\nabla\psi_J \times \hat{\xi}$: poloidal field due to plasma current

$B_2\hat{\xi}$: toroidal correction proportional to $r\cos\theta/R_0$ and diamagnetic correction

δ : expansion parameter

Note

(1) The stellarator magnetic field contributes to both the rotational transform and the helical curvature through κ(average curvature).

(2) Ordering for the amplitude of stellarator field:

$$\delta^2 \sim \epsilon \quad .$$

((3) μ and $D_\perp$ should be finite to obtain a saturated state of nonlinear interchange modes.)

LINEAR THEORY OF RESISTIVE INTERCHANGE MODE

Resistive MHD Equations

(1) $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

(2) $$\rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla P \quad \left(\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right)$$

(3) $$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0 \quad (\gamma : \text{ratio of specific heats})$$

(4) $$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \quad (\eta : \text{resistivity})$$

(5) $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

(6) $$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$

(7) $$\nabla \cdot \mathbf{B} = 0$$

Stellarator and Heliotron Devices

MASAHIRO WAKATANI

Chapter 5

MHD INSTABILITIES IN HERIOTRONS

New York Oxford
OXFORD UNIVERSITY PRESS
1998

Table 11-1

Ordering	O(1)	O(ϵ)	O(ϵ^2)
Beta value		β	
pressure		p	
poloidal flux		Ψ	
stream function		ϕ	
resistivity		η	
toroidal field	B_0	B_t	
poloidal field		$\nabla\Psi \times \hat{z}$	
perpendicular velocity		$\nabla\phi \times \hat{z}$	
parallel velocity			$V_{//}$
time derivative		$\frac{\partial}{\partial t}$	
perpendicular derivative	$\nabla_\perp$		
parallel derivative		$\nabla_{//}$	

K. Nishikawa and M. Wakatani

Plasma Physics

- Basic Theory with Fusion Applications -

(Springer - Verlag)

REDUCED MHD EQUATIONS

(1) $$(ds)^2 = (dr)^2 + (rd\theta)^2 + \left(1 + \frac{x}{R_0}\right)(dz)^2$$

for the toroidal coordinates $(r,\ \theta,\ \varphi)$, $x = r\cos\theta$ and $z = -R_0\varphi$.

(2) $$\mathbf{B} = B_0\hat{z} + \nabla\Psi \times \hat{z} + B_1\hat{z}\ ,$$

where $B_1 = I_1/R_0 - B_0 x/R_0$, and I_1/R_0 is a diamagnetic correction and $-B_0 x/R_0$ is a toroidal curvature.

(3) $$\mu_0\mathbf{J} = -\nabla_\perp^2\Psi\hat{z} + \nabla I_1/R_0 \times \hat{z}$$

Here $|\mathbf{u}_z| \ll |\mathbf{u}_\perp|$ and $\nabla\cdot\mathbf{u}_\perp \simeq 0$ gives

(4) $$\mathbf{u}_\perp = \nabla\phi \times \hat{z}\ .$$

From Ohm's law and Faraday's law,

(5) $$\frac{\partial\Psi}{\partial t} = \mathbf{B}\cdot\nabla\phi + \frac{\eta}{\mu_0}\nabla_\perp^2\Psi + E$$

Equation of motion gives

(6) $$0 = -\nabla_\perp P + \mathbf{J} \times B_0\hat{z}$$

(7) $$\nabla_\perp\left(P + \frac{I_1}{I_0}\frac{B_0^2}{\mu_0}\right) = 0$$

or

(7') $$I_1/I_0 = -\mu_0 P/B_0^2\ .$$

$\nabla\cdot(\mathbf{J}_\perp + \mathbf{J}_{||}) = 0$ gives

(8) $$\nabla\cdot\left[-\left(\rho\frac{d\mathbf{u}}{dt} + \nabla P\right) \times \frac{\mathbf{B}}{B^2} + \sigma\mathbf{B}\right] = 0\ ,$$

where $\sigma = -\nabla_\perp^2\Psi/\mu_0 B_0$.

(9) $$\mathbf{B}\cdot\nabla \times \frac{\rho}{B^2}\frac{d\mathbf{u}}{dt} = \mathbf{B}\cdot\nabla\sigma + \frac{(\nabla B^2 \times \nabla P)\cdot\mathbf{B}}{B^4}\ .$$

(10) $$\mathbf{B}\cdot\nabla \times \frac{\rho}{B^2}\frac{d\mathbf{u}}{dt} \simeq -\frac{\rho}{B_0^2}\frac{d}{dt}\nabla_\perp^2\phi\ .$$

By noting

(11) $$B^2 = B_0^2\left[1 - 2\left(\frac{\mu_0 P}{B_0^2} + \frac{x}{R_0}\right)\right]$$

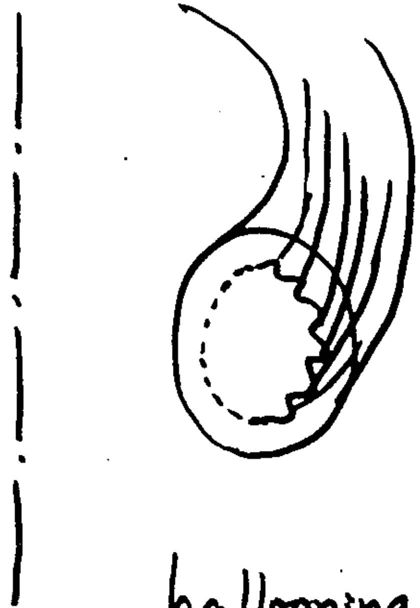
Thus,

(12) $$\rho\frac{d}{dt}\nabla_\perp^2\phi = \frac{1}{\mu_0}\mathrm{B}\cdot\nabla\nabla_\perp^2\Psi + \left(\nabla\frac{2x}{R_0}\times\nabla P\right)\cdot\hat{z}$$

Pressure evolution equation is

(13) $$\frac{\partial P}{\partial t} + (\nabla\phi\times\hat{z})\cdot\nabla P = 0$$

Equations (5), (12), (13) are a closed set of MHD equations called Redeced MHD Model.



ballooning mode

Energy conservation

$$\frac{\partial}{\partial t}\int\left[\frac{1}{2}(\nabla_\perp\Psi)^2 + \frac{1}{2}(\nabla_\perp\phi)^2 - xP\right]dV = -\eta\int(\nabla_\perp^2\Psi)^2 dV$$

$$+\frac{P}{\gamma-1}$$

For stellarators the average curvature can be produced by external helical windings. In this case $\nabla(2x/R_0)$ is replaced with $\nabla(2x/R_0+\Omega(r))$. For heliotrons, $|\nabla(2x/R_0)| \ll |\nabla\Omega|$. Thus, the average curvature effect can be included in a cylindrical geometry in case of heliotron plasmas.

$\Omega(r) = \overline{B_h^2}/B_0^2$, where B_h is a helical magnetic field and the bar denotes averaging over the field period in the toroidal direction.

NONLINEAR EVOLUTION OF INTERCHANGE MODES

– Reduced MHD Equations for Stellarator/Heliotron Devices –

(1) $$\rho\frac{\partial U}{\partial t} = -\rho\,\vec{V}_\perp\cdot\nabla U - \nabla_\parallel\cdot J_\parallel + \hat{z}\cdot(\nabla\Omega\times\nabla P) + \rho\mu\nabla_\perp^2 U$$

(2) $$\frac{\partial \Psi_J}{\partial t} = -R_0\nabla_\parallel\phi + R_0\eta J_\parallel$$

(3) $$\frac{\partial P}{\partial t} = -\vec{V}_\perp\cdot\nabla P - \frac{dP_0}{dr}V_r + D_\perp\nabla_\perp^2 P$$

(4) $$\vec{B} = -(\nabla(\psi_J+\psi_h)\times\hat{z})/R_0 + B_0\hat{z}$$

(5) $$\vec{V}_\perp = (\nabla\phi\times\hat{z})/B_0$$

(6) $$U = \nabla_\perp^2\phi$$

(7) $J_{\|} = -\nabla_\perp^2 \Psi_J / R_0$

(8) $\kappa \equiv \dfrac{d\Omega}{dr} = \dfrac{r}{R_0} B_0^2 V''$

(κ : averaged magnetic field line curvature,

and $V = \int d\ell / B$)

(9) $$\frac{\partial \langle P \rangle}{\partial t} = -\frac{\partial}{\partial t}\langle V_r P\rangle + D_\perp \frac{1}{r}\frac{d}{dr}\left(r\frac{d\langle P\rangle}{dr}\right) + S(r)$$

($r = const$ is a magnetic surface)

Dissipations η : resistivity

μ : viscosity

$D_\perp$: diffusion coefficient

Ordering $\beta = \langle P \rangle / (B_0^2/2\mu_0) \sim \epsilon \sim \delta^2$

Note For stellarator/heliotron, κ includes both toroidal curvature and helical curvature. If we neglect the helical curvature, Eqs.(1), (2), (3) become Strauss equations for high beta tokamaks.

Several dissipative terms are added to obtain saturated states

$$\frac{\partial}{\partial t}\nabla_\perp^2 \phi = [\phi,\ \nabla_\perp^2 \phi] + [\nabla_\perp^2 A,\ \Psi] + \frac{\partial}{\partial \zeta}\nabla_\perp^2 A + [\Omega,\ p] + \mu \nabla_\perp^2 u$$

$$\frac{\partial A}{\partial t} = [\phi,\ \Psi] + \frac{\partial \phi}{\partial \zeta} + \frac{1}{S}\nabla_\perp^2 A$$

$$\frac{\partial p}{\partial t} = [\phi,\ p] + \kappa_\perp \nabla_\perp^2 p + \kappa_{\|}\nabla_{\|}^2 p$$

μ : viscosity, S : magnetic Reynolds number

$\kappa_\perp$: perpendicular heat conductivity

$\kappa_{\|}$: parallel heat conductivity

Equilibrium, $\dfrac{\partial}{\partial t} = 0, \quad \phi = 0, \quad \dfrac{\partial}{\partial \zeta} = 0$

$$\nabla_\perp^2 A = -\Omega \frac{dp}{d\Psi} + G(\Psi)$$ (G-S equation)

($[f,\ g] = \nabla f \times \nabla g \cdot \hat{\zeta}$)

FLOW SHEAR EFFECTS ON IDEAL INTERCHANGE MODES

Linearized Reduced MHD Equations for Stellarators

$$(1)\quad \left(\frac{\partial}{\partial t}+\mathrm{v}_0\cdot\nabla\right)\nabla_\perp^2\tilde{\phi}+\tilde{\mathrm{v}}_0\cdot\nabla_r(\nabla_\perp^2\Phi_0)$$
$$=-\nabla_\parallel(\nabla_\perp^2\tilde{A})+\nabla\tilde{P}\times\nabla\Omega\cdot\hat{\zeta}$$

$$(2)\quad \left(\frac{\partial}{\partial t}+\mathrm{v}_0\cdot\nabla\right)\tilde{A}=-\nabla_\parallel\tilde{\phi}$$

$$(3)\quad \left(\frac{\partial}{\partial t}+\mathrm{v}_0\cdot\nabla\right)\tilde{P}+\tilde{\mathrm{v}}_0\cdot\nabla_r P_0=0$$

$$\nabla_\parallel=\frac{\partial}{\partial\zeta}+\nabla\psi_h\times\hat{\zeta}\cdot\nabla$$

$$\frac{d\Omega}{dr}=\frac{M\epsilon}{\ell}\frac{1}{r^2}\frac{d}{dr}(r^4\imath_h)\ \text{(large aspect ratio limit)}$$

ℓ: pole number, M: pitch number

$\imath_h$: rotational transform

For the cylindrical model;

(I) poloidal shear flow case

$$\mathrm{v}_0=v_E(r)\hat{\theta}=\frac{d\Phi_0(r)}{dr}\hat{\theta}$$

((II) toroidal shear flow case

$$\mathrm{v}_0=v_{\parallel0}(r)\hat{\zeta}\ .$$
)

By introducing $\omega_E=\frac{m}{r}v_E(r)\left(+\frac{n}{R}v_{\parallel0}(r)\right)$, the eigenmode equation is obtained as

$$(\omega-\omega_E)^2\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{m^2}{r^2}\right)\tilde{\phi}-(\omega-\omega_E)\frac{m}{r}(\nabla_\perp^2\Phi_0)'\tilde{\phi}$$
$$-k_\parallel(\omega-\omega_E)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{m^2}{r^2}\right)\left(\frac{k_\parallel\tilde{\phi}}{\omega-\omega_E}\right)+\frac{m^2}{r^2}P_0'\Omega'\tilde{\phi}$$
$$=0\ .$$

note : $\phi(r,\theta,\zeta)=\tilde{\phi}(r)\exp(im\theta-i\frac{n}{R}\zeta)$

For a radially localized marginal interchange mode at $\iota(r_0) = n/m$,

$$\frac{d^2\tilde{\phi}}{dx^2} + \frac{m^2}{r_0^2}\left(\frac{P_0'\Omega'}{[(\omega_E')^2 - (k_{||}')^2]x^2} - 1\right)\tilde{\phi} = 0 \ ,$$

where $x = r - r_0$, $k_{||} \simeq k_{||}'(r_0)x = m\iota_h'(r_0)x$, $\omega - \omega_E = -\omega_E'(r_0)x$ (note: $R_0 = 1$, $B_0 = 1$).

The necessary stability condition for the solution, $\tilde{\phi} \propto x^\nu$, is given by

$$-\frac{m^2 P_0'\Omega'}{(k_{||}')^2[1 - (\omega_E'/k_{||}')^2]r_0^2} < \frac{1}{4} \ .$$

If $\omega_E' = 0$, this inequality becomes Suydam condition.

Euler - Lagrange equation

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

which minimizes potential energy

$$\delta W = \int_0^a (f(\xi')^2 + g\xi^2)\,dr \, .$$

We expand f and g at $r \simeq r_s$ (resonant surface),

$$f \simeq \left[\frac{rF'^2}{k_0^2}\right]\Bigg|_{r=r_s} x^2 \, , \qquad F = \mathbf{k}\cdot\mathbf{B}, \quad x = r - r_s$$

$$g \simeq \left[\frac{2\mu_0 k^2 p'}{k_0^2}\right]\Bigg|_{r=r_s}$$

$$\frac{d}{dx}\left(x^2\frac{d\xi}{dx}\right) + D_s\xi = 0$$

$$D_s = -\left[\frac{2\mu_0 k^2 p'}{rF'^2}\right]\Bigg|_{r=r_s}$$

solution:

$$\xi = C_1 x^{S_1} + C_2 x^{S_2}$$

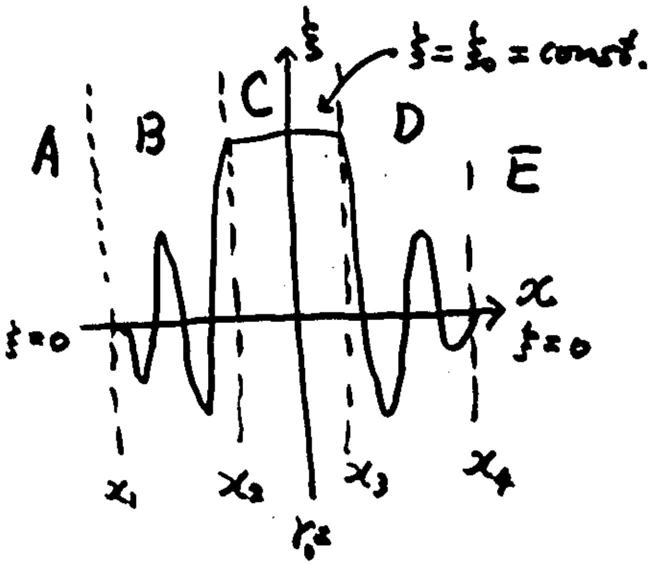
for $r \simeq r_s$, where

$$S_{1,2} = -\frac{1}{2} \pm \frac{1}{2}(1-4D_s)^{1/2}$$

For $1-4D_s < 0$,

$$\xi = \frac{1}{|x|^{1/2}}\left[C_1 \sin(k_r \ln|x|) + C_2 \cos(k_r \ln|x|)\right],$$

where $k_r = (4D_s - 1)^{1/2}/2$. This solution oscillates as $x \to 0$ rapidly.



$$\delta W(B) = \int_{x_1} (x^2\xi'^2 - D_s\xi^2)dx = x^2\xi\xi'\Big|_{x_1}$$

$$= 0.$$

$$\delta W(D) = 0.$$

$$\delta W(C) = \int_{x_2}^{x_3} (x^2\xi'^2 - D_s\xi^2)dx$$

$$= -D_s\xi_0^2(x_3 - x_2)$$

By the assumption of $D_s > 1/4$,

$$\delta W = \delta W(A) + \delta W(B) + \delta W(C) + \delta W(D) + \delta W(E) < 0.$$

$\delta W < 0$ corresponds to that the system is unstable.

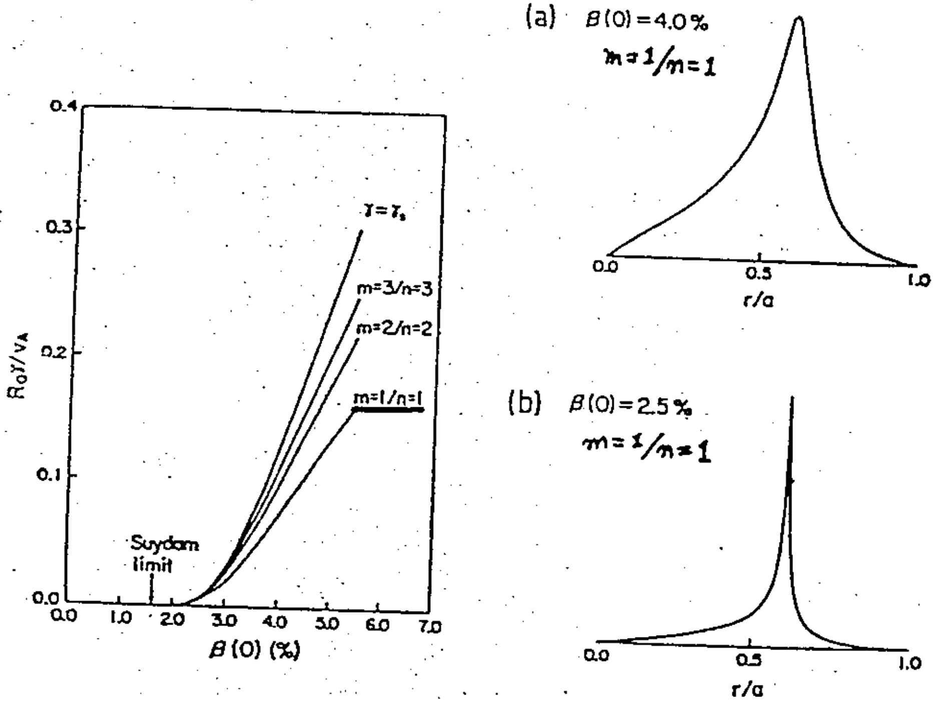
Suydam condition for instability:

Cylindrical Case

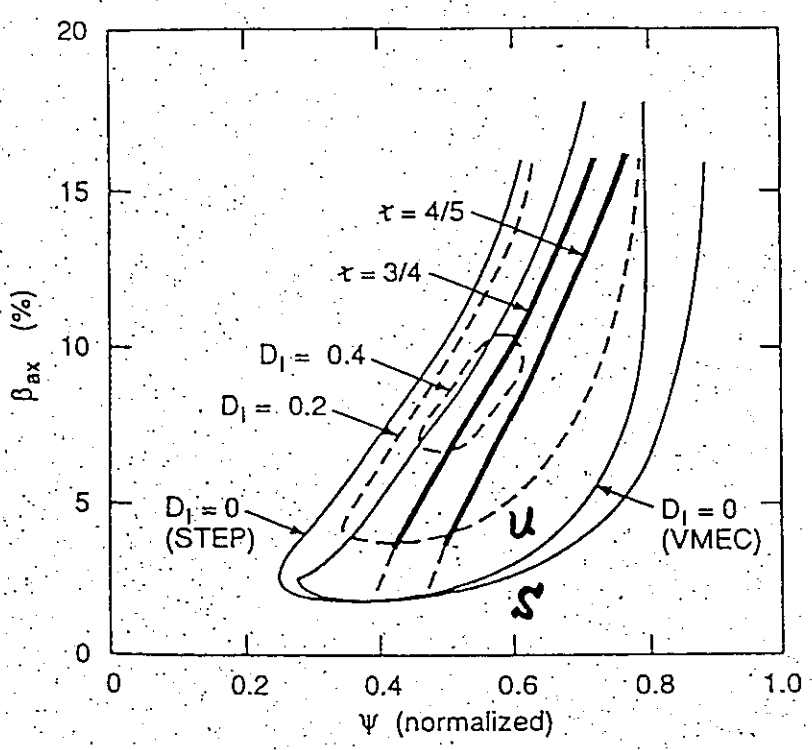
$$\frac{1}{4}\left(\frac{\iota'}{\iota}\right)^2 + \frac{1}{\iota^2}\frac{dP_0}{dr}\underline{\frac{d\Omega}{dr}} < 0$$

Growth rate of Suydam mode:

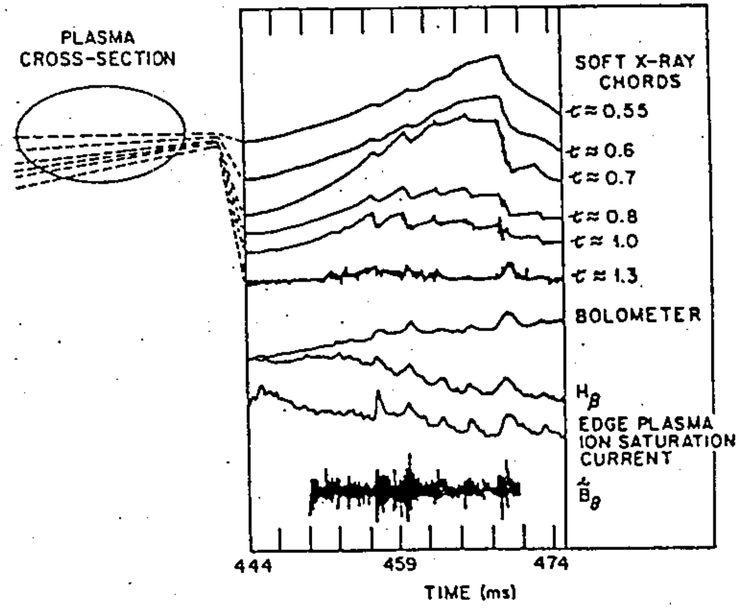
$$\gamma_s \propto \exp[-const/(\beta_{(0)}/\beta^c_{(0)} - 1)^{1/2}]$$

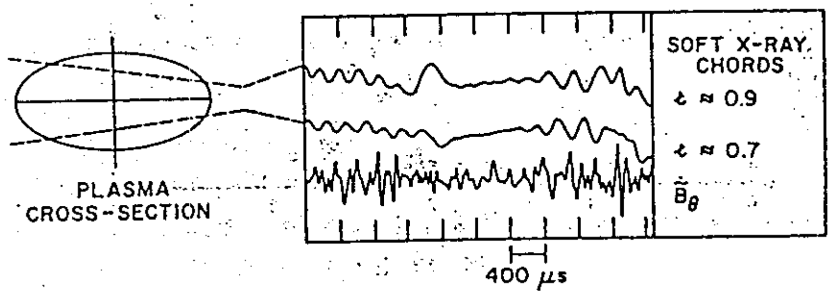


Suydam criterion



Mercier criterion





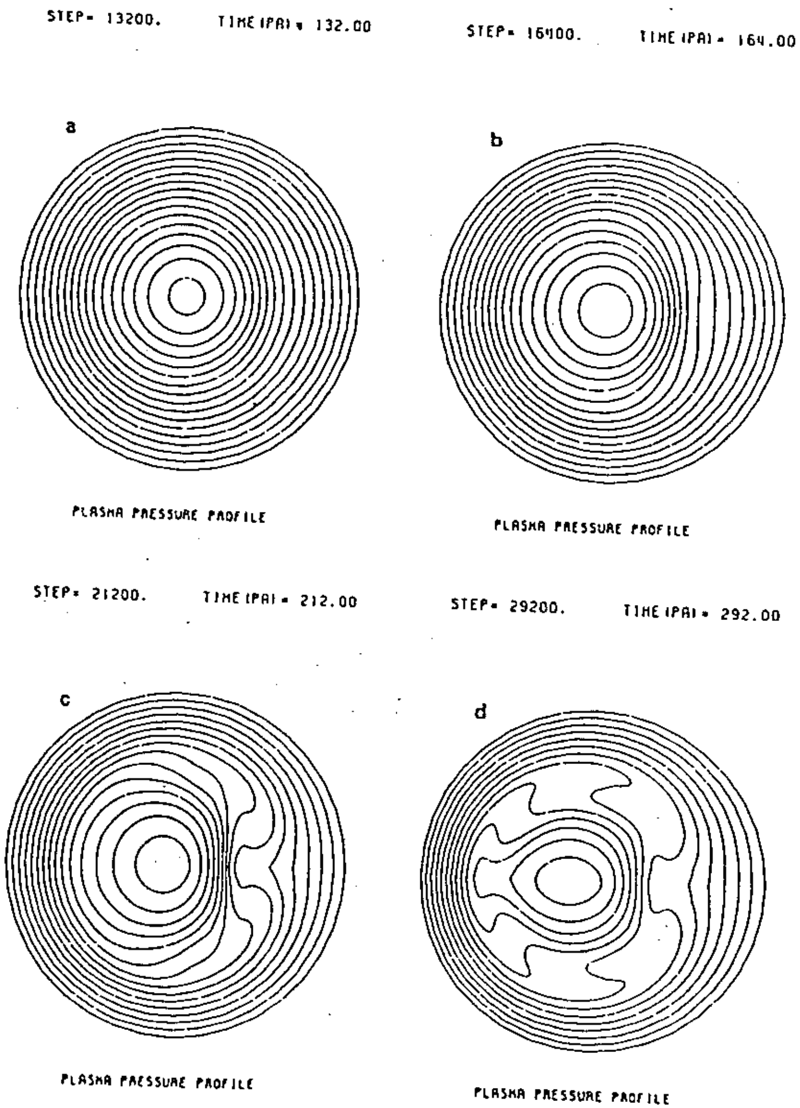


FIG.4. Pressure contours corresponding to Fig.3. (a) T = 132, (b) T = 164, (c) T = 212, (d) T = 292.

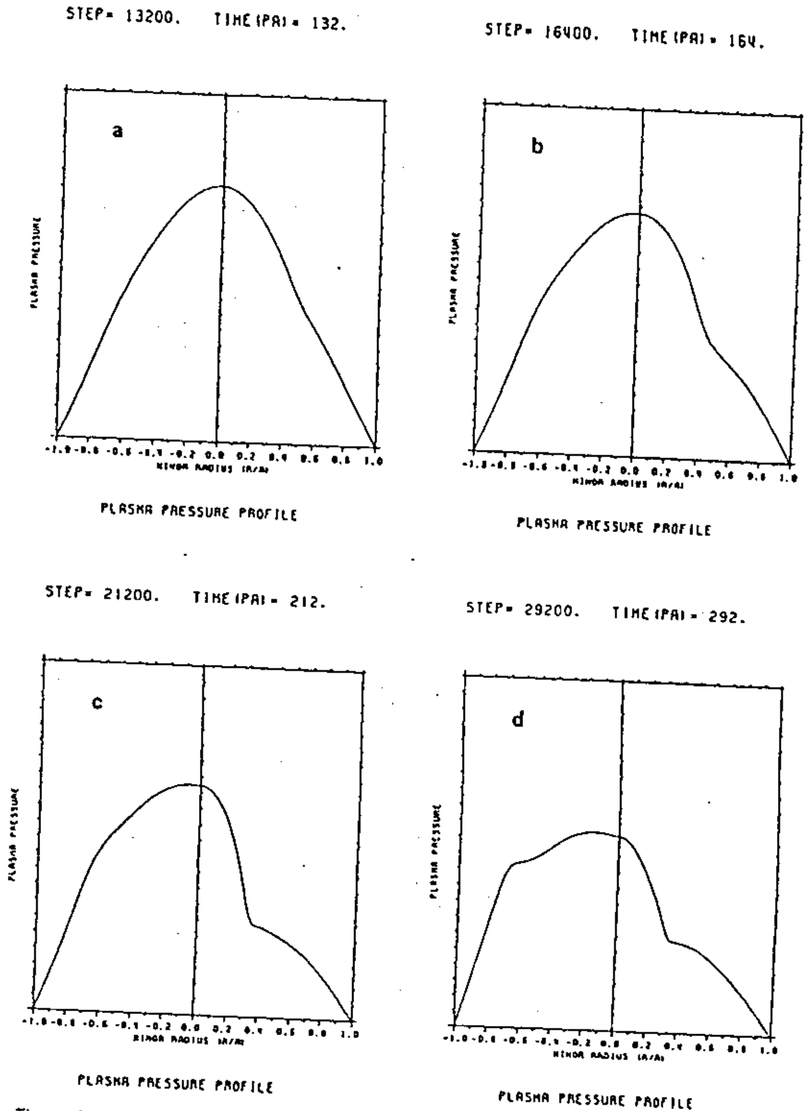


FIG.3. Time evolution of pressure profile along θ = 0 and θ = π (Case I). (a) T = 132, (b) T = 164, (c) T = 212, (d) T = 292.

Stability Criterion

The stability criteria for localized interchange modes are derived by Glasser, Greene and Johnson. These criteria can be examined by equilibrium quantities only.

- For ideal interchange modes

$$D_M(ideal) \equiv D_M(E) + D_M(F) + D_M(H) + D_M(shear) > 0 \tag{1}$$

- For resistive interchange modes

$$D_M(resistive) \equiv D_M(E) + D_M(F) - \left(\frac{D_M(H)}{\iota'}\right)^2 > 0 \tag{2}$$

$$= D_M(ideal) - \left(\frac{D_M(H)}{\iota'} + \frac{\iota'}{2}\right)^2 \quad > 0 \tag{3}$$

$$\left\{ \begin{array}{l} D_M(E) \equiv -(\iota')^2 E \\ D_M(F) \equiv -(\iota')^2 F \\ D_M(H) \equiv -(\iota')^2 H \\ D_M(shear) \equiv \frac{1}{4}\left(\iota'\right)^2 \end{array} \right. \tag{4}$$

Definitions of E, F and H

$$\left\{ \begin{array}{l} E \equiv -\frac{1}{(\iota')^2}\left\{P'V'' + \iota'\left(I_T' - \frac{\langle j\cdot B\rangle}{\langle B^2\rangle}\right)\right\}\left\langle \frac{B^2}{|\nabla\Phi_T|^2}\right\rangle \\ F \equiv -\frac{1}{(\iota')^2}\left[\left\langle \frac{j\cdot B}{|\nabla\Phi_T|^2}\right\rangle^2 - \left\{(P')^2\left\langle\frac{1}{B^2}\right\rangle + \left\langle \frac{(j\cdot B)^2}{B^2|\nabla\Phi_T|^2}\right\rangle\right\}\left\langle \frac{B^2}{|\nabla\Phi_T|^2}\right\rangle\right] \\ H \equiv -\frac{1}{\iota'}\left\{\frac{\langle j\cdot B\rangle}{\langle B^2\rangle}\left\langle \frac{B^2}{|\nabla\Phi_T|^2}\right\rangle - \left\langle \frac{j\cdot B}{|\nabla\Phi_T|^2}\right\rangle\right\} \end{array} \right. \tag{5}$$

$\boldsymbol{B}$: magnetic field
P : plasma pressure
ι : rotational transform
Φ_T : toroidal flux
j : current density
V : plasma volume
I_T : toroidal current

Prime denotes the derivative with respect to Φ_T
Flux surface average is expressed by angle brackets

$$\langle A\rangle \equiv \frac{\oint d\theta d\zeta\sqrt{g}A}{\oint d\theta d\zeta\sqrt{g}} \tag{6}$$

θ : poloidal angle
$\sqrt{g}$: Jacobian
ζ : toroidal angle