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MODEL MAGNETIC HAMILTONIANS

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These are preliminary lecture notes, intended only for distribution to participants.

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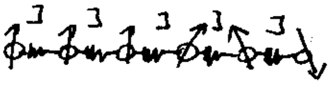
Introduction.



Let the atoms carry a magnetic moment. The key element that forces the system of magnetic moments to display a macroscopic order is that the magnetic moments of neighbouring atoms are linked by the exchange interaction. In this way, rotating one single magnetic moment might cost some energy. The simplest Hamiltonian capturing these essential facts was suggested in the early days of QM by Heisenberg (see also the book of P.A.M. Dirac on QM for an explicit derivation of this Hamiltonian):

$$H_{Heis} = -J \cdot \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j \qquad (1)$$

In (1) the sum runs over nearest neigbours $\langle ij \rangle$ and $\vec{S}_i$ are spin operators. The essential question about (1) is:

(i) does this Hamiltonian imply a macroscopic order, i.e. a macroscopic alignment of all spins?

(ii) if so, does it have a phase transition to a disordered state?

We will try to give an answer to this question and to provide some experimental insights on phase transitions of magnetic systems.

1. The Landau Ginzburg Wilson Hamiltonian.

It turns out that most essential steps toward the understanding of phase transitions were not obtained by studying (1). Instead, (1) was mapped onto the so called Landau Ginzburg Wilson (LGW) Hamiltonian. This mapping allows a more regular approach to phase transitions. Furthermore, it can be used to give a very intuitive insight into a famous theorem of 2d physics: The Mermin Wagner theorem.

Let us start with the map. We begin by rewriting (1) in the form

$$H = \frac{1}{2} J \cdot \sum_i \sum_{\vec{\delta}_i} \left[\vec{S}(\vec{x}_i) - \vec{S}(\vec{x}_i + \vec{\delta}_i) \right] - q \cdot \sum_i \vec{S}(\vec{x}_i)^2 \quad (2)$$

where $\vec{x}_i = (x_i^1, \dots, x_i^d)$ label the sites of a d-dimensional lattice. For convenience, we take the lattice to be hypercubic with co-ordination number $q = 2d$ and nearest neighbours unit vector $\vec{\delta}_i$ of length a. In (1) and (2) $|\vec{S}_i| = S \equiv 1$. The thermodynamics of the system described by (2) is contained in the partition function Z

$$Z = \sum_{\{|\vec{S}_i| = 1\}} e^{-\beta H} \qquad (3) \qquad 1/k_B T \equiv \beta$$

where $\sum_{\{\vec{S}_i\}}$ means that one has to sum over all possible configurations with the constraint $|\vec{S}_i| = 1$. We can explicitly take into account this constraint by rewriting (3) as

$$Z = \prod_i \int_{-\infty}^{\infty} d\vec{S}_i \cdot \delta(|\vec{S}_i| - 1) \cdot e^{-\beta H[\vec{S}_i]} \qquad (4)$$

The weight function $\delta(|\vec{S}_i| - 1)$ is very singular, and this is one reason for the immense difficulty one has in calculating the partition function of the Heisenberg model. One can obtain a more tractable problem see Wilson + Kogut, Physics Reports 12, 75 (1974) by replacing the singular weight function by

$$e^{-\beta \cdot f(|\vec{S}_i|^2)} \quad \text{with} \quad f(|\vec{S}_i|^2) = \frac{r}{2} \cdot \vec{S}_i^{\,2} + \frac{u}{4} \vec{S}_i^{\,4} \quad (5)$$

while relaxing the condition $|\vec{S}_i| = 1$ and letting $|\vec{S}_i|$ being a continuous variable

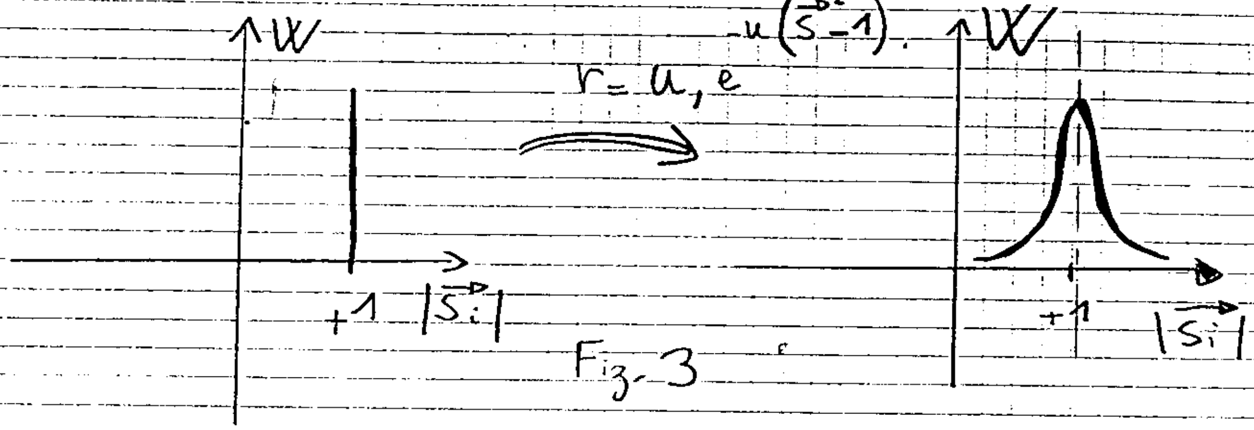


Fig. 3

The partition function becomes

$$Z = \prod_i \int d\vec{S}_i \, e^{-\beta H_{eff}}$$

$$H_{eff} = \frac{1}{2} J \sum_i \sum_{\vec{\delta}_i} \left| \vec{S}(\vec{x}_i) - \vec{S}(\vec{x}_i + \vec{\delta}_i) \right|^2 +$$

$$+ \sum_i \left[\left(\frac{r}{2} - q \right) \cdot \vec{S}_i^{\,2} + \frac{u}{4} \vec{S}_i^{\,4} \right] \quad (6)$$

The effective Hamiltonian, which incorporates the weight function, can be simplified by taking the continuum limes $a \to 0$, so that

$$\vec{S}_i^{\,\alpha}(\vec{x}_i) - \vec{S}^{\,\alpha}(\vec{x}_i + \vec{\delta}_i) \longrightarrow a (\vec{\delta} \cdot \vec{\nabla}) S^{\alpha}(\vec{x}) \quad (7)$$

where $\vec{\nabla}$ is a d dimensional gradient operator. In the continuum limes, the $\sum_i$ becomes an integral

$$H_{eff} = \frac{1}{2} J \cdot \frac{1}{a^d} a^2 \int d^d\vec{x} \sum_{\alpha=1}^{n} \left(\vec{\nabla} S^\alpha\right)^2 +$$

$$\frac{1}{a^d} \cdot \int d^d\vec{x} \left[\left(r/_2 - q\right) \vec{S}^2 + \frac{u}{4} S^4 \right] \quad (8)$$

By setting $\frac{r}{2} - q = \frac{r}{2}$ we obtain the LGW Hamiltonian

$$H_{LGW} = \frac{J}{2} a^{-d+2} \int d^d x \sum_\alpha (\nabla S_\alpha)^2 + a^{-d} \int d^d x \left[\frac{r}{2} \vec{S}^2 + \frac{u}{4} \vec{S}^4 \right] \quad (9)$$

which is the starting point of many fruitful insights into the behaviour of systems close to the phase transition point. In the literature, the LGW effective Hamiltonian is often called "Landau free energy", not to be confused with the "Helmholtz free energy" or the "Gibbs free energy". These are thermodynamic potentials, defined over variables like $T, V, M, p, \ldots$ H_{LGW} is a functional of a given spin field $\vec{S}(\vec{x})$, i.e. it associates to any $\vec{S}(\vec{x})$ a real number.

One first word about the choice of r and u: the singular weight function corresponding to the Heisenberg model is recovered by the special choice $r = u$ and $u \to \infty$, but Ken Wilson argued that one should really study the H_{LGW} Hamiltonian for all coupling constants r and u in order to learn about the critical behaviour. The reason is that if there is such a fundamental law as "universality", then the relevant critical quantities like critical exponents should be really independent of the original coupling constants. Hence, results obtained for arbitrary r and u might be closely related to those obtained for the special values $r = u$, $u \to \infty$.

2. The Landau theory.

This is the simplest approximate calculation of Z_{LGW}, originally suggested by Landau in the $30^{'s}$ (Phys. Z. Sowjet. $\underline{11}$, 545 (1937)). After the theory of Van der Waals, Landau theory represented a substantial advance in the understanding of phase transition. It is on this theory that Wilson founded his Renormalization Group approach to phase transitions, which solved the long standing problem of explaining phase transitions. Wilson himself defined the RG as a natural expansion of Landau ideas.

Within Landau theory, the partition function Z is calculated by replacing the functional integration with the maximum value of the integrand. This is equivalent to state that the equilibrium distribution $\vec{S}(\vec{x})$ is the result of the variational problem

$$\frac{\delta H_{LGW}}{\delta \vec{S}(\vec{x})} = 0 \qquad (10)$$

which typically leads to Euler-Lagrange equations.

It is instructive to carry out this programme for the simplest non trivial functional exhibiting a phase transition. This is constructed by setting homogeneous fields $\vec{S}(\vec{x}) \equiv \vec{S}$ in H_{LGW}, thus making the gradient term in (6) - which obscures the value of the integrand - vanishing. The Euler-Lagrange equation determining the equilibrium value for $\vec{S}$ is thus

$$r \cdot \underline{S_z} + u \cdot \underline{S_z}^3 = h \qquad (11)$$

whereby we have allowed for a magnetic field h to be applied. ($h = g\mu_B S \cdot H_z, \Rightarrow \underline{S_x} = S_y = 0$).

Eq. 11 must be considered the equation of state of the system. The simplest way of guessing a realistic value for the parameters r and u is to compare (11) with the equation of state resulting from Mean Field Theory:

$$\frac{M}{M_S} = B_S\left(2J\cdot S^2\cdot q\cdot\frac{M}{M_S}\cdot\frac{1}{T} + g\mu_B S\cdot H\cdot\frac{1}{T}\right) \quad (12)$$

which, for $S = 1/2$ and small M, H reduces to

$$h \approx \frac{M}{M_S}\cdot(T - T_c) + \left(\frac{M}{M_S}\right)^3\cdot\frac{T_c}{3} \quad (13)$$

Comparing (13) and (11) we immediately find that $r = (T - T_c)$ $u = T_c/3$, which is the result guessed by Landau on empirical grounds.
Eq. 11 or 13 provide a very general picture of the phase transition. The Helmholtz free energy can be obtained by integrating the right hand side of (11) over $\underline{S_z}$:

$$A(\underline{S_z}) = \int_0^{S_z} h(S_z')\, dS_z' = \frac{r}{2}\underline{S_z}^2 + \frac{u}{4}S_z^4 \quad (14)$$

The graph of this function depends on whether $r>0$ or $r<0$:

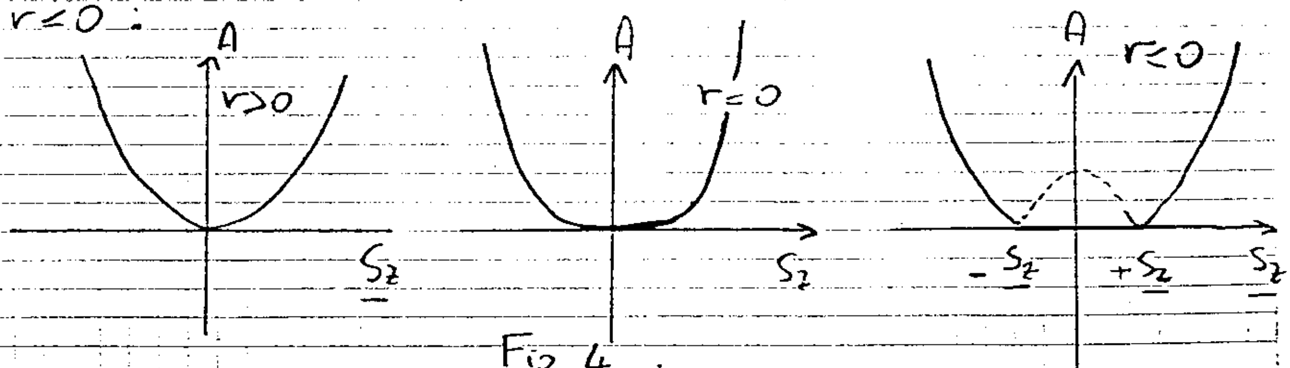
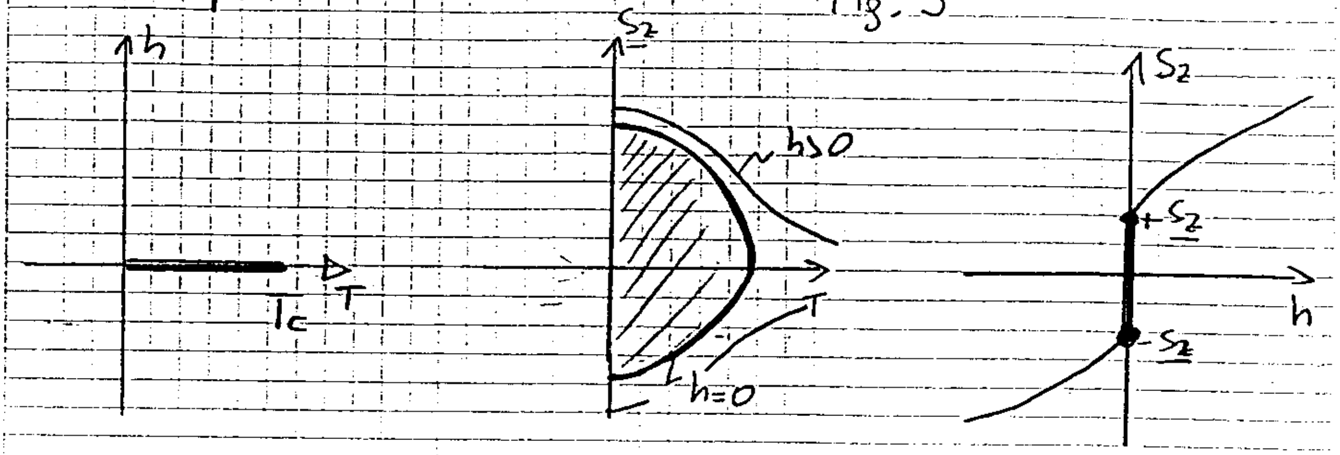


Fig. 4

The graph of $A(\underline{S_z})$ contains, for $r<0$, a portion (dotted line) where the convexity of the thermodynamic potential is violated. Convexity is restored by taking the convex envelope of the graph, leading to phase coexistence. Thus, the phase diagram contains a line of first order phase transitions terminating at a critical point

Fig. 5



The shape of the coexistence curve ($h=0$), i.e. the decay of $\underline{S_z}$ with r, can be found by solving eq. 11 for $h=0$:

$$\underline{S_z} = \begin{cases} 0 & r>0 \\ \pm\sqrt{\frac{-r}{u}} & r<0 \end{cases} \qquad (15)$$

leading to the power law decay

$$\underline{S_z}(T, h=0) \sim \left(1-\frac{T}{T_c}\right)^{\beta_L} \qquad \beta_L = 1/2 \qquad (16)$$

For the critical isotherm $\underline{S_z}(h, r=0)$ we obtain

$$\underline{S_z}(h, T=T_c) \sim h^{1/\delta_L} \qquad \delta_L = 3 \qquad (17)$$

To calculate the susceptibility $\chi(r^+) \doteq \left.\frac{\partial \underline{S_z}}{\partial h}\right|_{h=0} = \frac{1}{\left(\frac{\partial h}{\partial \underline{S_z}}\right)_{\underline{S_z}=0}}$

we differentiate Eq. 11 with respect to $\underline{S_z}$:

$$\left.\frac{\partial h}{\partial \underline{S_z}}\right|_{\underline{S_z}=0} = r + 3u\underline{S_z}^2\Big|_{\underline{S_z}=0} = r \qquad (18)$$

$$\Rightarrow \qquad \chi^+(T) \sim (T-T_c)^{-\gamma_L} \qquad \gamma_L = 1 \qquad (19)$$

Similarly,

$$\chi^-(r^-) = \frac{1}{\left(\frac{\partial h}{\partial s_z}\right)\Big|_{\underline{s_z}}} \Rightarrow \frac{1}{\chi^-(r^-)} = r + 3u\left(-\frac{r}{u}\right) = -2r$$

$$\Rightarrow \chi^-(T) \sim \frac{1}{2}\left(T_c - T\right)^{-\gamma_L'} \qquad \gamma_L' = 1 \qquad (20)$$

Notice that with $\chi^+ = C^+ r^{-\gamma}$ and $\chi^- = C^-(-r)^{-\gamma}$, it follows from (20) and (13⁺)

$$\frac{C_L^+}{C_L^-} = 2 \qquad (21)$$

C^+, C^- being so called critical amplitudes.

To obtain the correlation length within "Landau theory" we consider the response of $S_z(\vec{x})$ to a "test source" $h(x) = \lambda \cdot \delta(\vec{x})$. The Euler-Lagrange equation for inhomogeneous fields $\vec{s}(\vec{x})$ reads: $(d=3)$

$$\delta H_{eff} = \underbrace{\frac{1}{2}\frac{J}{a}\int d^3x\, \delta\left(\vec{\nabla} s_z\right)^2}_{\text{p.I.}} + \frac{1}{a^3}\int d^3x \left[r \cdot s_z + u \cdot s_z^3\right]\delta s_z \qquad (22)$$

$$-\frac{J}{a}\int d^3x\, \Delta s_z \cdot \delta s_z$$

Setting $\delta H_{eff} = 0$ we obtain the Euler-Lagrange equation

$$r \cdot \underline{s_z} + u \cdot \underline{s_z}^3 - J a^2 \Delta s_z = h(\vec{x}) \qquad (23)$$

For $r > 0$ and $s_z \sim 0$ we can neglect the $\underline{s_z}^3$ term, so that (23) reduces to

$$-Ja^2 \Delta S_z + r S_z = h(\vec{x}) = \lambda \cdot \delta(\vec{x}) \quad (24)$$

This equation can be solved by taking the Fourier-transform on both sides, to give

$$J a^2 \vec{k}^2 S_{\vec{k}} + r S_{\vec{k}} = \lambda$$

$$S_{\vec{k}} = \frac{\lambda}{r + J a^2 \vec{k}^{2-\eta}} \qquad \eta_L = 0 \quad (25)$$

and

$$\underbrace{S_z(\vec{x}) = \lambda \int d\vec{k} \frac{e^{i\vec{k}\vec{x}}}{r + J a^2 \vec{k}^2} \underset{x \to \infty}{\Rightarrow} \sim \frac{1}{|\vec{x}|} \cdot e^{-|\vec{x}|/\xi}}_{\text{Ornstein-Zernicke}}$$

with

$$\xi = \left(\frac{r}{J}\right)^{-\nu_L}, \quad \nu_L = 1/2 \quad (26)$$

We summarize the critical exponents found within Landau theory:

$$\boxed{\beta_L = 1/2 \quad \gamma = 1 \quad \delta = 3 \quad \nu = 1/2 \quad \eta = 0} \quad (27)$$

These values are sometimes called classical. They are not very consistent with experiments, although they give a pretty good idea about the expected power law behaviour.

The Landau exponents have a high degree of universality, for they are independent of the various coupling constants and of the dimensionality of the space. This is a too high degree of universality, as shown by experiments and more accurate theoretical models.

Exponent	TH	EXPT	MFT	ISING2	ISING3	HEIS3
α		0–0.14	0	0	0.12	−0.14
β		0.32–0.39	1/2	1/8	0.31	0.3
γ		1.3–1.4	1	7/4	1.25	1.4
δ		4–5	3	15	5	
ν		0.6–0.7	1/2	1	0.64	0.7
η		0.05	0	1/4	0.05	0.04
$\alpha + 2\beta + \gamma$	2	2.00 ±0.01	2	2	2	2
$(\beta\delta - \gamma)/\beta$	1	0.93 ±0.08	1	1	1	
$(2-\eta)\nu/\gamma$	1	1.02 ±0.05	1	1	1	1
$(2-\alpha)/\nu d$	1		$4/d$	1	1	1

(28)

[a] TH, theoretical values (from scaling laws); EXPT, experimental values (from a variety of systems); MFT, mean field theory; ISINGd, Ising model in d dimension; HEIS3, classical Heisenberg model, $d = 3$.
[b] For more details and documentation see A. Z. Patashinskii and V. L. Pokrovskii, *Fluctuation Theory of Phase Transitions* (Pergamon, Oxford, 1979), Table 3, pp. 42–43.

However, we do learn quite a lot from this simple theory. First, we learn that a regular function such as Landau free energy can lead to singular behaviour of thermodynamic quantities. Second, the universality of critical exponents suggests that critical phenomena are not so sensitive to details of the microscopic Hamiltonian, but depend possibly only on its symmetries, like dimensionality and the number of components of the order parameter. These results contributed significantly to the search of a general understanding of critical phenomena.

Before we end this chapter, we would like to discuss a simple symmetry of Eq. (11), which is often neglected. If we divide the equation of state by S_z we obtain an "Arrot" plot:

$$\frac{h}{\underline{S_z}} = r + u \cdot \underline{S_z}^2 \qquad (23)$$

(A. Arrot, Phys. Rev. 108, 1394 (1957))

which can be improved by correcting (28) in an obvious way
(A. Arrott and J. E. Noakes, Phys. Rev. Lett. 19, 786 (1967))

$$\left(\frac{h}{S_z}\right)^{1/\gamma} = r + u \cdot S_z^{1/\beta}$$

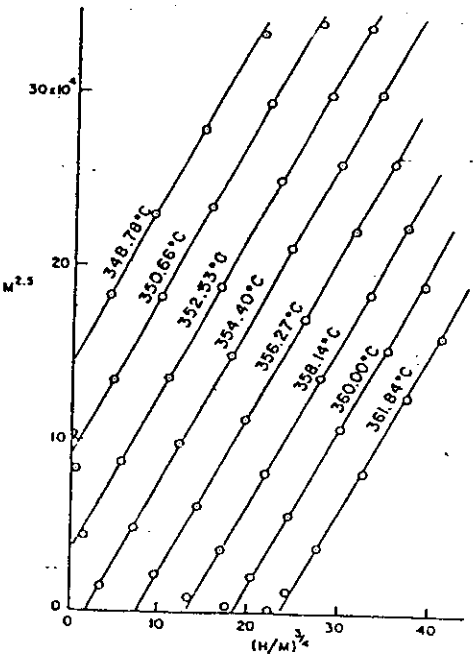


FIG. 3. A replot of the data of Weiss and Forrer to conform to the variables of Eq. (1).

Fig. 6

Arrott, however, missed the most important symmetry of Eq. 11, which becomes apparent if one divides by S_z^3:

$$\frac{h}{S_z^3} = \frac{r}{S_z^2} + 1 \qquad (30)$$

which we can generalize to

$$\frac{h}{S_z^\delta} = \frac{r}{S_z^{1/\beta}} + u \qquad (31)$$

This symmetry was discovered by Griffiths (Phys. Rev. 158, 176 (1967)) who established its profound general significance:

$$\frac{h}{S_z^\delta} = h\left(\frac{r}{S_z^{1/\beta}}\right) \qquad (32)$$

This equation means that by using well specific thermodynamic variables, all experimental data on a two-dimensional surface $S_2(r,h)$ "collapses" onto one single function, the so called "scaling" function: in Eq. 32, the left hand side is a single valued function of one single variable. This property is called "scaling".

Experimental confirmation of universality for a phase transition in two dimensions

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When a system is poised at a critical point between two macroscopic phases, it exhibits dynamical structures on all available spatial scales, even though the underlying microscopic interactions tend to have a characteristic length scale. According to the universality hypothesis[1,2], diverse physical systems that share the same essential symmetry properties will exhibit the same physical behaviour close to their critical points[1,3-5]; if this is so, even highly idealized models can be used to describe real systems accurately. Here we report experimental confirmation that the scaling behaviour of thermodynamic variables predicted by the universality hypothesis holds over 18 orders of magnitude. We show that the equation of state of a two-dimensional system (an atomic layer of ferromagnetic iron deposited on a non-magnetic substrate) closely follows the behaviour[6] of the two-dimensional Ising model[7]—the first and most elementary statistical model of a macroscopic system with short-range interactions[3].

We grow epitaxial Fe films at room temperature on top of a non-magnetic single-crystal W(110) surface. Pressure in the vacuum chamber during growth does not exceed 2×10^{-10} mbar. In the ferromagnetically ordered phase, the films have a uniaxial easy magnetization axis along the in-plane $[1\bar{1}0]$ direction. Along this direction the magnetization M versus applied field H follows a square hysteresis loop, that is, within experimental uncertainty ($< 1\%$) M at $H = 0$ coincides with the value for M in an applied magnetic field. Accordingly[8], such films are in a single domain state. M is measured by means of the magneto-optic Kerr effect[9].

As revealed by our scanning tunnelling microscopy study of the epitaxial growth, films between 1 and 2 atomic layers (AL) consist of one complete AL of Fe followed by a two-dimensional

(2D) network of irregular patches, (Fig. 1). Some third layer is already visible. Low-energy electron diffraction shows, in this thickness range, a sharp p(1 × 1) pattern indicating growth in registry with the substrate.

In the vicinity of the critical point $[t=(T/T_c-1), H]=[0, 0]$ the critical exponents β and δ for the zero-field magnetization curve $M(t, H=0)\propto(-t)^{\beta}$ and for the critical isotherm $M(H, t=0)\propto H^{1/\delta}$ are determined to be 0.13 ± 0.02 and 14 ± 5 respectively, (Fig. 2). The theoretical 2D Ising values are $\beta=1/8$ (exactly known[10]) and $\delta=15$ (conjectured from numerical calculations[11]).

The sharpness of the transition observed in Fig. 2 is consistent with the magnetic correlation length reaching values of ~1,000 nm (ref. 9). Evidently, the observed morphological defects—such as monatomic steps arising from the imperfect substrate (typical separation, 50–200 nm in the present sample) and the steps due to patches formation (Fig. 1) during growth—are not able to break the correlation length, that is, the correlations extend to much larger length than the spacing between defects. The picture emerging by comparing Fig. 1 and Fig. 2 is one where locally the number of coexisting layers (that is, the coordination number of Fe atoms) changes, thus causing local fluctuations of the strength of the various magnetic interactions. T_c, however, turns out to be sensitive to the average strength of the magnetic interactions over large distances rather than to the local fluctuations and thus is well defined. Notice that the lateral correlation length growing much larger than the film thickness is the key to observing a truly 2D phase transition.

Figure 3 shows a M versus t family of isochamps (curves with constant magnetic field). This figure shows the familiar picture $M(t)$ curves separating out according to the strength of the applied magnetic field. In agreement with the high value of the exponent δ, fields as small as 0.01 Oe (ref. 9) already introduce a measurable tail in the $M(t)$ curve. This strong response to minute magnetic fields is characteristic of 2D systems, for which $\delta=15$ (ref. 11), and is not found in three-dimensional systems for which $\delta=5$ (ref. 5).

The family of isochamps given in Fig. 3 collapses into just one single curve—the scaling function (that is, the equation of

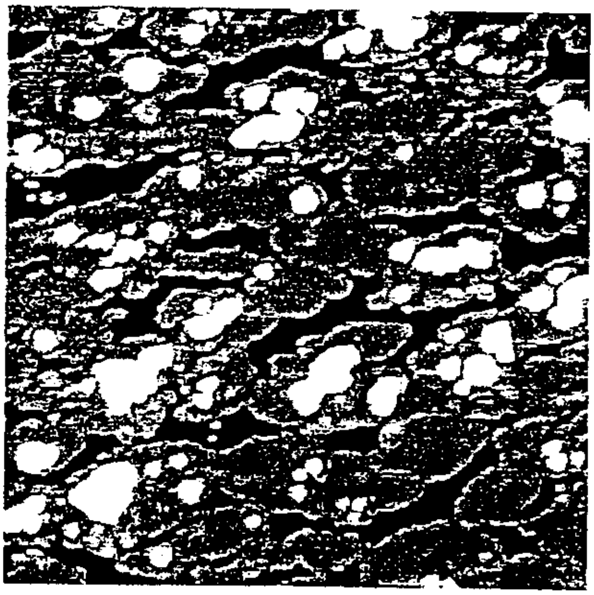

FIG. 1 STM topography of a 1.8 ± 0.1 AL of Fe on W(110). The image shows an area of 100 × 100 nm. The grey-scale range is 0.36 nm. Black corresponds to the first, grey to the second and white to the third Fe layer. A tunnelling current of 0.2 nA and a sample voltage of −620 mV were used to collect the image.

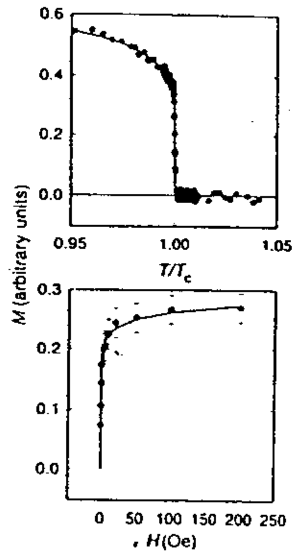

FIG. 2 Top, zero-field magnetization curve versus T/T_c ($H=0\pm10$ mOe) measured using the magneto-optical Kerr effect for an Fe film 1.7 ± 0.1 AL thick. The thickness was determined from the STM topography. Absolute temperatures are determined to within ±5 K, while relative temperatures are accurate to within 0.01 K. T_c (316.77 K in the present case) was determined separately, by locating the temperature at which $[M(T, H)-M(T, 0)]$ has a maximum[9], with a relative accuracy of ±0.07 K. This means that the origin of the t-axis is known with an accuracy of $\pm2\times10^{-4}$. The solid line is a power-law fit to the data $M\propto(1-T/T_c)^{\beta}$ yielding $\beta=0.13\pm0.02$. The error in β is due to the uncertainty in locating T_c. Bottom, magnetization as a function of the applied magnetic field at T_c. The solid line is a power-law fit to the data $M\propto H^{1/\delta}$, yielding $\delta=14\pm5$. The critical isotherm was obtained by plotting the maximum value of the family of curves $[M(t, H)-M(t, 0)]$ versus H.

state in the critical region)—when the variables M and t are scaled by $H^{1/\delta}$ and $H^{1/\beta\delta}$, respectively[12] (Fig. 4*a*). This collapsing is a manifestation of the scaling hypothesis[1,12]. Except for a discrete number of data points on the left-hand side of the figure, deviation from scaling is <10%.

Whereas the representation of Fig. 4*a*—introduced by Hankey and Stanley[12]—is most suited to visualize the data-collapsing, plotting[13] $h(x)=H/M^{\delta}$ versus $x=t/M^{1/\beta}$ allows the determination of the zero-field susceptibility critical exponents γ and of the ratio of the critical amplitudes Γ_+/Γ_-, defined as $\chi(T)=\Gamma_+(t)^{-\gamma}(T>T_c)$ and $\chi(T)=\Gamma_-=\Gamma_-(-t)^{-\gamma}(T<T_c)$, see Fig. 4*b*). This representation of scaling is designated as the $h(x)$ representation of the scaling function[13]. For large x we expect the leading term[14] in $h(x)$ to be x^{γ}/Γ_+ (the point $(h(\infty), \infty)$ corresponds to the critical isochore). In this representation, the point $(h(x_0)=0, x_0)$ corresponds to the zero-field magnetization curve shown in Fig. 2. In the vicinity of this point one expects[14] the leading term in $h(x)$ to be linear in $(x+x_0)$ with coefficient $\beta/\Gamma_-x_0^{(\gamma-1)}$. The point $(h(0), 0)$ is the critical isotherm shown in Fig. 2. By fitting the measured scaling function in the corresponding ranges according to the expected functional form we obtain $\gamma=1.74\pm0.05$ and $\Gamma_+/\Gamma_-=40\pm10$. The theoretical 2D Ising values are 7/4 (exactly known[15]) and $\Gamma_+/\Gamma_-\approx37.7$ (ref. 16).

The coincidence between the measured critical quantities β, δ, γ, Γ_+/Γ_- and the calculated ones for the 2D Ising model might seem surprising at first sight. Our system is certainly not—see Fig. 1—the lattice with perfect translational symmetry underlying the Ising hamiltonian and the calculations by Gaunt and Domb[6]. Also Fe does not have localized spins as the Ising hamiltonian, which imitates rather an insulating ferromagnet with localized moments than a broad-band metallic ferromagnet like Fe. Moreover, although our uniaxial ferromagnet might

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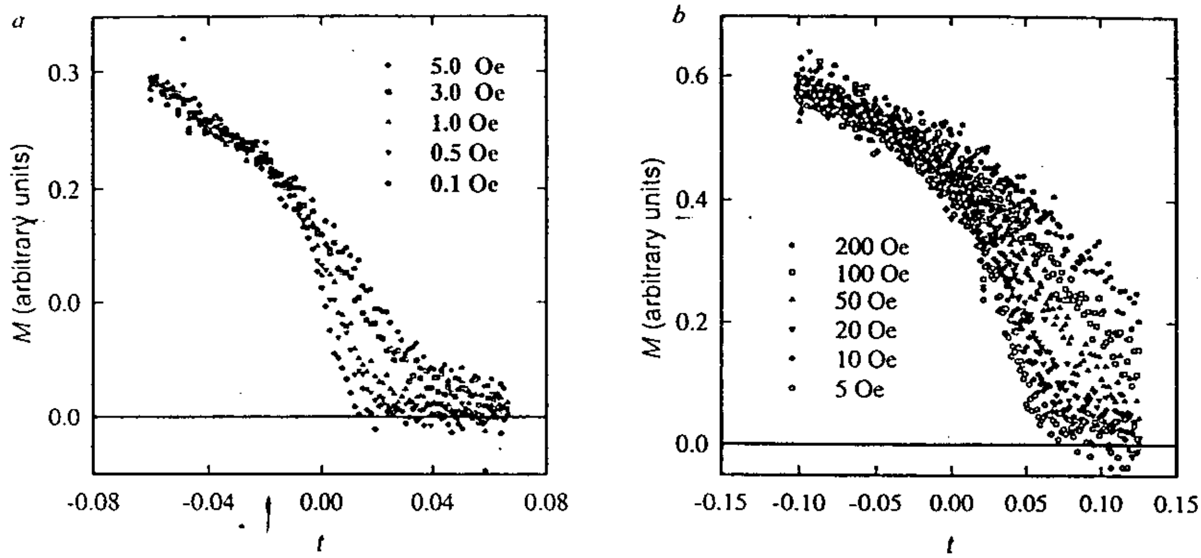

FIG. 3 Magnetization curves in constant magnetic field obtained in the vicinity of the Curie temperature as a function of t. For sample 1 (panel a) the applied field was varied between 0.1 and 5 Oe. For sample 2 (panel b) the field was varied between 5 and 200 Oe. The Curie temperatures were determined as in Fig. 2. The Curie temperature of sample 1 was 331.3 K. Sample 2 was slightly thicker and had a Curie temperature of 345 K. Notice that in this thickness range (~1.7 AL) the phase transition occurs at temperatures close to deposition temperature, so the morphology does not change during the thermal scan.

have the same rotational symmetry as the 2D Ising hamiltonian, the strength of the various magnetic anisotropies in our films is very small (of the order of 1 K, compared to $T_c \approx 300$ K). Accordingly, the individual Fe-spins can move easily away from the easy axis and thus have more degrees of freedom than the Ising spins, which can only be 'up' or 'down'. In fact, the temperature dependence of the magnetization far away from the critical point deviates strongly from the 2D Ising behaviour (ref. 9). Finally, a realistic magnetic interaction between two Fe spins $S_i S_j$ should be much more complicated than the scalar product (S_i, S_j) forthcoming in the Ising hamiltonian. Thus, the good agreement between the experimental and measured critical quantities should be regarded as a stringent test of the universality hypothesis[1,2]. According to this hypothesis, it is believed[14] that the critical exponents and the ratio of critical amplitudes depend only on (1) the spatial dimensionality of the system, (2) the number of components of the order parameter, (3) the symmetry of the hamiltonian and (4) the range of the micro-

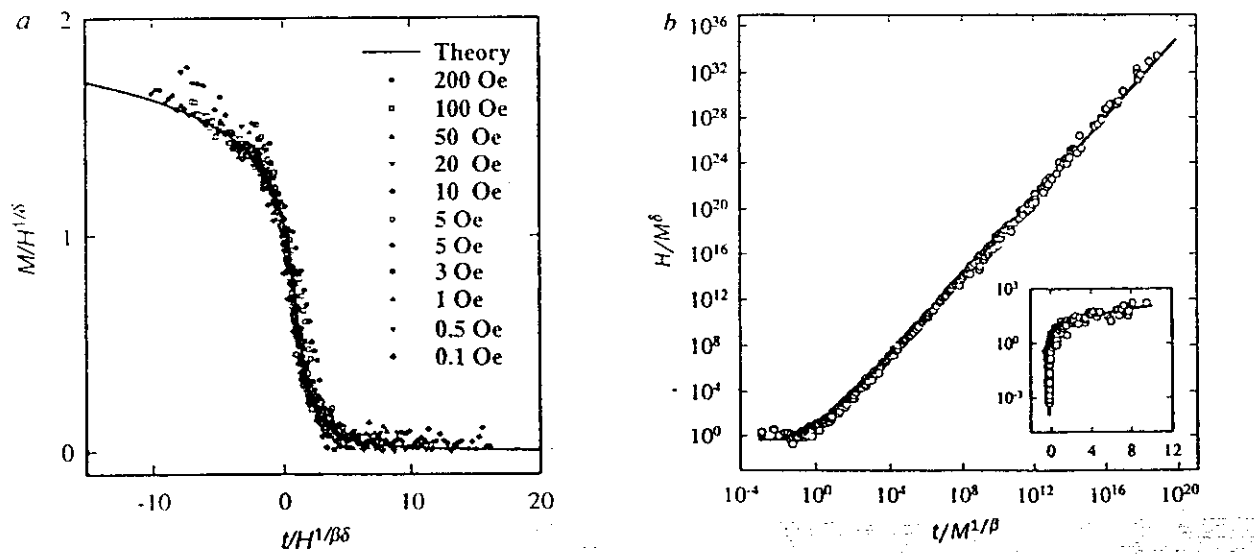

FIG. 4 a, Scaled magnetization $M/H^{1/\delta}$ versus scaled temperature $t/H^{1/\beta\delta}$ for the two films of Fig. 3. A linear scale is used. The solid line is the theoretical scaling function after ref. 6. For small fields, a discrete number of data points deviate from the scaling function. They are neglected in b. We used the value of β and δ determined from Fig. 2. For δ outside the range 12–16 the data collapsing is poorly realized. This allows us to reduce the experimental uncertainty for δ to ± 2. b, Plot of H/M^{δ} versus $t/M^{1/\beta}$ for the two films of Fig. 3. The data with $x > 0$ are represented as a log–log plot. The data points in the interval $-1 \times < 10$ are represented semilogarithmically in the inset. The solid line is the theoretical curve from ref. 6. All data points, regardless of their value of H, are shown with the same symbol.

scopic interactions responsible for the phase transition. Also the presence of defects is expected to alter the universality class[14]. Apparently, the morphological defects observed in our system (Fig. 1) are not able to bring our system outside the Ising universality class. This might be related to the fact that such defects are not effective in limiting the correlation length.

A deeper level of universality was predicted[1,3,5,14]. At this level, provided that the non-universal scales for M and H are specified, the resulting scaling function should coincide within one universality class. This level of universality is also known historically as the Van der Waals Law of Corresponding States[4]: using properly normalized thermodynamic variables, all systems within the same universality class have the same equation of state. Thus, by suitably rescaling M and H (or, equivalently, $h(x)$ and x) it should be possible to bring the experimental scaling function and the theoretical one to coincidence. This rescaling is performed in Fig. 4, where the solid line is the theoretical scaling function of ref. 6 (the algebraic expressions used for constructing the solid lines in Fig. 4 are given explicitly in the Appendix of ref. 6). The theoretical and experimental results are normalized by the non-universal critical parameters $h(0)$ and x_0, as suggested in ref. 12. The agreement between the experimental scaling function and the calculated one extends over 18 orders of magnitude of the variable $x = t/M^{1/\beta}$.

The exact solution of the 2D Ising model in an applied magnetic field is still missing[2], so that one has to resort to approximate solutions. Our results confirm the validity of such calculations and represent a test of universality and scaling over a wide range of parameters. Further experimental work on this system along the lines of for example, refs 17–21 (including the experimental determination of the scaling behaviour of the correlation function[14] by, for example, a new generation of neutron spectrometers) should improve our knowledge of cooperative phenomena beyond the pure numerical awareness. □

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6. – Scaling and universality.

There is not only one form of expressing the scaling hypothesis [2, 6]. HANKEY and STANLEY [2] proposed a form which is illustrated in Fig. 2 and summarized by the equation of state

$$M/H^{1/\delta} = g(y) , \qquad y \equiv \varepsilon/H^{1/\beta\delta} . \tag{15}$$

It reads as follows: « The magnetization M scaled by $H^{1/\delta}$ is a function of one and only one argument $y \equiv \varepsilon/H^{1/\beta\delta}$, *i.e.* ε scaled by $H^{1/\beta\delta}$ ». One can easily

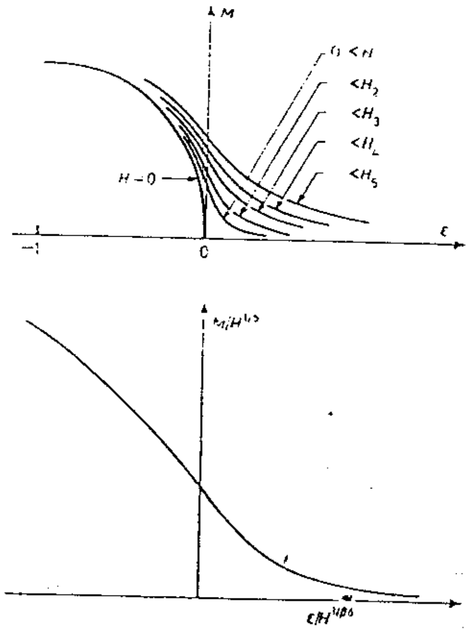

Fig. 2. – Schematic illustration of the scaling equation of state, eq. (15). The family of the isochamps (curves with constant magnetic field H), such that $H_i < H_{i+1}$, gives just one curve when the variables ε and M are scaled by $H^{1/\beta\delta}$ and $H^{1/\delta}$ respectively.

verify that the scaling function $g(y)$ can be expressed by $h(x)$, and *vice versa*:

$$g = h^{-1/\delta}, \quad y = x/h^{1/\beta\delta}. \tag{16}$$

It turns out that the plotting of current experimental data in the form « $M/H^{1/\delta}$ against $\varepsilon/M^{1/\beta\delta}$ » is advantageous over the plotting « H/M^{δ} against $\varepsilon/M^{1/\beta}$ », since in the former case the data points are neither clustered in some small regions nor spread over an extremely large region. Hence, we decided to compare the calculated and experimental forms $h^{-1/\delta}$ of the scaling functions.

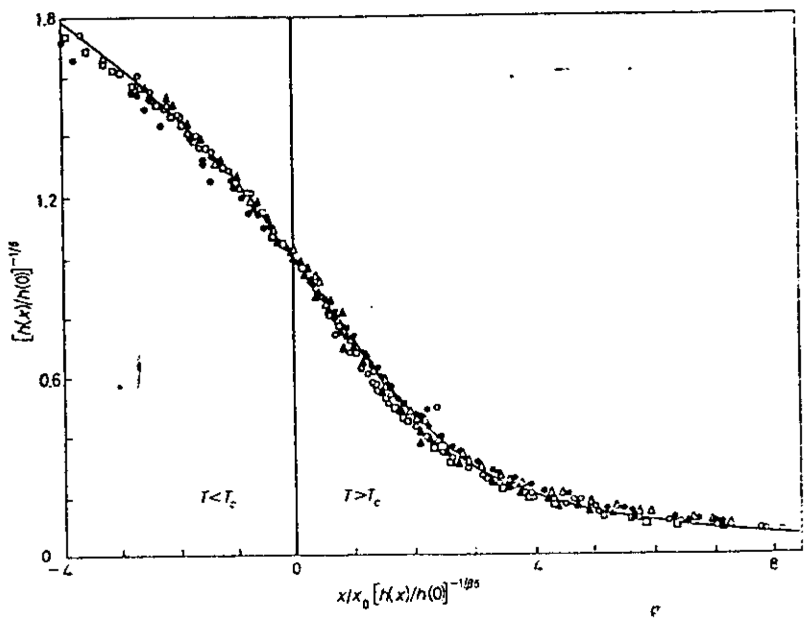

Fig. 3. – Comparison of the calculated scaling function for the $S = \frac{1}{2}$ Heisenberg model with the experimental data for various magnets. The Pd_3Fe data correspond to the disordered state of the alloy. △ $CrBr_3$, ○ EuO, ▲ Ni, ● Pd_3Fe, □ YIG.

In Fig. 3 we present the comparison of the $S = \frac{1}{2}$ Heisenberg-model scaling function with the experimental data for different magnets. The theoretical and experimental results are normalized by the corresponding critical amplitudes $C = h(0)$ and $B = x_0^{-\beta}$ (see eqs. (1), (2) and (4)), which are considered to be irrelevant in the critical region [22]. The agreement between the two kinds of results is satisfactory enough. It should be emphasized that the solid curve in Fig. 3 was obtained for a microscopic Hamiltonian via calculations which did not include adjustable parameters whatsoever. The whole Fig. 3 resembles the Guggenheim plot made for fluids [1,23], because of the perception that data for many different materials fall roughly on only one curve.

However, there is a profound difference between the two pictures. The Guggenheim plot compares *coexistence curves* of different fluids near the critical point (*i.e.* it is a comparison of data taken along a specific path in the critical region), whereas Fig. 3 presents experimental data for magnets taken virtually out of every accessible part of the critical region.

$CrBr_3$ is a rhombohedral two-sublattice ferromagnet with $S=\frac{3}{2}$ [8]. Ni is a metallic ferromagnet. EuO is a semiconducting magnet with $S=\frac{7}{2}$ and with a considerable next-nearest-neighbor interaction [9]. Pd_3Fe is a conducting ferromagnetic alloy [10] (in Fig. 3 we present comparison with the data for the disordered alloy only, cf. [17]). YIG is a cubic ferrimagnet insulator ($Y_3Fe_5O_{12}$) with $S=\frac{5}{2}$. The solid curve in Fig. 3 represents the calculated scaling for the nearest-neighbor isotropic Heisenberg model on the f.c.c. lattice with $S=\frac{1}{2}$ (the elementary magnetic moments are localized and the model imitates an insulating magnet). Therefore, the good agreement observed in Fig. 3 between the experimental data and the theoretical results supports the *universality hypothesis* [22], which asserts that many specific properties of materials are irrelevant in the critical region.

There are two immediate questions « What are the properties which dominate the critical behavior of a system? » and « May one consider Fig. 3 as a virtual proof of the universality hypothesis? » Two major characteristics of systems which make their critical phenomena distinguishable are the lattice dimensionality d (there is a perceivable difference between the two-dimensional and three-dimensional Ising scaling functions [14]) and the range of the interacting forces between spins (the apparent difference appears between systems with finite-range and those with infinite-range interactions). Secondly, we are tempted to accept Fig. 3 merely as a graphical exposition of the universality hypothesis, since the necessary proviso of its correctness—all considered materials and the theoretical model should have the same critical-point exponents—is not fulfilled (cf. Table II).

TABLE II. – *The critical-point exponents used in the displaying experimental and theoretical results in* Fig. 3.

	$CrBr_3$	EuO	Ni	Pd_3Fe (disordered)	YIG	$S=\frac{1}{2}$ Heisenberg model
β	0.368	0.368	0.378	0.364	0.370	0.385
δ	4.28	4.46	4.58	4.61	4.65	4.71

The universality hypothesis has been supported more specifically within the model calculations than by the comparison of the theoretical and experimental results. Calculations of the Ising-model scaling functions [14, 16] and the $S=\frac{1}{2}$ and $S=\infty$ Heisenberg-model scaling functions [16, 21, 24] for different cubic lattices support the conjecture that the normalized scaling function is independent of detailed lattice structure. Furthermore, comparison of the calculated scaling functions for the $S = \frac{1}{2}$ and $S = \infty$ Heisenberg model suggests that the scaling function may be spin-independent as well [17, 21]. However, it is evident that the scaling function does depend on the spin dimensionality n, as the change in $h(x)$ on going from the Ising model ($n = 1$) to the Heisenberg model ($n = 3$) is quite substantial [17].

7. – Corollary.

The static scaling hypothesis has been found to be correct within the few percents of experimental errors, as well as within the same percentage of the computational error during the calculatiens of the scaling functions. The calculated scaling functions for the Heisenberg models agree reasonably well with the experimental results [17, 21]. Besides, the agreeable Heisenberg-model scaling functions were also obtained by applying the Fisher droplet model [25, 26], and by utilizing a Monte Carlo procedure [24] and thus they support the correctness of the above-described method. As for the universality hypothesis, both experimental and numerical results are giving good hints of its plausibleness, but still it is about ten percent less plausible than the scaling hypothesis.

Recently WILSON [27, 28] has initiated a new attractive approach to the justification of scaling and universality. Applying the renormalization group

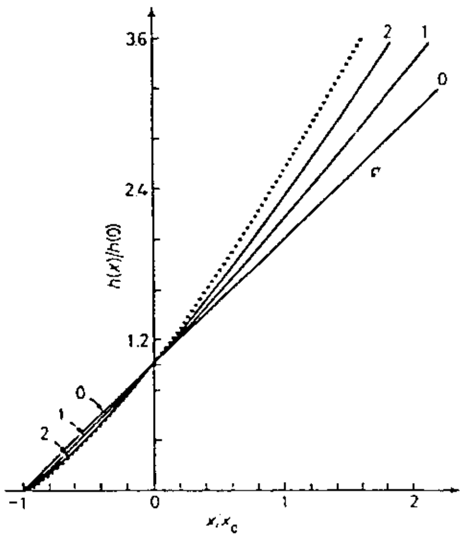

Fig. 4. – Comparison of the $S = \infty$ Heisenberg-model scaling function presented in Table I with the scaling function calculated using the ε-expression, ref. [33]. The numbered curves correspond to the zeroth, first and second order in the ε-expansion.

technique he was able to provide an explanation of the scaling hypothesis and to demonstrate the universality. Further success was observed when WILSON and FISHER have introduced the $\varepsilon = 4 - d$ parameter which measures « departure » from the four-dimensional space, where the classical critical-phenomena theories are supposed to be correct [29]. Assuming that ε is small enough, they derived expansions in ε of the critical-point exponents up to order ε^2, which even in the case $\varepsilon = 1$ exhibited a good agreement with the numerical results. Afterwards BREZIN, WALLACE and WILSON [30], and independently AVDEYEVA and MIGDAL [31], have used the ε-expansion and the Feynman-diagram technique to calculate, up to order ε^2, the Ising-model scaling function and the $S = \infty$ Heisenberg scaling function as well [32-33]. The agreement with the numerical results is better in the case of the Ising model than in the case of the $S = \infty$ Heisenberg model. However, in both cases the ε-expansion equations of state are not substitutive for the numerical equations, as far as the comparison with experimental results is concerned (see Fig. 4). Therefore, we may conclude, at this moment, that introduction of the renormalization group technique into the critical-phenomena studies opened the way for a deeper understanding and appreciating of scaling and universality, but still the relevance to the reality of the specific results is measured by comparison with the painfully obtainable numerical predictions.

* * *

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These results stress the need to account for dimensionality within the theoretical picture. This can be achieved e.g. by the Renormalization Group.

3. The Renormalization Group.

To introduce the ideas of the RG it is suitable to work with a scalar spin field $S(\vec{x})$. The H_{LGW} simplifies to the effective Hamiltonian for the Ising model:

$$H_{LGW} = \frac{J}{2} \cdot a^{2-d} \int d^d x \left(\vec{\nabla} m(\vec{x})\right)^2 + \frac{1}{a^d} \int d^d x \left[\frac{r}{2} \cdot m^2(\vec{x}) + \frac{u}{4} m^4(\vec{x})\right] \quad (1$$

The corresponding microscopic Hamiltonian is derived by setting $\vec{S}_i = (0, 0, S_{zi})$ in Eq. 1. The Ising model plays a special role in the history of phase transition, because it has been solved analytically in 2d by Onsager. Moreover, we know from the previous chapter that it is an appropriate model to describe uniaxial epitaxial films. As a matter of fact, many problems of physics but not only of physics can be mapped onto an Ising Hamiltonian, so that Eq. 1 is a very general model.
The partition function of this model which is called in the literature the S^4 model for reason we will soon understand writes

$$Z(r, u, J) = \prod_{\vec{x}} \int dm(\vec{x})\, e^{-\beta H_{eff}[m(\vec{x})]} \quad (2)$$

multiple integrals over the continuous real value $m(\vec{x})$ can be extended from $-\infty$ to $+\infty$.
Within the RG theory, one considers $m(\vec{x})$ as the superposition of two fields, $S(\vec{x})$ and $\sigma(\vec{x})$, where $S(\vec{x})$ and $\sigma(\vec{x})$ are a "slowly" varying field and a rapidly varying field. By "slowly" and "rapidly" varying we mean that their Fourier transforms contain components with "long" and "short" wavelenghts, respectively.

$$\sigma(\vec{x}) = \sum_{\frac{\pi}{La} < |k| < \frac{\pi}{a}} \sigma(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

$$S(\vec{x}) = \sum_{|k| < \frac{\pi}{La}} S(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} \quad (3)$$

The operation

$$Z[S] = \sum_{\{\sigma\}}{}' e^{\beta H_{eff}[S,\sigma]} \qquad (4)$$

is a convenient way of integrating out the short wavelength fluctuations and defines the effective Hamiltonian governing the behaviour of the field S by the equation

$$H_{eff}[S] = -\frac{1}{\beta} \ln Z[S] \qquad (5)$$

(4) is a "smoothing" operation supposed to leave behind a "smoother" field $S(\vec{x})$ which can be possibly handled by conventional methods like Landau theory or MFA.
It might turn out that $H_{eff}[S]$ has, under some approximations, the same form as $H_{eff}(m)$. In this case the model is said to be renormalizable. We will see that the S^4 model belongs to this class. The smoothing operation might not change the form of H_{eff}, but is most likely to renormalize the coupling constants. This eventually leads to corrections of the critical exponents away from their classical values. Exactly this process we want to illustrate in detail. It turns out that working out the mathematical details helps very much to understand the very spirit of the RG.

We start by writing, on very general grounds,

$$H_{eff}[S,\sigma] = H_S[S] + H_\sigma[\sigma] + H_{int}[S,\sigma] \qquad (6)$$

This leads to

$$Z[S] = \sum_{\{\sigma\}}{}' e^{-\beta\left[H_S[S] + H_\sigma[\sigma] + H_{int}[S,\sigma]\right]} =$$

$$= e^{-\beta H_s[s]} \cdot \sum_{\{\sigma\}} e^{-\beta H_\sigma[\sigma]} \cdot \sum_n \frac{(-\beta)^n}{n!} \cdot H_{int}^n =$$

$$= e^{-\beta H_s[s]} \cdot \sum_n \frac{(-\beta)^n}{n!} \cdot \sum_{\{\sigma\}} e^{-\beta H_\sigma[\sigma]} \cdot H_{int}^n \qquad (7)$$

We define the operation

$$\sum_{\{\sigma\}} e^{-\beta H_\sigma[\sigma]} \cdot H_{int}^n[\sigma, s] \equiv \langle H_{int}^n[\sigma, s] \rangle_\sigma \qquad (8)$$

to find

$$\underline{\underline{H_{eff}[s]}} = H_s[s] - \frac{1}{\beta} \ln \sum_n \frac{(-\beta)^n}{n!} \langle H_{int}^n[\sigma, s] \rangle_\sigma$$

$$= H_s[s] - \frac{1}{\beta} \ln \left[1 + \frac{(-\beta)}{1} \langle H_{int}[\sigma, s] \rangle_\sigma + \frac{\beta^2}{2} \langle H_{int}^2 \rangle_\sigma + \dots \right]$$

$$= H_s[s] - \frac{1}{\beta} \left[-\beta \langle H_{int} \rangle_\sigma + \frac{\beta^2}{2} \langle H_{int}^2 \rangle - \frac{\beta^2}{2} \langle H_{int} \rangle^2 + \dots \right]$$

$$\underline{\underline{= H_s[s] + \underbrace{\langle H_{int}[\sigma, s] \rangle_\sigma}_{(a)} - \beta/2 \cdot \underbrace{\left[\langle H_{int}[\sigma, s]^2 \rangle - \langle H_{int}[\sigma, s] \rangle_\sigma^2 \right.}_{(b)}}} + \dots \qquad (9)$$

Eq. 9 represents the perturbation theoretical approach to the smoothing operation: It produces a series which can be cut according to the desired accuracy. At 0-th order, $(n=0)$ $H_{eff}[s] = H_s[s]$. Studying $n=1$ and $n=2$ gives rise

to the important modifications of the classical exponents sought for.
We now have to define $H_s[s]$, $H_\sigma[\sigma]$ and $H_{int}[\sigma, s]$ for the S^4 model.

Inserting $m = s + \sigma$ in (1) we obtain

$$H_{LGW} = \frac{J}{2} a^{2-d} \int d^dx \left[(\vec{\nabla} s)^2 \right] + \frac{r}{2} \frac{1}{a^d} \int d^dx\, s^2 \longleftarrow H_s[s]$$

$$(10) \qquad + \frac{J}{2} a^{2-d} \int d^dx (\vec{\nabla}\sigma)^2 + \frac{r}{2} \frac{1}{a^d} \int d^dx\, \sigma^2 \longleftarrow H_\sigma[\sigma]$$

$$+ \frac{u}{4} \cdot \frac{1}{a^d} \int d^dx \left[s + \sigma \right]^4 \longleftarrow H_{int}[\sigma, s]$$

The odd terms with $\vec{\nabla} s \cdot \vec{\nabla}\sigma$ and $s \cdot \sigma$ need not to be considered: after Fourier transforming them they vanish as s and σ are defined over two different ranges of $\vec{k}$-vectors.
The $H_s[s]$ and $H_\sigma[\sigma]$ represent the Hamiltonian for the Gaussian model and do not contain any coupling. The term with the fourth power instead couples s and σ and is referred to as the <u>interaction Hamiltonian</u>.

We now turn to the calculation of $H_{eff}[s]$ within the perturbational approach (3). For this purpose, it is useful to go to the Fourier space for calculating $\langle \ \rangle_\sigma$, so that we will soon encounter <u>Feynman graphs</u>. Let us first rewrite the model in $\vec{k}$ space:

Appendix . The $\vec{k}$ - representation of the S^4 - model

With $S(\vec{x}) = \frac{1}{\sqrt{N}} \sum_{\vec{k}} S_{\vec{k}} e^{i\vec{k}\vec{x}}$

we obtain

$$H_s[S] = \frac{J a^2}{2N} \int dx \sum S_k S_{k'} ik ik' e^{i(k+k')x}$$

$$+ \frac{1}{N}\frac{r}{2} \int dx \sum S_k S_{k'} e^{i(k+k')x}$$

$$= \frac{J}{2} a^2 \sum S_k S_{k'} ik ik' \int e^{i(k+k')x} \cdot \frac{1}{N}$$

$$\frac{r}{2} \sum S_k S_{k'} \underbrace{\int e^{i(k+k')x}}_{N \cdot \delta_{k,-k'}} \cdot \frac{1}{N}$$

$$= \sum \left[\frac{J}{2} a^2 k^2 + \frac{r}{2}\right] S_k S_{-k} \qquad (11)$$

Similarly

$$H_a[a] = \sum_q \left[\frac{J}{2} a^2 k^2 + \frac{r}{2}\right] a_q a_{-q} \qquad (12)$$

$$H_{int}[a,S] = \frac{u}{4} \int (S+a)^4 = \frac{u}{4} \int S^4 + \frac{u}{4} \cdot 6 \cdot \int S^2 a^2 + \frac{u}{4} \int a^4$$

$$+ \text{ Odd terms in } a.$$

$$= \frac{u}{4} \cdot \frac{1}{N} \sum_{k_1}^{k_4} S_{k_1} \cdots S_{k_4} \delta_{k_1 + \cdots k_4, 0}$$

$$+ \frac{u}{4} \cdot \frac{6}{N} \cdot \sum_{k_1}^{q_2} S_{k_1} S_{k_2} a_{q_1} a_{q_2} \cdot \delta_{k_1 + \cdots + q_2, 0}$$

$$+ \frac{u}{4} \cdot \frac{1}{N} \cdot \sum_{q_1}^{q_4} a_{q_1} \cdots a_{q_4} \delta_{q_1 + \cdots q_4, 0} \qquad (13)$$

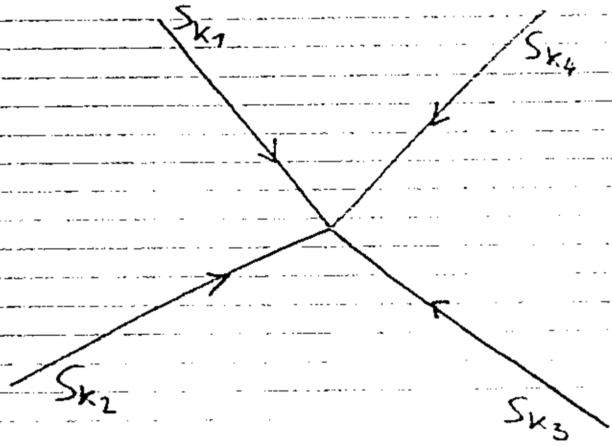
(a) The smoothing operation on H_{int} gives three terms:

1. $\frac{u}{4}\cdot\sum\langle S_{k_1}\cdots S_{k_4}\rangle\,\delta_{k_1+\dots k_4,0}$

2. $\frac{u}{4}\cdot 6\cdot\sum\langle S_{k_1}\cdots\sigma_{q_2}\rangle\,\delta_{k_1+\dots q_2,0}$

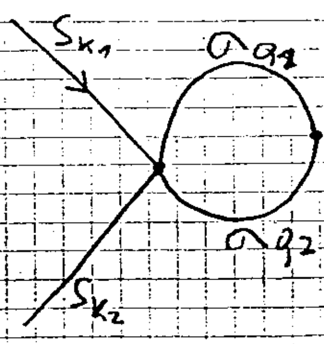
3. $\frac{u}{4}\cdot\sum\langle\sigma_{q_1}\cdots\sigma_{q_4}\rangle\,\delta_{q_1+\dots q_4,0}$ (14)

We can illustrate 1. 2. and 3. by simple graphs.
1. will correspond to four lines meeting in a point. The lines carry the momenta $k_1,\dots,k_4$, the sum of which is zero. Each line represents a field S_{k_i}. This graph is called a vertex. Its magnitude is given by the coupling constant $\frac{u}{4}\cdot\frac{1}{N}$



This term adds a fourth order term to $H_S[S]$
$\frac{u}{4}\cdot\frac{1}{N}\cdot\sum S_{k_1}\cdots S_{k_4}\cdot\delta_{k_1\dots k_4}$

2. gives rise to the following graph, whereby two fields σ_{q_1} and σ_{q_2} must be contracted because of the $\langle\ \rangle_\sigma$ operation:



This vertex contains only two external lines: it contributes an extra term to the S^2 coupling in $H_S[S]$.
The strength of this coupling is
$\frac{u}{4}\cdot 6\cdot\frac{1}{N}\cdot\langle\sigma_{q_1}\sigma_{q_2}\rangle_\sigma$.

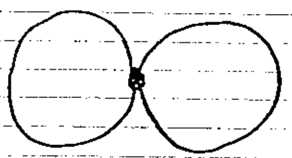
As a consequence of this graph, the coupling constant $r/2$ is modified by the H_{int} term — a very important result which is not contained in Landau theory. This establishes a very important fact: averaging over short wave length fluctuations modifies the coupling constant of your Hamiltonian.

3. Wick theorem can be invoqued to simplify $\langle \sigma_{q_1} \cdots \sigma_{q_4} \rangle$. Accordingly

$$
\begin{aligned}
\langle \sigma_{q_1} \cdots \sigma_{q_4} \rangle = \; & \sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{q_4} \\
+ \; & \sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{q_4} \\
+ \; & \sigma_{q_1}\sigma_{q_2}\sigma_{q_3}\sigma_{q_4}
\end{aligned}
\qquad (15)
$$

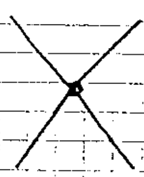
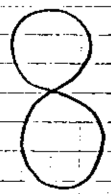
where ⌐¬ means contraction.

3. is therefore represented by 3 graphs of the form

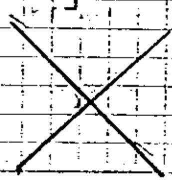


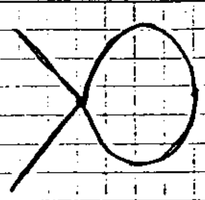
This topological factor accounts for all possible ways one can realize 2 contraction with 4 different fields. Notice that this graph do not involve external lines, so that it does not affect $H_s[s]$. It contributes only to the regular part of the free energy, which is not relevant for the phase transition.

Summing up, the first order correction to $H_s[s]$ can be represented diagrammatically as follows:

$\langle H_{int} \rangle_\sigma =$  $+ \; 6 \cdot 1$  $+ \; 1 \cdot 3$  (16)

whereby

 adds a S^4 term to $H_s[S]$ and



adds an S^2 contribution to $H_s[S]$. It turns out that for producing the required modification of classical exponents, it is necessary to include 2 order diagrams.

(b) The number of terms in (b) is conspicuous, and the bookkeeping correspondingly complicated. The difficulty comes from the fact that H^2_{int} is already some complicated polynomial. Consider for instance in

$$\frac{1}{2}\beta\cdot\left(\frac{u}{4}\right)^2 \int (S+a)^4 \int (S'+a')^4 \qquad (17)$$

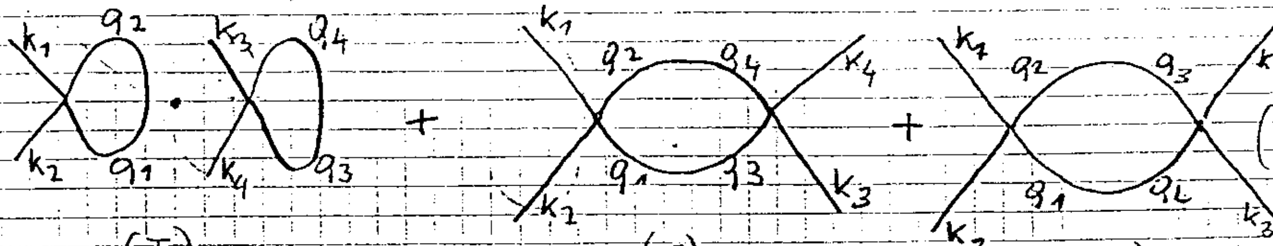
the term

$$\frac{\beta}{2}\cdot\left(\frac{u}{4}\right)^2\cdot 6^2\cdot \int S^2 a^2 \int S'^2 a'^2 \qquad (18)$$

The Fourier transform of this term amounts to

$$\frac{\beta}{2}\cdot\left(\frac{u}{4}\right)^2\cdot 36\cdot\sum_{k,q} S_{k_1}S_{k_2}a_{q_1}a_{q_2}a_{q_3}a_{q_4}S_{k_3}S_{k_4}\cdot \delta_{k_1+k_2-q_1+q_2,0}\cdot\delta_{k_3-k_4-q_3-q_4,0} \qquad (19)$$

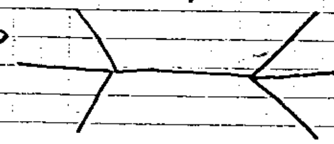
Because of Wick's theorem, this term is the sum of following diagrams:

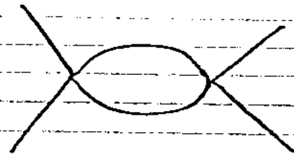
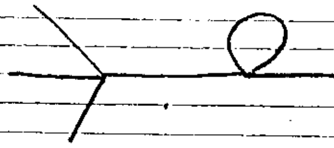


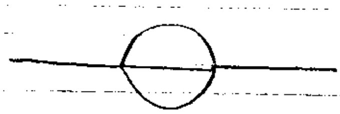
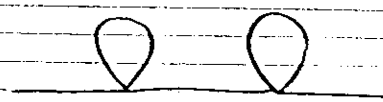
(I) (II) (III) (20)

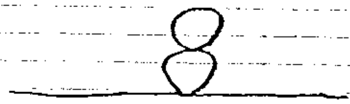
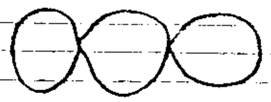
A graph of type (I) is said to be disconnected. Wilson has shown that all disconnected diagrams in $\langle H^2_{int}\rangle$ exactly cancel the diagrams of $\langle H_{int}\rangle^2$, which are all disconnected. We therefore have only to consider connected diagrams in order to find the 2-order corrections to $H_S[s]$. As shown by II and III there are exactly two ways of contracting a's, so that a diagram of the type  has a weight of 2.

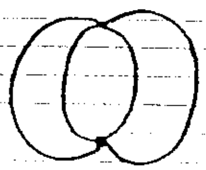
The contribution of (b) can be summarized as follows:

$$\frac{\beta}{2}\left[\langle H_{int}^2\rangle - \langle H_{int}\rangle^2\right] = 8$$ 

$+\ 36$  $+\ 48$ 

$+\ 48$  $+\ 72$ 

$+\ 72$  $+\ 36$ 

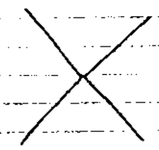
$+\ 12$  (21)

The topological factors accompanying each graph arises from the polynomial expression of $(s(x)+a(x))^4 \cdot (s(x')-a(x'))^4$ and from the possible ways one can contract the a-fields by keeping the graph connected. The working out of these topological factors becomes increasingly complicated at higher orders, becoming an intriguing combinatorial problem.

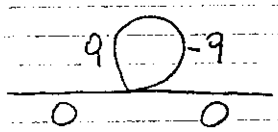
We now turn to the evaluation of the single graphs. We follow a set of simplified rules suggested by Polyakov. These rules slightly simplify the problem without affecting the physics too much.

1. The external momenta are small by principle: set them to 0.
2. Set the internal momenta so that the sum of the momenta at each vertex is 0 (momentum conservation, required by the δ-functions associated with each vertex)
3. Each internal line gives the propagator $\langle \sigma_q \sigma_{-q} \rangle_0 = \frac{1}{Ja^2q^2+r}$
4. Integrate over the internal momenta.

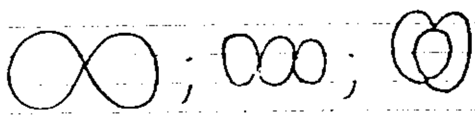
Let us now consider all graphs worked out so far:



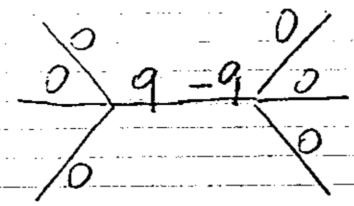
S^4 coupling



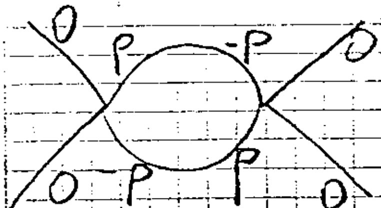
S^2 coupling



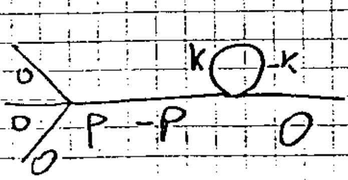
no external lines



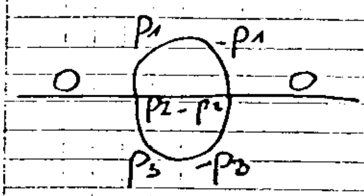
S^6 term, forbidden because of momentum conservation



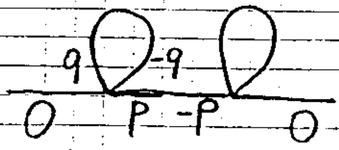
S^4 coupling



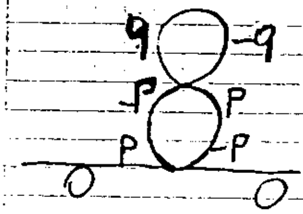
forbidden because of momentum conservation



forbidden because of momentum conservation



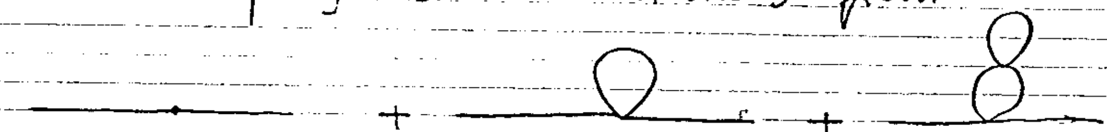
forbidden



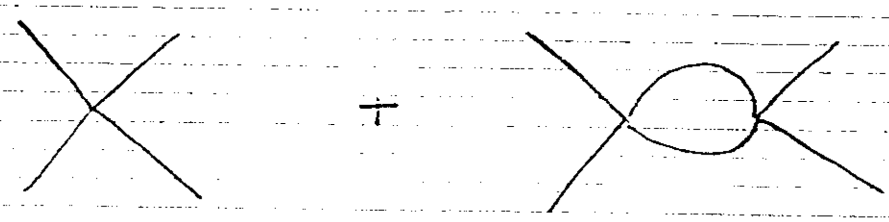
S^2 - coupling.

Summarizing:

The S^2 - coupling receives contributions from



The S^4 - coupling receives contributions from (22)



The $\vec{\nabla}$ - coupling is left unchanged by Hint, approximately.

Appendix: calculation of $\langle a_{q_1} a_{q_2} \rangle_a$

We have now to perform explicitly the calculation of the propagator, which formally is the functional integration

$$\langle a_{q_1} a_{q_2} \rangle = \int (Da) \cdot e^{-H_a[a]} \cdot a_{q_1} a_{q_2} . \qquad (23)$$

The functional integration can be written in realspace as a multiple integral

$$\int (Da) = \prod_{\vec{x}} \int_{-\infty}^{\infty} da(\vec{x}) \qquad (24)$$

where the product is taken over all the lattice sites $\vec{x}$. Another way to define the functional integral is to integrate each Fourier component $a(k)$ separately and independently:

$$\int (Da) = \prod_{\vec{q}} \int_{-\infty}^{\infty} da_{\vec{q}} \qquad (25)$$

(25) is particularly useful when $H_a[a]$ is expressed in Fourier space. Accordingly, (23) becomes

$$\langle a_{q_1} a_{q_2} \rangle = \frac{\prod_K \int da_K \, e^{-\sum_{q} \underbrace{\left(\frac{J}{2} a^2 q^2 + \frac{r}{2}\right)}_{\alpha_q} \overbrace{a_q a_{-q}}^{a_q^2} \cdot \beta} \, a_{q_1} a_{q_2}}{\prod_K \int da_K \, e^{-\sum_q \left[J/2\, a^2 q^2 + r/2\right] a_q a_{-q} \cdot \beta}} =$$

$$= \begin{cases} 0, & \text{if } q_1 \neq -q_2 \\ \dfrac{\int da_1 \cdot e^{-\alpha_1 \cdot a_1^2 \cdot \beta} \cdot \ldots \cdot \int da_{q_1} \cdot e^{-\alpha_{q_1} \cdot a_{q_1}^2 \cdot \beta} \cdot a_{q_1}^2 \cdot \ldots}{\int \quad '' \quad \cdot \int da_{q_1} e^{-\alpha_{q_1} a_{q_1}^2 \cdot \beta} \cdot 1 \cdot \ldots} , & q_2 = -q_1 \end{cases}$$

The integral remaining are Gaussian integrals and are all tabulated:

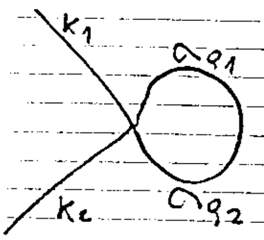
$$\int da_{q_1}\, e^{-\alpha_{q_1} a_{q_1}^2}\, a_{q_1}^2 = \frac{\sqrt{\pi}}{4\left[\alpha_{q_1}\right]^{3/2}\cdot\beta^{3/2}}$$

$$\int da_{q_1}\, e^{-\alpha_{q_1} a_{q_1}^2} = \frac{\sqrt{\pi}}{2\left[\alpha_{q_1}\right]^{1/2}\cdot\beta^{1/2}} \qquad (26)$$

This leads to

$$\boxed{\langle a_{q_1} a_{q_2}\rangle = \delta_{q_1+q_2,0}\cdot\frac{1}{Ja^2q_1^2+r}\,\frac{1}{\beta}} \qquad (27)$$

Let us now apply (27) to calculate e.g. 2. in Eq. (14)



$$= \frac{u}{4}\cdot 6\cdot\frac{1}{N}\cdot\sum_{k_1k_2q_1q_2} S_{k_1}S_{k_2}\langle a_{q_1}a_{q_2}\rangle\,\delta_{k_1+k_2+q_1+q_2,0}$$

$$= \frac{u}{4}\cdot 6\cdot\frac{1}{N}\cdot\sum_{\substack{k_1k_2\\ q_1q_2}} S_{k_1}S_{k_2}\,\delta_{k_1+k_2+q_1+q_2,0}\cdot\delta_{q_1+q_2,0}\cdot\frac{1}{Ja^2q_1^2+r}$$

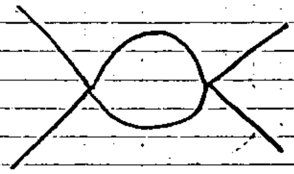
$$= \frac{u}{4}\cdot 6\cdot\frac{1}{N}\cdot\sum_{k,q} S_k S_{-k}\cdot\frac{1}{Ja^2q^2+r} \qquad \sum_q \rightarrow \frac{N\cdot a^d}{(2\pi)^d}\int dq$$

$$= \frac{u}{4}\cdot 6\cdot\frac{a^d}{(2\pi)^d}\cdot\int d^dq\cdot\frac{1}{Ja^2q^2+r}\cdot\sum_k S_k S_{-k}$$

$$= \frac{1}{2}\left[12\cdot\frac{u}{4}\cdot\frac{a^d}{(2\pi)^d}\cdot\int d^dq\,\frac{1}{Ja^2q^2+r}\right]\sum_k S_k S_{-k}$$

The first order result leads to

$$\boxed{r^{(1)} = r^{(0)} + \frac{12\cdot u}{4\,\beta}\frac{a^d}{(2\pi)^d}\int d^dq\cdot\frac{1}{[Ja^2q^2+r]}} \qquad (28)$$



$$= \beta\cdot 36\cdot\left(\frac{u}{4}\right)^2\frac{1}{N^2}\sum_{k,q} S_{k_1}\cdot S_{k_2}\cdot S_{k_3}\cdot S_{k_4}$$

$$\cdot\langle a_{q_1}a_{q_3}\rangle\cdot\langle a_{q_2}a_{q_4}\rangle\cdot\delta_{k_1+k_2+q_1+q_2,0}\cdot\delta_{k_3+k_4+q_3+q_4,0} =$$

$$\beta\cdot 36\cdot\left(\frac{u}{4}\right)^2\cdot\frac{1}{N^2}\cdot\sum_{k,q} S_{k_1}\cdots S_{k_4}\frac{1}{Ja^2q_1^2+r}\cdot\frac{1}{Ja^2q_2^2+r}\cdot\delta_{k_1+k_2+q_1+q_2,0}\cdot$$

$$\cdot\delta_{k_3+k_4+q_3+q_4,0}\cdot\delta_{q_1+q_3,0}\cdot\delta_{q_2+q_4,0} =$$

$$= \beta\cdot 36\cdot\left(\frac{u}{4}\right)^2\cdot\frac{1}{N^2}\cdot\sum_{k,q} S_{k_1}\cdots S_{k_4}\cdot\frac{1}{Ja^2q_1^2+r}\cdot\frac{1}{Ja^2q_2^2+r}\cdot\delta_{k_1+k_2+q_1+q_2,0}$$

$$\cdot\delta_{k_3+k_4-q_1-q_2,0} =$$

$$= \beta\cdot 36\cdot\left(\frac{u}{4}\right)^2\cdot\frac{1}{N^2}\cdot\sum_{k,q}\cdot S_{k_1}\cdots S_{k_4}\cdot\frac{1}{Ja^2q^2+r}\cdot\frac{1}{Ja^2[k_3+k_4-q]^2+r}\cdot$$

$$\cdot\delta(k_1+\dots+k_4) =$$

$$\beta\cdot 36\cdot\left(\frac{u}{4}\right)^2\cdot\frac{1}{N}\cdot\sum_{k}\frac{a^d}{(2\pi)^d}\int d^dq\cdot\frac{1}{(Ja^2q^2+r)(Ja^2[k_3+k_4-q]^2+r)}S_{k_1}S_{k_2}\dots S_{k_4}$$

$$\delta(k_1+\dots k_4).$$

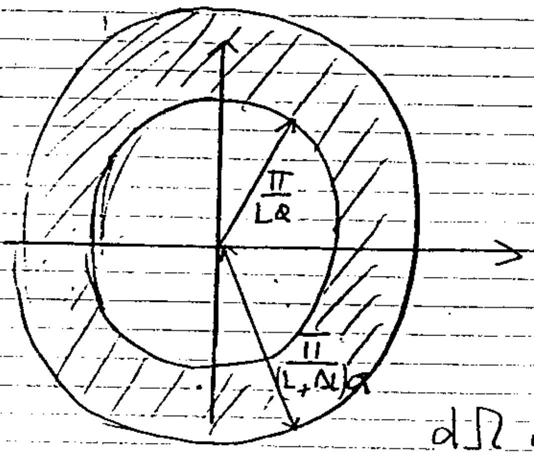
Polyakov

$$\stackrel{!}{=}\frac{1}{4}\left[\frac{\beta\cdot 36\cdot u^2}{4}\frac{a^d}{(2\pi)^d}\int d^dq\frac{1}{(Ja^2q^2+r)^2}\right]\frac{1}{N}\sum_k S_k\dots\delta_k\dots$$

$$\boxed{u^{(2)} = u^{(1)} - \frac{\beta}{4}\cdot\frac{36\,u^2a^d}{\beta^2(2\pi)^d}\int d^dq\frac{1}{(Ja^2q^2+r)^2}} \qquad (29)$$

Appendix Calculation of

$\frac{a^d}{(2\pi)^d}\int d^dq \frac{1}{Ja^2q^2+r}$. Here it is where the smoothing comes to work. In general, we consider performing the smoothing over a slice of the Brillouin zone.



Because the integrand depends only on q, we turn to polar coordinates. The integral then becomes

$$\frac{a^d}{(2\pi)^d}\int d\Omega \int q^{d-1}dq \cdot \frac{1}{Ja^2q^2+r}$$

$d\Omega$ is the differential solid angle in d dimensions: $\frac{1}{(2\pi)^d}\int d\Omega \equiv C_d$. The remaining integral gives

$$\int q^{d-1}dq \cdot \frac{1}{Ja^2q^2+r} \cong \int q^{d-1}dq \cdot \frac{1}{Ja^2q^2}\left[1-\frac{r}{Ja^2q^2}\right] =$$

$$\int q^{d-1}dq \cdot \frac{1}{Ja^2q^2} - \int q^{d-1}dq \frac{r}{Ja^2q^2} =$$

$$= \frac{1}{Ja^2}\int dq\, q^{d-3} - \frac{r}{J^2a^4}\cdot\int dq\, q^{d-5}$$

Finally:

$$\frac{a^d}{(2\pi)^d}\int d^dq \frac{1}{Ja^2q^2+r} \cong \frac{C_d}{Ja^2}\int dq\, q^{d-3} - \frac{C_d \cdot r}{J^2a^4}\int dq\, q^{d-5} \quad (30)$$

An immediate results of (30) is that by this integral r becomes explicitely dependent on d, in contrast to Landau theory.

To the same degree of accuracy we find

$$\frac{(a)^d}{(2\pi)^d}\int d^dq \frac{1}{(Ja^2q^2+r)^2} \cong \frac{C_d}{J^2a^4}\int dq\, q^{d-5} \quad (31)$$

To lowest order we obtain

$$r_{L+\Delta L} = r_L + \frac{12}{4}\frac{u}{\beta}\frac{C_d}{J a^2}\int_{\frac{\pi}{(L+\Delta L)a}}^{\frac{\pi}{La}} dq\, q^{d-3} \qquad (22)$$

$$u_{L+\Delta L} = u_L$$

$$\int_{\frac{\pi}{(L+\Delta L)a}}^{\pi/La} dq\, q^{d-3} = a^d \frac{1}{d-2}\, q^{d-2}\Big|_{\frac{\pi}{(L+\Delta L)a}}^{\pi/La} = \frac{a^d}{d-2}\left[\left(\frac{\pi}{La}\right)^{d-2} - \left(\frac{\pi}{(L+\Delta L)a}\right)^{d-2}\right]$$

According to the lore of RG, one consider an infinitesimally thin slice of the BZ, i.e. ΔL very small. This step improves the otherwise linearized RG transformations. Then

$$\frac{(a)^d}{d-2}\cdot\left(\frac{\pi}{La}\right)^{d-2}\left[1-\left[1-\frac{\Delta L}{L}\right]^{d-2}\right] = \frac{a^2}{d-2}\left(\frac{\pi}{L}\right)^{d-2}\cdot(d-2)\cdot\frac{\Delta L}{L}$$

$\Rightarrow$

$$r_{L+\Delta L} = r_L + \frac{12}{4}\frac{u}{\beta}\frac{C_d}{J}\cdot\pi^{d-2}\cdot L^{1-d}\cdot\Delta L \qquad (23)$$

$$u_{L+\Delta L} = u_L$$

From this transformation rule over a very thin slice we can construct a differential equation of $r(L)$, which is called the Gell-Mann-Low RG equation:

$$r_{L+\Delta L} - r_L = \frac{12}{4}\frac{u}{\beta}\frac{C_b\pi^{d-2}}{J}\cdot L^{1-d}\, dL \qquad (24)$$

$$u_{L+\Delta L} - u_L = 0$$

$$\frac{dr_L}{dL} = \frac{12}{4}\frac{u}{\beta^3} C_d \pi^{d-2} \cdot L^{1-d}$$

$$\frac{du_L}{dL} = 0 \qquad (25)$$

The brilliant idea of Wilson the existence of a characteristic length at which renormalization stops – up to this length one ought to integrate (25). This length is ξ – the correlation length.
Integrating (25) we obtain

$u(\xi) = u$ $\qquad \boxed{d>2!}$

$$r^{(1)}(\xi) = r_1 + \frac{12}{4}\frac{u}{\beta^3} C_d \pi^{d-2} \int_1^{\xi} L^{1-d}\, dL =$$

$$= r + \frac{12}{4}\frac{u}{\beta^3} c_d \pi^{d-2} \left[\frac{\xi^{2-d}}{2-d} - \frac{1}{2-d}\right] \qquad (26)$$

The physical significance of this additive correction to r is the following: as $\xi^{2-d} \ll 1$

$$r^{(1)}(\xi) \cong r + \frac{12}{4}\frac{u}{\beta^3} c_d \pi^{d-2} \frac{1}{d-2} \qquad (27)$$

Thus, the main effect of this correction is that $r^{(1)}$ becomes vanishing at some temperature $T^* \neq T_c$. Therefore, the main effect of the RG at this lowest order is a shift in the transition temperature. We can incorporate this correction into Landau theory by redefining r: the critical exponents are not affected.
Notice that for $d \leq 2$ the additive correction is diverging: this signals that $d = 2$ must be some special value at which the picture of phase transitions, as we know it for Landau theory, breaks down! We will see later how to deal with $d \leq 2$.
Having shifted T_c, we proceed with higher order corrections.
We have to find the Gell-Mann Low equations to the transformation rules

$$r_{L+\Delta L} = r_L - \frac{12}{4} u_L \cdot r_L \cdot \frac{C_d}{\beta J^2 2^4} \int dq\, q^{d-5}$$

$$u_{L+\Delta L} = u_L - \frac{36}{4} u_L^2 \cdot \frac{C_d}{\beta J^2 2^4} \int dq\, q^{d-5} \qquad (28)$$

We arrive at the coupled Diff. Eq.

$$\frac{dr}{dL} = -\frac{12}{4} u \cdot r \frac{C_d}{\beta J^2} \cdot L^{3-d}$$

$$\frac{du}{dL} = -\frac{36}{4} u^2 \cdot \frac{C_d}{\beta J^2} \circ L^{3-d} \qquad (29)$$

Solution: $(d \neq 4)$

$$u(L) = u_0 \frac{1}{1 + u_0 \frac{36}{4} C_d \cdot \frac{1}{J^2\beta} \frac{L^{4-d}-1}{4-d}} \Longrightarrow \sim u_0 \cdot L^{d-4} \qquad (30)$$

$$\frac{dr}{r} = -\frac{12}{4} \cdot \frac{C_d}{\beta J^2} \cdot u_0 \cdot \frac{1}{1 + u_0 \frac{36\, C_d}{4 J^2 \beta} \cdot \frac{L^{4-d}-1}{4-d}} \cdot L^{3-d}\, dL$$

$$\ln \frac{r}{r_0} = -\frac{1}{3} \ln\left[1 + \frac{u_0\, 36\, C_d}{4 J^2 \beta} \cdot \frac{L^{4-d}-1}{4-d}\right]$$

$$r \simeq r_0 \cdot L^{\frac{d-4}{3}} \qquad (31)$$

We are now able to write Heff [5] inserting the renormalized coupling constants according to (30) and (31).

$$H_{eff}[S] = \frac{J \cdot a^2}{2} \int dx \left(\vec{\nabla} S\right)^2 + \int dx \left[\frac{r_0}{2} \cdot \xi^{\frac{d-4}{3}} \cdot S^2 + \frac{u_0}{4} \cdot \xi^{\frac{d-4}{3}} \cdot S^4 \right] \quad (32)$$

One can try to find the relevant quantities by applying Landau theory to $H_{eff}[S]$. This gives e.g.

$$\xi \sim \left(\frac{J}{r}\right)^{1/2} = \left(\frac{J}{r_0 \cdot \xi^{\frac{d-4}{3}}}\right)^{1/2} \sim \xi^{\frac{4-d}{3}} \left(T - T_c\right)^{-1/2} \quad (33)$$

which is an implicit equation for ξ! The solution of (33) is

$$\xi \sim \left(\frac{J}{T - T_c}\right)^{\frac{1}{2 - \frac{4-d}{3}}} \quad (34)$$

The critical exponent ν has no longer the classical value ν_L:

$$\nu = \frac{1}{2 - \frac{4-d}{3}} \qquad \nu = 1/2 \quad (35)$$

which illustrate the power of RG.

If we insert (34) in (32) we understand the major achievement of RG: in contrast to Landau suggestion, the coupling constant in $H_{eff}[S]$ are not analytic functions of T. This is the true revolutionary content of RG!

For the critical exponent β we obtain $\beta = \frac{1}{2} + \frac{d-4}{[illegible]}$, $\beta_L = 1/2$.

	Landau	$d=4$	$d=3$	$d=2$	exp (3)	exp (2)
β	$1/2$	$1/2$	0.36	0.25	0.33	0.125
ν	$1/2$	$1/2$	0.6	0.75	0.6-0.7	1

We discuss finally the case $d>4$ and $d=4$.

i) $d>4$

$$u_L \xrightarrow{\sim} u_0$$

$$\ln\frac{r}{r_0} = -\frac{1}{3}\ln\left(1+\frac{u_0\, 36 C_d}{4J^2\beta\,(d-4)}\right) \Rightarrow r\sim r_0$$

Just as u so is r independent of ξ. As a consequence, for $d>4$ the exponents remain classical.

ii) $d=4$ is a borderline, as the evaluation of the integrals show.

$$\underline{u_L \sim} \; u_0 \cdot \frac{1}{1+u_0\cdot\frac{36 C_d}{4\beta J^2}\ln L} \underline{\sim (\ln L)^{-1}}$$

$$\frac{dr}{dL} = -\frac{12}{4}\frac{C_d}{\beta J^2}\, u_0 \,\frac{1}{1+u_0\frac{36 C_d}{4\beta J^2}\ln L}\, L^{-1} dL$$

$$\frac{r}{r_0} = -\frac{1}{3}\ln\left[1+\frac{u_0\, 36 C_d}{4J^2\beta}\cdot \ln L\right]$$

$$\underline{\underline{r \cong r_0\cdot(\ln L)^{-1/3}}}$$

In $d=4$ u and r are renormalized by logarithmic corrections as multiplicative factors. However, log. corrections do not affect critical exponents.

This chapter is essentially a personal view of the ideas exposed by K. Wilson in his Nobel lecture, see Physica $\underline{73}$, 119 (1974).

4. The Mermin-Wagner theorem

We have found in chapter 3 that $d=2$ is some kind of limiting dimension, at which the RG group correction of Landau theory fails. Yet experimental results on two dimensional systems suggest that they behave "properly". The physics of truly 2d system is truly difficult indeed, just because of these facts.

One of the key results of 2d physics is a theorem proved by D. Mermin and H. Wagner (PRL 17, 1133 (1966)). Instead of following the original work, we will present the results of the MW-theorem in an heuristic approach.

We start from the Landau functional of the Heisenberg model

$$\mathcal{L}(\vec{\phi}) = l(\vec{\phi}^2) - \vec{h}\cdot\vec{\phi} \qquad (1)$$

It is not difficult to guess the topology of the functional $l(\vec{\phi}^2)$, see Fig. 4 of Chapter 2:

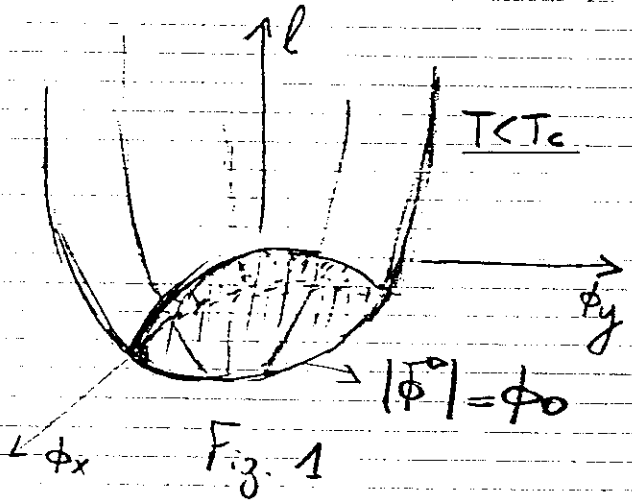


Fig. 1

"Mexican hat"

Accordingly, the Euler-Lagrange equation lead to the equation of state

$$\vec{h} = 2\vec{\phi}\cdot l'(\vec{\phi}^2) \qquad (2)$$

with solution

$$\vec{\phi} = \vec{h}\cdot\phi_h$$

For $\vec{h}\to 0$ the spontaneous magnetization is given by the solution of the equation

$$l'(\vec{\phi}^2) = 0$$

which gives all possible solutions $\vec{\phi}$ with $|\vec{\phi}| = \phi_0$.
These solutions correspond to a non-countable infinity of

solutions represented by all the points on the surface of a sphere. This is a direct consequence of the symmetry group of the Heisenberg Hamiltonian being SO_3, the continuous group of rotations in space. An important consequence of the mexican hat construction is that going from one minimum of $\mathcal{L}$ to another minimum does not require going over any energy barrier, as it is the case for the scalar field corresponding to the Ising model.

One question is natural to ask is whether the Landau functional is stable against excitations. In fact, as a consequence of degeneracy, there emerges a type of excited states in which the local ground state changes very gradually over space, so as to form a wave of very long wavelength. Such an excited state is locally one of the possible minima and one can imagine that its energy can be infinitely close to the actual minimum of $\mathcal{L}$, provided its wavelength is very long. This type of excitations allowed by the continuous symmetry of the problem are called "Goldstone" excitations. Once the system has been kicked into one of the excited states, to which minimum will it decay afterwards? to the original minimum it was kicked into? or to a completely different minimum? There is nothing to prevent a minimum decaying into a totally different minimum. This calls for an accurate analysis of the stability of the $\mathcal{L}$ functional against Goldstone excitations.

Let us introduce an excitation by adding $\vec{\delta}$ to $\vec{\Phi}_h$, with $\vec{\delta}\cdot\vec{\Phi}_h = 0$. In this way $|\vec{\Phi}_h + \vec{\delta}| = |\vec{\Phi}_h|$ to first order in $\vec{\delta}$. The change in Landau free energy introduced by such fluctuations is

$$\delta\mathcal{L} = \mathcal{L}(\vec{\Phi}+\vec{\delta}) - \mathcal{L}(\vec{\Phi}) = \frac{J}{2}(\vec{\nabla}\vec{\delta})^2 + \ell\left((\vec{\Phi}+\vec{\delta})^2\right)^2 -$$

$$- \vec{h}(\vec{\Phi}+\vec{\delta}) - \ell(\vec{\Phi}^{2})^2 + \vec{h}\cdot\vec{\Phi} -$$

$$\frac{J}{2}(\vec{\nabla}\vec{\delta})^2 + f'(\vec{\phi}_n^{2})\cdot\vec{\delta}^2 = \frac{J}{2}(\vec{\nabla}\vec{\delta})^2 + \frac{h}{2\phi_n}\cdot\vec{\delta}^2$$

or
$$\delta\mathcal{F} = \frac{J}{2}\,a^{2-d}\int d^d x\,(\vec{\nabla}\vec{\delta})^2 + \frac{1}{2}\frac{h}{\phi_n}\,a^{-d}\int d^d x\,\vec{\delta}^2 \qquad (3)$$

The quantity we are aiming to calculate is $\langle\vec{\delta}^2\rangle$, which is the mean square deviation from the thermal equilibrium average $\vec{\phi}_n$. The probability for such a fluctuation to occur is

$$W(\vec{\delta}) = \frac{1}{Z}\cdot e^{-\beta\,\delta\mathcal{F}(\vec{\delta})}$$

so that

$$\langle\vec{\delta}^2\rangle = \frac{\int D\vec{\delta}\cdot\vec{\delta}^2\cdot e^{-\delta\mathcal{F}\beta}}{\int D\vec{\delta}\cdot e^{-\delta\mathcal{F}\beta}} \qquad (4)$$

With

$$\vec{\delta} = \frac{1}{\sqrt{N}}\sum_k e^{i\vec{k}\vec{x}}\cdot\vec{\delta}_{\vec{k}},$$

$$\delta\mathcal{F} = \frac{1}{2}\sum_{\vec{k}}\left[\frac{J}{2}a^2\vec{k}^2 + \frac{h}{\phi}\right]\vec{\delta}_k^2$$

we obtain

$$\langle\vec{\delta}_{\vec{q}}^{\,2}\rangle = \frac{\prod_k\int d\vec{\delta}_{\vec{k}}\,\vec{\delta}_q^{\,2}\cdot e^{-\delta\mathcal{F}(\vec{\delta}_{\vec{k}'})\cdot\beta}}{\prod_k\int d\vec{\delta}_{\vec{k}}\cdot e^{-\delta\mathcal{F}(\vec{\delta}_{\vec{k}'})\cdot\beta}}$$

$$\underset{k=k'=q}{=}\ \frac{\int d\delta_{\vec{k}}\cdot\delta_k^2\cdot e^{-\beta\left[\frac{1}{2}Ja^2k^2+\frac{h}{\phi}\right]\delta_k^2}}{\int d\delta_{\vec{k}}\cdot e^{-\beta\left[\frac{1}{2}Ja^2k^2+\frac{h}{\phi}\right]\cdot\delta_k^2}} =$$

$$= \frac{k_B T}{\frac{1}{2} J a^2 k^2 + \frac{1}{2}\frac{h}{\phi}} . \qquad (5)$$

As a consequence:

$$\langle \vec{\delta}^2(\vec{x}^0) \rangle = \frac{1}{N} \cdot \int dx \, \langle \vec{\delta}^2(\vec{x}^0) \rangle = \frac{1}{N} \sum_{\vec{k}^0} \frac{k_B T}{\frac{1}{2}\frac{h}{\phi} + \frac{J}{2} a^2 k^2} .$$

$$= \frac{a^d}{(2\pi)^d} \int_{\Pi} d^d k \cdot \frac{k_B T}{\frac{1}{2} h/\phi + \frac{J}{2} a^2 k^2} \qquad (6)$$

For $d=3$ the integral is convergent for $h \to 0$: the spontaneous magnetization $\phi_0 \neq 0$ is stable against fluctuations and the mexican hat picture is valid.
For $d = 2$ or 1 the integral diverges for $h \to 0$ and $\phi \to \phi_0 \neq 0$. One can avoid this divergence by letting $\phi_0 \to 0$ as $h \to 0$ at any finite temperature (Mermin-Wagner theorem!).
The divergence of the integral occurring for $d=2$ in the thermodynamic limit is due to the lower integral range approaching 0 as $N \to \infty$ and is referred to in the literature as infrared divergence.
This infrared divergence plays a key role in determining physical properties of any 2d-system, even if interactions breaking the exact continuous symmetry occur in real systems.
One of the most subtle manifestations of this infrared divergence is the reorientation transition in ultrathin magnetic films, see Pescia, Pokrovsky, PRL 65, 2599 (1990). This is a very complicated problem which has to be explained in small steps. We will introduce here some of these steps and explain, as a preliminary exercise, the existence of a phase transition in 2d of a

finite ϕ_0 at finite temperature in truly 2d.

5 The non-linear σ-model in 2d.

As the previous chapter shows, only at $T=0$ a phase with $|\vec{\phi}_0| \neq 0$ can exist in a 2d-Heisenberg model. This call for important modification of H_{eff} in 2d, like dropping the $\vec{\phi}^2$ and $\vec{\phi}^4$ terms. The effective Hamiltonian is therefore more suitable described by

$$H = \frac{\Gamma}{2} \int d^2x \sum_{\mu=1}^{2} \left(\partial_\mu \vec{n}\right)^2 \qquad n \geq 2 \qquad (1)$$

where $\vec{n}(\vec{x})$ is, generally, a n-component field with $|\vec{n}|=1$. This model is called the two-dimensional isotropic $O(n)$-model or simply non-linear sigma model. The non-linearity is due to the constraint $|\vec{n}|=1$, which is absent in the LGW-Hamiltonian. The reason for introducing this constraint is that we would like to consider only those fluctuations which rotate locally the spin but keep its length constant: those fluctuations so called transverse fluctuations are those responsible for the infrared divergence. Polyakov (Phys. Lett. 59 B, 73 (1975)) was the first to develop a RG procedure for this Hamiltonian. He started by introducing an orthonormal basis set $(\vec{n}_0, \vec{e}_1, \dots, \vec{e}_{n-1})$ with $\vec{n}_0 \cdot \vec{e}_i = 0$ and parametrizing $\vec{n}$ according to

$$\vec{n} = \vec{n}_0 \sqrt{1 - \vec{\phi}^2} + \vec{\phi}$$

$$\vec{\phi}(\vec{x}) = \sum_{a=1}^{n-1} \phi_a \cdot \vec{e}_a(\vec{x}) \qquad (2)$$

so that $|\vec{n}| = |\vec{n}_0|$. Within the spirit of RG he sets $\vec{n}_0$ to vary slowly in space, i.e. its Fourier transform contains the vectors $0 \leq |\vec{k}| \leq \frac{\pi}{La}$. The transverse field $\vec{\phi}$ is defined over the components $\frac{\pi}{La} \leq |\vec{k}| \leq \frac{\pi}{a}$. Inserting this parametrization in (1) we obtain

$$H_{Heis} = \frac{1}{2} T \int d^2x \left\{ \sum_{\mu=1}^{2} \left(\partial_\mu \left(\vec{n}_0(\vec{x}) \sqrt{1 - \vec{\phi}^2} + \vec{\phi} \right) \right)^2 \right\} =$$

$$= \frac{1}{2} T \int d^2x \left\{ \sum_{\mu=1}^{2} \left[\partial_\mu \vec{n}_0(\vec{x}) \cdot \sqrt{1 - \vec{\phi}^2} + \vec{n}_0(\vec{x}) \cdot \partial_\mu \sqrt{1 - \vec{\phi}^2} + \right.\right.$$

$$\left.\left. + \partial_\mu \vec{\phi} \right]^2 \right\} =$$

$$= \frac{1}{2} T \int d^2x \left\{ \sum_{\mu=1}^{2} \left[\partial_\mu \vec{n}_0 \cdot \sqrt{} - \cancel{\underbrace{\vec{n}_0 \cdot \frac{1}{\sqrt{}}}_{0}} \cdot \vec{\phi} \cdot \partial_\mu \vec{\phi} + \right.\right.$$

$$\left.\left. + \partial_\mu \vec{\phi} \right]^2 \right\} =$$

$$= \frac{1}{2} T \int d^2x \left\{ \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{n_0} \cdot \left(1 - \vec{\phi}^{\,2}\right) + 2 \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{\phi} \cdot \sqrt{1-\vec{\phi}^{\,2}} + \right.$$

$$\left. + \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} \right\}$$

(convention: 2 equal indices $\Leftrightarrow \Sigma$).

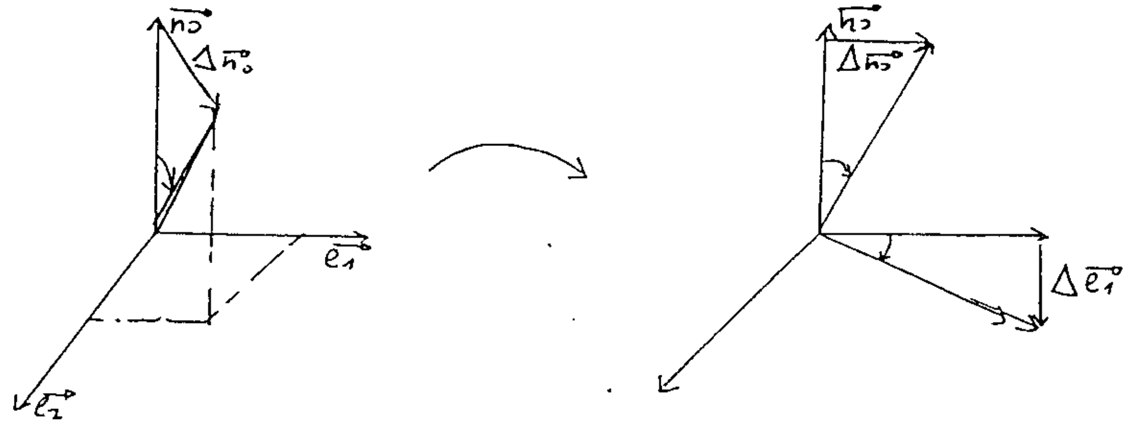
$$= \frac{1}{2} T \int d^2x \left\{ \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{n_0} \cdot \left(1 - \phi^2\right) + 2 \cdot \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{\phi} \cdot \sqrt{} + \right.$$

$$\left. + \partial_\mu \sum_a \phi_a e_a(x) \cdot \partial_\mu \sum_b \phi_b \, e_b(x) \right\} . \qquad (3)$$

We have to evaluate $\partial_\mu \vec{n_0}$ and $\partial_\mu e_a$. We can set

$$\partial_\mu \vec{n_0} = c_{\mu a} \vec{e_a}$$

without loosing generality. Then we expect, for a given μ, that the set $\{\vec{e_1}, \dots \vec{e_{n-1}}\}$ can be choosen so that



$$\partial_\mu \vec{e_a} = - c_{\mu a} \vec{n_0} \qquad (4)$$

(see Figure).

Evidently, it is not possible to choose $\{e_1, \dots e_{n-1}\}$ so that eq. 4 is fullfilled simultaneously for each μ. In general we have

$$\partial_\mu \vec{e_a} = - c_{\mu a} \cdot \vec{n_0} + f^b_{\mu,a} \cdot \vec{e_b} \qquad (5)$$

What we can do, however, is to choose the set $\{\vec{e_1}, \dots \vec{e_{n-1}}\}$ which minimizes $f^b_{\mu,a}$, so that

$$f^b_{\mu,a} \sim \partial_{\mu_1} \partial_{\mu_2} \vec{n_0} \qquad (6)$$

i.e $f^b_{\mu a}$ small with respect to $\vec{n_0}$ and $\partial_\mu \vec{n_0}$
Because of (4), (5) and (6) we have (up to irrelevant odd powers of $\partial_\mu \phi_a$)

$$\partial_\mu \vec{n_0} \cdot \partial_\mu \vec{\Phi} = \sum_a \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{e_a} \cdot \phi_a =$$

$$= \sum_{a,b} c_{\mu a} \cdot (- c_{\mu_b} \vec{n_0}) \vec{e_a} \phi_a$$

$$= 0 \qquad (7)$$

and

$$\partial_\mu \vec{\Phi} \cdot \partial_\mu \vec{\Phi} = \sum_{a,b} \partial_\mu (\phi_a e_a) \cdot \partial_\mu \phi_b e_b =$$

$$= \sum_{a,b} (\phi_a \partial_\mu e_a + e_a \partial_\mu \phi_a)(\phi_b \partial_\mu e_b + e_b \partial_\mu \phi_b)$$

$$= c_{\mu a} \cdot c_{\mu b} \cdot \phi_a \cdot \phi_b + \partial_\mu \phi_a \cdot \partial_\mu \phi_a \tag{8}$$

so that

$$\underline{\underline{H_{Heis} = \frac{1}{2} \Gamma \cdot \int d^2x \left\{ \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{n_0} \cdot (1 - \vec{\phi}^{\,2}) + \right.}}$$

$$\underline{\underline{\left. + \; c_{\mu a} \cdot c_{\mu b} \phi_a \cdot \phi_b + \partial_\mu \phi_a \cdot \partial_\mu \phi_a \right\}}} \tag{9}$$

Next we perform the smoothing operation

$$e^{-\frac{H}{T}[\vec{n_0}]} = \int D\vec{\phi} \; e^{-H_{eis}/T} =$$

$$= \int D\vec{\phi} \; e^{-\frac{\Gamma}{2T}\int d^2x \, \partial_\mu \phi_a \partial_\mu \phi_a} \cdot \sum_n g_n \tag{10}$$

$$g_n = (-1)^n \cdot \left(\frac{1}{2}\frac{\Gamma}{T}\right)^n \cdot \frac{1}{n!} \cdot \left[\int d^2x \, (\partial_\mu \vec{n_0})^2 \cdot (1 - \vec{\phi}^{\,2}) + c_{\mu a} \cdot c_{\mu b} \cdot \phi_a \cdot \phi_b\right]^n \tag{11}$$

To first order $n = 1$

$$H[\vec{n_0}] = \left\langle \frac{1}{2} \Gamma \int d^2x \left[(\partial_\mu \vec{n_0})^2 (1 - \vec{\phi}^{\,2}) + c_{\mu a} c_{\mu b} \phi_a \phi_b \right] \right\rangle_{\vec{\phi}} \quad (1$$

$$\int D\vec{\phi} = \prod_{\substack{\frac{\pi}{La} < |\vec{k}| < \frac{\pi}{a} \\ i = 1, .., n-1}} \int d\phi_{i,\vec{k}}$$

$$\langle \phi_a \phi_b \rangle = \frac{\int_{-\infty}^{\infty} \prod_{\underline{k},i} d\phi_i(\underline{k}) \cdot e^{-\frac{T}{2T}\sum_{j,\underline{k}} k^2 |\phi_k^j|^2} \cdot \frac{1}{V}\sum_{k_1,k_2} e^{i(k_1+k_2)\vec{x}} \cdot \phi_a^{k_1}\phi_b^{k_2}}{\int \prod d\phi_i(\underline{k})\, e^{-T/_{2T}\sum_{j,k} k^2 |\phi_k^j|^2}}$$

$i=j=a=b \leftarrow$

$$= \frac{\int_{-\infty}^{\infty} d\phi_a(k)\, e^{-\frac{T}{2T} k^2 \phi_a^2(\underline{k})} (\phi_a(\underline{k}))^2}{\int_{-\infty}^{\infty} e^{-\frac{T}{2T} k^2 |\phi_a(k)|^2}\, d\phi_a(k)}$$

$$= \frac{1}{n-1} \cdot \langle \vec{\phi}^{\,2} \rangle .$$

i.e.

$$\langle \phi_a \phi_b \rangle = \delta_{ab} \cdot \frac{1}{n-1} \cdot \langle \vec{\phi}^{\,2} \rangle \qquad (13)$$

so that

$$\langle c_{\mu a} c_{\mu b} \phi_a \phi_b \rangle = \frac{1}{n-1} \langle \vec{\phi}^{\,2} \rangle \cdot \underbrace{c_{\mu a} \cdot c_{\mu a}}_{\partial_\mu \vec{n}_0 \partial_\mu \vec{n}_0}$$

$$= \partial_\mu \vec{n}_0 \cdot \partial_\mu \vec{n}_0 \cdot \frac{\langle \vec{\phi}^{\,2} \rangle}{n-1} \qquad (14)$$

$H(\vec{n_0})$ becomes

$$H(\vec{n_0}) = \frac{T}{2} \cdot \int d^2x \, \partial_\mu \vec{n_0} \cdot \partial_\mu \vec{n_0} \left(1 - \langle \vec{\Phi}^2 \rangle + \frac{\langle \vec{\Phi}^2 \rangle}{n-1}\right)$$

$$= \frac{T}{2} \int d^2x \, \partial_\mu \vec{n_0} \, \partial_\mu \vec{n_0} \left(1 - \frac{n-2}{n-1} \langle \vec{\Phi}^2 \rangle\right) \quad (15)$$

It is not difficult to see, by the methods already explored, that (Eq. 4.6)

$$\langle \vec{\Phi}^2 \rangle = \frac{(n-1)}{(2\pi)^2} \cdot \int_{\frac{\pi}{L}}^{\frac{\pi}{\Lambda}} d^2k \cdot \frac{T}{T' k^2} =$$

$$= \frac{(n-1)}{2\pi T'} . T . \ln L/a \quad (16)$$

so that

$$\boxed{H(\vec{n_0}) = \frac{T}{2} \left(1 - (n-2) \cdot \frac{T}{2\pi T'} \ln L/a\right) \cdot \int d^2x \, \partial_\mu \vec{n_0} \, \partial_\mu \vec{n_0}} \quad (17)$$

The net result is that the long wave length field $\vec{n_0}(\vec{x})$ is described by the ~~the~~ same Heisenberg Hamiltonian with renormalized coupling constant

$$T_L = T\left(1 - (n-2)\cdot\frac{T}{2\pi T}\ln L/a\right) \qquad (18)$$

The philosophy of RG_method goes a little step further than 18 _ a step invented by Gell_Mann and Low. Eq. 18 is to be considered a <u>recursion formula</u> bringing from a field of characteristic <u>scale L_1 to</u> <u>a field with characteristic scale L_2 infinitesimally</u> <u>larger than L_1.</u> $\ln R_2/R_2$ can be considered as $d\xi$ and $T_R - T$ as dT. From the recursion formula one obtains the true RG_equation

$$dT = -(n-2)\cdot\frac{T}{2\pi}\, d\xi$$

with initial condition $T(\xi_0 = \ln a) = T$

$$\frac{dT}{d\xi} = -(n-2)\cdot\frac{T}{2\pi}$$ has the

solution

$$\underline{\underline{T(L)}} = -(n-2)\cdot\frac{T}{2\pi}\cdot\frac{\xi}{\xi_0} + T(\xi_0)$$

$$= \underline{\underline{T\left(1 - (n-2)\frac{T}{2\pi}\ln L/a\right)}} \qquad (19)$$

which, by coincidence, coincide with the recursion equation (18). This equation contains, on a solid basis, important results of 2d-physics.

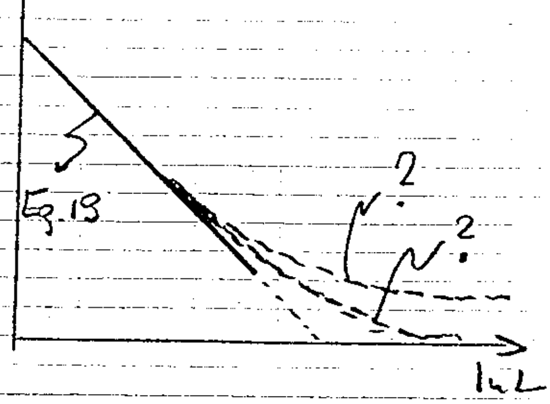
$n>2$: There exists a correlation length L_Γ, defined by the equation $\Gamma(L)=0$ inside which spin waves propagate and the harmonic approximation holds. Beyond this length spin waves don't exist as there is no barrier to the creation of fluctuations. Thus over sufficiently large scales the spins are uncorrelated. Within regions of length

$$L_\Gamma = a \cdot e^{\frac{2\pi\Gamma}{(n-2)T}} \qquad (20)$$

instead, the system has a large amount of order. It is a sort of "superparamagnet" with fluctuations continuously changing the spin configurations while keeping the spins over spin blocks aligned.

$n=2$: For this marginal case (the so called planar or x-y model) Γ is independent on the length-scale: there still is no long range magnetic order, but the spins, even on large scales, have a finite stiffness against fluctuations. This suggests that $O(2)$-systems behave quite differently than $n>2$ - 2d magnets. The peculiarity of $n=2$ systems became more clear within Kosterlitz-Thouless theory.

Notice that Eq. 13 sums explicitely only the leading logarithmic terms of the form $(T \ln L)^n$. The remaining terms introduce corrections which have not been completely calculated so far. Thus, we are still left with the puzzling behaviour of Γ for large L.



6. Spontaneous magnetization and phase transition in 2d.

The continuous symmetry of the non-linear σ-model is the key to the infrared divergence, which destroy long-range order and renormalize Γ to zero over sufficiently large distances. Let us now introduce a small, symmetry breaking interaction of the form $\lambda \int d^2x \, n_z^2$. Depending on the sign of λ, this interaction favours the x-y plane ($\lambda > 0$) or the z-direction ($\lambda < 0$), respectively. The Hamiltonian modifies to

$$\frac{\Gamma}{2} \int d^2x \sum_\mu (\partial_\mu \vec{n})^2 + \frac{\lambda}{2} \int d^2x \, n_z^2 \quad (1).$$

We now ask how does the extra term renormalize. This is equivalent to calculate $\langle n_z^2 \rangle_{\vec{\phi}}$.

$$n_3^2 = n_{03}^2 \left(\sqrt{1-\vec{\phi}^2}\right)^2 + \sum_{a,b} \vec{e_{a3}}\, \vec{e_{b3}}\, \phi_a \phi_b + 2 n_{03} \sqrt{1-\vec{\phi}^2} \cdot \sum_a \phi_a \vec{e_{a3}} \quad (2)$$

Following the methods of the previous chapter, we need $\langle n_3^2 \rangle_{\vec{\phi}}$. We obtain

$$\langle n_3^2 \rangle_{\vec{\phi}} = n_{03}^2 \left(1 - \langle \vec{\phi}^2 \rangle\right) + \sum_{a,b} \vec{e_{a3}}\, \vec{e_{b3}} \langle \phi_a \phi_b \rangle \quad (3)$$

Because

$$\langle \phi_a \phi_b \rangle = \delta_{ab} \cdot \frac{\langle \vec{\phi}^2 \rangle}{n-1} \quad (4)$$

we obtain further

$$\langle n_3^2 \rangle = n_{03}^2 \left(1 - \langle \vec{\phi}^2 \rangle\right) + \frac{\langle \vec{\phi}^2 \rangle}{n-1} \cdot \sum_a e_{a3}^2 \quad (5)$$

Some help from linear algebra is now necessary. It is appropriate to consider $(\vec{n_0}, \vec{e_1}, \vec{e_2})$ as a set of orthonormal basis vectors arising from $(\vec{e_x}, \vec{e_y}, \vec{e_z})$ though a basis transformation

$$\begin{pmatrix} \vec{n_0} \\ \vec{e_1} \\ \vec{e_2} \end{pmatrix} = \underbrace{\begin{pmatrix} n_{0x} & n_{0y} & n_{0z} \\ e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \end{pmatrix}}_{R} \begin{pmatrix} \vec{e_x} \\ \vec{e_y} \\ \vec{e_z} \end{pmatrix} \qquad (6)$$

The transformation matrix has obviously the welcome property

$$R \cdot R^T = R^T \cdot R = \mathbb{1} \qquad (7)$$

which means

$$n_{0\mu} \cdot n_{0\nu} + \sum_a \vec{e_{a\mu}}\, \vec{e_{a\nu}} = \delta_{\mu,\nu} \qquad (8)$$

In other words

$$\sum_a e_{az}^2 + n_{0z}^2 = 1$$

$$\sum_a e_{az}^2 = 1 - n_{0z}^2 \qquad (9)$$

Inserting into we obtain

$$\langle r_3^2 \rangle = n_{0z}^2 \left(1 - \langle \vec{\phi} \rangle^2\right) + \frac{\langle \phi \rangle^2}{n-1} \cdot \left(1 - n_{0z}^2\right) \quad \text{(up to a constant)}$$

$$= n_{0z}^2 \left[\left(1 - \langle \vec{\phi} \rangle^2\right) + \langle \vec{\phi} \rangle^2 /_{n-1} \cdot (-1)\right] =$$

$$= n_{0z}^2 \left(\frac{n-1 - \langle \vec{\phi}^{\,2} \rangle (n-1) - \langle \vec{\phi}^{\,2} \rangle}{n-1} \right) =$$

$$= n_{0z}^2 \left(1 - \frac{n}{n-1} \langle \vec{\phi}^{\,0\,2} \rangle \right) \qquad (10)$$

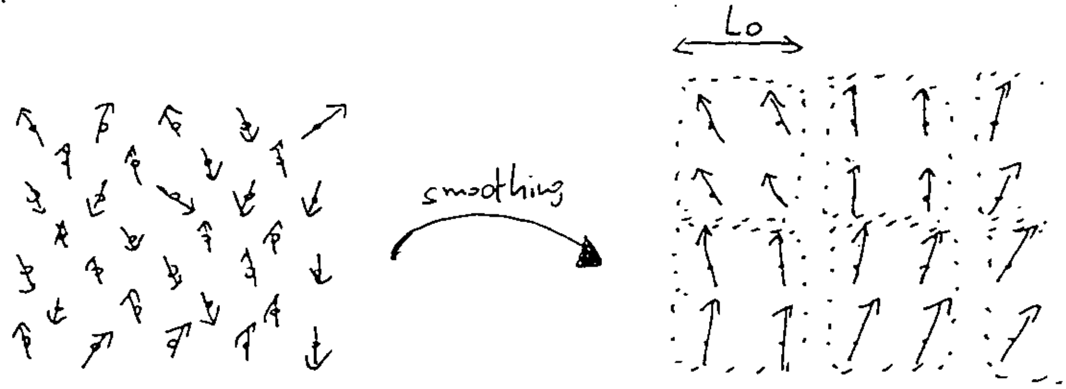
and

$$H_\lambda(\vec{n}_0) = + \frac{\lambda}{2} \cdot \left(1 - \frac{n}{n-1} \langle \vec{\phi}^{\,2} \rangle \right) \int d^2x \; n_{0z}^2 \qquad (11)$$

This leads to a renormalized coupling constant λ_L

$$\lambda_L = \lambda \cdot \left(1 - \frac{n}{n-1} \langle \vec{\phi}^{\,0\,2} \rangle \right) = \lambda \cdot \left(1 - n \frac{T}{2\pi\Gamma} \cdot \ln L \right) \qquad (12)$$

The smoothing operation leaves the system in a so called "Kadanoff" block spin state by integrating out the small scale fluctuations _ the so called coarse graining operation.



One important consequence of the smoothing operation is that, in general, the lenght of the field $\vec{n}_0$ is altered, as $\vec{n}_0$ only contains a limited number of Fourier components. To restore the original length, one has to rescale $\vec{n}_0$ by a factor $Z(L)$. The special parametrization used by Polyakov, however, allows $|\vec{n}| = |\vec{n}_0|$, so that $Z(L) = 1$ in this particular case.

Furthermore, the smoothing operation generate a new lattice with lattice constant $a' = L \cdot a$. The effective Hamiltonian of $\vec{n}_0$ is formally identical to the effective Hamiltonian of $\vec{n}'$ only

$$a^{g(d)} \int d^d x = \left(\frac{a'}{L}\right)^{g(d)} \int d^d x = L^{-g(d)} \cdot a'^{g(d)} \int d^d x \,.$$

This introduces a further rescaling of the coupling constants by $L^{-g(d)}$. The renormalization of the coupling constants receives the contribution of three operations:

$$\alpha_L = S(L) \cdot Z(L) \cdot L^{-g(d)} \qquad (13)$$

where $S(L)$ is the smoothing operation. By (13) we restore formally the original lattice and the original effective Hamiltonian. By the combination of these three operation one generate a serie of effective Hamiltonian which are formally similar but have renormalized coupling constants. The idea of RG is that <u>exactly at the phase transition point the coupling constants are invariant upon the RG transformation (13)</u>

Let us apply this hypothesis to the Hamiltonian (1).
The first term is unaffected by rescaling for $d = 2$. Thus

$$\Gamma_{L'} = \Gamma_L \left(1 - (n-2) \frac{T}{2\pi \Gamma_L} \ln \frac{L'}{L}\right)$$

$$\lambda_{L'} = \lambda_L \left(1 - n \cdot \frac{T}{2\pi \Gamma} \ln \frac{L'}{L}\right) \cdot \left(\frac{L'}{L}\right)^2 \qquad (14)$$

To extract the Gell-Mann-Low equations from these recurrence relations we set

$$\left(\frac{L'}{L}\right)^2 = e^{2 \ln \frac{L'}{L}} = 1 + 2 \ln \frac{L'}{L} \quad \text{and} \quad \ln L' - \ln L = d\xi \qquad (15)$$

so that

$$\boxed{\frac{d\Gamma}{d\xi} = -(n-2) \cdot \frac{T}{2\pi} \qquad \frac{d\lambda}{d\xi} = \left(2 - \frac{nT}{2\pi\Gamma}\right)\lambda} \qquad (16)$$

So

(see also Pelcovitz and Nelson, Phys. Lett. 57a, 23 (1976)).
According to the lore of RG, a phase transition point is a fixed point of (16):

$$\begin{cases} \dfrac{d\Gamma}{d\xi} = 0 \\ \dfrac{d\lambda}{d\xi} = 0 \end{cases} \iff T = \lambda = 0 \qquad (17)$$

which is the isotropic fixed point. In the vicinity of this fixpoint, a RG invariant can be constructed, namely

$$\lambda \cdot e^{\frac{4\pi\Gamma}{T(n-2)}} . \qquad (18)$$

Proof:

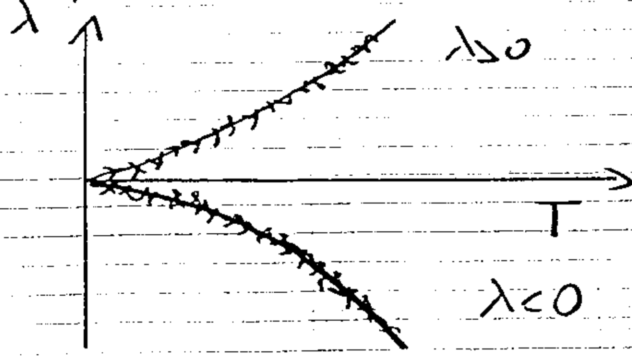
$$\frac{d\left(\lambda \cdot e^{\frac{4\pi\Gamma}{(n-2)T}}\right)}{d\xi} = \left[\frac{d\lambda}{d\xi} + \lambda \frac{d}{d\xi}\frac{4\pi\Gamma}{T(n-2)}\right] \cdot e^{\frac{4\pi\Gamma}{T(n-2)}} =$$

$$= \left[2\lambda + \lambda \frac{4\pi}{T(n-2)} \cdot \frac{d\Gamma}{d\xi}\right] e^{4\pi\Gamma/T(n-2)}$$

$$= \left[2\lambda + \lambda \cdot \frac{4\pi}{T(n-2)} \cdot -(n-2)\frac{T}{2\pi}\right] e^{4\pi\Gamma/T(n-2)}$$

$$= 0 + O(\lambda \cdot \Gamma)$$

The existence of a renormalization invariant defines two lines of critical points in the T-λ-plane, given by the implicit equation. The lines end at the bicritical point (0,0).



$$\lambda_c\, e^{\frac{4\pi\Gamma_c}{T_c(n-2)}} = \text{const}$$

$$\Rightarrow T_c = \frac{4\pi\Gamma_c}{(n-2)} \cdot \frac{1}{\ln\frac{\text{const}}{\lambda}} \qquad (19)$$

What kind of phase transition do we expect?

In going to larger scales, Γ weakens smoothly while λ becomes very large. If $\lambda < 0$ the minima of the effective, large scale Hamiltonian, are strongly peaked around the spin configuration

$$(n_x = 0, n_y = 0, n_z = \pm 1) \qquad (19).$$

so that the system slowly crosses over to a Ising model. If $\lambda > 0$, the system crosses over to an x-y-model. There is a more intuitive approach to Eq. 19. Consider e.g. an Heisenberg system with a very weak anisotropy λ favouring the x-y-plane. Clearly, the ground state will be one where all the spins are within this plane. Let us suppose to introduce a fluctuation, by e.g. rotating one spin perpendicular to the plane. Because the spins are highly correlated, a region of spins of the order of $a \cdot e^{4\pi\Gamma/T}$ will also align perpendicular to the plane. This fluctuation costs an energy

$$\Delta E = \lambda \cdot e^{4\pi\Gamma/T} \qquad (20)$$

On the other side, there is a gain in entropy which is $\sim \ln 2$ corresponding to the two directions one can align the spin block. The change of free energy associated with this fluctuation is therefore

$$\Delta F = \lambda e^{4\pi\Gamma/T} - T \ln 2 \qquad (21).$$

There exists a critical temperature T_c at which $\Delta F = 0$, i.e. one does not have to pay any free energy cost to introduce such fluctuation. At this temperature, nothing prevents the proliferation of such fluctuations, which eventually will disorder the system. From $\Delta F = 0$ we obtain

$$T_c = \frac{4\pi\Gamma}{(n-2)} \cdot \frac{1}{\ln \Gamma/\lambda} \qquad (22)$$

which is just (13).

Scaling, 2d- Ising model and Universality

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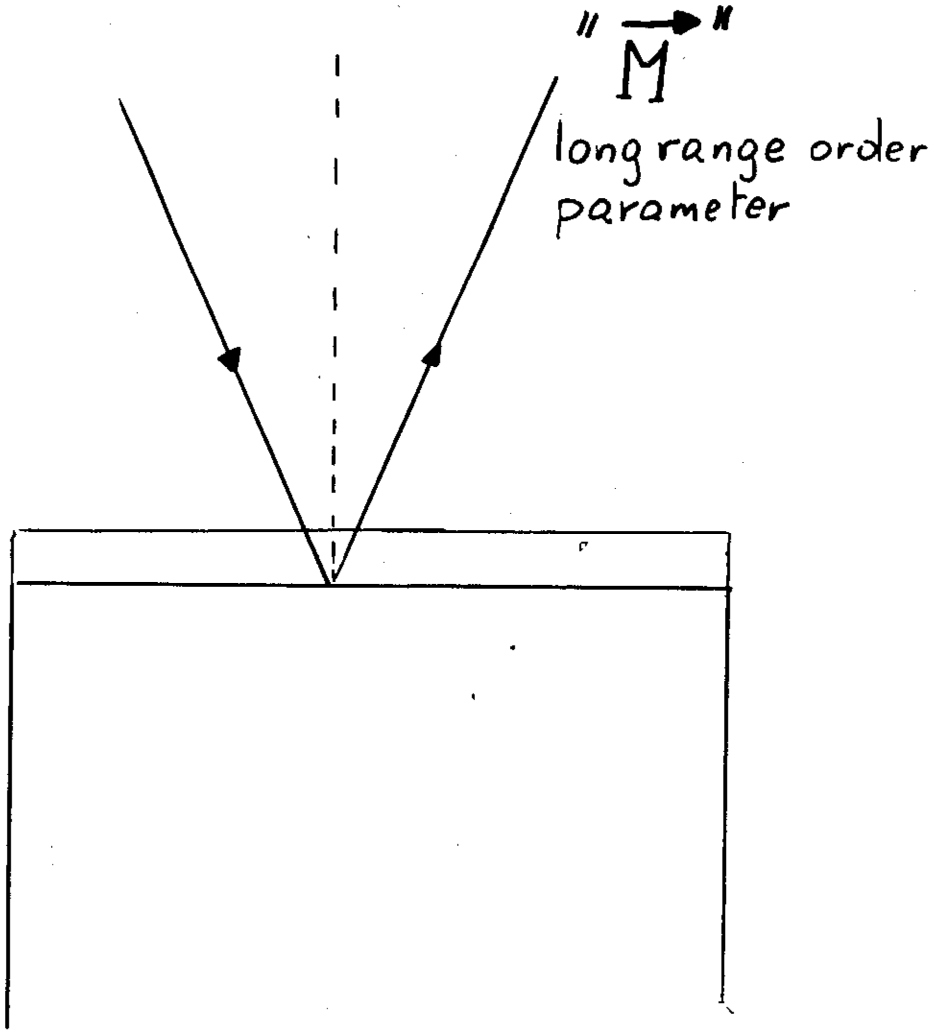
- **Scaling**

- **2d Ising model and universality**

a

Magnetooptical Kerr effect



Within **Landau** theory, the equation of state close to the critical point is (**Arrot's plots**)

$$H(t, M)/H_0 = t \cdot M/M_0 + (M/M_0)^3 \qquad (1)$$

–M is the magnetisation

–H the applied magnetic field

–t the reduced temperature $(T - T_C)/T_C$ and

–H_0, M_0 material dependent scale factors.

Fundamental symmetry of Eq.1 (**not recognized by Arrot**):

$$\frac{H/H_0}{(M/M_0)^3} = 1 + \frac{t}{(M/M_0)^2} \qquad (2)$$

Griffiths (scaling hypothesis):

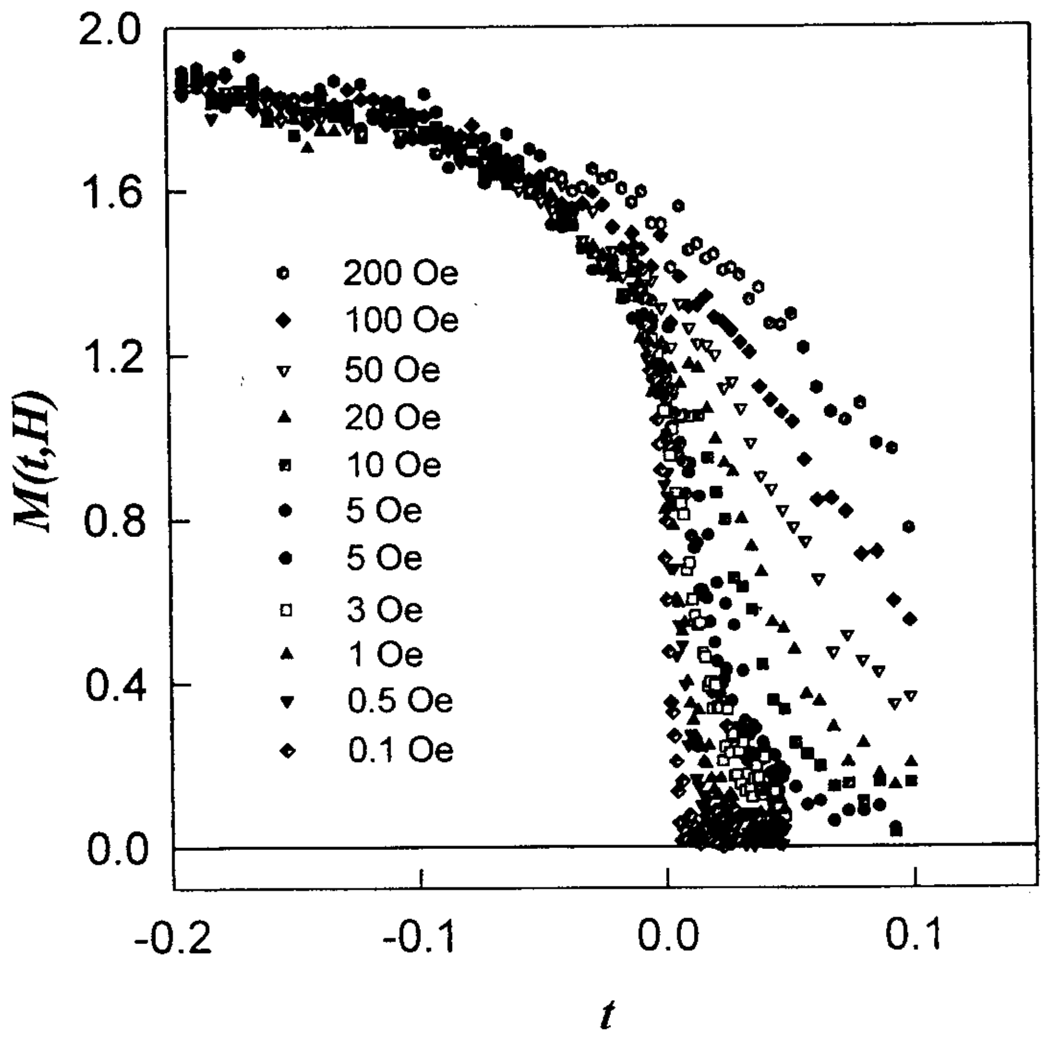
$$\frac{H/H_0}{(M/M_0)^\delta} = h[\frac{t}{(M/M_0)^{1/\beta}}] \qquad (3)$$

$-h(x)$ is the (non-necessarily linear) scaling function

$-\beta$, δ critical exponents.

Kadanoff: $h(x)$, β and δ **universal**

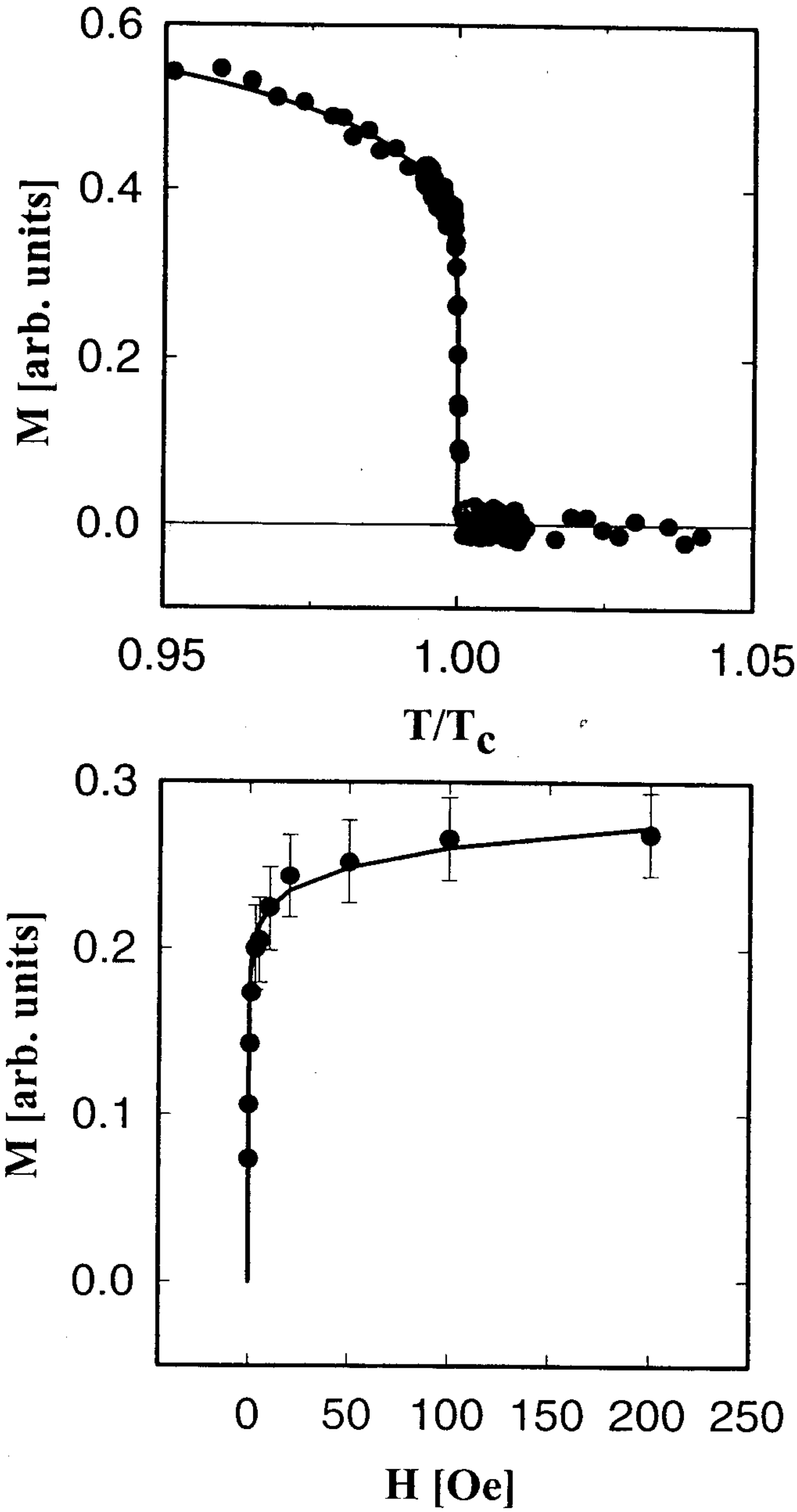
d



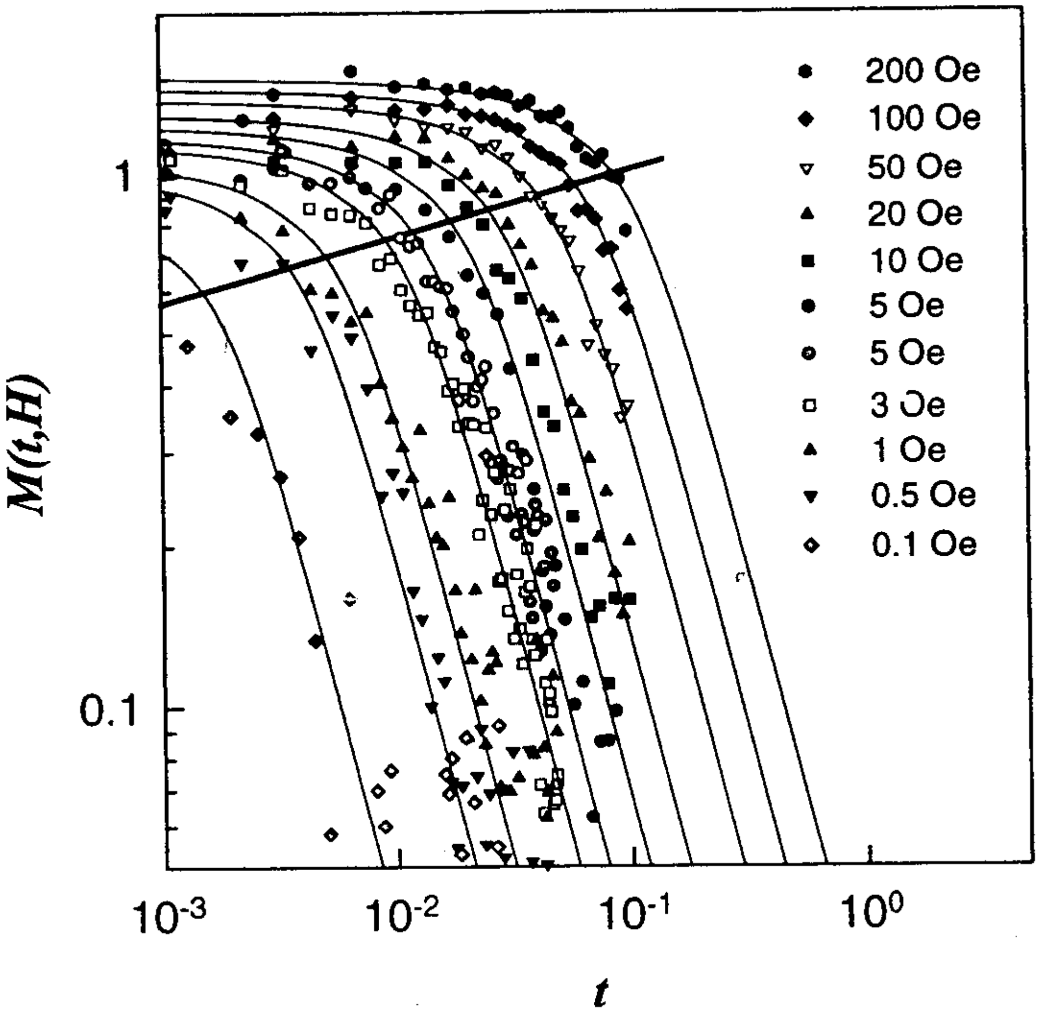
2.0
1.6
1.2
0.8
0.4
0.0
M(t,H)
200 Oe
100 Oe
50 Oe
20 Oe
10 Oe
5 Oe
5 Oe
3 Oe
1 Oe
0.5 Oe
0.1 Oe
-0.2
-0.1
0.0
0.1
t

e

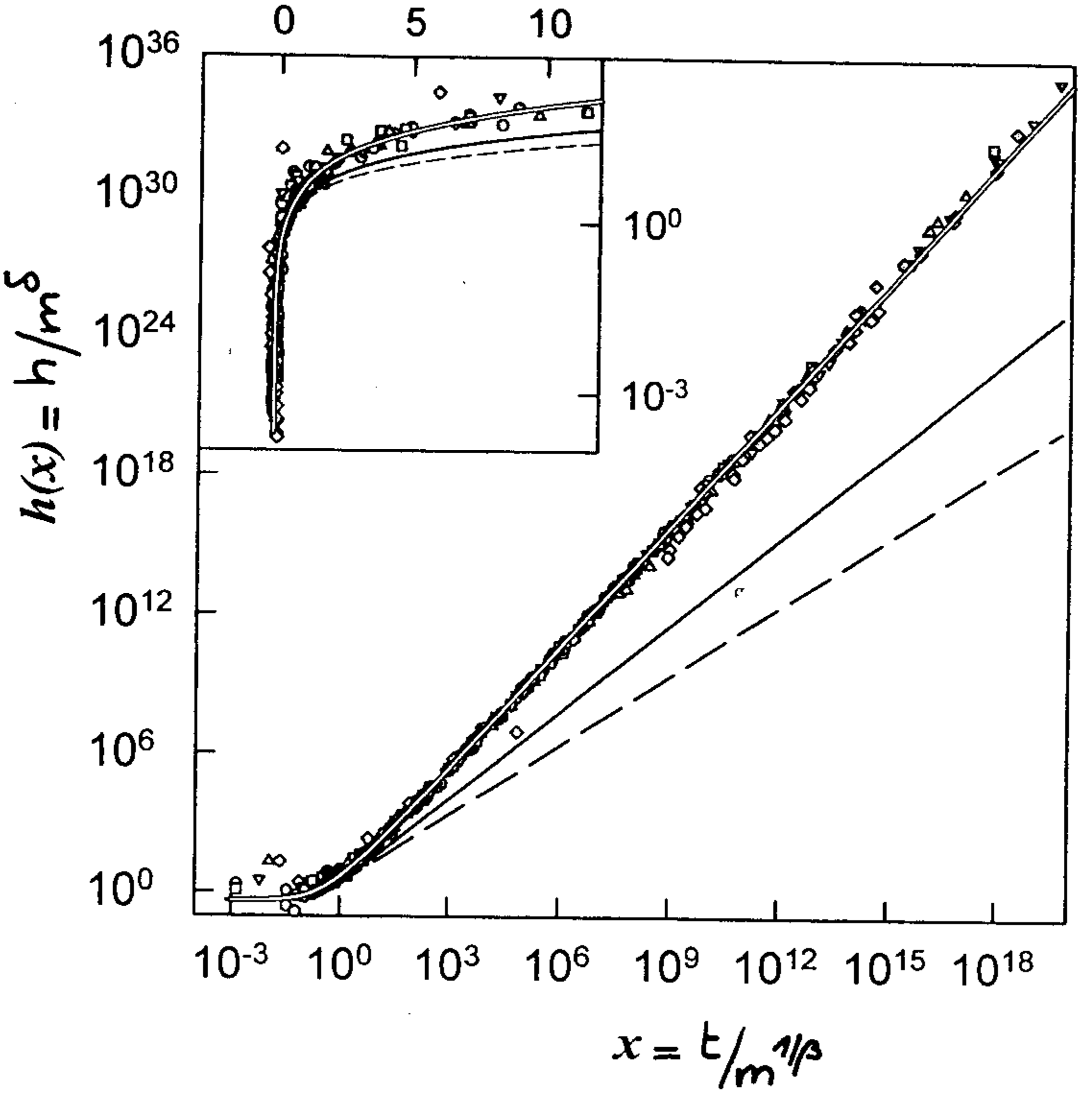
m(t)_icm.spw


0.6
0.4
0.2
0.0
M [arb. units]
0.95
1.00
1.05
T/T_c
0.3
0.2
0.1
0.0
M [arb. units]
0
50
100
150
200
250
H [Oe]

f


200 Oe
100 Oe
50 Oe
20 Oe
10 Oe
5 Oe
5 Oe
3 Oe
1 Oe
0.5 Oe
0.1 Oe
M(t,H)
1
0.1
10-3
10-2
10-1
100
t

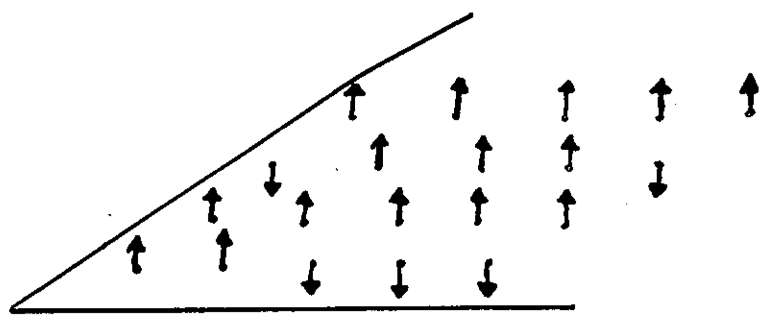
m(t)_icm.spw


0
5
10
10^{36}
10^{30}
10^{24}
10^{18}
10^{12}
10^{6}
10^{0}
10^{0}
10^{-3}
$h(x) = h/m^{\delta}$
10^{-3}
10^{0}
10^{3}
10^{6}
10^{9}
10^{12}
10^{15}
10^{18}
$x = t/m^{1/\beta}$

h

Figure 1

- The Two-Dimensional Ising Model.
- Universality



- $H = -J \cdot \sum_{\langle i,j \rangle} s_i s_j + h \cdot \sum_i s_i$

∴ Onsager (1944): exact solution for $h = 0$

∴ C. Domb (1970): numerical equation of state $m = m(T, h)$

(a)

	β	δ	γ	Γ_+/Γ_-
Exp.	0.13 ± 0.02	14 ± 2	1.74 ± 0.05	40 ± 10
2d-Ising	1/8	15	7/4	37.7

(b) scaling function

$\longrightarrow$ universality

ℓ

Magnetism at the mesoscopic scale

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- **Stripes**

- **Dots**

m

THE MEASURING TECHNIQUE

S canning

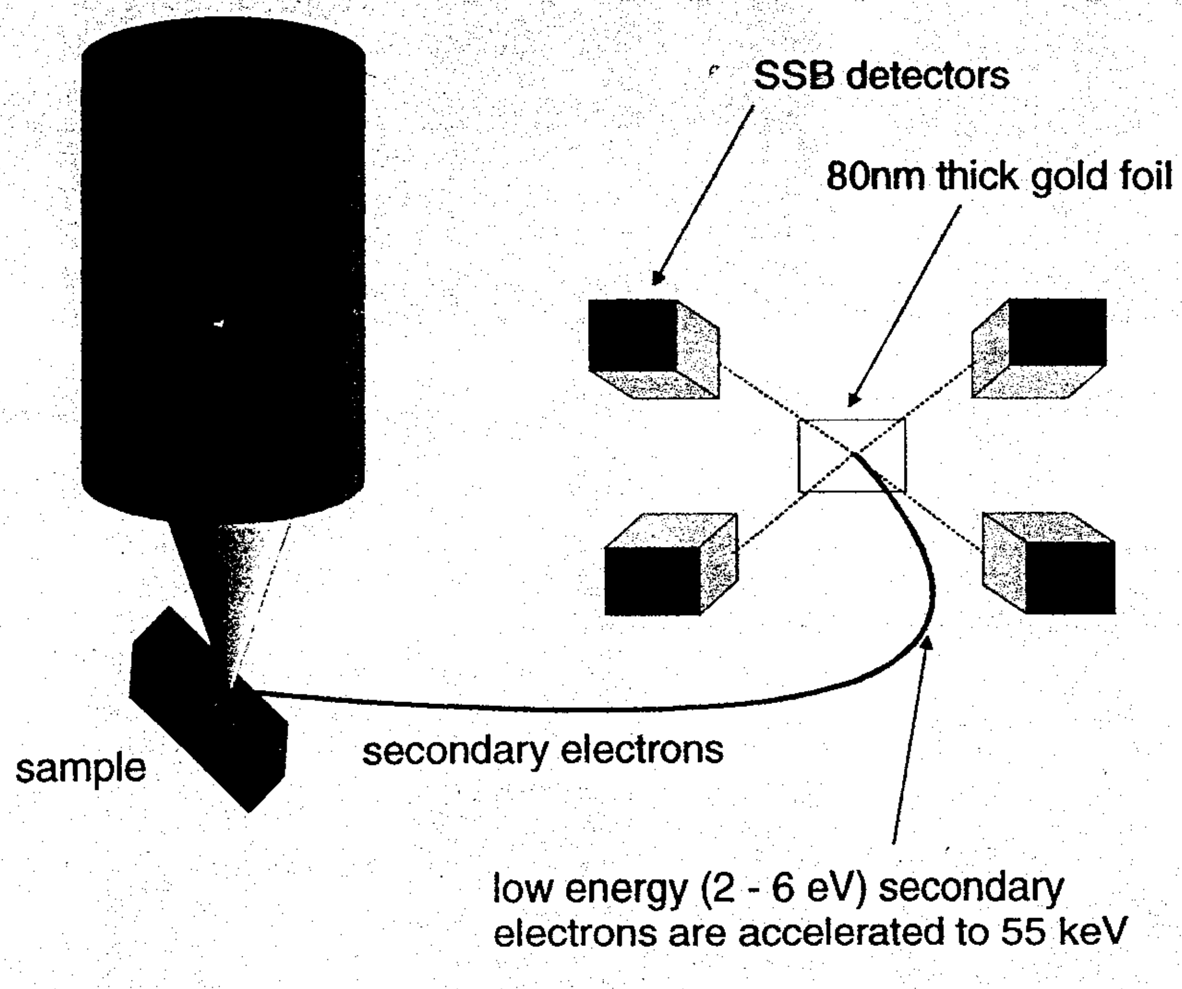
E lectron

M icroscope with

P olarization

A nalysis of the secondary Electrons

Hitachi 4100 S Electron Microscope
in UHV



n

Phase diagram of ultrathin ferromagnetic films with perpendicular anisotropy

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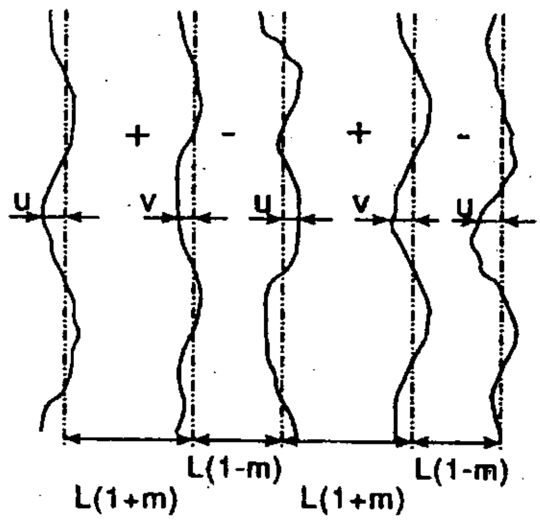


FIG. 1. The geometry of the stripe-domain structure.

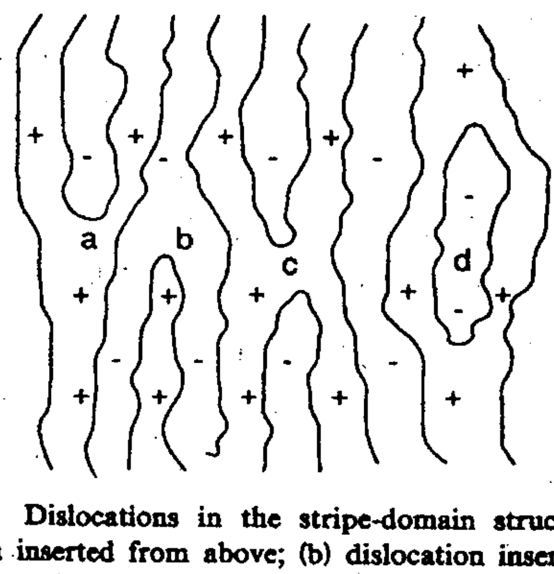
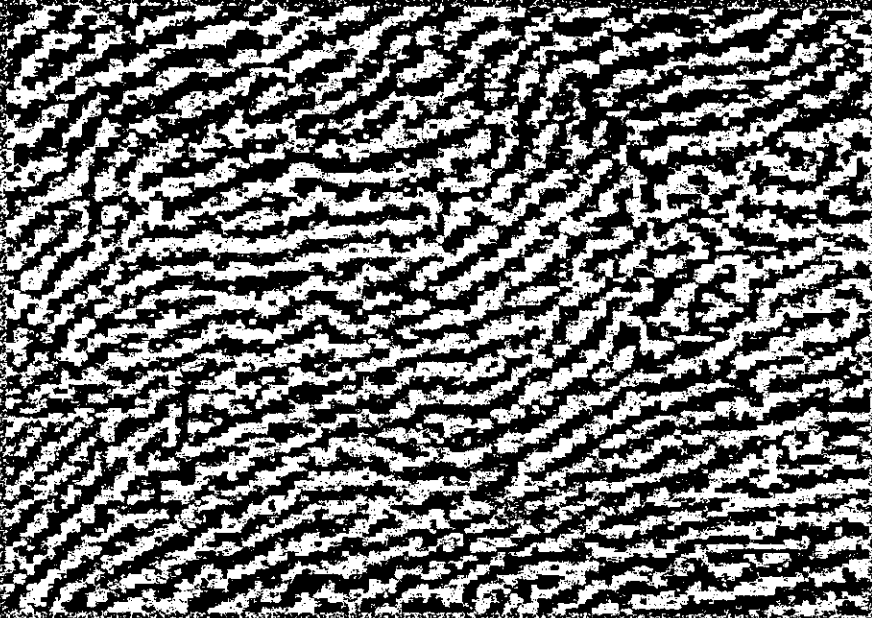
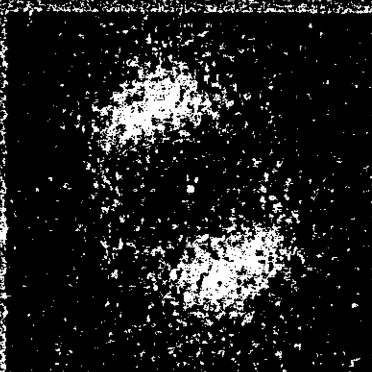


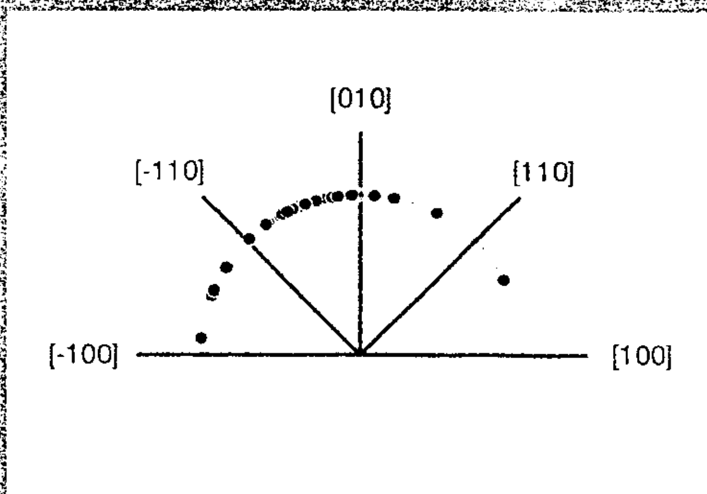
FIG. 2. Dislocations in the stripe-domain structure. (a) dislocation inserted from above; (b) dislocation inserted from below; (c) "strait," or "passage," due to two dislocations, one inserted from above and one from below; (d) "island," due to two dislocations inserted in the center, one ending above and one ending below the center.

O

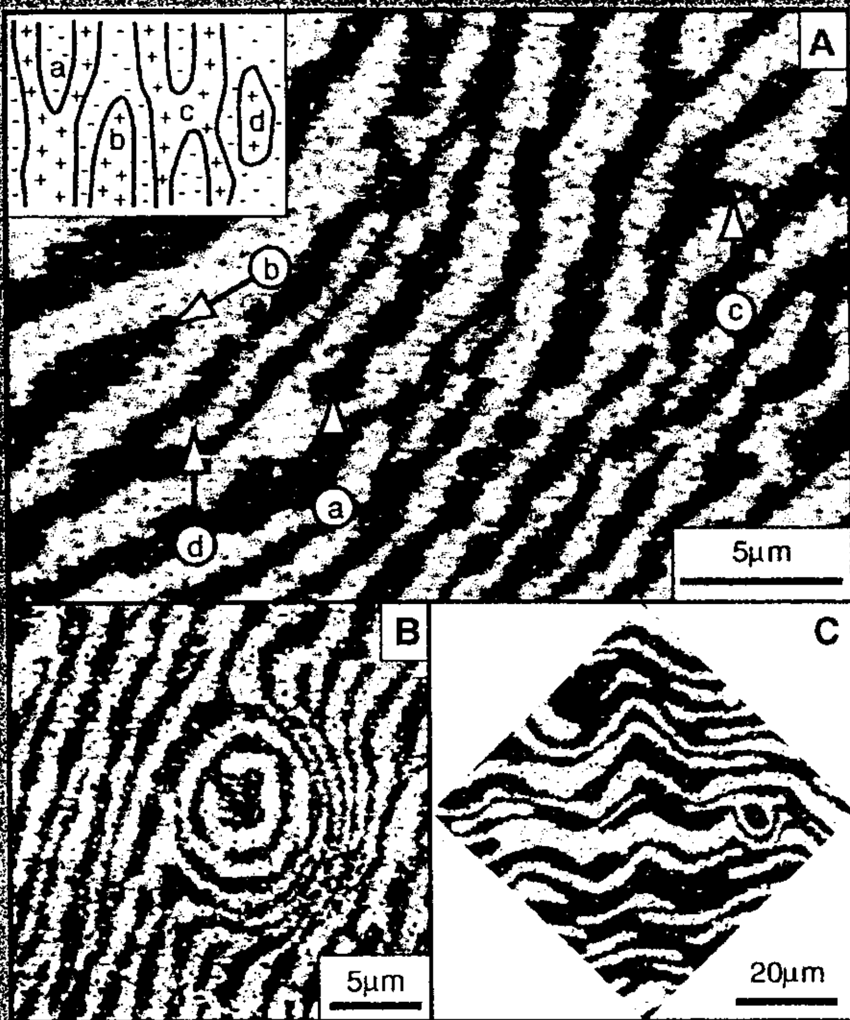
Meandering of the stripe phase



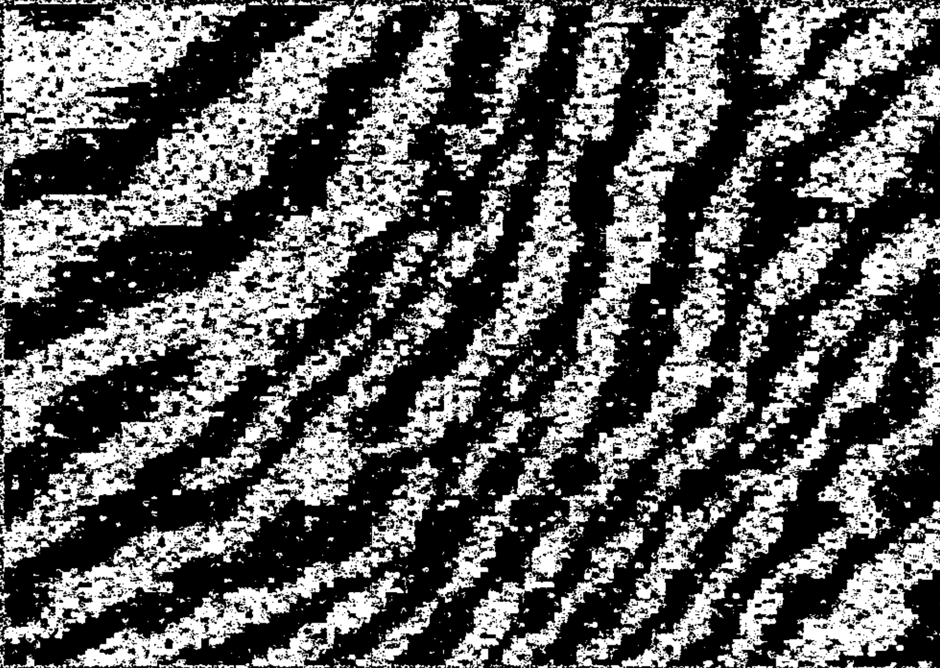




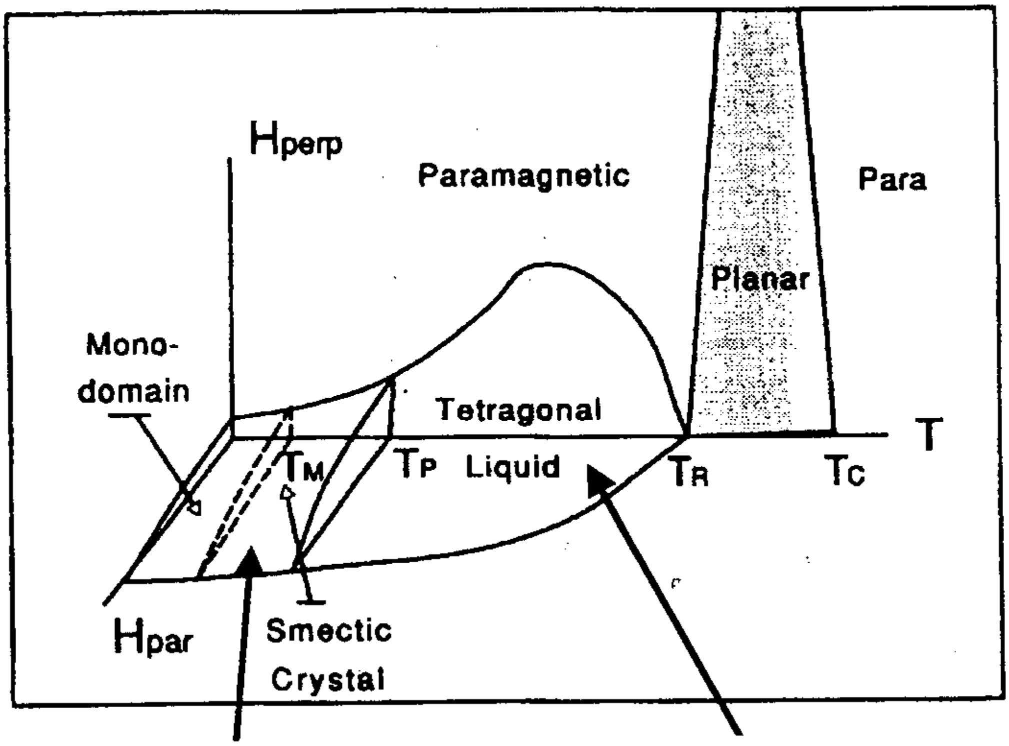
Dislocations of the stripe phase



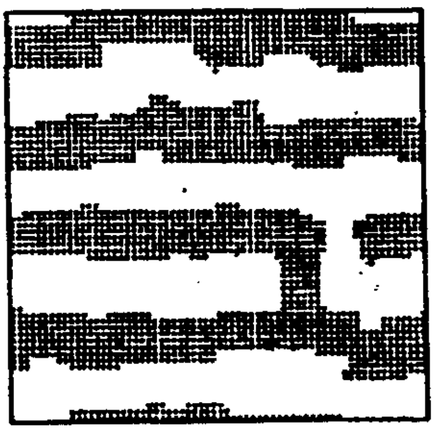
STRIPES

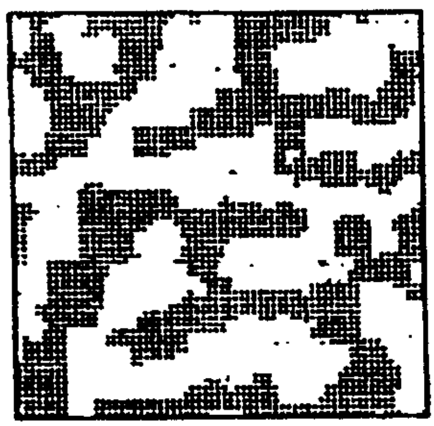


The Phase Diagram

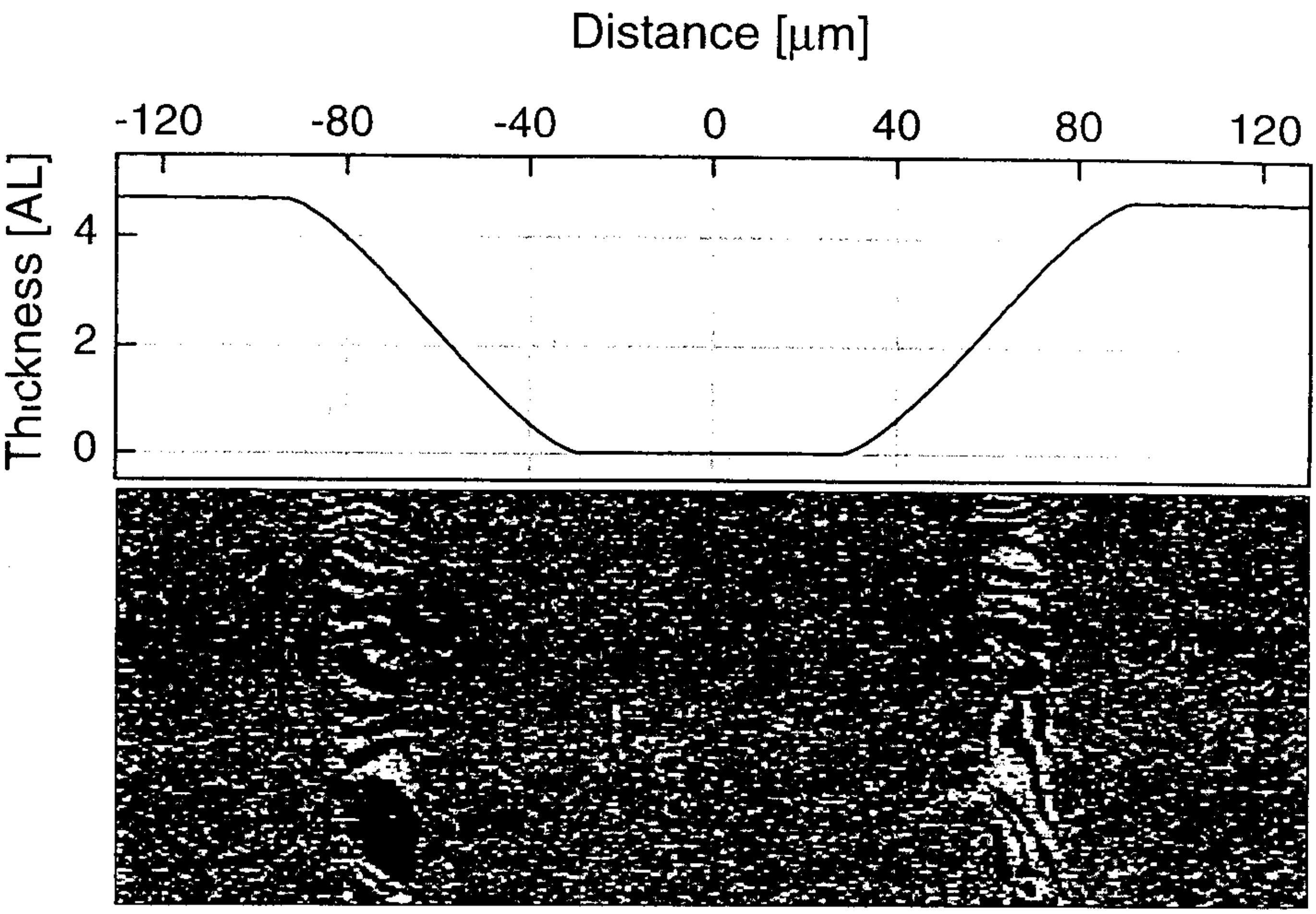


Ar. Abanov et al. Phys. Rev. B **51**, 1023 (1995)



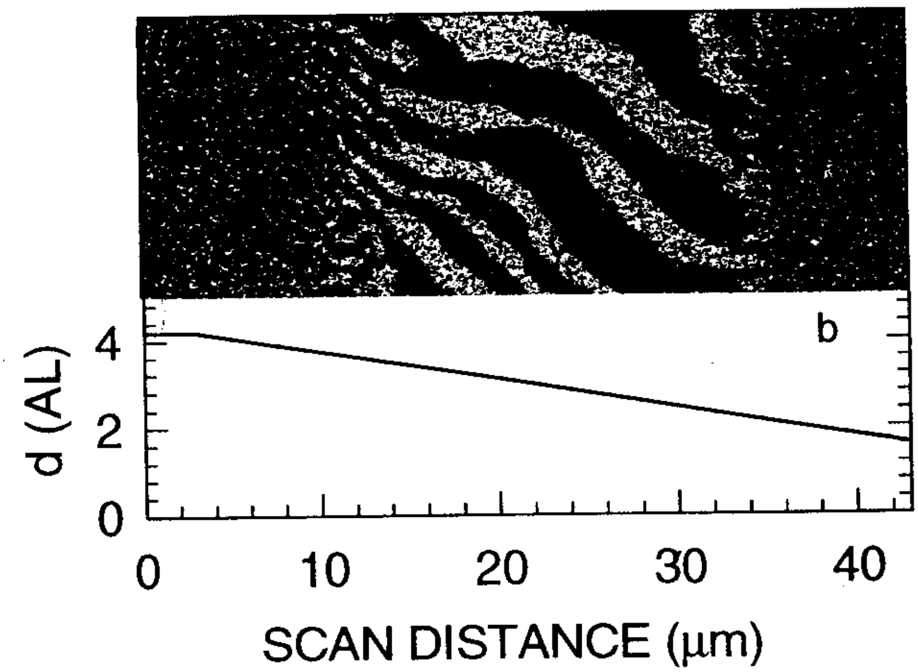


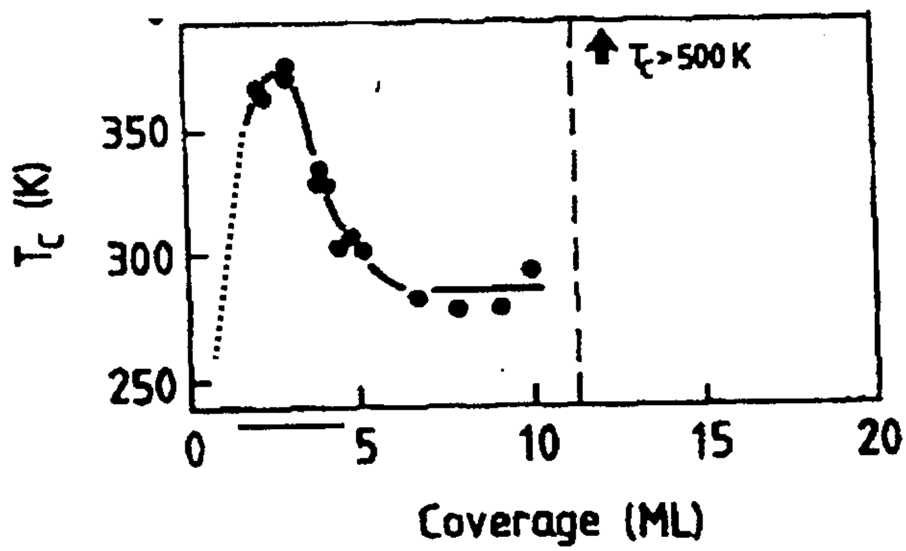
I. Booth, et al. Phys. Rev. Lett. **75**, 950 (1995)



t

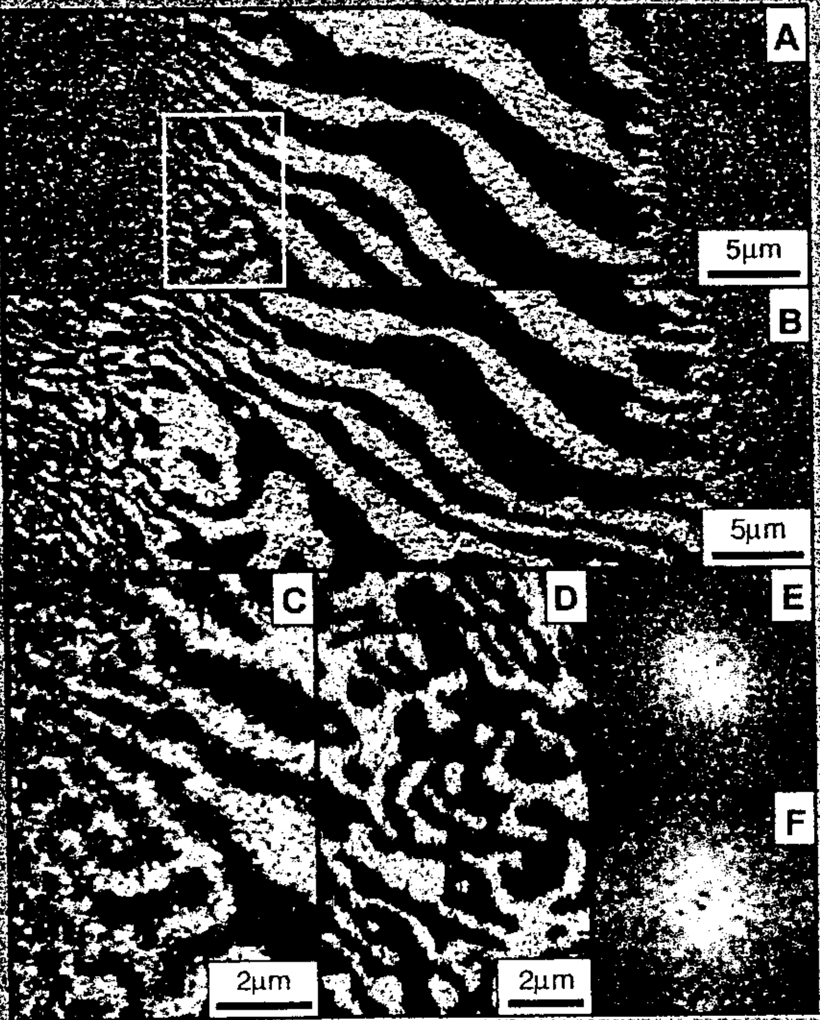
Wedge of Fe grown on Cu(001)



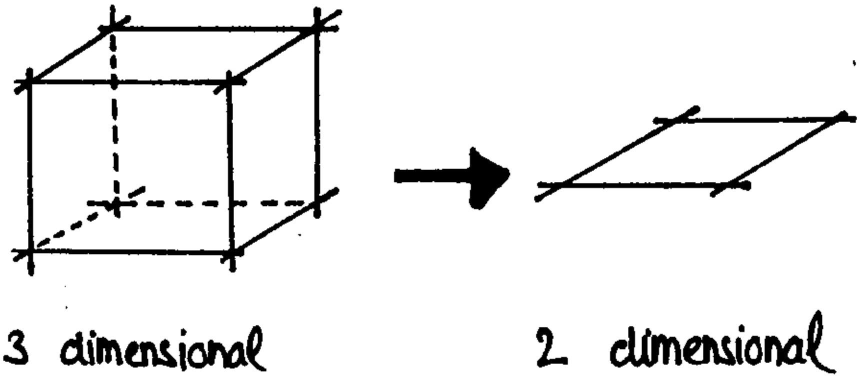


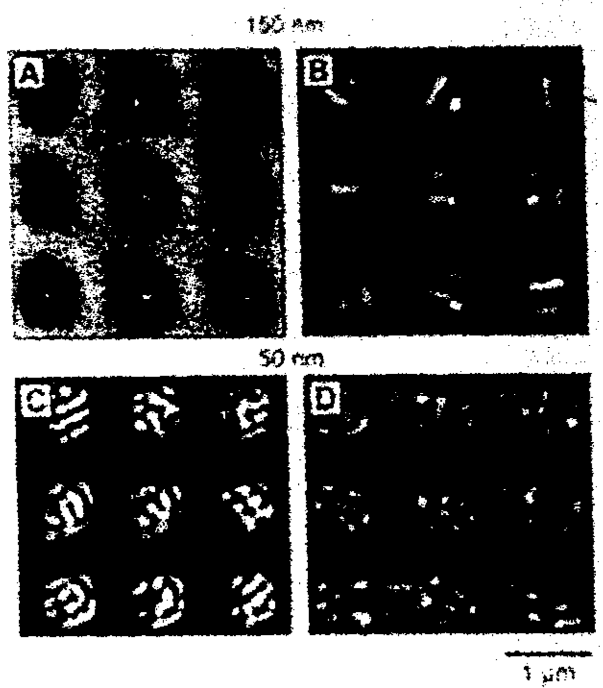
J. Thomassen, et al. Phys. Rev. Lett. **69**, 3831 (1992)

Imaging the Phase Transition



TWO DIMENSIONAL PARTICLES



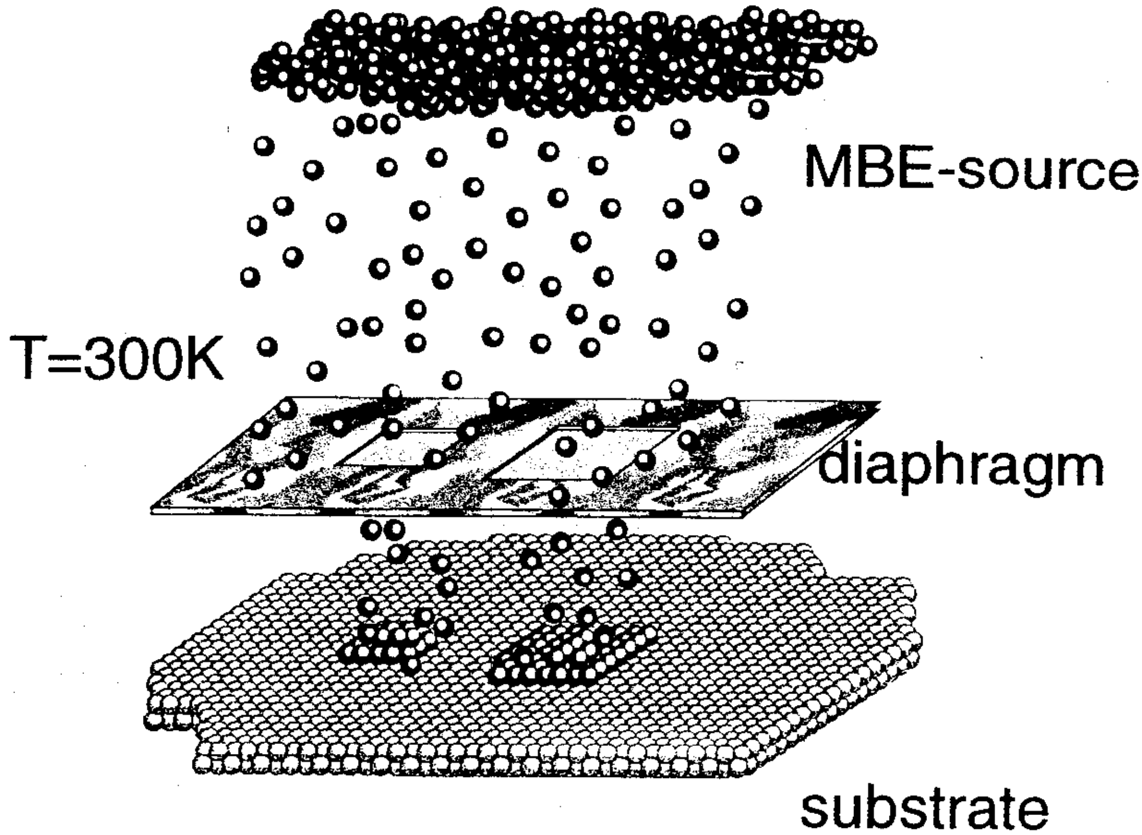


3dimensional particles do have a size and shape dependent domain pattern

M. Hehn, et al. Science **272**, 1782 (1996)

W

The growth of flat (two dimensional) particles

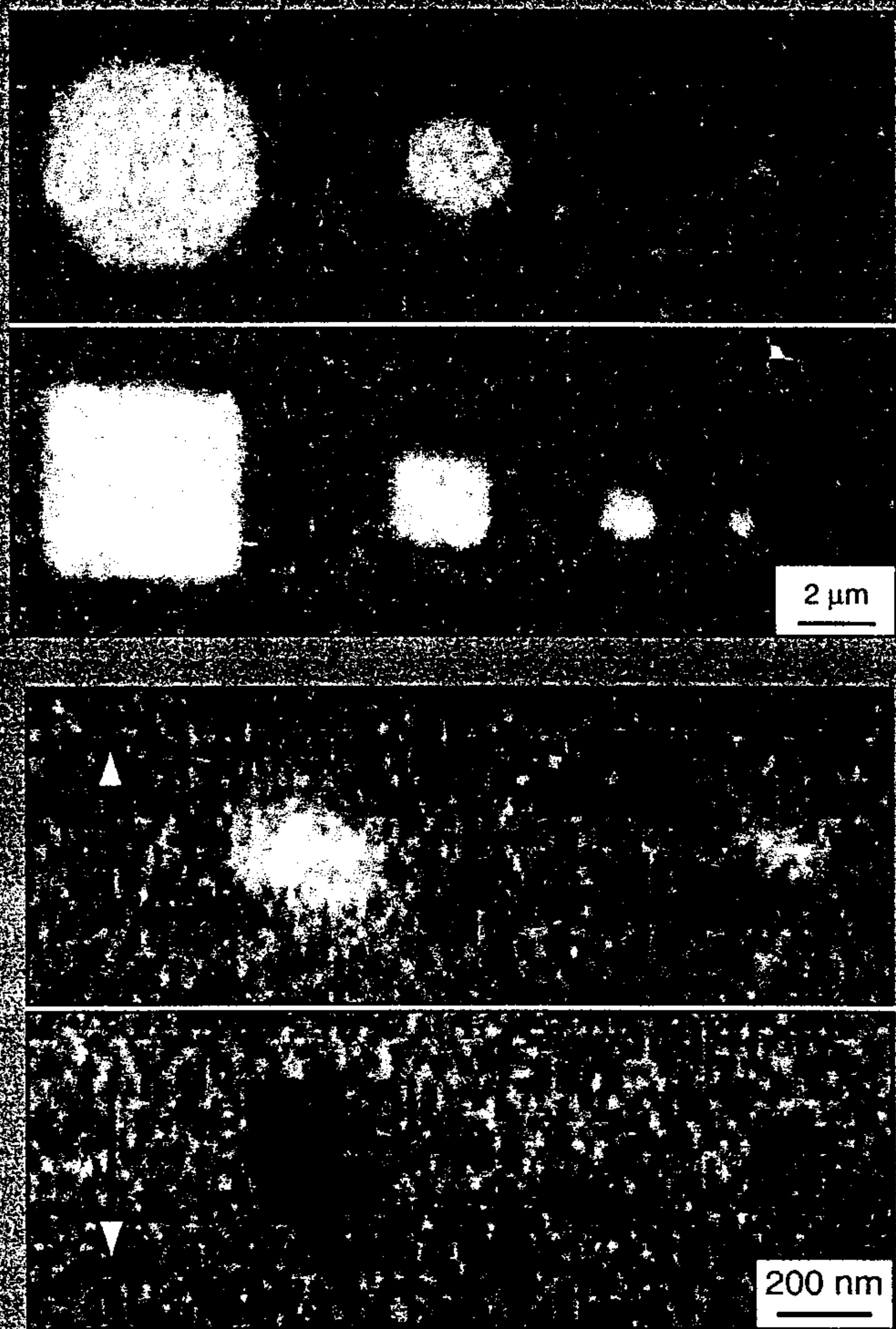


– thickness: 1AL<d<10AL

– lateral dimension: ≥100 nm

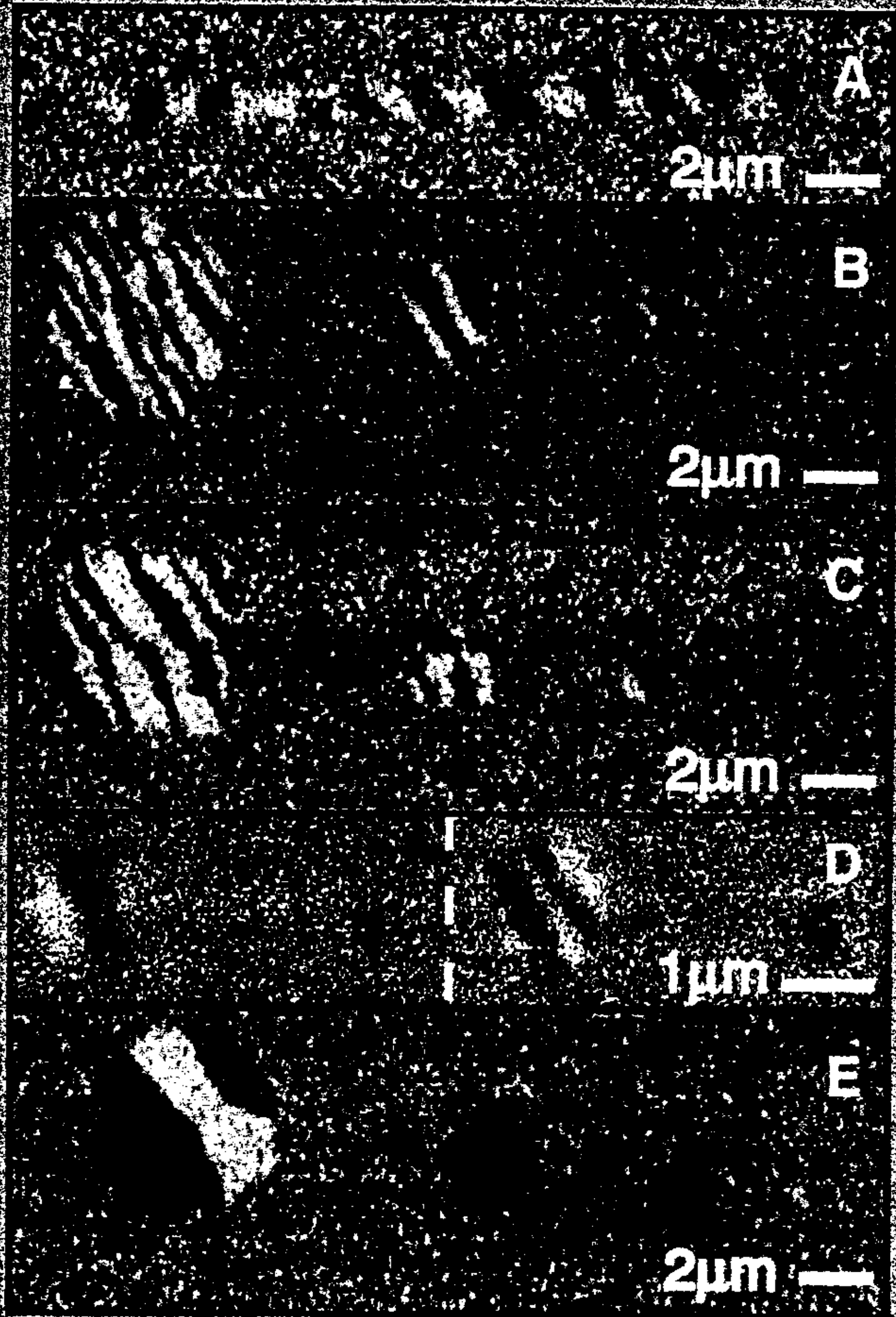
INPLANE MAGNETIZED

Co on Cu(001)



Thickness between 2AL and 3AL

PERPENDICULAR MAG-NETIZED Fe on Cu(001)



size and shape have no influence on the domain pattern

SUMMARY

In the first part of this talk we had a look at the domain pattern of a perpendicularly magnetized thin film and found a remarkable agreement between experiment and theoretical predictions.
(*accepted for publication in Phys. Rev. Lett. 6.1.2000*)

In the second part we did introduce a new class of magnetic particles and found that their properties differ significantly from classical three dimensional particles.
(*Science* ***282****, 449 (1998) and to be published*)