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SCHOOL ON DIFFERENTIAL GEOMETRY

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Symplectic geometry and Gromov-Witten invariants

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Symplectic geometry and Gromov-Witten invariants

(M,ω) : symplectic manifold, $d\omega=0$, $\omega^n=0$
where $\dim_{\mathbb{R}} M = 2n$.

There are many examples of symplectic manifolds.

1. Riemann surfaces
2. Kähler manifolds
3. connected sum along [redacted] symplectic submanifolds

Gompf: Given two symplectic manifolds (M_1,ω_1), (M_2,ω_2) with corresponding symplectic submanifolds X_1, X_2, respectively.

Assume that (X_1,ω_1) is symplectically diffeomorphic to (X_2,ω_2) and $N_{X_1|M_1} \cong N^{-1}_{X_2|M_2}$ over $X_1 \cong X_2$

Then one can construct a symplectic manifold $M = M_1 \#_{X_1 \cong X_2} M_2$ by gluing M_1, M_2 along $X_1 \cong X_2$. This symplectic structure depends only on the isotopy class of the symplectic map between X_1 and X_2.

Basic facts:

1) Darboux lemma: any (M,ω) is locally equivalent to (U,ω_0) where $U \subset \mathbb{R}^{2n}$, $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$

2) Moser's theorem: let M be a closed manifold, ω_t be a family symplectic structures on M with $[\omega_t] = \text{const.}$, $t \in [0,1]$. Then $\exists$ isotopy $\varphi_t : M \circlearrowleft$ s.t. $\varphi_t^* \omega_t = \omega_0$

Goal of these lectures: use J-holomorphic maps to construct symplectic invariants of (M,ω).

Almost complex structures:

$J : TM \circlearrowleft$, $J^2 = -\mathrm{id}$, J : a tensor of type $(1,1)$.

ω-compatibility: $\omega(Ju, Jv) = \omega(u,v)$ for any $u, v \in TM$.

Lemma. $\exists$ ω-compatible almost complex structure J on M.

$\mathcal{J} = \{$all ω-compatible almost complex structures on $M\}$ $\neq \phi$.

Lemma. $\mathcal{J}$ is path connected.

Reason: $\mathcal{J} = \{$all sections $\sigma : M \to Sp(TM)/U(TM)\}$

where $Sp(TM)/U(TM)\big|_{x \in M} \cong Sp(2n)/U(n)$ is contractible.

Definition of J-holomorphic maps.

Let Σ be a Riemann surface and j be its complex structure, then $f : \Sigma \to M$ is J-holo. if

$$df \cdot j = J \cdot df$$

or

$$df + J \cdot df \cdot j = 0$$

This is the generalization of the classical Cauchy-Riemann equation.

Consider the moduli space

$$\mathcal{M}_A(J) = \left\{ (f,j) \;\middle|\; \begin{array}{l} j \text{ is a complex structure on } \Sigma, \\ f \text{ is J-holo. w.r.t. } j, \\ [f_* \Sigma] = A \end{array} \right\} \Big/ \sim$$

$(f,j) \sim (f',j')$ if $\exists$ a conformal map $\sigma : (\Sigma, j) \to (\Sigma', j')$ s.t. $f' \circ \sigma = f$.

We like to study some invariant properties of this moduli space $\mathcal{M}_A$.

<u>Questions</u>. 1. Compactness of $\mathcal{M}_A(J)$

2. Local[redacted] properties of $\mathcal{M}_A(J)$

3. Dependence of $\mathcal{M}_A(J)$ on J.

We will assume $\Sigma = S^2$ for simplicity.

<u>Local properties of $\mathcal{M}_A(J)$</u>:

Given any $f \in \mathcal{M}_A(J)$, look at the "tangent space" of $\mathcal{M}_A(J)$ at f,

$$T_f \mathcal{M}_A(J) = \{ \xi \in \Gamma(\Sigma, f^* TV) \mid L_f \xi = 0 \}$$

where L_f is the linearization of the generalized Cauchy-Riemann equation at f.

$$L_f \xi = \bar{\partial}_u \xi + \tfrac{1}{2} N(\cdot, \xi)$$

where $\bar{\partial}_u(J\xi) = J\bar{\partial}_u \xi$, N is the Nijenhuis tensor of J.

Let L_f^* be the adjoint operator of L_f with respect to the metric induced by ω and J.

<u>Definition</u>. f is regular if $\ker(L_f^*) = \{0\}$.

<u>Theorem</u>: If $f \in \mathcal{M}_A(J)$ is regular, then there is a neighborhood V of f in $\mathcal{M}_A(J)$ such that

1) $V \cong \{ \xi \in \operatorname{Ker}(L_f) \mid \|\xi\|_{C^2} \leq \varepsilon \} / \operatorname{Aut}(f)$

2) $\dim_{\mathbb{C}} V = c_1(V)(A) + (3-n)(g-1)$

3) V carries a natural orientation.

(A simple case: J is integrable, then L_f is J-inv., so $T_f \mathcal{M}_A(J)$ has a canonical orientation. The general case can be reduced to this case)

Examples of regular J-holomorphic maps.

1. M is an algebraic manifold with ample tangent bundle (e.g. $M = \mathbb{CP}^n$), J is the integrable almost complex structure, $g(\Sigma) = 0$.

Then $\quad \mathrm{Ker}(L_f) = H^1(\Sigma, f^*TM) = 0$, that is, all J-holo. maps of genus 0 are regular.

2. Let $n = 2$, i.e., (M, ω) is a 4-dimensional symplectic manifold. For simplicity, we first assume that $g(\Sigma) = 0$.

<u>Proposition</u>: Let $f : \Sigma \to M$ be a J-holo. map, immersive, $c_1(M)(f_*\Sigma) > 0$, then f is regular.

Pf. Let $\zeta \in \mathrm{Ker}(L_f^*)$, decompose

$$\zeta = \zeta^T + \zeta^\perp, \quad \zeta^T \parallel T\Sigma, \quad \zeta^\perp \perp T\Sigma$$

- $L_f^* \zeta^\perp \perp T\Sigma$, since S^2 has no nonzero holomorphic quadratic diff.
- $\zeta^\perp \in \Gamma^{0,1}(\Sigma, N_\Sigma) = \Gamma(\Sigma, \overline{\det(TM)})$, locally,

$$L_f^* \zeta^\perp = 0, \quad \zeta^\perp = \overline{fe}, \quad e : \text{a frame}.$$
$$\Updownarrow$$
$$\bar\partial f + af + a'\bar f = 0$$

$\Longrightarrow$ # zeroes of $\zeta^\perp$ counted with sign ≤ 0

$$\Vert$$

$$\int_{f(\Sigma)} c_1(TM) > 0 \quad \text{if } \zeta \not\equiv 0.$$

So $\mathrm{Ker}(L_f^*) = 0$. //

In general, $\mathrm{Ker}(L_f^*) \neq \{0\}$. However, by using the Sard-Smale theorem, we can show that any simple $f \in \mathcal{M}_A(J)$ is regular for a generic $J \in \mathcal{J}$.

Let us say a few more words about it.

<u>Definition</u>. A J-holomorphic map f is simple if [redacted] for all $x \in \Sigma$ except finitely many ones, $df(x) \neq 0$, $f^{-1}(f(x)) = \{x\}$.

<u>Proposition</u>: For every nonsimple J-holo. map f, there is a branch cover $\sigma : \Sigma \to \Sigma'$ and a simple J-holomorphic map $f' : \Sigma' \to M$ such that $f = f' \circ \sigma$.

For its proof, see "J-holomorphic maps and quantum cohomology" by D. Mcduff and D. Salamon.

<u>Theorem</u>. For a generic J, [redacted] every simple J-holomorphic map $f \in \tilde{\mathcal{M}}_A(J)$ is regular, in particular, $\tilde{\mathcal{M}}_A(J)$ is smooth near any simple J-holomorphic map.

<u>Pf</u>: Consider universal moduli space $(\Sigma = S^2)$

$$\tilde{\mathcal{M}}_A^{\circ} = \{(f, J) \mid f \text{ is simple } J\text{-holomorphic map}, [f_*\Sigma] = A, J \in \mathcal{J}\}$$

$$\cap$$

$$Map_A(S^2, M)$$

Then $\tilde{\mathcal{M}}_A^{\circ}$ is smooth. Here we used the simpleness of f.

The projection $\pi : \tilde{\mathcal{M}}_A^{\circ} \longrightarrow \mathcal{J}$, $(f, J) \to J$ is a Fredholm operator of index $=$ index(L_f) for any $f \in \tilde{\mathcal{M}}_A(J)$, i.e.,

$$\text{index}(\pi) = 2\big(c_1(M)(A) + (3-n)(g-1)\big)$$

Then the theorem follows from the Sard–Smale transversality theorem.

<u>Example</u>: Simpleness is necessary.

If M is a Calabi-Yau 3-fold, then expected dimension of $\mathcal{M}_A(J)$ is zero. However, if $f \in \mathcal{M}_A(J)$ is a multiple map of deg d, then $\mathcal{M}_A(J)$ near f is of $\dim_{\mathbb{C}}$ $2d-2$. So it is not regular when $d \geq 2$. In this case, deforming J will not ~~change~~ reduce multiplicity of f.

More generally, one can introduce

<u>Definition.</u> $f \in \widetilde{\mathcal{M}}_A(J)$ is strongly regular if for every branch covering $\sigma: S^2 \to S^2$ set the composition $\tilde{f} = f \circ \sigma : S^2 \to M$, then $\dim \operatorname{Ker}(L_{\tilde{f}}^*) = \max\{(2-r)(d-1), 0\}$ where $r = c_1(M)(A)$.

- For a generic J, almost every $f \in \widetilde{\mathcal{M}}_A(J)$ should be strongly regular.
- If J is integrable, regular $\Leftrightarrow$ strongly regular. But in nonintegrable cases, they are not necessarily the same.

Compactness. (Uhlenbeck, Gromov, Parker + Wolfson, Ye [redacted], so on)

Given any J-holomorphic map $f: \Sigma \to M$, we have

$$\int_\Sigma |\partial f|^2 dV = \int_\Sigma f^*\omega = \omega(A)$$

so the energy is a topological invariant.

- Monotonicity: for any δ_0 sufficiently small, if $\delta < \delta_0$, $\partial(S \cap B_\delta(p)) = S \cap \partial B_\delta(p)$ where $S = f(\Sigma)$,

then $\quad \mathrm{Area}(S \cap B_\delta(p)) \geq \pi \delta^2 e^{-c\delta}$, $\quad c =$ uniform constant.

Pf. When δ_0 is sufficiently small, for each $p \in M$, there is a coordinate system $(x_1, \cdots, x_{2n})$ in $B_{2\delta_0}(p)$ such that $x_i(p) = 0$ and

$$\omega = \sum_{i=1}^n dx_i \wedge dx_{i+n}$$

$$g_J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)(p) = \delta_{ij},$$

then $\quad \omega = d\alpha, \quad \alpha = \frac{1}{2}\sum_{i=1}^n x_i\, dx_{i+n} - x_{i+n}\, dx_i$, so

$$|\alpha(q)| \leq \frac{1}{2} d(p,q) + O(d(p,q)^2)$$

It follows

$$\mathrm{Area}(S \cap B_\delta(p)) = \int_{S \cap B_\delta(p)} \omega = \int_{S \cap B_\delta(p)} d\alpha$$

$$= \int_{S \cap \partial B_\delta(p)} \alpha \leq \frac{1}{2}(1 + c\delta)\, L(S \cap \partial B_\delta(p))$$

$$= \frac{1}{2}(1 + c\delta) \frac{d}{d\delta} \mathrm{Area}(S \cap B_\delta(p))$$

Then the inequality follows from integrating this. //

- **Corollary**: If D_r is the disk of radius r in $\mathbb{C}$, $u : D \to M$ is J-holomorphic with $\int_{D_r} |\partial u|^2 dV < \varepsilon$, then $u(D_{\frac{r}{2}}) \subset B_{30\sqrt{\varepsilon}}(p)$ for some $p \in M$.

- **Theorem** (Removable of singularity) Let J be ω-compatible almost complex structure on M with associated metric g_J. If $u : D \setminus \{0\} \to M$ is a J-holomorphic map with $E(u) < \infty$, then u extends to ~~be~~ a smooth map from D into M, where $E(u) = \int_D |\partial u|^2 dV$.

Proof: Let us outline a proof here. First we show that u has a continuous extension. For any $\varepsilon > 0$, choose $\delta > 0$ such that $\int_{D_\delta} |\partial u|^2 dV \le \left(\frac{\varepsilon}{30}\right)^2$. This is possible since $E(u) < \infty$.

By the above Corollary, $u(D_\delta) \subset B_\varepsilon(p_\delta)$ for some p_δ. This implies that for any $x, y \in D_\delta$,

$$d_M(u(x), u(y)) \le 2\varepsilon$$

where d_M denotes the distance function of M induced by the metric g_J. Therefore, u has a continuous extension.

Next we show that $|\partial u|$ is uniformly bounded on D. Let (ρ, θ) be the polar coordinate of $\mathbb{C}$. Then the Cauchy-Riemann equation becomes

$$\frac{\partial u}{\partial \rho} + \frac{1}{\rho} J(u) \frac{\partial u}{\partial \theta} = 0$$

so

$$0 = \int_{D_r} \left| \frac{\partial u}{\partial \rho} + \frac{1}{\rho} J \frac{\partial u}{\partial \theta} \right|^2 \rho \, d\rho \, d\theta$$

$$= \int_{D_r} |\partial u|^2 dV + 2 \int_{D_r} \left\langle \frac{\partial u}{\partial \rho}, \frac{1}{\rho} J \frac{\partial u}{\partial \theta} \right\rangle \rho \, d\rho \, d\theta$$

(integration by parts) $= \int_{D_r} |\partial u|^2 dV + 2 \int_0^{2\pi} \left\langle u - k, J \frac{\partial u}{\partial \theta} \right\rangle (r, \theta) \, d\theta$ (k = const determined later)

$$- 2 \int_{D_r} \left\langle u - k, \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(J \frac{\partial u}{\partial \theta} \right) \right\rangle dV$$

$$= \int_{D_r} |\partial u|^2 dV + 2\int_0^{2\pi} \langle u-\lambda, J\frac{\partial u}{\partial \theta}\rangle (r,\theta)\, d\theta$$

$$-2\int_{D_r} \langle u-\lambda, \frac{1}{\rho}(\nabla_{\frac{\partial u}{\partial \rho}} J)\frac{\partial u}{\partial \theta}\rangle dV - 2\int_{D_r} \langle u-\lambda, \frac{1}{\rho} J \frac{\partial^2 u}{\partial \rho \partial \theta}\rangle dV$$

(integration by parts) $$\geq \int_{D_r} |\partial u|^2 dV + 2\int_0^{2\pi} \langle u-\lambda, J\frac{\partial u}{\partial \theta}\rangle (r,\theta)\, d\theta \quad c\sup_{D_r}|u-\lambda| \int_{D_r} |\partial u|^2 dV$$

$$+ 2\int_{D_r} \underbrace{\langle \frac{\partial u}{\partial \theta}, J\frac{\partial u}{\partial \rho}\rangle \rho\, d\rho\, d\theta}_{= -\langle \frac{\partial u}{\partial \rho}, \frac{1}{\rho} J \frac{\partial u}{\partial \theta}\rangle dV}$$

it follows

$$\int_{D_r} |\partial u|^2 dV \leq \left| \int_0^{2\pi} \langle u-\lambda, J\frac{\partial u}{\partial \theta}\rangle (r,\theta)\, d\theta \right| + c\sup_{D_r}|u-\lambda| \int_{D_r} |\partial u|^2 dV$$

Now choose $\lambda = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta$, then by the Poincaré inequality,

$$\int_0^{2\pi} |u-\lambda|^2 d\theta \leq \int_0^{2\pi} \left|\frac{\partial u}{\partial \theta}\right|^2 d\theta = \frac{r^2}{2}\int_0^{2\pi} |\partial u|^2 d\theta$$

$$\text{So} \left(1 - c\sup_{D_r}|u-\lambda|\right)\int_D |\partial u|^2 dV \leq \frac{r}{2}\int_0^{2\pi} |\partial u|^2 (r,\theta)\, r\, d\theta$$

$$= \frac{r}{2}\frac{d}{dr}\left(\int_{D_r} |\partial u|^2 dV\right)$$

i.e.,

$$\left(1 - c\left(\sup_{D_r} u - \inf_{D_r} u\right)\right)\int_{D_r} |\partial u|^2 dV \leq \frac{r}{2}\frac{d}{dr}\left(\int_{D_r} |\partial u|^2 dV\right)$$

From this, one can deduce

$$\int_{D_r} |\partial u|^2 dV \leq C r^2 \int_{D_{1/2}} |\partial u|^2 dV \quad , r \leq \tfrac{1}{2}$$

similarly, one can do it with center at any $x \in D_{\frac{1}{2}}$, i.e., for $r \leq \frac{1}{2}$

$$\int_{B_r(x)} |\partial u|^2 dV \leq C r^2 \int_{B_{\frac{1}{2}}(x)} |\partial u|^2 dV \leq C r^2 \int_D |\partial u|^2 dV$$

Therefore, $|\partial u|$ is bounded in $D_{\frac{1}{2}}$. Then by the standard theory for elliptic PDE, u is smooth in D.

The same arguments also yield

- <u>Energy estimate</u>: There exists $\varepsilon = \varepsilon(M, \omega, J)$ such that for any J-holomorphic map $u: D_r \to M$ with $\int_{D_r} |\partial u|^2 dV < \varepsilon$,

$$\sup_{x \in D_{\frac{r}{2}}} |\partial u|^2(x) \le \frac{c}{r^2} \int_{D_r} |\partial u|^2 dV$$

<u>Theorem</u>. Assume that there are no J-holomorphic maps $u: S^2 \to M$ with $0 < \int_{S^2} u^*\omega < [\omega](A)$. Then $\mathcal{M}_A(J)$ is compact.

Here we say that $f_\nu \in \mathcal{M}_A(J)$ converge to f in $\mathcal{M}_A(J)$ if there are conformal transformations σ_ν of S^2 such that $f_\nu \circ \sigma_\nu$ converge to f in the C^2-topology.

<u>Proof</u>. First we make a conversion in this proof. Usually, $|\partial f_\nu|$ depends on the choice of the metric on S^2, here we fix the metric to be the standard metric on S^2 and $|\partial f_\nu|$ denotes the norm of ∂f_ν with respect to this metric.

Write $S^2 = \underset{\overset{\Psi}{z}}{\mathbb{C}} \cup \{\infty\}$, then the metric is $\frac{|dz|^2}{(1+|z|^2)^2}$, so

$$|\partial f_\nu|^2 = (1+|z|^2)^2 \left| \frac{\partial f_\nu}{\partial z} \right|^2$$

where $|\cdot|$ on the right is just the absolute value.

Define $\lambda_\nu = \sup_{S^2} |\partial f_\nu| = |\partial f_\nu|(x_\nu)$. By using isometries of S^2, we may assume that $x_\nu = 0$.

Put $\sigma_\nu : \mathbb{C} \to \mathbb{C}$, $\sigma_\nu(z) = \lambda_\nu^{-1} z$, then

$$|\partial(f_\nu \cdot \sigma_\nu)|^2(z) = (1+|z|^2)^2 \lambda_\nu^{2} \left|\frac{\partial f}{\partial z}\right|^2 (\lambda_\nu z)$$

$$= \left(\frac{1+|z|^2}{1+\lambda_\nu^2|z|^2}\right)^2 \lambda_\nu^{-2} |\partial f_\nu|^2 (\lambda_\nu z)$$

$$\leq \left(\frac{1+|z|^2}{1+\lambda_\nu^2|z|^2}\right)^2$$

so $|\partial(f_\nu \circ \sigma_\nu)| \leq C_K$ for any compact subset $K \subset \mathbb{C} \subset S^2$.

It follows that by taking a subsequence if necessary, we may assume that $f_\nu \circ \sigma_\nu$ converge to a J-holomorphic map f on $\mathbb{C}$ (at least in C^0-topology). By the Fatou lemma,

$$\int_{\mathbb{C}} |\partial f|^2 dV \leq \varliminf_{\nu \to \infty} \int_{\mathbb{C}P^1} |\partial(f_\nu \circ \sigma_\nu)|^2 dV = [\omega](A)$$

By the Removable singularity theorem, f extends to a smooth J-holomorphic map from S^2 into M.

By our normalization, $|\partial(f_\nu \circ \sigma_\nu)|(0) = 1$ for all ν, so

$$|\partial f|(0) = 1$$

It follows that $0 < \int_{S^2} |\partial f|^2 dV \leq [\omega](A)$

Our assumption assures $\int_{S^2} |\partial f|^2 dV = [\omega](A)$.

For any $\varepsilon > 0$, choose $\delta > 0$, such that

$$-\int_{S^2 \setminus B_\delta(\infty)} |\partial f|^2 dV + [\omega](A) = \int_{B_\delta(\infty) \subset S^2} |\partial f|^2 dV < \varepsilon^2,$$

so when ν is sufficiently large,

$$\int_{B_\delta(\infty)} |\partial(f_\nu \circ \sigma_\nu)|^2 dV = -\int_{S^2 \setminus B_\delta(\infty)} |\partial(f_\nu \circ \sigma_\nu)|^2 + [\omega](A)$$
$$\leq \varepsilon^2$$

By the Corollary on Page 8, $f_\nu \circ \sigma_\nu (B_\delta(\infty)) \subset B_{30\varepsilon}(p_\nu)$ for some $p_\nu \in M$.

Then one conclude that $f_\nu \circ \sigma_\nu$ converge to f on S^2. //

Before we go to the compactness theorem in general cases, we give a simple application of the above theorem.

<u>Definition</u>. Let (M, ω) be a symplectic manifold and $N \subset M$ be a submanifold. We say that N is a symplectic submanifold of M if $\omega|_N$ is non-degenerate, i.e., $(N, \omega|_N)$ is a symplectic manifold by itself.

Now we assume that M is a symplectic 4-manifold and ask if two homotopic, embedded symplectic surfaces $S_1, S_2 \subset M$ are isotopic, namely, there are a continuous family of smooth embedded surfaces in M joining S_1 to S_2. In general, ~~there are many examples~~ this question has a negative answer. For example, R. Fint and R. Stern have constructed many examples of embedded symplectic surfaces in certain symplectic 4-manifolds which are homotopic to each other, but not isotopic. Therefore, we need further constraints on M in order to have an affirmative answer to such an isotopy problem. Here is a simple example.

<u>Proposition</u>. Let J be any almost complex structure, which is ω-compatible, on $M = \mathbb{CP}^2$ with standard symplectic form ω. Let $A \in H_2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$ be the class of complex lines. Then $\mathcal{M}_A(J)$ is diffeomorphic to $\mathbb{CP}^2$.

<u>Proof</u>. For any complex line $\ell \subset \mathbb{CP}^2$,

$$\int_\ell \omega = 1,$$

i.e., $[\omega](A) = 1$. So we can apply last theorem to conclude that

$\mathcal{M}_A(J)$ is compact.

Next we ~~claim without proof~~ know that any J-holomorphic map $f \in \mathcal{M}_A(J)$ is embedded, otherwise self-intersect # of $f(S^2)$ is more [redacted] than one, which is impossible. Then by the proposition on page 4, each f in $\tilde{\mathcal{M}}_A(J)$ is smooth. It follows that $\bar{\mathcal{M}}_A(J)$ is smooth.

Now we join J to the standard complex structure J_0 on $\mathbb{C}P^2$ by a smooth family of almost complex structures J_t with $J_1 = J$. Then we have a smooth family of compact, smooth manifolds $\mathcal{M}_A(J_t)$. It follows that $\mathcal{M}_A(J)$ is diffeomorphic to $\mathcal{M}_A(J_0)$, which is simply $\mathbb{C}P^2$. //

<u>Corollary</u>: Let $M = \mathbb{C}P^2$ with standard symplectic structure ω and A be the class of complex lines. Then any two symplectic surfaces in M of homology class A are isotopic.

<u>Proof</u>: Given any symplectic surface $S \subset \mathbb{C}P^2$ of class A, one can construct an ω-compatible almost complex structure J such that S is parametrized by a J-holomorphic map. So S is isotopic to a complex line (use the above proposition). Then this corollary follows.

With extra efforts, one can prove the same statement in the above corollary for A being $d[\ell]$, where $d \leq 4$ and ℓ is a complex line. [redacted] When d increases, the difficulties increase, partly because $\mathcal{M}_A(J)$ is not compact and its compactification contains [redacted] J-holomorphic curves with more complicated singularities.

Nevertheless, Siebert and I proposed

<u>Conjecture</u>: If M is either $\mathbb{C}P^2$ or a complex ruled surface with positive first Chern class, two symplectic surfaces are isotopic if and only if they are homotopic.

Now [redacted] we return to the compactness in general cases.

First we give

<u>Definition</u>: A stable J-holomorphic map with k marked points consist of a tuple $(f, \Sigma, x_1, \cdots, x_k)$ satisfying:

1) Σ is a connected with only nodes as singularities. Roughly speaking, Σ can be obtained by identifying pairs of distinct points in a union of Riemann surfaces $\tilde{\Sigma}_i$ $(i=1,2,\cdots,\ell)$, where ℓ is the number of components in Σ. Note that $\tilde{\Sigma}_i$ can be thought as the normalization of Σ_i.

2) For each irreducible component Σ_i of Σ, the restriction of f to Σ_i lifts to a J-holomorphic map from $\tilde{\Sigma}_i$ into M.

3) $x_1, \cdots, x_k \in \Sigma \setminus \{\text{nodes}\}$ are distinct.

4) For each irreducible component Σ_i of Σ, either the Euler number of $\Sigma_i \setminus \{\text{nodes}\} \cup \{x_j\}_{1\le j\le k}$ is negative or $f|_{\Sigma_i}$ is not a constant map.

We say that two stable maps (f, Σ, x_i) and (f', Σ', x_i') are equivalent if there is a conformal map $\sigma: \Sigma \to \Sigma'$ such that $\sigma(x_i) = x_i'$ and $f' \circ \sigma = f$. We also denote by $\mathrm{Aut}(f, \Sigma, x_i)$ the automorphism group

$$\{\sigma: \Sigma \to \Sigma \mid \sigma \text{ conformal}, \ \sigma(x_i) = x_i, \ f\circ\sigma = f\}.$$

The stability implies that $\mathrm{Aut}(f, \Sigma, x_i)$ is finite.

For each stable J-holomorphic map (f, Σ, x_i) $\overset{\shortparallel}{\mathcal{C}}$, we can associate a dual graph $G(\mathcal{C})$ as follows: for each irreducible component Σ_i of Σ, we assign a vertex v_i in $G(\mathcal{C})$; For any node [redacted] connecting Σ_i to Σ_j (i may be the same as j), we join v_i to v_j by an edge. If $\Sigma_i = \Sigma_j$, then this edge is in fact a loop in G; For each marked point x_α in Σ_i, we attach a leg to v_i marked by α; We will assign a marking (A_i, g_i) to each vertex v_i of $G(\mathcal{C})$, where

$A_i = [f(\Sigma_i)] \in H_2(M, \mathbb{Z})$, g_i is the geometric genus of Σ_i.

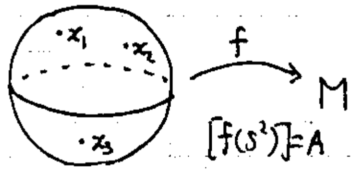
We define 1) the genus $g(G)$ of a dual graph G as

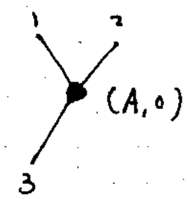
$$\sum_{v_i \in G} g_i + \#\text{ of holes in } G$$

2) the homology class $A = A(G)$ to be $\sum_i A_i$.

For convenience, we also denote by $L(G)$ the number of legs in G.

Examples: (f, Σ, x_i) — dual graph G

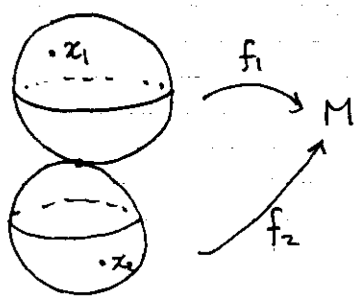


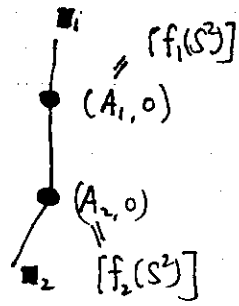


$g(G) = 0$

$A(G) = [f(S^2)]$

$L(G) = 3$

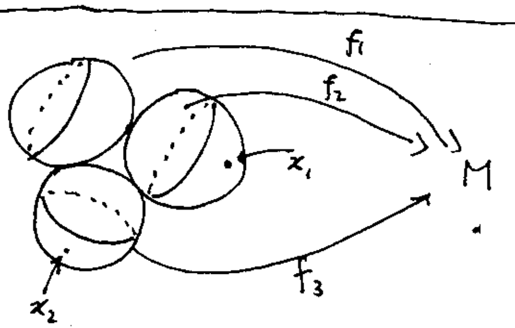


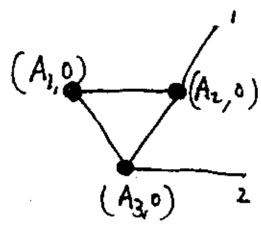


$g(G) = 0$

$A(G) = [f_1(S^2)] + [f_2(S^2)]$

$L(G) = 2$

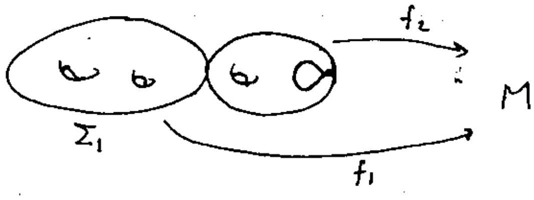


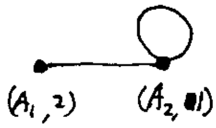


$g(G) = 1$

$A(G) = \sum_i [f_i(S^2)]$

$L(G) = 2$





$g(G) = 4$

$A(G) =$ [redacted] $[f_1(\Sigma_1)] + [f_2(\Sigma_2)]$

$L(G) = 0$

Simple observations: 1) A dual graph of a stable map is connected.

2) A dual graph is the same for any stable map in [redacted] an equivalence class.

3) If v[redacted] is a [redacted] vertex of a dual graph with marking $(0,0)$, then the number of legs and edges from v is at least three. Note that a loop at v will be counted as two edges from v.

For each dual graph G, we define

$$\mathcal{M}_G(J) = \{\text{all}\ {}^{\text{stable}}\ J\text{-holomorphic maps } \zeta = (f, \Sigma, x_i) \mid G(\zeta) = G\}/\sim$$

where $\sim$ is the equivalence relation on stable maps.

Clearly, $\mathcal{M}_A(J)$ defined before is just the moduli $\mathcal{M}_{G_0}(J)$ with

$$G_0 = \quad \bullet\ (A, 0) \quad .$$

We define

$$\overline{\mathcal{M}}_{A,g,k}(J) = \bigcup_{\substack{L(G)=k \\ g(G)=g \\ A(G)=A}} \mathcal{M}_G(J) \quad .$$

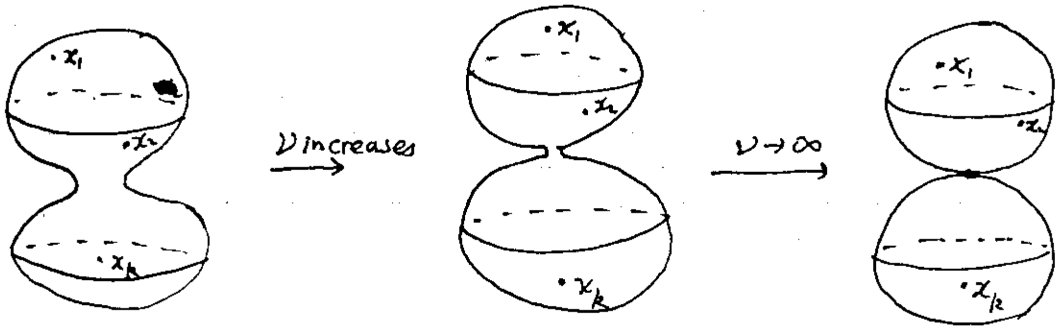
It is easy to check that there are at most finitely many nonempty $\mathcal{M}_G(J)$ with fixed $L(G)$, $g(G)$ and $A(G)$. This is because if $\mathcal{M}_G(J) \neq \phi$, then $\omega(A_i) \geq 0$ and $\omega(A_i) = 0$ implies # legs from v_i + # edges from $v_i \geq 3 - 2g_i$.

Now we can state the main theorem on compactness first studied by Gromov.

<u>Theorem</u>. $\overline{\mathcal{M}}_{A,g,k}(J)$ is compact in an appropriate topology.

We are not going to give precise definition of this topology except a few remarks.

- Let $\zeta_\nu = (f_\nu, \Sigma_\nu, x_{\nu i}) \in \bar{\mathcal{M}}_{A,g,k}(J)$. Suppose that ζ_ν converge to $\zeta = (f, \Sigma, x_i)$. If Σ is [redacted] homeomorphic to Σ_ν, then the convergence simply means that there are conformal transformations σ_ν of Σ_ν such that $f_\nu \circ \sigma_\nu$ converge to f in the C^2-topology and $\sigma_\nu^{-1}(x_{\nu i})$ converge to x_i. If Σ is not homeomorphic to Σ_ν, then we choose a [redacted] metric [redacted] h_ν on Σ_ν in the given conformal class such that h_ν converge to h outside nodes of Σ, and $(f_\nu, \Sigma_\nu, x_{\nu i})$ converge to (f, Σ, x_i) in a similar way as described in previous case except that the convergence of Σ_ν to Σ is measured in terms of h_ν. Here is an example: [redacted] $\Sigma_\nu \cong S^2$, $\Sigma =$ [redacted] $S^2 \vee S^2$, then metrics h_ν are chosen such that Σ_ν look like



- The topology [redacted] used in last theorem implies at least that $f_\nu(\Sigma_\nu)$ converge to $f(\Sigma)$ in the Hausdorff topology for sets in M and $f_\nu(x_{\nu i})$ converge to $f(x_i)$ in M. This later weaker property suffice for constructing the Gromov-Witten invariants. [redacted]

Now we are ready to define invariants. First we observe that there is an evaluation map

$$\begin{array}{ccc} ev: \bar{\mathcal{M}}_{A,g,k}(J) & \longrightarrow & M^k \\ \Downarrow & & \Downarrow \\ (f, \Sigma, x_i) & \longrightarrow & (f(x_i)) \end{array}$$

<u>Theorem</u>. For each compact symplectic manifold (M,ω), we can associate a sequence of homology classes $\{\chi^{M,\omega}_{A,g,k}\}$, where $A \in H_2(M,\mathbb{Z})$, $g \geq 0$, $k \geq 0$, satisfying:

1) For each triple (A,g,k), $\chi_{A,g,k} = \chi^{M,\omega}_{A,g,k} \in H_r(M^k, \mathbb{Q})$, where
$$r = 2\left(c_1(M)(A) + (3-n)(g-1) + k\right).$$

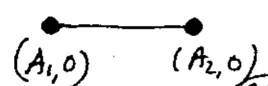
2) $\chi^{M,\omega}_{A,g,k} = \chi^{M',\omega'}_{A,g,k}$ if (M,ω) is symplectically equivalent to (M',ω'), namely, there is a diffeomorphism $\phi: M \to M'$ such that $\phi^*\omega'$ can be connected to ω by a smooth family of symplectic forms ω_t on M.

<u>Proof</u>. The idea of the proof is to construct representatives for $\chi_{A,g,k}$ by using $ev(\overline{M}_{A,g,k}(J))$. In particular, if each stratum [redacted] $M_G(J)$ in $\overline{M}_{A,g,k}(J)$ is a smooth manifold of expected dimension, then $\chi_{A,g,k}$ is indeed represented by the $\mathbb{Q}$-cycle $ev(\overline{M}_{A,g,k}(J))$.

The expected dimension of $M_G(J)$ is equal to $r - \#$ edges.

This can be proved for a generic J under further conditions on M. In general, one has to use theory of constructing virtual moduli cycles developped by a number of people including J. Li, G. Tian, Fukaya–Ono, Siebert.

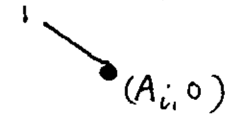
Let us prove this theorem for $g=0$, $k=0$ under the following stronger conditions:
$\overline{M}_{A,0,[redacted]0}(J)$ has only two strata $M_A(J)$ and $M_G(J)$, where G is given by



$\in H_2(M,\mathbb{Z})$

moreover, $\omega(A_i) > 0$ and there [redacted] are no B_i such that $A_i = d_i B_i$ for $d_i \geq 2$.
$\omega(A) > 0$ and there [redacted] are no $B \in H_2(M,\mathbb{Z})$ such that $A = dB$ for $d \geq 2$.

These conditions ensure that all J-holomorphic maps in $\mathcal{M}_A(J)$, $\mathcal{M}_{A_i}(J)$ are simple. Then by the Sard-Smale theorem, we can conclude as before that $\mathcal{M}_A(J)$, $\mathcal{M}_{G_i}(J)$ are smooth manifolds of expected dimension, where G_i is the dual graph



We denote by (f_i, S^2, x_i) a point in $\mathcal{M}_{G_i}(J)$, then one can easily see

$$\mathcal{M}_G(J) = \{ (f_1, S^2, x_1) \times (f_2, S^2, x_2) \in \mathcal{M}_{G_1}(J) \times \mathcal{M}_{G_2}(J) \mid f_1(x_1) = f(x_2) \}$$

Therefore, when J is generic, $\mathcal{M}_G(J)$ is a smooth manifold of dimension

$$\begin{aligned} &\dim \mathcal{M}_{G_1}(J) + \dim \mathcal{M}_{G_2}(J) - 2n \\ = \; & 2(c_1(M)(A_1) + n-3+1) + 2(c_1(M)(A_2) + n-3+1) - 2n \\ = \; & 2(c_1(M)(A) + n-3) - 2 . \end{aligned}$$

It follows that $ev(\overline{\mathcal{M}}_{A,0,0}(J))$ is a well-defined $\mathbb{Q}$-cycle.

To prove the independence of $[ev(\overline{\mathcal{M}}_{A,0,0}(J))]$ on J, one repeat the above arguments for

$$\overline{\mathcal{M}}_{A,0,0}(P) = \bigcup_{J_t \in P} \overline{\mathcal{M}}_{A,0,0}(J_t)$$

where $P = \{J_t\}$ is a path joining J to J'. As before, if the path P is generic among all paths with fixed ends, then $ev(\overline{\mathcal{M}}_{A,0,0}(P))$ provides a cobordism between $ev(\overline{\mathcal{M}}_{A,0,0}(J))$ and $ev(\overline{\mathcal{M}}_{A,0,0}(J'))$, so they represent the same homology class. //

<u>Remarks</u>: 1. This theorem was first proved by Ruan-Tian for semi-positive symplectic manifolds and any symplectic manifolds of complex dimension ≤ 3 ("A mathematical theory of quantum cohomology", JDG, 1995).

2. The semi-positivity was later removed by several

group of people including Fukaya-Ono, Li-Tian, Siebert, ...

(K. Fukaya-Ono: Arnold conjecture and GW-inv., 1996 preprint)

(Li-Tian. Virtual moduli cycles and GW-inv. for general symplectic manifolds, First Int. Press Lec. Series, I, edited by R. Stern. IP press, 1998)

(B Siebert, GW-inv. for general symplectic manifolds, alg-geom/9608005)

3) The classes $\chi_{A,g,k}$ for algebraic manifolds can be constructed purely by algebraic methods, which in fact give more information on $\chi_{A,g,k}$. For those readers who are interested in it, please check the following references:

J. Li-Tian: 'Virtual moduli cycle and GW-inv. of algebraic manifolds', J. of AMS, vol. 11, no. 1 (1998).

Behrend-Fantechi: The intrinsic normal cone, Inv. Math., 128 (1997).

4) For semi-positive symlectic manifolds, particularly, symplectic manifolds of complex dimension ≤ 3, the virtual moduli cycle $\chi_{A,g,k}$ is in fact defined over $\mathbb{Z}$. In general, it is not known if it can be defined over $\mathbb{Z}$ in a suitable sense.

As a simple corollary, we can derive from the above

<u>Theorem</u> Define

$$\Psi_{A,g,k} : H^*($$[redacted]$$M, \mathbb{Q})^k \longrightarrow \mathbb{Q}$$

$$(\alpha_1, \cdots, \alpha_k) \longrightarrow \int_{\mathcal{X}_{A,g,k}} \pi_1^*\alpha_1 \wedge \cdots \wedge \pi_k^*\alpha_k$$

where $\pi_i : M^k \to M$ is the i^{th} projection. Then $\Psi_{A,g,k}$ are symplectic invariants.

<u>Remarks</u> 1) These $\Psi_{A,g,k}$ can be used to distinguish symplectic manifolds. For example, Ruan proved in early 90's that $X \times \mathbb{C}P^1$ is symplectically different from $Y \times \mathbb{C}P^1$, though they are diffeomorphic, where X is the complex surface obtained by blowing up $\mathbb{C}P^2$ at 8 points and Y is the so called Barlow surface.

2) In fact, the GW-inv. $\Psi_{A,g,k}$ can be defined on larger space $H^*(\overline{\mathcal{M}}_{g,k}, \mathbb{Q}) \times H^*(M, \mathbb{Q})^k$ $(2g+k \geq 3)$, where $\overline{\mathcal{M}}_{g,k}$ denotes the Deligne-Mumford compactification of genus g, k marked stable curves. Formally speaking,

$$\Psi_{A,g,k}(\beta, \alpha_1, \cdots, \alpha_k) = \int_{\overline{\mathcal{M}}_{A,g,k}(J)} c^*\beta \wedge ev^*(\pi_1^*\alpha_1 \wedge \cdots \wedge \pi_k^*\alpha_k)$$

where $c : \overline{\mathcal{M}}_{A,g,k}(J) \to \overline{\mathcal{M}}_{g,k}$ is the standard contraction map.

Next we give an application of GW-invariants. We will construct quantum cohomology for symplectic manifolds.

For simplicity, we assume $H^*(M,\mathbb{Q}) = H^{even}(M,\mathbb{Q})$. Fix a basis $\{\beta_i\}$ of $H^*(M,\mathbb{Q})$ as a graded vector space. We write a general element $t \in H^*(M,\mathbb{C})$ as $\sum_i t_i \beta_i$, $t_i \in \mathbb{C}$.

Define a generating function

$$\Phi(t) = \Phi^M(t) = \sum_{k\geq 3} \sum_A \frac{1}{k!} \Psi_{A,0,k}(t,\cdots,t)$$

We [redacted] will not discuss its convergence problem here. Assume that it is a formal series.

In general, we define $\alpha \bullet_t \beta$ by

$$\langle \alpha \bullet_t \beta, \gamma \rangle = \sum_{k\geq 0} \sum_A \frac{1}{k!} \Psi_{A,0,k+3}(\alpha,\beta,\gamma,t,\cdots,t)$$

where $\langle\ ,\ \rangle$ denotes the [redacted] inner product on $H^*(X,\mathbb{C})$ induced by the cup product.

Under assumptions on $H^*(M,\mathbb{Q}) = H^{even}(M,\mathbb{Q})$, we have

[redacted] $$\beta_i \bullet_t \beta_j = \eta^{kl} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_l} \beta_k$$

where $\eta_{ij} = \langle \beta_i, \beta_j \rangle$ and $(\eta^{ij}) = (\eta_{ij})^{-1}$.

<u>Theorem</u>. The quantum product $\bullet_t$ ($\forall t$) is associative, i.e., for any α, β, γ,

$$(\alpha \bullet_t \beta) \bullet_t \gamma = \alpha \bullet_t (\beta \bullet_t \gamma).$$

This was first proved by Ruan-Tian for semi-positive symplectic manifolds, generalized to any symplectic manifolds later by those people mentioned before.

A simple computation shows that the associativity is equivalent to

$$\frac{\partial^3\phi}{\partial t_i \partial t_j \partial t_p}\,\eta^{pq}\,\frac{\partial^3\phi}{\partial t_k \partial t_l \partial t_q} = \frac{\partial^3\phi}{\partial t_k \partial t_j \partial t_p}\,\eta^{pq}\,\frac{\partial^3\phi}{\partial t_i \partial t_l \partial t_q}, \quad \forall\, i,j,k,l.$$

The generating function ϕ also satisfies some homogeneity equation, i.e., if we set $E = \sum_i \left(1-\frac{1}{2}\deg\beta_i\right) t_i \frac{\partial}{\partial t_i} + \sum_i r_i \frac{\partial}{\partial t_i}$, where $\sum_i r_i \beta_i = c_1(M)$, then

$$E\phi = (n-1)\,\phi + \text{quadratic polynomials}.$$

So we have constructed many solutions of the WDW equation in Dubrovin's talk.

<u>Examples</u>. 1. $M = \mathbb{C}P^2$, one can compute

$$\phi(t) = \frac{1}{2}\left(t_0^2 t_2 + t_0 t_1^2\right) + \sum_{d\geq 1} \frac{t_2^{3d-1}}{(3d-1)!}\, e^{d_1 t}$$

where $\deg\beta_0 = 0$, $\deg\beta_1 = 2$, $\deg\beta_2 = 4$. In particular, the "small" quantum cohomology given by $\circ_t$ for $t=0$ is

$$QH^*(\mathbb{C}P^2, \mathbb{C}) = \frac{\mathbb{C}[z]}{(z^3-1)}$$

Recall the classical cohomology ring is

$$H^*(\mathbb{C}P^2, \mathbb{C}) = \frac{\mathbb{C}[z]}{(z^3)}.$$

2. M = a quintic hypersurface in $\mathbb{C}P^4$. Then $c_1(M) = 0$ and

$$\underset{\substack{\cup\\ d}}{H^2(M,\mathbb{Z})} = H^2(\mathbb{C}P^4, \mathbb{Z}) = \mathbb{Z}$$

Assume that [redacted] β_1 is the positive generator of $H^2(M,\mathbb{Z})$, so $\deg \beta_i \neq 2$ for $i \neq 1$. One can show

$$\psi_{A,0,k+3}(\beta_1,\beta_1,\beta_1,\beta_{i_1},\cdots,\beta_{i_k}) = 0 \text{ unless } \sum_l \deg \beta_{i_l} = k .$$

It follows
$$\frac{\partial^3 \phi}{\partial t_1^3} = 5 + \sum_{d\geq 1} \psi_{d,0,3}(\beta_1,\beta_1,\beta_1)\, e^{dt_1}$$
$$= 5 + \sum_{d\geq 1} \frac{d^3 n_d\, e^{dt_1}}{1-e^{dt_1}}$$

where $n_d \in \mathbb{Q}$ are defined by
$$\psi_{d,0,3}(\beta_1,\beta_1,\beta_1) = \sum_{k|d} k^3 n_k$$
$$= d^3 n_d + \left(\frac{d}{2}\right)^3 n_{\frac{d}{2}} + \cdots$$

where we assume $n_\alpha = 0$ if α is not a positive integer. These n_d were discussed extensively in B. Lian's talk. Here we just provide the geometric intepretation. It is still unclear [redacted] how to intepret n_d precisely.

Known: for $d \leq 9$, using a result of T. Johnson & S. Kleiman, one can show that $n_d = \#$ of embedded rational curves in M

However, $n_{10} \neq \#$ of embedded rational curves in M.

This was told to me by S. Katz.

In general, it is very interesting to see if all n_d are integers.