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WORKSHOP ON "MODELLING REAL SYSTEMS: A HANDS-ON FIRST ENCOUNTER WITH INDUSTRIAL MATHEMATICS

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"Asymptotic Analysis for Differential Equations"

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ASYMPTOTIC ANALYSIS FOR DIFFERENTIAL EQUATIONS

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WHY ARE WE INTERESTED IN THIS SUBJECT?

(BIASED) OVERVIEW: In general, the world works according to systems of *continuous* laws.

(eg death, disease, rain, football, shares, radios, food, sex.....)

Thus the world is governed by

- ODEs (Ordinary Differential Equations)
- PDEs (Partial Differential Equations)
- IEs (Integral Equations)
- IDEs (Integro-Differential Equations)
- etc. etc. etc.

Assume that WHENEVER WE DEAL WITH SUCH EQUATIONS we non-dimensionalize them first (anybody who does not, we judge as being insane).

OK then, we END UP with a lot of terms, all multiplied by dimensionless parameters.

What is the probability that these are all about 1?

If you think about it, it's almost zero!

This is confirmed by practical experience: almost always we find that some parameters are large or small.

For ease, let's think *only* about small parameters (if one is large: consider its reciprocal!)

THE KEY QUESTION:

How can we take advantage of the smallness of some of these parameters?

This is what we will discuss here.

NOTE: for simplicity we will confine our discussion *only* to ODEs. But all the other important sorts of equations can be similarly dealt with (though things may get a little more complicated).

NOTATION: we will always try to find $y(x)$, a dash will represent d/dx, and the small parameter(s) will be $\epsilon \ll 1$.

Historical: Methods started to be developed in the 1950's and 1960's by people like Julian Cole, Bob O'Malley, Joe Keller etc.

Many advances since.........

Note also: typical books:

A.H. Nayfeh, Perturbation Methods, John Wiley, 1973.

M. Van Dyke, Perturbation Methods in Fluid Mechanics, Academic Press 1964.

A. Erdelyi, Asymptotic Expansions, Dover 1956.

E.J. Hinch, Perturbation Methods, Cambridge University Press, 1991.

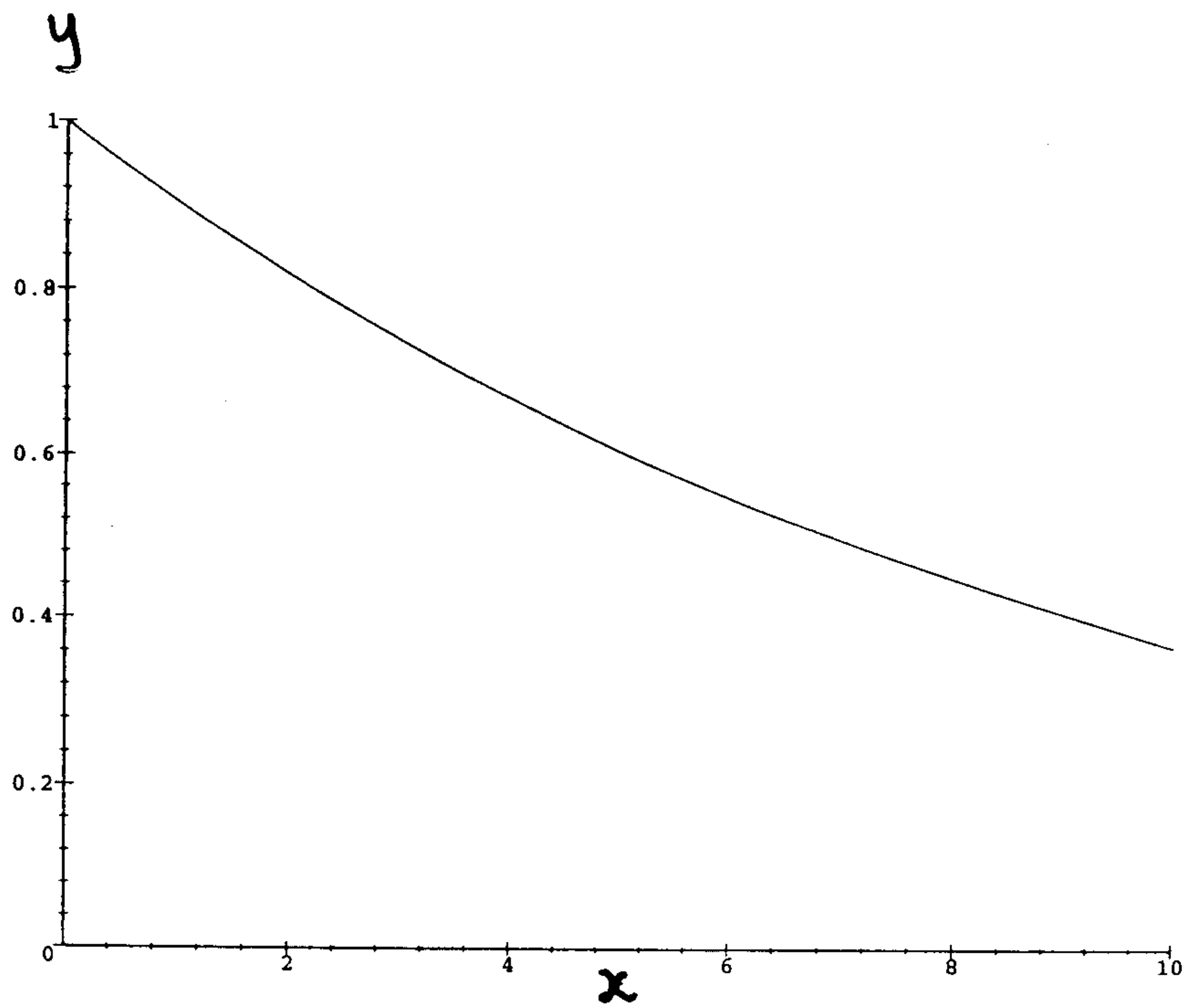
Everybody will tell you that "there are many many different sorts of asymptotic methods". This is true, but it complicates things.

Basically all asymptotic methods are "refinements" on the following simple example:

$$y' + \epsilon y = 0, \quad (y(0) = 1)$$

This has exact solution $y = e^{-\epsilon x}$, and a nice smooth solution.

NOTE: Usually we will illustrate things with equations that can be solved EXACTLY. This is so that we can compare the asymptoptic to the exact solution. But all of these asymptotic methiods STILL work even when the equations CANNOT be solved in closed form.



$y' + \varepsilon y = 0 \qquad y(0) = 1$

$(\varepsilon = 1/10)$

To solve the equation asymptotically:

Assume that since ϵ is small, there is a nice simple power-series solution:

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + ...$$

Put this into the equation:

$$(y_0' + \epsilon y_1' + \epsilon^2 y_2') + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2) = 0$$

Equate powers of ϵ: (AND THE BC!)

$O(\epsilon^0) : y_0' = 0, \qquad y_0(0) = 1 \qquad \boxed{y_0 = 1}$

$O(\epsilon^1) : y_1' + y_0 = 0, \qquad y_1(0) = 0 \qquad \boxed{y_1 = -x}$

$O(\epsilon^2) : y_2' + y_1 = 0, \qquad y_2(0) = 0 \qquad \boxed{y_2 = \frac{x^2}{2}}$

$O(\epsilon^3) \qquad$ ....

So the "perturbation solution" is:

$$y = 1 + (-x)\epsilon + (-x^2/2)\epsilon^2 + ...$$

OF COURSE this is right, since if we expand the exact solution, we get

$$e^{-\epsilon x} = 1 - \epsilon x + \epsilon^2 \frac{x^2}{2!} + ...$$

This is known as the "method of regular perturbations".

NOTE: this is NOT an "approximate method".

It's a carefully controlled *asymptotic procedure*.

At every stage we knew the approximations that were being made, the errors involved, etc. etc.

If we'ed wanted to we could have computed many more terms.

We stress again: THIS VERY OFTEN WORKS WHEN THE ODE *CANNOT* BE SOLVED IN CLOSED FORM!

This is very nearly all there is to asymptotic methods for ODE's......

EVERYTHING is really based on the previous example.

The procedure is basically the same for every sort of equation that we have to deal with.

Any fool could do it!

But there are some questions that we have to ask. Most importantly,

WHEN DOES THIS ALL GO WRONG?

We shall take a slightly unorthodox approach: we shall propose equations that can be solved exactly, (or numerically) look at graphs of their solutions, and try to see if we might expect to reproduce these graphs with a regular perturbation method.

consider this problem:

$$\epsilon y'' + y' + y = 0 \qquad (y(0) = 1, \;\; y(1) = 0)$$

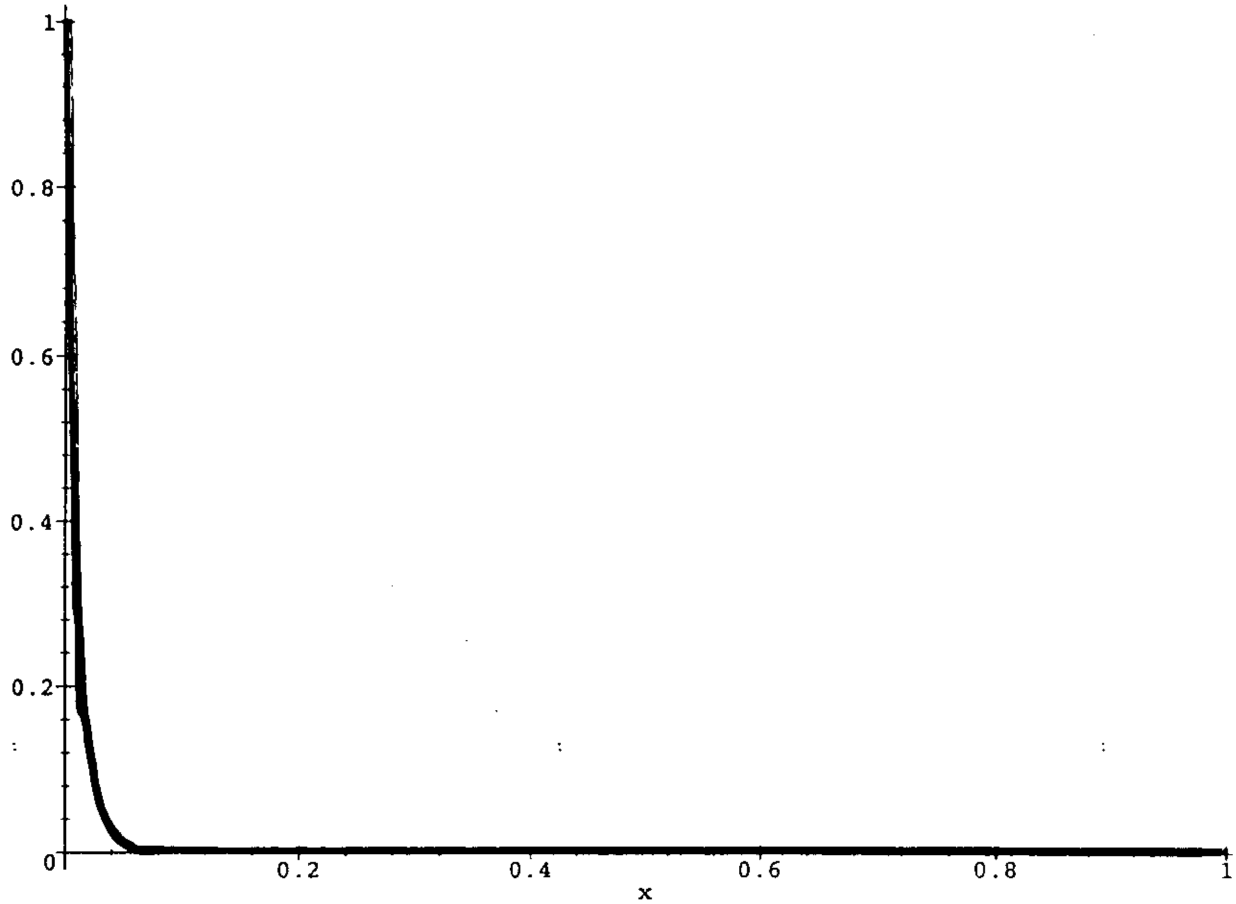
Here's the exact solution:

$$y(x) = \frac{\exp(x(-1+\sqrt{1-4\epsilon})/2\epsilon) - k^2 \exp(-x(1+\sqrt{1-4\epsilon})/2\epsilon)}{k^2 - 1}$$

$$k = \exp\left(\frac{\sqrt{1-4\epsilon}}{2\epsilon}\right)$$

Not easy to interpret........

Here's a graph of the exact solution for a very small value of ϵ (actually $\epsilon = 1/100$)



$$\varepsilon y'' + y' + y = 0 \qquad y(0) = 1 \quad y(1) = 0$$

$$\varepsilon = \frac{1}{100}$$

It's clear that some thing funny is going on here. But let's just try regular perturbations anyway.....We put

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \dots.$$

into

$$\epsilon y'' + y' + y = 0 \qquad (y(0) = 1, \;\; y(1) = 0)$$

and straight away we find that we get

$$O(\epsilon^1): \qquad \boxed{y_0' + y_0 = 0, \quad y_0(0) = 1, \;\; y_0(1) = 0.}$$

So we need $y_0 = Ae^{-x}$ with $A = 0$ AND $A = 1$!. Oh dear.

What has happened here is that there is a ***boundary layer.*** This is a thin region where things change very rapidly.

For the moment, let us forget about the "boundary layer" and ignore the boundary condition at $x = 0$. If we do this, then we just need to solve

$$y_0' + y_0 = 0, \quad y_0(1) = 0.$$

EASY - the solution is $\underline{y_0 = 0}$!

Looking at the graph of the exact solution we know that this is correct.

We call this the OUTER solution

i.e. AWAY from the boundary layer.

But what's going on near $x = 0$ where somehow we have to "save ourselves" and allow $y(0) = 1$ to be satisfied?

To sort things out we have to look INSIDE the boundary layer ("mathematical magnifying glass"). Obviously from the picture of the exact solution the boundary layer is near $x = 0$.

So put $X = x/\epsilon$. (Then when x is small X is order 1).

The equation becomes

$$\frac{d^2y}{dX^2} + \frac{dy}{dX} + \epsilon Y = 0$$

and so to leading order

$$\frac{d^2y}{dX^2} + \frac{dy}{dX} = 0$$

which has solution (call it the "inner" solution)

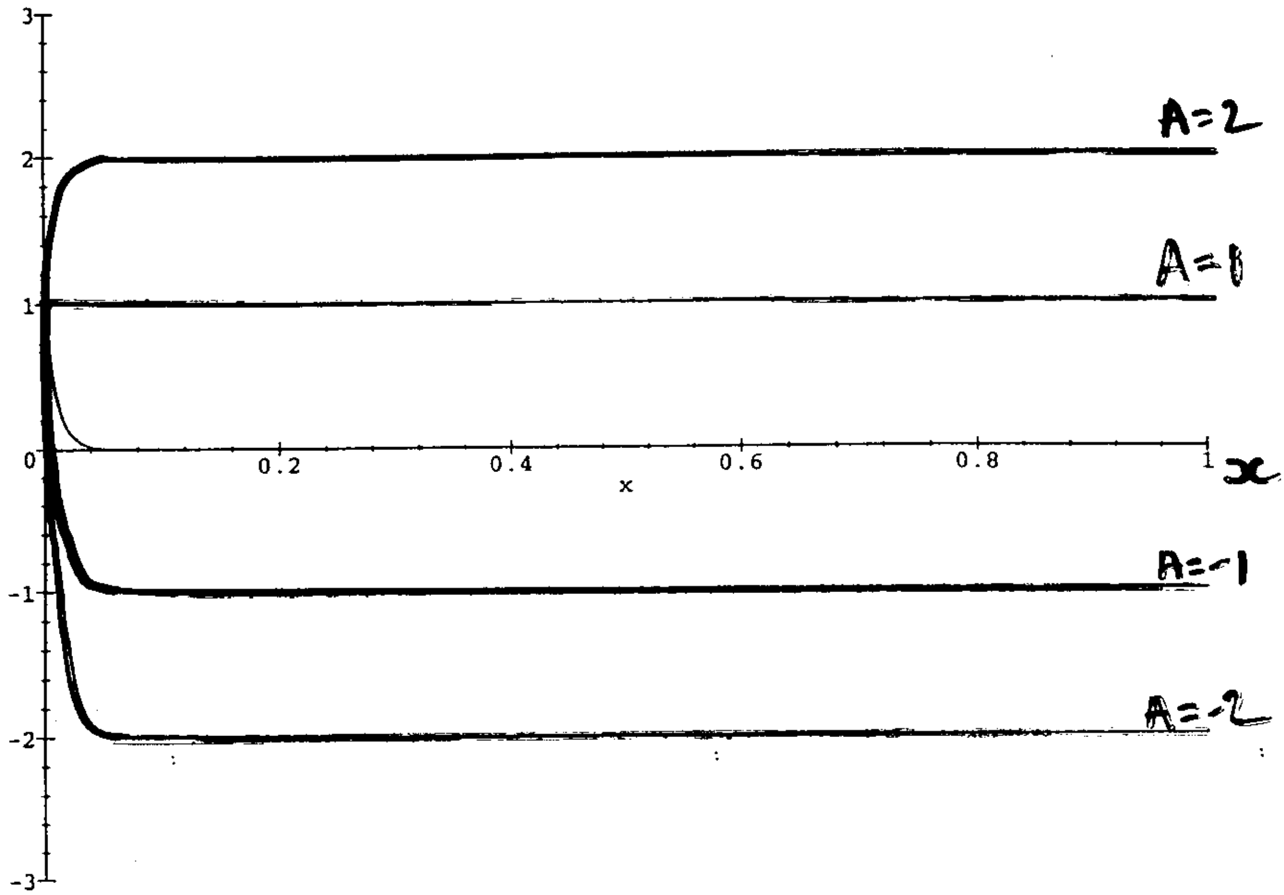
$$y_i = A + Be^{-X}.$$

Now near to $x = 0$, we of course want to satisfy $y(0) = 1$. So

$$y_i = A + (1 - A)e^{-X}$$

The question is: How do we choose A?

We do NOT use the boundary condition at $x = 1$, since this is FAR away from the boundary layer. Instead, let's plot the outer solution and the inner solution for some choices of A.



$$y_i = A + (1-A)e^{-x}$$

$$(\varepsilon = 1/100)$$

OBVIOUSLY TO "MATCH" WITH $y_0 = 0$ NEED $A = 0$

$$y_i = e^{-x}.$$

NOTES:

- This is the method of "matched asymptotic expansions"
- The matching can be taken up to higher orders of ϵ if required.
- By combining the inner and outer solutions we can form a *uniformly valid* solution

$$y = e^{-x/\epsilon}$$

- This happened because the small parameter multiplied the highest derivative in the problem.
- It may not be clear where the boundary layer is!
- This is a "singular perturbation" problem.

Here's another thing that can mess us up.....

<u>VAN DER POL OSCILLATOR</u>

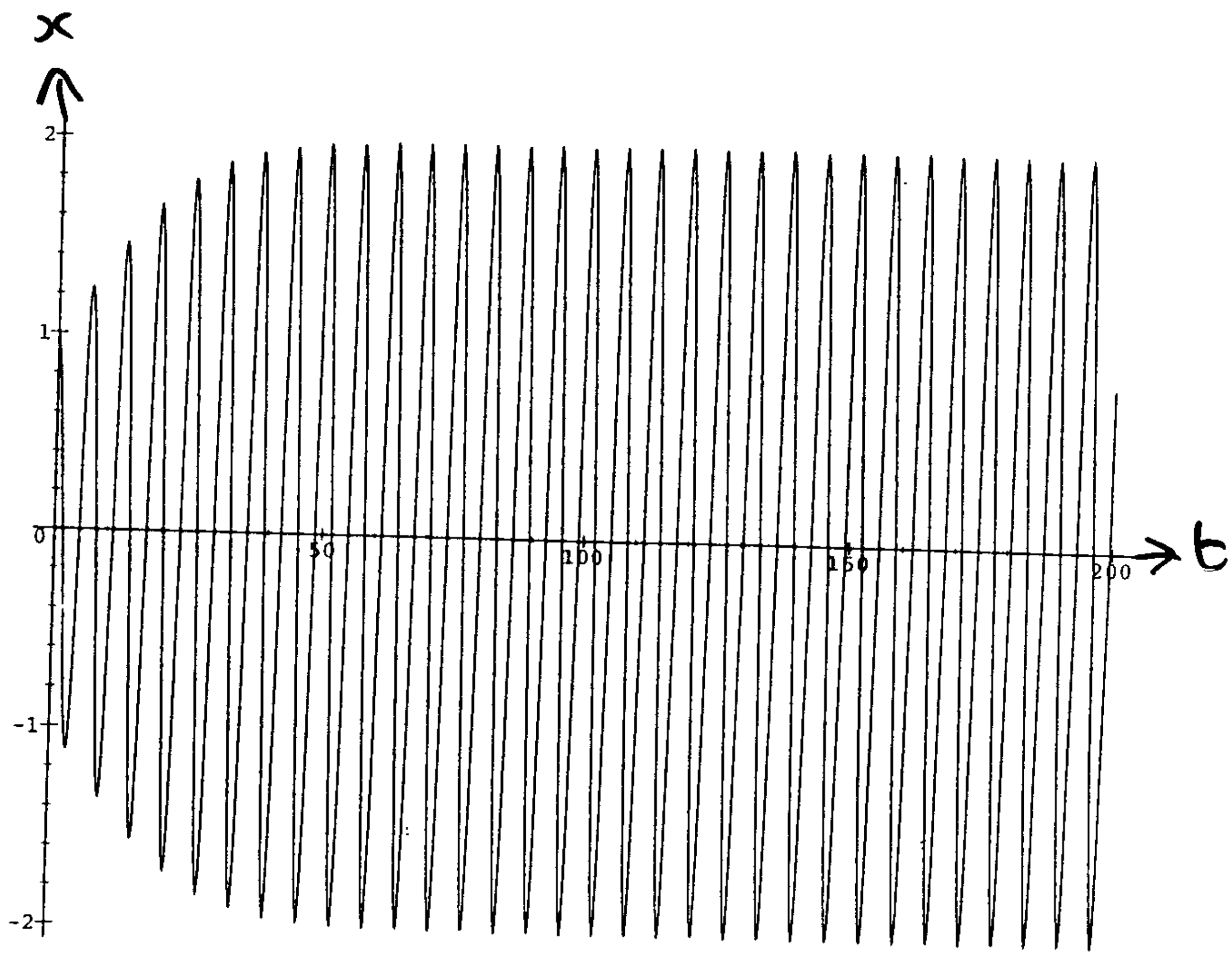
(nb change from variables $y(x)$ to $x(t)$ as it's supposed to be a model for a *real* oscillator)

$$\ddot{x}+\epsilon\dot{x}(x^2-1)+x=0, \qquad x(0)=1, \quad \dot{x}(0)=0$$

(OSCILLATOR WITH NONLINEAR FRICTION)

No general exact solution is known. (But easy to solve numerically).

So let's look at the numerical solution:



"EXACT" SOLUTION

$$\ddot{x} + \varepsilon \dot{x}(x^2 - 1) + x = 0$$

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

$\varepsilon = 1/10$

OK so let's just do it by regular perturbations. We have

$$x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots..$$

so that when we put it into

$$\ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0$$

we get

$$(\ddot{x}_0 + \epsilon\ddot{x}_1) + \epsilon(\dot{x}_0 + \epsilon\dot{x}_1)(\dot{x}_0^2 - 1) + (x_0 + \epsilon x_1) = 0$$

equate coefficients:

$$O(\epsilon^0): \qquad \ddot{x}_0 + x_0 = 0, \qquad x_0(0) = 1, \quad \dot{x}_0(0) = 0$$

$$O(\epsilon^1): \; \ddot{x}_1 + \dot{x}_0(\dot{x}_0^2 - 1) + x_1 = 0, \; x_1(0) = 0, \; \dot{x}_1(0) = 0$$

and so

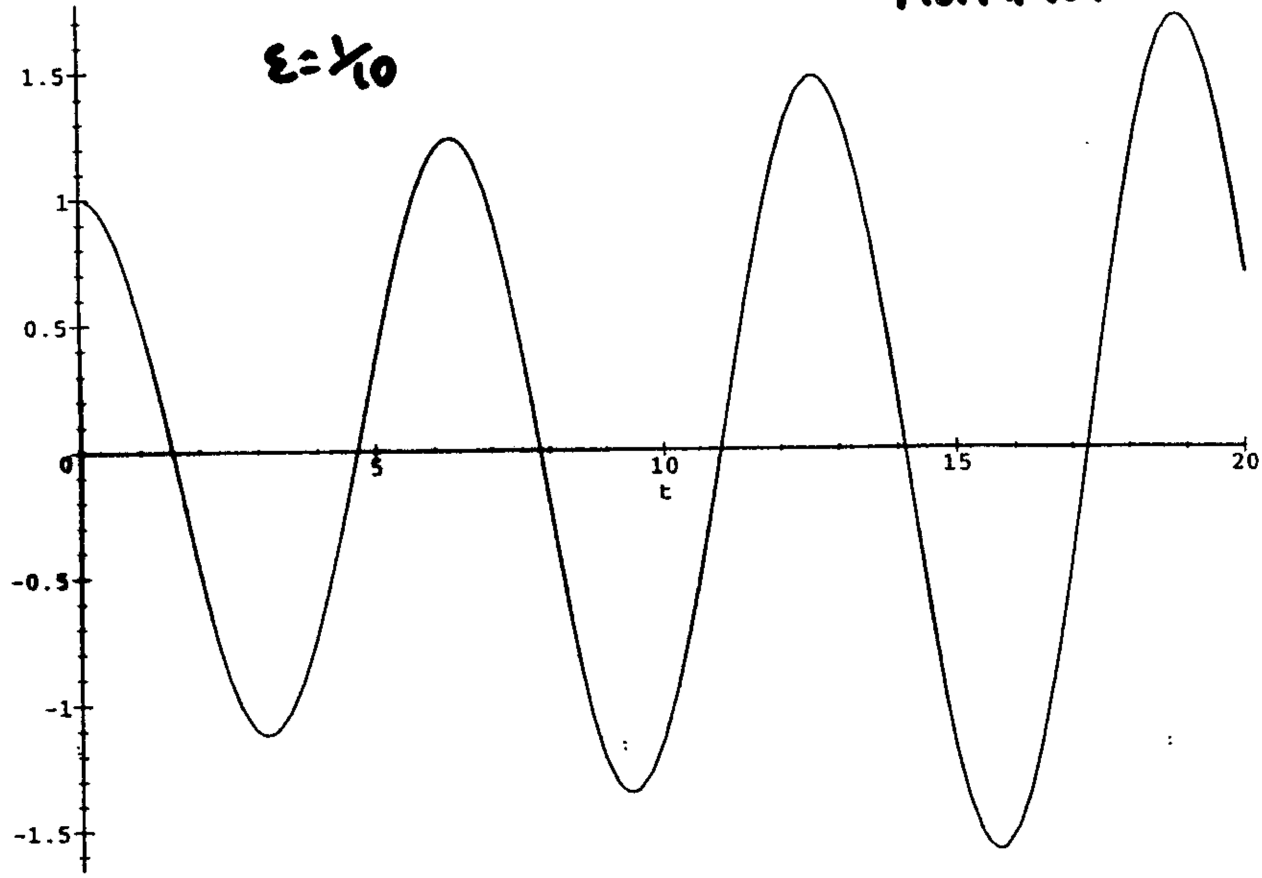
$$x_0 = \cos t, \qquad x_1 = \frac{3}{8}(t\cos t - \sin t) - \frac{1}{32}(\sin 3t - 3\sin t)$$

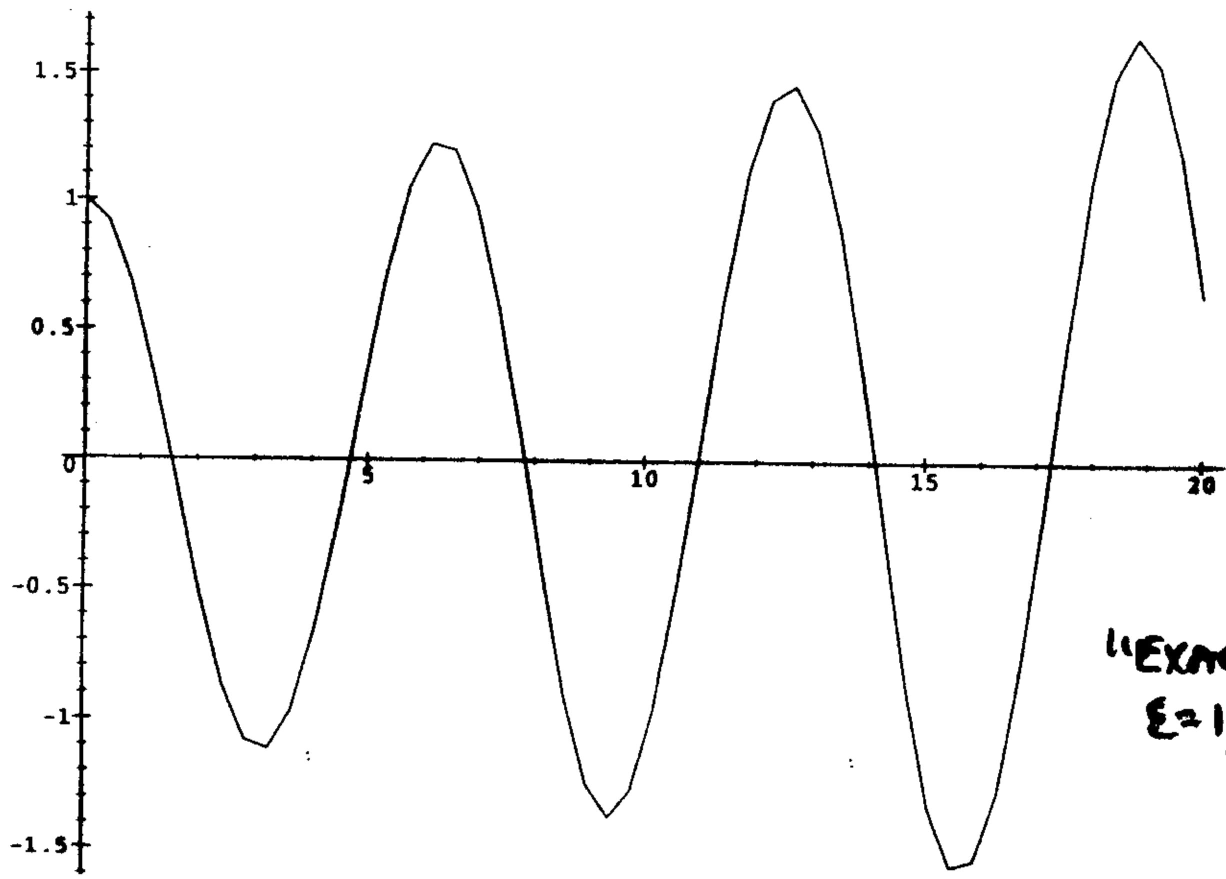
so how does this asymptotic solution perform?

$\cos t + \varepsilon\left(\frac{3}{8}(t\cos t - \sin t) - \frac{1}{32}(\sin 3t - 3\sin t)\right)$

ASYMPTOTIC

$\varepsilon = 1/10$

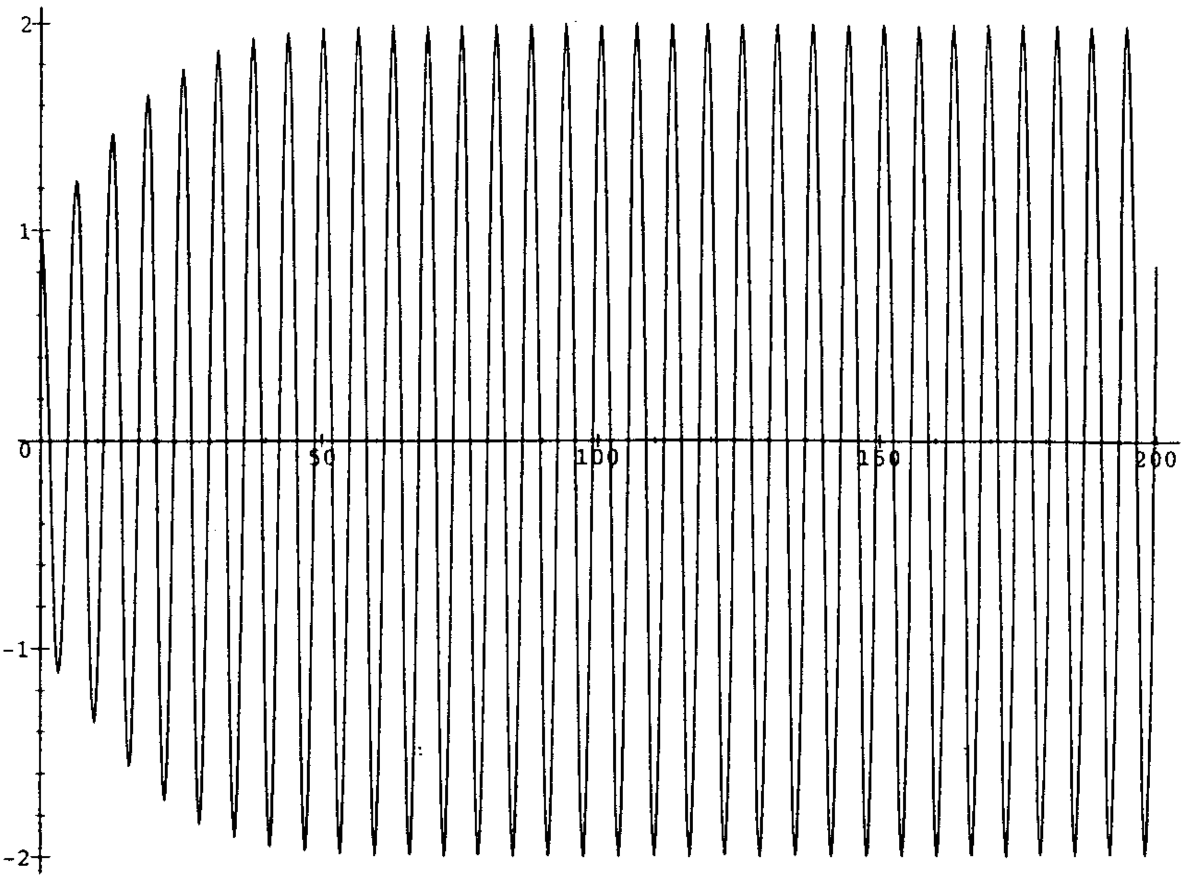


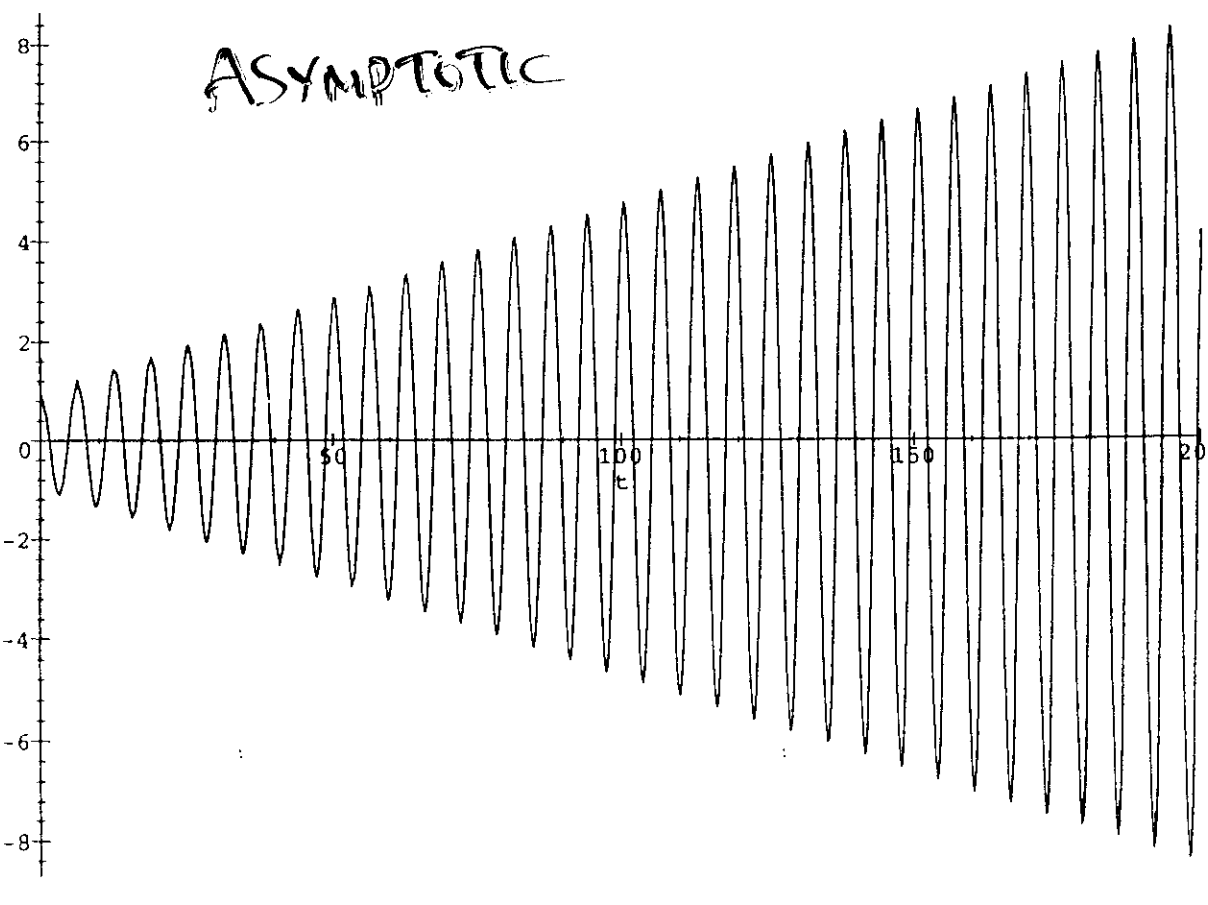


"EXACT"

$\varepsilon = 1/10$

"EXACT"





Oh dear. The asymptotics are OK until t gets large, at which point the amplitude of the oscillations goes all wrong.

PHYSICAL OBSERVATION:

From looking at the "exact" solution we can see that there are really TWO time scales here:

(i) time between oscillations

(ii) time over which amplitude grows.

Some how we have to take account of BOTH of these.

To do this, introduce

(Fast) oscillation time $\tau = t$

(Slow) aplitude drift time $T = \epsilon t$

Now we have (for example)

$$\frac{d}{dt} = \frac{\partial}{\partial \tau} + \epsilon \frac{\partial}{\partial T}$$

and so (e.g.)

$$\ddot{x} = x_{\tau\tau} + 2\epsilon x_{\tau T} + \epsilon^2 x_{TT}$$

Actually, we are proceeding by "simplifying" and ODE to a PDE!

we seek

$$x(t;\epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + \ldots$$

Putting it in and equating power of ϵ:

$$O(\epsilon^0): \quad \frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0, \quad \left(x_0 = 1, \ \frac{\partial x_0}{\partial \tau} = 0 \ \text{at} \ t = 0\right)$$

this gives

$$x_0 = R(T)\cos(\tau + \theta(T))$$

where R and θ satisfy $R(0) = 1$ and $\theta(0) = 0$ but are otherwise unknown.

General rule: if things are not determined, go to the next order!

This gives:

$$O(\epsilon^1): \quad \frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -\frac{\partial x_0}{\partial \tau}(x_0^2 - 1) - 2\frac{\partial^2 x_0}{\partial \tau \partial T}$$

with initial (t = 0) conditions

$$x_1 = 0, \quad \frac{\partial x_1}{\partial \tau} = -\frac{\partial x_0}{\partial T} = -\frac{\partial R}{\partial T}$$

NOW HERE's THE KEY POINT:

When we substitute x_0 is the RHS of the above, we get

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = 2R\frac{\partial \theta}{\partial T}\cos(\tau+\theta) + (2\frac{\partial R}{\partial T} + \frac{1}{4}R^3 - R)\sin(\tau+\theta) + \frac{1}{4}R^3 \sin 3(\tau+\theta)$$

Now we KNOW that the cos and sin terms with argument $(\tau + \theta)$ will give rise to terms proportional to τ which will mess things up for large time. So we conclude ("Poincare condition") that

$$\frac{\partial \theta}{\partial T} = 0, \quad \frac{\partial R}{\partial T} = \frac{1}{8}R(4 - R^2)$$

This gives

$$\theta = 0, \qquad R = 2(1 + 3e^{-T})^{-1/2}$$

and now everything is OK; the amplitude comes out right.

"METHOD OF MULTIPLE SCALES"

Here's one final wrinkle.......

Suppose we consider high frequency vibrations of a variable density string.

Equation:

$$y'' + \omega^2 r(x) y = 0 \qquad y(0) = y(1) = 0.$$

We want to consider ω^2 large. $(= 1/\epsilon)$

We want to find the eigenfunctions.

There's a simple trick here:

although exact solutions are known for hardly any $r(x)$, between any two successive zeros of the eigenfunctions $r(x)$ is nearly constant for large ω.

If r WAS constant then we'ed have

$$y = Ae^{\pm i\omega x\sqrt{r}}.$$

So the trick is to try

$$y(x, \omega) = e^{\omega g(x,\omega)}$$

where

$$g = g_0 + \frac{1}{\omega} g_1 + \frac{1}{\omega^2} g_2 + \ldots.$$

This is the WKB method

(Named after Wentzel, Kramers, Brillouin, but invented by Green, Jeffreys, Carlini and Liouville!)

We find that it is most convenient to set

$$g = \int h$$

and then we find that

$$h_0 = \pm i\sqrt{r}(x)$$

$$h_1 = -\frac{1}{4}[\ln r(x)]'$$

$$h_2 = \pm\frac{1}{8\sqrt{r}}\left[\frac{5}{4}\left(\frac{r'}{r}\right)^2 - \frac{r''}{r}\right]$$

etc etc

Analysis of this method can be extended to Stokes lines, hyperasymptotics, and all sorts of very complicated things!

Other methods to deal with various problems

(No time to explain them)

- Method of strained coordinates
- Lighthill technique
- Averaging methods
- Krylov-Bogoliubov-Mitroploski technique
- etc. etc. etc.