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"Approximate & Numerical Methods for Analysis & Modelling of Optical Fibers & Waveguides"

A. SHARMA
Department of Physics
IIT - Delhi
India

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APPROXIMATE AND NUMERICAL METHODS FOR

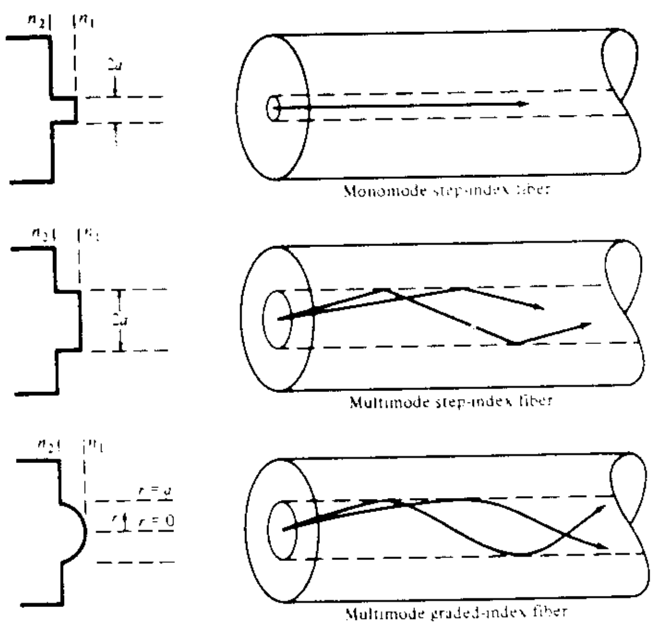
ANALYSIS AND MODELLING OF OPTICAL FIBERS AND WAVEGUIDES

ANURAG SHARMA

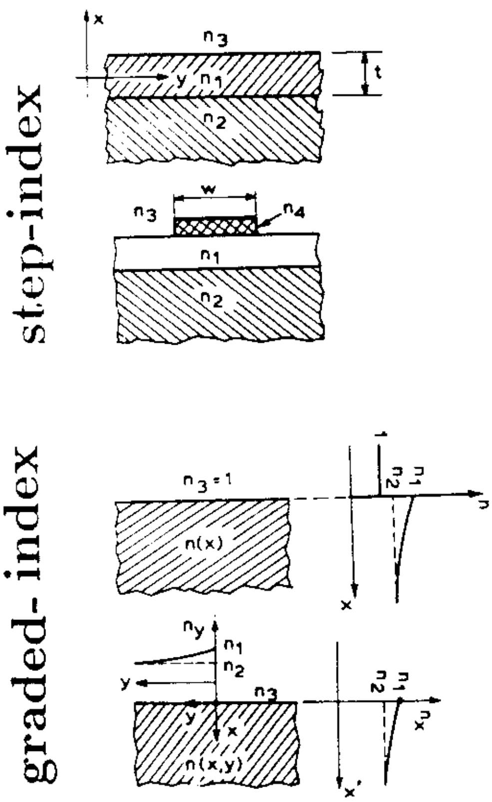
FIBER OPTICS GROUP
PHYSICS DEPARTMENT
INDIAN INSTITUTE OF TECHNOLOGY DELHI
NEW DELHI–110 016
INDIA

OPTICAL WAVEGUIDES

- An optical waveguide is a cylindrical dielectric structure consisting of a high-index region (core or film) surrounded by a relatively lower index region (cladding or substrate/cover).
- Fiber waveguides: cross-section is circularly symmetric: $n^2(r)$
- Integrated Optical Waveguides: cross-section is rectangular (non-circular): $n^2(x, y)$.
- The index may be piecewise homogeneous in diffrent regions: step-index waveguides.
- Or, it may be continuously varying: graded-index waveguides or diffused waveguides.



Fiber waveguides



Integrated Optical Waveguides

Wave Propagation through Optical Waveguides

- Optical Waveguides are dielectric structures and the wave propagation through them is governed by Maxwell's equations.
- Maxwell's equation in a charge-free, non-magnetic medium can be transformed into a vectorial wave equation

$$\nabla^2 \mathbf{E} + k_0^2 n^2 \mathbf{E} + \nabla \left(\frac{\nabla n^2}{n^2} \cdot \mathbf{E} \right) = \mathbf{0}$$

  with time dependence as $e^{i\omega t}$ and $k_0 = \omega / c$.
- The term $\frac{\nabla n^2}{n^2}$ represents relative variation of the refractive index and if the index varies slowly such that its relative variation over a wavelength is small, this term can be neglected. Then,

$$\nabla^2 \mathbf{E} + k_0^2 n^2 \mathbf{E} = 0$$

- Assuming the cartesian system for vectors, it follows that each component of $\mathbf{E}$ satisfies the same equation

$$\nabla^2 \Psi + k_0^2 n^2 \Psi = 0$$

  Here Ψ is one of the cartesian components (say, E_x) and the others (E_y and E_z) are obtained using Maxwell's equations.
- Thus, under the slow variation of the index, the fields can be expressed in terms of a scalar field Ψ which satisfies the scalar wave equation or the Helmholtz equation

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + k_0^2 n^2(x, y, z) \Psi(x, y, z) = 0$$

Modes of Uniform Waveguides

- If the waveguide is uniform along the direction of propagation, z, then

$$n^2(x,y,z) \equiv n^2(x,y) = n^2(r,\phi)$$

- The scalar wave equation then takes the form

$$\nabla_t^2\Psi + k_0^2 n^2(x,y)\Psi(x,y,z) = -\frac{\partial^2\Psi}{\partial z^2}$$

- Separating the z-dependence as $\Psi(x,y,z) = \psi(x,y)Z(z)$, we can obtain

$$Z(z) = \exp(\pm i\beta z)$$

and the scalar field $\psi(x,y)$ satisfies the wave equation

$$\nabla_t^2\psi + [k_0^2 n^2(x,y) - \beta^2]\psi(x,y,z) = 0$$

- Thus, $\Psi(x,y,z,t) = \psi(x,y)\ \mathrm{e}^{i(\omega t-\beta z)}$ represents a forward $(+z)$ propagating mode [for a backward propagating $(-z)$ mode, the argument of the exponential has a +ve sign in front of β].

"A mode is defined as a field configuration which propagates along the waveguide without any change in polarisation or in the field distribution except for a change in phase."

The constant β is called the propagation constant which defines the phase velocity of the mode and the function $\psi(x,y)$ is the mode field distribution or the modal field.

Characteristics of Modes

- The scalar wave equation for modes

$$[\nabla_t + k_0^2 n^2(x,y)]\psi(x,y) = \beta^2\psi(x,y,z)$$

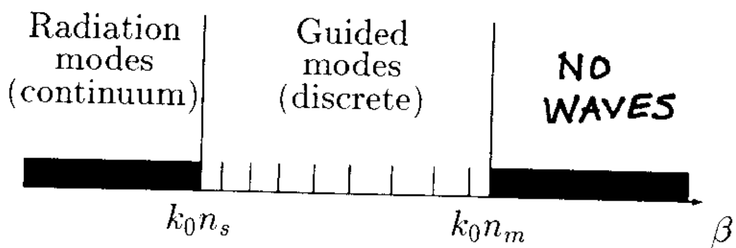
is, in fact, an eigenvalue equation with β^2 as the eigenvalues and $\psi(x,y)$ as the corresponding eigenfunctions of the operator $[\nabla_t + k_0^2 n^2(x,y)]$ defined by the refractive index distribution of the waveguide (the boundary conditions to be imposed are: ψ and its first spatial derivatives should be continuous everywhere).

- In general, a waveguide has a higher refractive-index region surrounded by a lower index region which is usually uniform:

$$n^2(x,y) = n_s^2 + \delta n(x,y) \leq n_m$$

where n_s is the lower uniform index of the surrounding region, n_m is the highest index and $\delta n(x,y)$ represents the variation of the index of the waveguide.

- The eigenvalues, β, have only distrete values in the range $k_m n_0 > \beta > k_0 n_s$ and the corresponding modes are called the **guided modes**. For $\beta < k_0 n_s$, a continuum of modes, the **radiation modes**, exists.



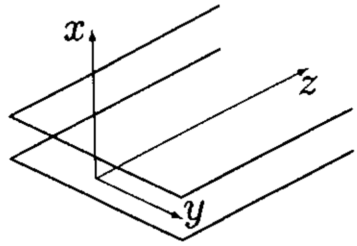
- The modal field is oscillatory in nature when $n(x,y) > \beta/k_0$ and monotonic when $n(x,y) < \beta/k_0$.

- It is possible, in general, to choose such index variation and dimensions that above a certain wavelength only one guided mode exists. Such waveguides are called single mode or monomode waveguides. Multimode waveguide, on the other hand, support a large number of guided modes.

Different Waveguide Geometries

Planar Geometry

$n^2(x, y, z) \equiv n^2(x)$



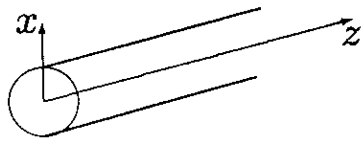
Extends from $-\infty$ to ∞ in y and z directions.

$\Psi(x, y, z, t) = \psi(x)\mathrm{e}^{i(\omega t - \beta z)}$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + [k_0^2 n^2(x) - \beta^2]\psi(x) = 0$$

Circular Geometry

$n^2(x, y, z) = n^2(r, \phi, z) \equiv n^2(r)$

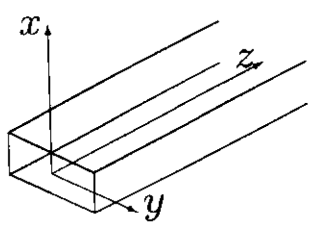


$\Psi(r, \phi, z, t) = \psi(r)\ \mathrm{e}^{il\phi}\ \mathrm{e}^{i(\omega t - \beta z)}$

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} - \frac{l^2\psi}{r^2} + [k_0^2 n^2(x) - \beta^2]\psi(r) = 0$$

Rectangular Geometry

$n^2(x, y, z) \equiv n^2(x, y)$



$\Psi(x, y, z, t) = \psi(x, y)\mathrm{e}^{i(\omega t - \beta z)}$

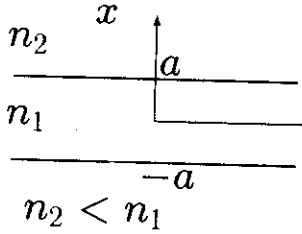
$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\psi}{\mathrm{d}y^2} + [k_0^2 n^2(x) - \beta^2]\psi(x, y) = 0$$

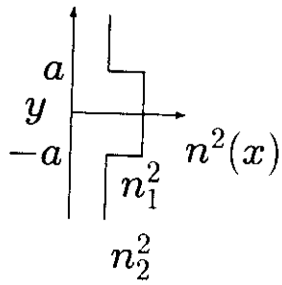
Exactly Solvable Profiles
(Guided Modes)

Planar Step-Index Waveguide

- Profile:

$$n^2(x) \begin{aligned} &= n_1^2 \quad |x| < a \\ &= n_2^2 \quad |x| > a \end{aligned}$$





- Symmetric Modes:

$$\begin{aligned} \psi_s(x) &= A_s \cos(Ux/a) & |x| \leq a \\ &= A_s \cos(U) \exp[-W(|x|/a - 1)] & |x| \geq a \end{aligned}$$

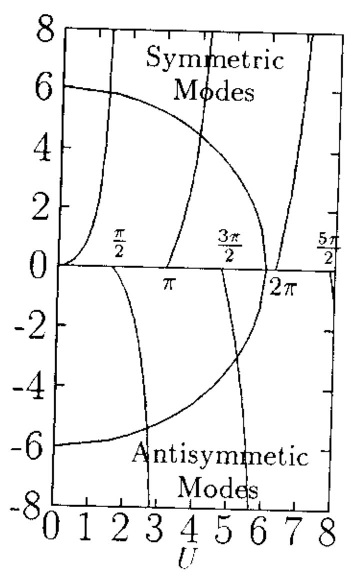
The eigenvalue equation (obtained using continuity of $d\psi(x)/dx$):

$$U \tan(U) = W = \sqrt{V^2 - U^2}$$

$U = a\sqrt{k_0^2 n_1^2 - \beta^2}$; $W = a\sqrt{\beta^2 - k_0^2 n_2^2}$, and $V = k_0 a \sqrt{n_1^2 - n_2^2}$

- Antisymmetric Modes:

$$\begin{aligned} \psi_a(x) &= A_a \sin(Ux/a) & |x| \leq a \\ &= A_a \sin(U)\, e^{-W(x/a-1)} & x \geq a \\ &= -A_a \sin(U)\, e^{W(x/a+1)} & x \leq -a \end{aligned}$$



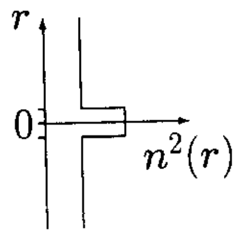
The eigenvalue equation (obtained using continuity of $d\psi(x)/dx$):

$$U \cot(U) = -W$$

Exactly Solvable Profiles
(Guided Modes)

Step-Index Fiber

- The Index Profile:



$$n(r) = n_1 \qquad 0 < r < a \qquad \text{core}$$
$$= n_2 \qquad r > a \qquad \text{cladding}$$

- The modal field, $\psi(r,\phi)$ for the LP_{lm}-modes

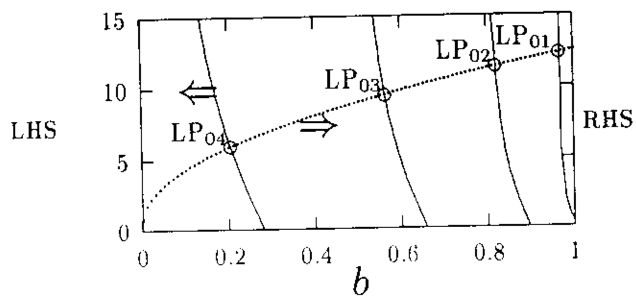
$$\psi(r,\phi) = \mathrm{e}^{il\phi} A J_l(UR) \qquad 0 < R < 1$$
$$= \mathrm{e}^{il\phi} A \frac{J_l(U)}{K_l(W)} K_l(WR) \qquad R > 1$$

$$R = \frac{r}{a}; \qquad U = a\sqrt{k_0^2 n_1^2 - \beta^2}; \qquad V = k_0 a \sqrt{n_1^2 - n_2^2}$$

$$k_0 = \frac{2\pi}{\lambda}; \qquad W = a\sqrt{\beta^2 - k_0^2 n_2^2}; \qquad b = W^2/V^2 = \frac{\beta^2 - k_0^2 n_2^2}{k_0^2(n_1^2 - n_2^2)}$$

- The eigenvalue equation (obtained from continuity of $\mathrm{d}\psi/\mathrm{d}r$ at $R = 1$):

$$\sqrt{1-b}\frac{J_{l+1}(V\sqrt{1-b})}{J_l(V\sqrt{1-b})} = \sqrt{b}\frac{K_{l+1}(V\sqrt{b})}{K_l(V\sqrt{b})}$$



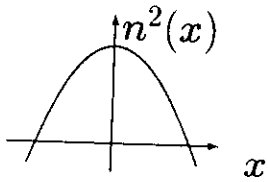
$V = 12, \quad l = 0$

Exactly Solvable Profiles
(Guided Modes)

Parabolic-Index Medium

- One-dimensional (1-D) Medium
  - Profile:

    $n^2(x) = n_1^2[1 - 2\Delta(x^2/a^2)]$

    Δ - grading parameter
  - The modal field, $\psi(x)$:

$$\psi(x) = N_m\ H_m(\xi)\ \mathrm{e}^{-\frac{1}{2}\xi^2}$$

$$\xi = \alpha x, \quad \alpha = (k_0^2 n_1^2 2\Delta/a^2)^{1/4} = \sqrt{V}/a, \quad V = k_0\ a\ n_1\sqrt{2\Delta}$$

  - Eigenvalue equation: $U^2 = V(2m+1)$

- Two-dimensional (2-D) Medium (Cartesian)
  - Profile: $n^2(x,y) = n_1^2[1 - 2\Delta(x^2+y^2)/a^2]$
  - The modal field, $\psi(x,y)$:

$$\psi(x,y) = N_m\ N_n\ H_m(\xi)\ H_n(\eta)\ \mathrm{e}^{-\frac{1}{2}(\xi^2+\eta^2)}, \qquad \eta = \alpha y$$

  - Eigenvalue equation: $U^2 = 2V(m+n+1)$

- Two-dimensional (2-D) Medium (Polar)
  - Profile: $n^2(r) = n_1^2[1 - 2\Delta(r^2/a^2)]$
  - The modal field, $\psi(r)$:

$$\psi(r) = N_{lm}\ r^l L_m^l(\alpha r^2)\ \mathrm{e}^{-\frac{1}{2}\alpha r^2}$$

  - Eigenvalue equation: $U^2 = 2V(2m+n+1)$

GENERAL WAVEGUIDES

- Arbitrary profiles

$$n^2(r) \quad \text{or} \quad n^2(x) \quad \text{or} \quad n^2(x,y)$$

- Quantities of interest

ψ – modal field

β – propagation constant

$\beta'(\lambda), \beta''(\lambda)$ – derivatives of β(for dispersion)

- Effective-index $n_{\text{eff}} = \beta/k_0$

$$n_1 < \beta/k_0 < n_2$$

$$n_1 < n_{\text{eff}} < n_2$$

Typically $n_1 - n_2 \sim 10^{-3}$ (fibers).
Hence accuracy required in $n_{\text{eff}} > 10^{-5}$.
For, β' & β'' the accuracy should be $> 10^{-6}, 10^{-7}$.

- Various methods used are:

WKB	:	multimode waveguides for β
Perturbation	:	for individual modes for β
Variational	:	for single mode/multimode for β & ψ and also β' & β''
Numerical (direct)	:	ψ. β. β' & β'' (few mode)

VARIATIONAL METHOD

- The wave-equation is

$$\nabla_t^2\psi + \left[k_0^2 n^2(x,y) - \beta^2\right]\psi(x,y) = 0$$

- The integral form can be written as

$$\beta^2 = \frac{\iint k_0^2 n^2(x,y)\,|\psi|^2\;dx\,dy - \iint |\nabla_t\psi|^2\;dx\,dy}{\iint |\nabla_t\psi|^2\;dx\,dy}$$

- The right hand side is STATIONARY with respect to variations in $\psi(x,y)$.

- If we substitute a function $\psi_t(x,y)$ for $\psi(x,y)$, the stationary expression gives an estimate β_t^2 for β^2 which is always such that [for fundamental mode]

$$\beta_t^2 < \beta_{\text{exact}}^2$$

- If we try a number of trial functions $\psi_t(x,y)$ and estimate corresponding β_t^2, then the largest value of β_t^2 would be closest to the exact value and the corresponding, $\psi_t(x,y)$ would represent the best approximation for the modal field, $\psi(x,y)$.

- A variational estimate of β^2, thus, represents a lower bound for the propagation constant.

ALGORITHM FOR VARIATIONAL METHOD

- Set-up a trial field

$$\Psi_t(x, y; p_1, p_2, p_3 \cdots p_n)$$

  $p_1, p_2, \ldots p_n$ are n-parameters which are adjustable.

- The functional dependence of Ψ_t on x, y is chosen in such a way that it resembles the modal field as far as possible.

- This field is then used in the stationary expression and β_t^2 is maximized with respect to $p_1, p_2, \cdots p_n$.

- The maximum value of β_t^2 is the estimate for the propagation constant & the function $\psi_t(x, y; p_1, \cdots p_n)$ with optimized values of $p_1, p_2, \cdots p_n$ is the approximation for the modal field.

- Generally, by increasing the number of parameters, in a suitable fashion, the accuracy of a trial field can be increased. But, **a better trial field with a smaller number of parameters is always sought for**.

VARIATIONAL METHOD (contd.)

EQUIVALENT WAVEGUIDES

- The optimized trial field $\psi_t(x, y)$ and the propagation constant, β_t^2, when used in the wave equation give an index profile, $n_t^2(x, y)$:

$$n_t^2(x, y) = \frac{1}{k_0^2}\left[\beta_t^2 - \frac{1}{\psi_t}\nabla\psi_t\right]$$

- Within the accuracy of the trial field, the index profile $n_t^2(x, y)$ represents a waveguide EQUIVALENT to the given waveguide as far as the mode used is concerned. This is useful for single mode waveguide.

PERTURBATION METHOD AS A VARIATIONAL METHOD

- The variational estimate obtained by simply substituting a trial field, $\psi_P(x, y)$ without optimization is same as the one obtained using the FIRST ORDER PERTURBATION METHOD with the unperturbed profile as $n_P^2(x, y)$ (corresponding to $\psi_P(x, y)$).
- The optimized $\psi_t(x, y)$ is such that the correction obtained by first order perturbation theory is zero!
- So, the first order perturbation method can be regarded as UNOPTIMIZED variational method

Rayleigh-Ritz Method

- TRIAL FIELD:

$$\psi_t(x,y) = \sum_{m=1}^{M} C_m \phi_m(x,y) \qquad m = 1, M$$

- $\phi_m(x,y)$ are orthonormal set of functions:

$$\iint \phi_m^*(x,y)\,\phi_{m'}(x,y)\;dx\,dy = \delta_{mm'}$$

- The stationary expression then becomes:

$$\beta^2 \sum_m \sum_{m'} c_m c_{m'} \iint \; \phi_m^* \phi_{m'} \; dx\,dy$$

$$= \sum_m \sum_{m'} c_m c_{m'} \iint \; \left(k_0^2 n^2(x,y) \phi_m^* \phi_{m'} - |\nabla_t \phi_m^* \cdot \nabla_t \phi_{m'}|\right) \; dx\,dy$$

- Maximizing w.r.t c_m: $\partial\beta^2/\partial c_m = 0$ leads to

$$HC = \beta^2 C$$

where

$$\begin{aligned} H &= \left\{H_{mm'} \equiv \iint \; \left[k_0^2 n^2(x,y)\phi_m^*\phi_{m'} - |\nabla_t \phi_m^* \cdot \nabla_t \phi_{m'}|\right] \; dx\,dy\right\} \\ C &= \{C_m\} \end{aligned}$$

- The matrix eigenvalue problem gives M eigenvalues of which some may correspond to guided modes & others to discrete representations of radiation modes. However, only higher values of β^2 have good accuracy.

DIRECT NUMERICAL METHODS

PLANAR WAVEGUIDES

- **Wave Equation**:

$$\frac{d^2\psi}{dx^2} + \left[k_0^2 n^2(x) - \beta^2\right] \psi(x) = 0$$

  - Second order homogeneous ordinary differential equation.
  - One could use:
    - Runge-Kutta Method (self starting)
    - Predictor-Corrector Methods (require a starter)
  - Boundary conditions
    - ψ & $d\psi/dx \to 0$ at $x \to \pm\infty$
    - both ψ & $d\psi/dx$ are continuous everywhere.

- **Eigenvalue Determination:**
  - Since the equation is homogeneous one of ψ & $\psi'(x)$ is arbitrary.
  - For a given β, the equation can be solved for an initial ψ'/ψ & ψ/ψ'.
  - The boundary condition at $x \to +\infty$ would be satisfied by only certain β, which are the eigenvalues.
  - So, by varying β, those β are found for which the boundary conditions are matched.

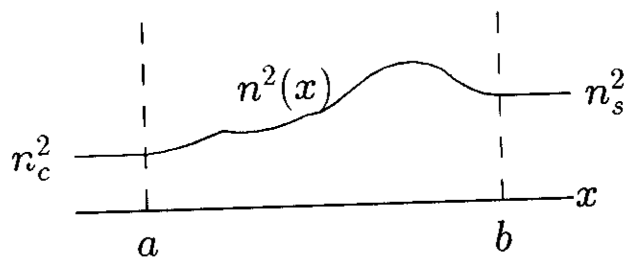
Algorithm

- Select a point $x = a$ such that the refractive index is constant for $x < a$ or at least can be assumed to be so. Let the index be n_c.

  The solution for $x < a$ is then

$$\psi(x) = A \exp\left[+\sqrt{\beta^2 - k_0^2 n_c^2}\,(x-a)\right]$$

$$\psi'(x) = \sqrt{\beta^2 - k_0^2 n_c^2}\,\psi(x)$$



- Similarly, select a point $x = b$ such that the index is constant or can be assumed to be so for $x > b$. Let the index be n_s. Then for $x \geq b$

$$\psi'(x)/\psi(x) = -\sqrt{\beta^2 - k_0^2 n_s^2}$$

- Starting with $\psi(x = a) = 1$ and $\psi'(x = 1) = -\sqrt{\beta^2 - k_0^2 n_c^2}$ one can numerically solve the wave equation to obtain $\psi'(x)/\psi(x)|_{x=b}$. This quantity would be a function of β. Thus,

$$F(\beta) = \left.\frac{\psi'(x)}{\psi(x)}\right|_{x=b}$$

- The eigenvalue (transdental) equation for determining β is

$$F(\beta) = -\sqrt{\beta^2 - k_0^2 n_s^2}$$

Convergence Tests

- There are two convergences to be tested:

  a) Convergence of the numerical solution of the differential equation with h, the extrapolation interval in the Runge-Kutta method.

  b) Convergence of β.

- The transcendental equation can be solved by either bisection method or secant method (since it is not easy to compute the derivative of the function w.r.t β).

Riccati Transformation

- The transformation:

$$G(x) = \psi'(x)/\psi$$

  may be used to obtain

$$G'(x) = -G^2 - \left[k_0^2 n^2(x) - \beta^2\right]$$

  - Ist order differential equation
  - leads to shorter computation time
  - useful for lower order modes particularly the fundamental mode

- For higher order modex, as $\psi(x) \to 0$, $G(x) \to \infty$ and hence beyond $G(x) = 1$. one should switch to $F(x) = 1/G(x)$ & use:

$$F'(x) = 1 + F^2\left[k_0^2 n^2(x) - \beta^2\right]$$

OPTICAL FIBERS

- For optical fibers, the equation is

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \frac{l^2\psi}{r^2} + \left[k_0^2 n^2(r) - \beta^2\right] \psi(r) = 0$$

- The index profile is

$$\begin{aligned} n^2(x) &= n^2(r) \qquad r < a \\ &= n_{cl} \qquad\;\; r > a \end{aligned}$$

- For $r > a$, the solution is $[W = a\sqrt{\beta^2 - k_0^2 n_{cl}^2}]$

$$\psi(r) = A\,\frac{K_l(W\,r/a)}{K_l(W)}$$

$$\psi'(x) = A\,\frac{W}{a}\,\frac{K_l'(W\,r/a)}{K_l(W)}$$

$$\text{or,} \quad \psi'/\psi\,(\mathrm{r}) = \frac{W}{a}\,\frac{K_l'(W\,r/a)}{K_l(W)}$$

- The boundary condition at $r = 0$ is

$$\begin{aligned} \psi'(r)/\psi(r) &= 0 \qquad \text{for} \qquad l = 0 \\ \psi(r)/\psi'(r) &= 0 \qquad \text{for} \qquad l \neq 0 \end{aligned}$$

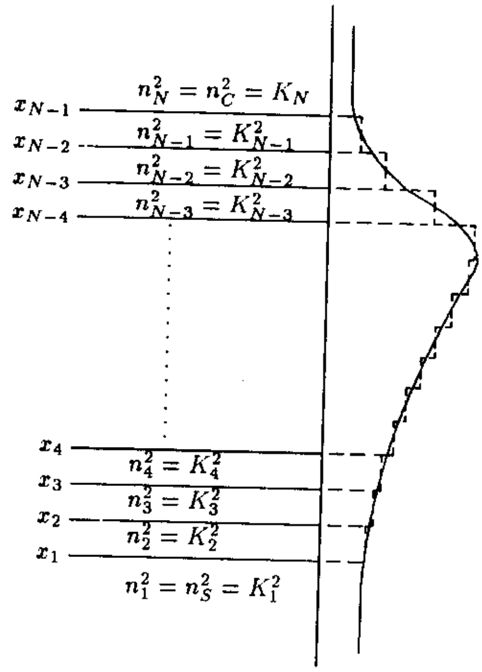
- It is better to start the solution at $r = a$ and solve backwards to obtain the equation (for $l = 0$)

$$\left.\frac{\psi'}{\psi}\right|_{r=0}(\beta) = 0$$

which is transcendental eigenvalue equation.

- Riccati transformation can be helpful here too!

STAIRCASE (OR MULTILAYER) METHOD



- The graded part is divided in N-parts such that the interfaces are at $x_1, x_2, x_3 \ldots x_{N-1}$ and the index between x_m and x_{m+1} is $n_m^2 = K_m$.
- The lowest layer (substrate) between $-\infty < x \leq x_1$ has an index $n_s^2 = K_1$ and upper most layer (cover) between $x_{N-1} < x < \infty$ has an index $n_c^2 = K_N$
- We start with an exponentially decaying solution in the substrate:

$$\psi_1(x) = A_1 \exp[\gamma_1 x] \quad -\infty < x \leq x_1 = 0$$

- The solution in the m^{th} layer would be

$$\psi_m(x) = A_m \cos[\gamma_m (x - x_m)] + B_m \frac{\sin[\gamma_m (x - x_m)]}{\gamma_m}$$
$$x_{m-1} \leq x \leq x_m$$

with $\gamma_m = k_0 [K_\beta - K_m]^{1/2}$ & $K_\beta = (\beta/k_0)^2 = n_{\text{eff}}^2$

$\Rightarrow$ If $K_\beta < K_m$ is a layer then $\gamma_m = i\kappa_m$ and in the field expression simply replace cos by cosh and $\sin[\gamma_m \ldots]/\gamma_m$ by $\sinh[\kappa_m \ldots]/\kappa_m$

- The boundary conditions: ψ & $d\psi/dx$ continuity is applied at each interface
- In the final layer $\gamma_N = i\kappa_N$ and only a decaying solution should exist & hence

$$A_N + (B_N/\kappa_N) = 0 \qquad \Rightarrow \text{eigenvalue equation}$$

EVALUATION OF COEFFICIENTS A_m, B_m

- Using the continuity condition, one can obtain a simple recurrence scheme to evaluate A_m & B_m:

$$A_1 = \text{Arbitrary} = 1(\text{say})$$

$$A_2 = A_1 \quad B_2 = \gamma_1 A_1$$

$m \geq 2$

$$A_{m+1} = A_m \cos(\gamma_m d_m) + B_m \gamma_m^{-1} \sin(\gamma_m d_m)$$

$$B_{m+1} = [-A_m \gamma_m \sin(\gamma_m d_m) + B_m \cos(\gamma_m d_m)]$$

- Using above recurrence relations one can obtain,

$$F_N = A_N + (B_N/\kappa_N)$$

which is a function of β: $F_N = F_N(\beta)$.

- The eigenvalues are then given by as the solution of

$$F_N(\beta) = 0$$

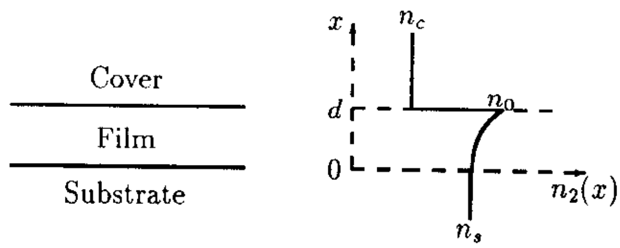
- Convergence has to be tested with respect to N and for the value of β.

- The above procedure is valid *even* when one or more layers have complex refractive index (absorbing/amplifying media). All the quantities, then become complex and a two dimensional search is required to obtain β, since it is also complex.

PLANAR WAVEGUIDES

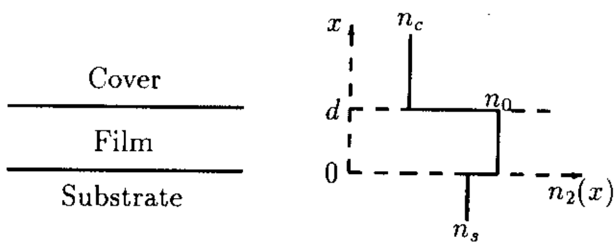
- General structure

$$n^2(y) = n_c^2 \qquad y < 0$$
$$\quad\;\; = n_f^2(y) \qquad y > 0$$



- Step-index waveguides

$$n^2(y) = n_c^2 \qquad y < 0$$
$$\quad\;\; = n_f^2 \qquad 0 < y < a$$
$$\quad\;\; = n_s^2 \qquad y > a$$



- Analytical solutions of the wave equation:

$$\psi(y) = \begin{cases} \cos(U\sigma)\; e^{W_c\, y/a} & \infty < y \leq 0 \\ \cos\left[U\,(y/a - \sigma)\right] & 0 \leq y \leq a \\ \cos\left[U\,(1-\sigma)\right]\; e^{-W_s\left(\frac{y}{a}-1\right)} & a \leq y < \infty \end{cases}$$

$$U = a\sqrt{k_0^2 n_f^2 - \beta^2},\; W_c = a\sqrt{\beta^2 - k_0^2 n_c^2},\; W_s = a\sqrt{\beta^2 - k_0^2 n_s^2}$$

- The continuity of $d\psi/dy$ at $y = 0$ and $y = a$ gives

$$\tan U = \frac{U\,(W_c + W_s)}{U^2 - W_s W_c}$$

and $\sigma = \frac{1}{U}\tan^{-1}(W_c/U)$

- $x = \sigma$ is the point where the field has a peak.

ARBITRARY INDEX PROFILES

- The general structure of graded-index planar waveguides can be written as

$$\begin{aligned} n^2(y) &= n_s^2 + 2n_s\Delta n\, f(y) \quad & y > 0 \\ &= n_c^2(y) & y < 0 \end{aligned}$$

- These waveguides are generally made by diffusion of ions into substrates of glass, $LiNb0_3$, etc. and generally have profiles which can be well modeled through:

$$\begin{aligned} f(y) &= \exp\left(-y/D\right) \quad & \text{exponential} \\ &= \exp\left(-y^2/D^2\right) & \text{Gaussian} \\ &= \text{erfc}\left(y/D\right) & \text{complementary error function} \end{aligned}$$

$D-$diffusion depth (a constant)

- Propagation constant is generally expressed as

$$B = \left[n_{\text{eff}}^2 - n_s^2\right]/2n_s\Delta n$$

where $n_{\text{eff}} = \beta/k_0$ is called the effective index of the mode.

- The values of n_s are $\sim 1.5 - 1.7$(glass) 2.2-2.3 ($LiNb0_3$)
Typical values of $\Delta n \sim 0.002 - 0.01$

- The error in $B \rightarrow \Delta B$ can be written as

$$\Delta B = \Delta n_{\text{eff}}^2/2n_s\Delta n \approx \Delta n_{\text{eff}}/\Delta n$$

Hence, for an accuracy of 10^{-5} in n_{eff}, the accuracy required in $\Delta B \sim 10^{-3}$.

VARIATIONAL METHODS FOR PLANAR WAVEGUIDES

- The stationary expression is

$$\beta_t^2 = \frac{\int\limits_{-\infty}^{\infty} k_0^2 n^2(y) \left|\psi_t(y)\right|^2 \, dy - \int\limits_{-\infty}^{\infty} \left|\frac{d\psi_t}{dy}\right|^2 \, dy}{\int\limits_{-\infty}^{\infty} \left|\psi_t(y)\right|^2 \, dy}$$

- By making different ansatz for $\psi_t(y)$ and maximizing β_t^2, one obtains different models:
  - Hermite-Gauss Model (HG) Korotky et al.(1982)
  - Cosine-Exponential Model (CE) Mishra & Sharma (1985)
  - Secant-Hyperbolic Model (SH) Sharma & Bindal (1934/94)

A. Hermite-Gauss (HG) Model

Trial field

$$\begin{aligned} \psi_t(y) &= 0 & y \leq 0 \\ &= A(y/d)\, e^{-y^2/d^2} & y \geq 0 \end{aligned}$$

- A simple function
- Only one variational parameter: d
- Assume that field vanishes in the cover. Due to large index difference between the film & the cover at $y = 0$, this assumption has been made.
- But the field varies as a Gaussian deep in the substrate rather than as an exponential.

B. Secant-Hyperbolic (SH) Model

Trial field

$$\begin{aligned}\psi_t(y) &= A\sin(y/D)\,\mathrm{sech}^{\tau}(y/D) && y \geq 0 \\ &= 0 && y \leq 0\end{aligned}$$

- Simple function
- Only one variational parameter: τ
- Also assume vanishing field in the cover.
- The field deep in the substrate varies more like exponential as it should indeed be.

C. Cosine-Exponential (CE) Model

Trial field

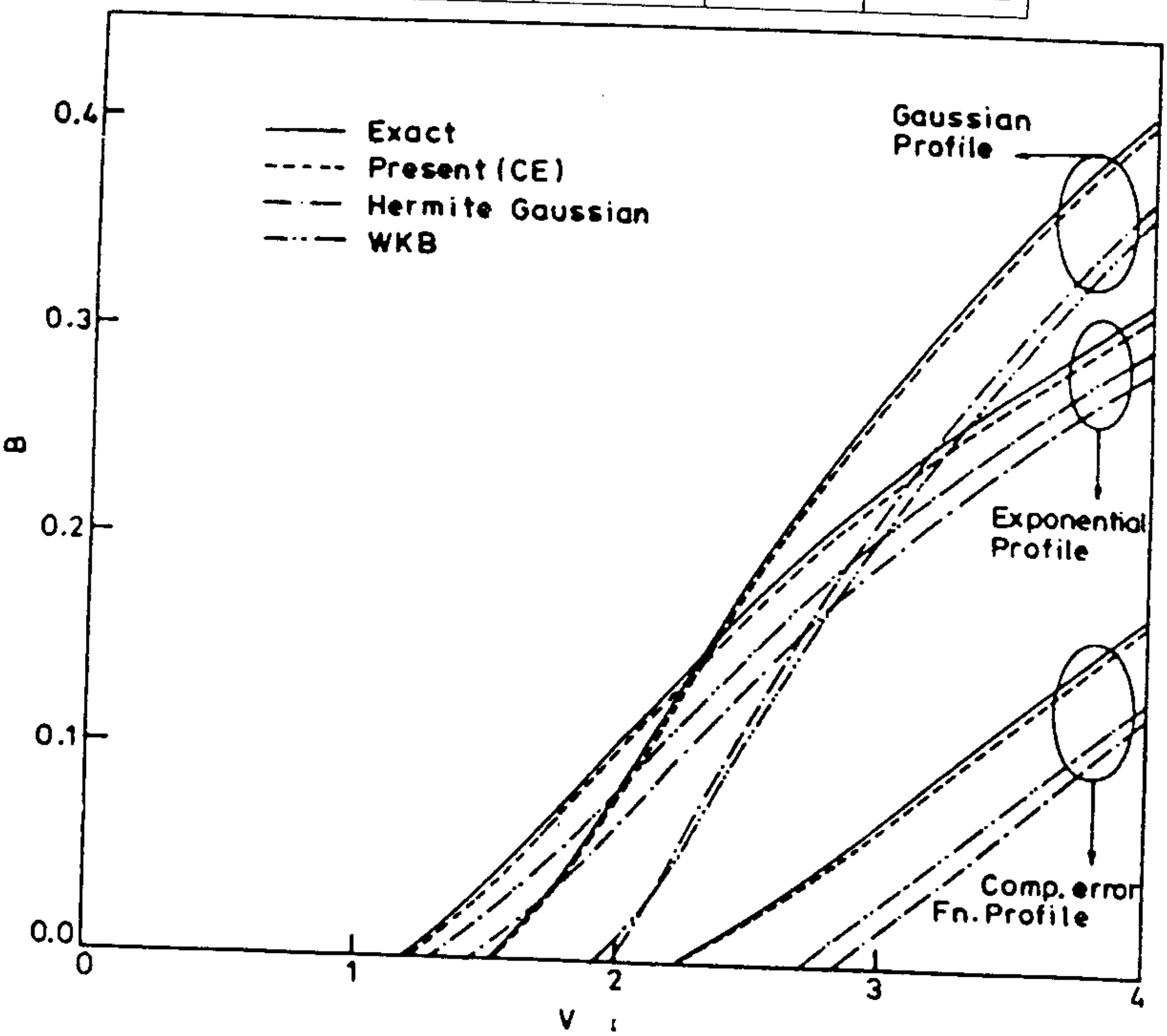
$$\begin{aligned}\psi_t(y) &= A\cos(pr)\,e^{p\tan pr\cdot(y/D)} && y \leq 0 \\ &= A\cos\left[p\left\{\tfrac{y}{D}-\sigma\right\}\right] && 0 \leq y \leq \xi D \\ &= A\cos\left[p\left\{\xi-\sigma\right\}\right] e^{-p\tan\{p(\xi-\sigma)\}[(y/D)-\xi]} && y \geq \xi D\end{aligned}$$

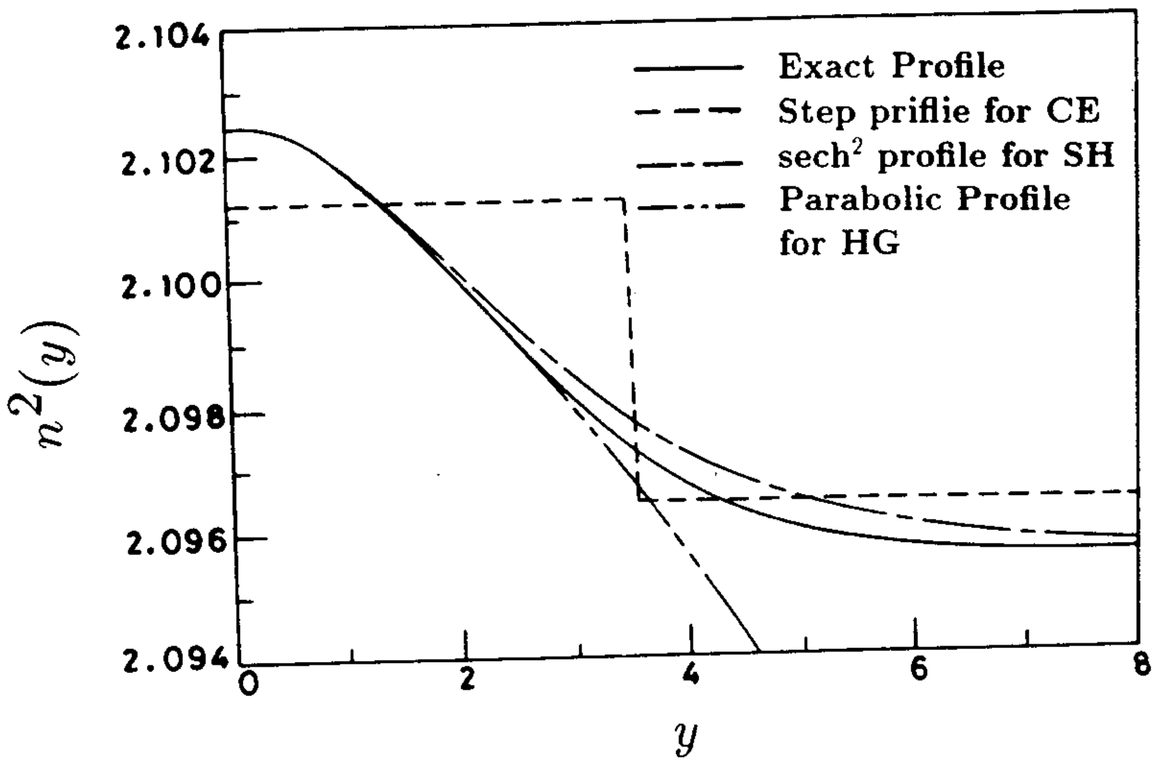
- $\psi_t(y)$ is assumed to be the mode of an asymmetric 3-layer waveguide.
- Relatively complex function
- Three variational parameters: p, ξ, σ
- No assumption of vanishing field in the cover

RESULTS FOR PLANAR WAVEGUIDES

Values of $B = [(\beta/k_0)^2 - n_s^2]/2n_s\Delta n$

g(y)	V	Exact	HG	CE
$\exp(-y^2/D^2)$	2.0	0.082	0.005	0.078
	3.0	0.275	0.216	0.270
	4.0	0.413	0.370	0.408
$\exp(-y/D)$	2.0	0.105	0.066	0.100
	3.0	0.229	0.193	0.223
	4.0	0.321	0.289	0.316
erfc(y/D)	3.0	0.068	0.015	0.064
	4.0	0.169	0.121	0.164





Exact Profile
Step priflie for CE
sech^2 profile for SH
Parabolic Profile
for HG
2.104
2.102
2.100
2.098
2.096
2.094
$n^2(y)$
0
2
4
6
8
y

SINGLE MODE OPTICAL FIBERS

- Only one mode LP_{01}
- The field is symmetric without any zeroes
- The profile:

$$\begin{aligned} n^2(r) &= n_1^2 f(r) \quad r < a \quad \text{(core)} \\ &= n_2^2 \qquad\quad r > a \quad \text{(cladding)} \end{aligned}$$

- The field

$$\begin{aligned} \psi(r) &= F(r) \qquad\qquad\qquad r \leq a \\ &= F(a) \frac{K_0(Wr/a)}{K_0(W)} \quad r \geq a \end{aligned}$$

- In the single mode region a sizable fraction of power flows in the cladding and hence the field in the cladding, particularly near $r = a$ region is important.
- The modal field $\psi(r)$ and the propagation constant, β^2, have been obtained through approximations:
  - Gaussian
  - Gaussian Exponential(GE)
  - Gaussian exponential Hankel(GEH)
  - Equivalent Step-index(ESI)
- The dispersion has been obtained using
  - Direct numerical method
  - Rayleigh-Ritz method

APPROXIMATE MODELS FOR SMFs

A. GAUSSIAN APPROXIMATION

- The simplest & earliest model (Marcuse, 1976)
- Trial field

$$\psi_{\mathrm{GA}}(r) = A\, e^{-r^2/w^2}$$

- $w \rightarrow$ spot-size of the width of the Gaussian field.
- w is obtained either by maximizing the coupling efficiency of the field, $\psi_{\mathrm{GA}}(r)$ to the fiber mode (Marcuse, 1976) or by the variational expression (Snyder & Love, 1983).
- The approximation works well for fiber operating near the edge of the single mode region, but is very poor for lower V-values since, the field in the cladding decays too rapidly that the $K_0(Wr/a)$ function.

B. GAUSSIAN-EXPONENTIAL(GE) APPROXIMATION

- Replaced the Gaussian field in the cladding by a matched exponential function (Sharma & Ghatak, 1981).
- Trial field

$$\begin{aligned}\psi_{\mathrm{GE}}(r) &= Ae^{-\alpha r^2} & r \leq d \\ &= Ae^{-\alpha d(2r-d)} & r \geq d\end{aligned}$$

  Variational Parameter: α & d.
- The field is Gaussian upto $r = d$ and exponential for $r > d$. The $r = d$, the field and its derivative are matched
- The performance improves for smaller V values.

C. GAUSSIAN-EXPONENTIAL HANKEL(GEH) APPROXIMATION

- To improve the field in the cladding further the GE was modified to replace the exponential in the cladding by K_0 function.
- Trial Field

$$\begin{aligned} \psi_{\text{GEH}}(r) &= Ae^{-\alpha r^2} & r \leq d \\ &= Ae^{-\alpha d(2r-d)} & d \leq r \leq a \\ &= Ae^{-\alpha d(2a-d)} \frac{K_0(\gamma r)}{K_0(\gamma a)} & r \geq a \end{aligned}$$

The continuity of derivative at $r = a$ gives

$$\alpha = \frac{\gamma}{2d} \frac{K_1(\gamma\, a)}{K_0(\gamma\, a)}$$

- The variational parameters are γ & d.
- Performance was extremely good even when $V \approx 0$ & the field in the cladding was also modelled quite accurately.

D. EQUIVALENT STEP-INDEX(ESI) APPROXIMATION

- The field is assumed to correspond to *some* step-index fiber.
- The variational parameters are usually $\Delta_e = \left(n_1^2 - n_2^2/2n_2^2\right)$ and a_e (the radius of core) of the step-index fiber.
- The field:

$$\begin{aligned} \psi(r) &= A \frac{J_0(ur/a_e)}{J_0(u)} & r \leq a_e \\ &= A \frac{K_0(wr/a_e)}{K_0(w)} & r \geq a_e \end{aligned}$$

u & w correspond to $V_e = k_0 a_e n_2 \sqrt{2\Delta_e}$; n_2 = cladding index of the given fiber.

PERFORMANCE OF MODELS FOR SINGLE MODE FIBERS

			Gaussian Approx.		ESI-Model		GE-Model		GEH-Model	
q	V	U_{exact}	U	Error(%)	U	Error(%)	U	Error(%)	U	Error(%)
0.25	7.76	7.14467	7.16364	0.27	7.15736	0.18	7.14980	0.072	7.14976	0.071
2.0	1.5	1.46587	1.49637	2.10	1.46608	0.015	1.46830	0.17	1.46591	0.002
	2.5	2.14316	2.16728	1.12	2.14563	0.11	2.14362	0.02	2.14340	0.01
	3.5	2.61244	2.62145	0.34	2.61932	0.30	2.61268	0.01	2.61268	0.01
	5.0	3.15473	3.15658	0.058	3.16952	0.47	3.15483	0.003	3.15484	0.003
∞	1.5	1.31689	1.34571	2.18	1.31689	0.00	1.31829	0.11	1.31737	0.04
	2.0	1.52811	1.54476	1.08	1.52811	0.00	1.53107	0.19	1.53084	0.17
	2.4	1.64528	1.65859	0.80	1.64528	0.00	1.65160	0.38	1.65153	0.37

DISPERSION IN SINGLE MODE OPTICAL FIBERS

- Dispersion is due to wavelength variation of the effective index $n_e = \beta / k_0$.
- The dispersion coefficient (ps/km-nm) is defined as (chromatic dispersion)

$$\Delta\tau_{\text{ch}} = \frac{-\lambda}{c n_e} \frac{d^2}{d\lambda^2}(n_e) \simeq \frac{-1}{0.0003\lambda}\left(\lambda^2 \frac{d^2 n_e}{d\lambda^2}\right) \frac{\text{ps}}{\text{km.nm}}$$

$$(\lambda \rightarrow \mu\text{m})$$

- Assuming:

$$n^2(r) = n_2^2 + \left(n_1^2 - n_2^2\right) f(r) \quad \begin{cases} f(r)\,\text{indep.} \\ \text{of}\,\lambda \end{cases}$$

$$\Rightarrow \left(-\frac{c n_e}{\lambda}\right) \Delta\tau_{\text{ch}} = (1-b)\,\nu_2 + b\nu_1 + 2b'\phi + \frac{1}{2}b''\theta$$

$$-\frac{1}{n_e^2}\left(n_2 n_2' + b\phi + \frac{1}{2}b'\theta\right)^2$$

with

$$\begin{aligned} \nu_i &= n_i n_i'' + n_i'^2 \quad i = 1,2 \\ \phi &= n_1 n_1' - n_2 n_2'; \quad \theta = n_1^2 - n_2^2 \\ b' &= \tfrac{d}{d\lambda}(b) = bV\left(\tfrac{\phi}{\theta} - \tfrac{1}{\lambda}\right) \\ b'' &= \dot{b}V\left(\tfrac{\nu_1 - \nu_2}{\theta} - \tfrac{\phi^2}{\theta^2} - \tfrac{2\phi}{\lambda\theta} + \tfrac{2}{\lambda^2}\right) \\ &\quad + \ddot{b}V^2\left(\tfrac{\phi}{\theta} - \tfrac{1}{\lambda}\right)^2 \end{aligned}$$

$' \equiv d/d\lambda$ & $^{\bullet} \equiv d/dv$

Required quantities: b, $\dot{b} = \frac{db}{dv}$, $\ddot{b} = \frac{d^2b}{dv^2}$, and n_1', n_1'', n_2', n_2'' which are obtained from Sellmeier's coefficients.

- Since $\Delta = \frac{n_1^2 - n_2^2}{2n_2^2}$,

$$\begin{aligned} \Rightarrow n_1^2 &= n_2^2(1 + 2\Delta) \\ \Rightarrow n_1 &= n_2\sqrt{1 + 2\Delta} \end{aligned}$$

- An approximation often made is that Δ is independent of λ, then the λ-dependence of n_1 is related to that of n_2, which is usually silica.

- For silica, the λ-dependence is modelled as

$$n^2(\lambda) = 1 + \frac{A\lambda^2}{\lambda^2 - B^2} + \frac{C\lambda^2}{\lambda^2 - D^2} + \frac{E\lambda^2}{\lambda^2 - F^2}$$

where A, B, C, D, E, F are Sellmeier's coefficients:

$$A = 0.6961663;\ B = 0.0684043;\ C = 0.4079426$$
$$D = 0.1162414;\ E = 0.8974794;\ F = 9.8961610$$

[Malitson, *J.Opt.Soc.Am.* **55**, 1205-1209(1965)]
The empirical formula is valid for the entire λ-range used in optical communications.

- Waveguide dispersion:
If n_1 & n_2 are assumed to be independent of λ, then we have waveguide dispersion

$$\Delta\tau_w \cong -\frac{n_2\Delta}{0.0003\lambda} V \frac{d^2(Vb)}{dV^2} \quad \text{ps/km.nm}$$

- Material Dispersion (of cladding)

$$\Delta\tau_m \approx -\frac{1}{0.0003\lambda}\left(\lambda^2 \frac{d^2 n_2}{d\lambda^2}\right) \quad \text{ps/km.nm}$$

- To a good approximation:

$$\Delta\tau_{ch} \approx \Delta\tau_w + \Delta\tau_m \,(\text{for } \ n_2)$$

DIRECT NUMERICAL METHOD FOR DISPERSION IN SMFs

- Wave equation

$$\frac{d^2\psi}{dr^2} + \frac{1}{2}\frac{d\psi}{dr} + \left[k_0^2 n^2(r) - \beta^2\right]\psi = 0$$

- Riccati transformation: $G(r) = (d\psi/dr)/\psi(r)$

$$\frac{dG}{dR} + G^2 + \frac{G}{R} + V^2 f(R) - W^2 = 0 \qquad (1)$$

B.C: $G(0) = 0$;$G(1) = -\frac{WK_1(W)}{K_0(W)} = F(W)$

with $R = r/a$ $\quad V = k_0 a\sqrt{n_1 - n_2} \quad W = \omega\sqrt{\beta^2 - k_0^2 n_2^2}$

- Differentiating w.r.t V: $\quad {}^{\bullet} \equiv d/dV$

$$\frac{d\dot{G}}{dR} + 2G\dot{G} + \frac{\dot{G}}{R} + 2Vf(R) - 2W\dot{W} = 0 \qquad (2)$$

B.C: $\dot{G}(0) = 0$; $\dot{G}(1) = \dot{F}(W) = \dot{W}\frac{dF}{dW}$

- Further,

$$\frac{d\ddot{G}}{dR} + 2\dot{G}^2 + 2G\ddot{G} + \frac{\ddot{G}}{R} + 2f(r) - 2\dot{W}^2 - 2W\ddot{W} = 0 \qquad (3)$$

B.C: $\ddot{G}(0) = 0$; $\ddot{G} = +\ddot{W}\frac{dF}{dW} + \dot{W}^2\frac{d^2F}{dW^2}$

- Solve (1) to obtain: W & $G(R)$
  Solve (2) to obtain: $\dot{W}$ & $\dot{G}(R)$ using $W, G(R)$
  Solve (3) to obtain: $\ddot{W}$ using $W, \dot{W}, G, \dot{G}(R)$

- Finally:

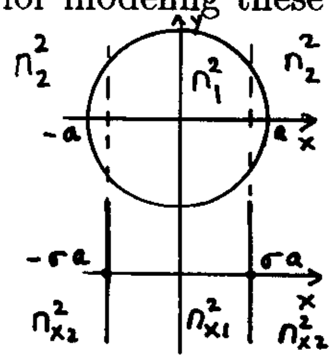
$$\begin{aligned} b &= W^2/V^2 \\ \dot{b} &= \frac{db}{dV} = 2\left(\dot{W}\sqrt{b} - b\right)/V \\ \ddot{b} &= \frac{d^2b}{dV^2} = 2\left(\dot{W}^2 + W\ddot{W} - 2\dot{b}V - b\right)/V^2 \end{aligned}$$

$\Rightarrow \tau_{ch}$: the chromatic dispersion.

Equivalent Slab Model for Single Mode Optical Fibers

- In a number of devices the circular symmetry of the fiber is broken and one has to model these devices in cartesian coordinates. Examples include directional couplers, polished fiber devices such as couplers half blocks and devices based on D-fibers.

- Using the cosine-exponential (CE)-fields, we have developed a step-index slab equivalent for a single mode step-index fiber [*J. Lightwave Technol.*, **8**, 143-151 (1990)], which is very useful for modeling these devices.



- The fiber profile:

$$\begin{aligned} n^2(r) &= n_1^2, \qquad & r < a \\ &= n_2^2, \qquad & r > a \end{aligned}$$

- The equivalent slab is defined as

$$\begin{aligned} n^2(x) &= n_{x1}^2 = n_1^2 - (U^2 - p^2)/(k_0 a)^2, \qquad & |x| < \sigma a \\ &= n_{x2}^2 = n_{x1}^2 - p^2 \sec^2(p\sigma)/(k_0 a)^2, \qquad & |x| > \sigma a \end{aligned}$$

  where p and σ are obtained using the variational method.

- However, simple empirical formula for p and σ are obtained as

$$\begin{aligned} p^3 &= -1.3528 + 1.6880\,V - 0.1894\,V^2 \\ \sigma &= 0.8404 + 0.0251\,V - 0.0046\,V^2 \end{aligned}$$

  where $V = k_0 a\sqrt{n_1^2 - n_2^2}$ and the maximum error in the empirical formula in the range $1.5 \le V \le 2.5$ is

$$(p)_{\text{error}} \le 0.03\%, \qquad (\sigma)_{\text{error}} \le 0.008\%$$

- The value of U in the above formula could be obtained using Neumann's approximation

$$U = \sqrt{V^2 - (1.1428\,V - 0.996)^2}$$

  which has an error of less than 0.1% in the above mentioned V-range.

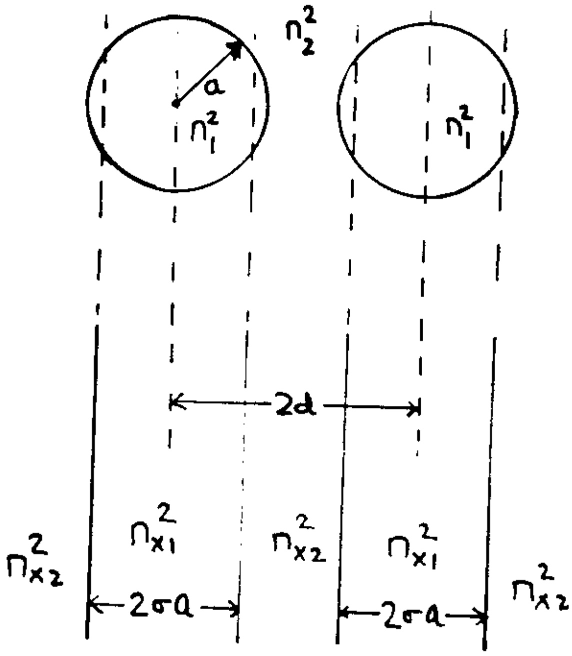
Equivalent Slab Model for Fibers: Applications

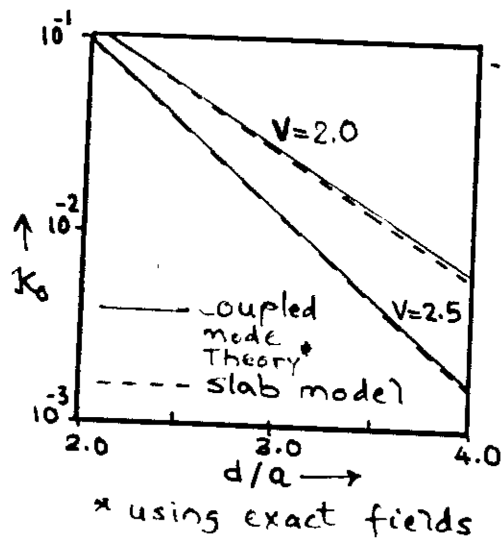
- **Fiber Directional Coupler:**

  A directional coupler consisting of two parallel fibers can be modelled as a directional coupler made of two slab waveguides and the coupling length can be obtained from an analytical expression:

$$\kappa_0 = \frac{\pi \beta_0 a^2}{V l_c} = \frac{2p^2 \sin^2(p\sigma)}{V[1 + p\tan(p\sigma)]} \cdot \exp[-2p\tan(p\sigma)\{(d'/a) - \sigma\}]$$

where β_0 is the propagation constant of the mode of the isolated fiber, l_c is the coupling length and κ_0 represents the normalized coupling coefficient.



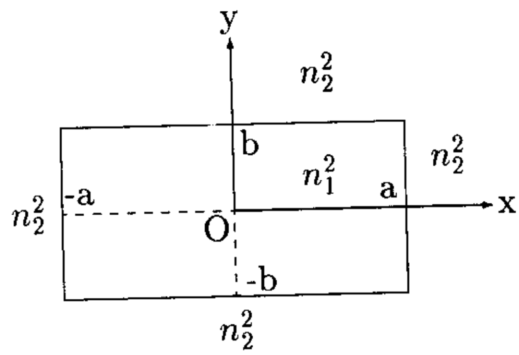


- In a similar way, a directional coupler with a buffer layer can be modelled. This would require the solution for the first two modes of a multi-layered planar waveguiding structure.
- Further, polished half block devices can also be modeled as 4-layer planar waveguides and the effects of dielectric and metallic overlays can be modelled.

RECTANGULAR WAVEGUIDES

- Profile

  $$n^2(x,y) = n_1^2 \quad |x| < a; |y| < b$$
  $$= n_2^2 \text{ otherwise}$$
  $(n_1 > n_2)$



- The wave equation

  $$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + [k_0^2 n^2((x,y) - \beta^2]\,\psi(x,y) = 0$$

  – not solvable analytically for any profile.

- Approximations:
  The main assumption is

  $$\psi(x,y) \approx \psi_x(x)\,\psi_y(y)$$

  In the framework of this assumption following methods have been developed:

  - Marcatili's method (1969)
  - Perturbation correction (1983) Kumar et al.
  - Variational method (1983) Sharma et al.

  Another method commonly used is

  - Effective index method (1970) Knox & Toulios

A. MARCATILI'S METHOD

- Assume that both $\psi_x(x)$ and $\psi_y(y)$ are the symmetric modes of the slab waveguides obtained by ignoring confinement in the other direction.
- Thus

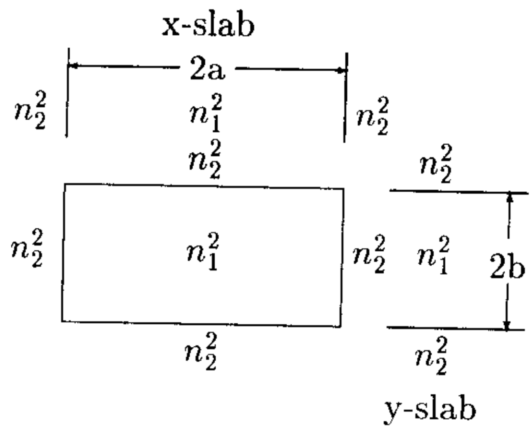
$\psi_x(x)$=fundamental mode of $n_x^2(x)$

$$n_x^2(x) = n_1^2 \quad |x| < a$$
$$= n_2^2 \quad |x| > a$$

&

$\psi_y(y)$=fundamental mode of $n_y^2(y)$

$$n_y^2(y) = n_1^2 \quad |y| < a$$
$$= n_2^2 \quad |y| > a$$



- The field $\psi(x, y) = \psi_x(x) \cdot \psi_y(y)$ is then the modal field of the profile

$$\widetilde{n}^2(x, y) = n_x^2(x) + n_y^2(y) - n_1^2$$

- If β_x^2 and β_y^2 are the propagation constants obtained for x- and y- slabs, then for the $\widetilde{n}^2(x, y)$ the propagation constant would be

$$\widetilde{\beta}^2 = \beta_x^2 + \beta_y^2 - k_0^2 n_1^2$$

– The profile $\widetilde{n}^2(x, y)$:

$2n_2^2 - n_1^2$	n_2^2	$2n_2^2 - n_1^2$
n_2^2	n_1^2	n_2^2
$2n_2^2 - n_1^2$	n_2^2	$2n_2^2 - n_1^2$

- In the corner regions:

$$2n_2^2 - n_1^2 = n_2^2 + (n_2^2 - n_1^2) \approx n_2^2$$

if $n_1 - n_2 \ll n_1$.
Thus, the error α $n_2^2 - n_1^2$.

- If the mode is well guided, then the power fraction in these corners is small and a reasonable accuracy can be obtained.
- For single mode fibers, however, a relatively large fraction of power flows in these corners & the accuracy is not sufficiently good.

B. PERTURBATION CORRECTION

- The small difference in $n^2(x, y)$ & $\widetilde{n}^2(x, y)$ is ideally suited for a perturbation correction.
- The correction does improve the value of β^2 particularly near the limit of the single mode region.

C. VARIATIONAL METHOD

- In Marcatili's method, the slab waveguides, have the same index distribution as in the given waveguide.
- If the indices in the core & cladding of the x- & y-slabs are made variational parameters, accuracy can be improved.
- Further, since scalar mode field shape depends only on the index difference, only the difference of indices of the core and the cladding need be made parameters.
- **Cosine Exponential (CE) Model**
  Trial field:

$$\begin{aligned}
\psi(x,y) &\cong \psi_x(x)\,\psi_y(y) & \\
\psi_x(x) &= A_1 \cos(px) & |x| \le a \\
&= A_1\,\cos(pa)\,e^{-p\tan(pa)\cdot[|x|-a]} & |x| \ge a \\
\psi_y(x) &= A_2 \cos(qy) & |y| \le b \\
&= A_2\,\cos(qb)\,e^{-q\tan(qb)\cdot[|y|-b]} & |y| \ge b
\end{aligned}$$

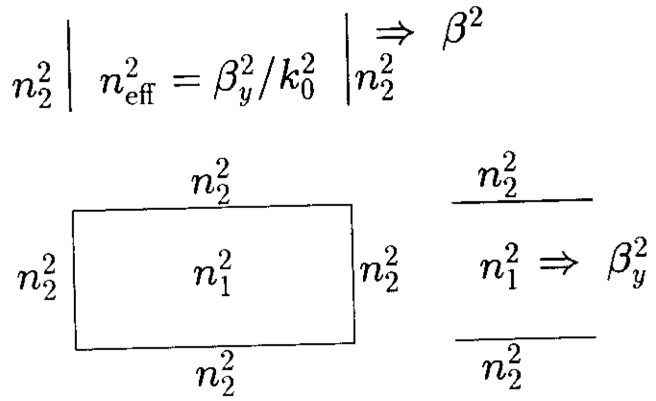
$\Rightarrow$ p, q are the variational parameters.

- $\beta^2(p,q)$ is obtained through the stationary expression is maximized w.r.t p, q:

$$\beta^2 = \frac{\iint k_0^2 n^2(x,y)\,|\psi|^2\;dx\,dy - \iint |\nabla_t\psi|^2\;dx\,dy}{\iint |\nabla_t\psi|^2\;dx\,dy}$$

D. EFFECTIVE-INDEX METHOD

- In this method, first the slab, which has shorter width is solved to obtain β.
- This $\beta/k_0 = n_{eff}$ is then used as the core index of the other slab and the cladding index is kept unchanged. This slab is then solved to obtain β which is taken as the β for the rectangular waveguide.



- The effective index method overestimates the propagation constant in most cases.

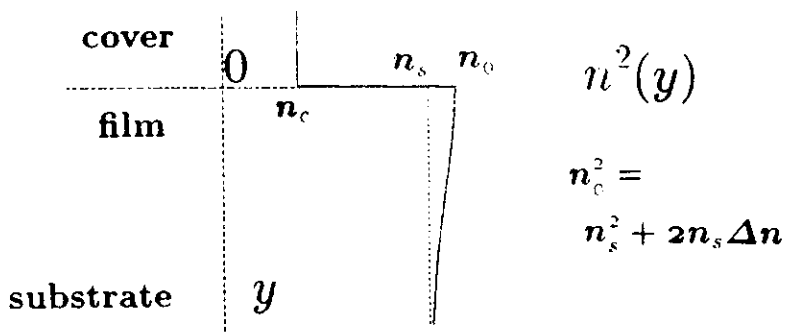
COMPARISON OF METHODS

- Marcatili's
  - simple
  - low accuracy
  - extendable to other modes.
- Perturbation
  - only improved β on Marcatili's method.
- Variational
  - better accuracy
  - field is also more accurate
  - generally not extendable to other modes
  - requires optimization
- Effective index
  - simple
  - low accuracy
  - extendable to other modes

DIFFUSED WAVEGUIDES

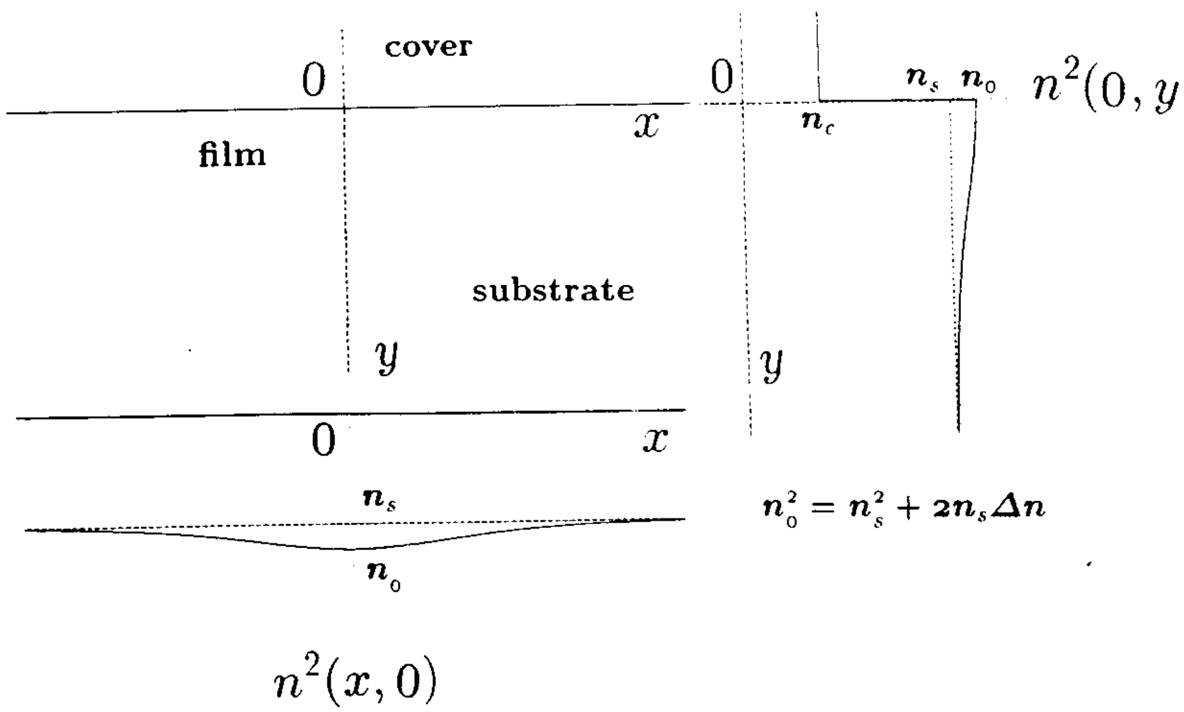
- Fabricated by diffusion of ions in $LiNbO_3$ and glass
- Index profile obtained is inhomogeneous; actual shape depends on the conditions of diffusion
- PLANAR WAVEGUIDES

$$n^2(y) = \begin{cases} n_s^2 + 2n_s\Delta n g(y) & y > 0 \\ n_c^2 & y < 0 \end{cases}$$



- CHANNEL WAVEGUIDES

$$n^2(x,y) = \begin{cases} n_s^2 + 2n_s\Delta n f(x)g(y) & y > 0 \\ n_c^2 & y < 0 \end{cases}$$



Single Mode Diffused Channel Waveguides

- Index Profile

$$n^2(x,y) = \begin{cases} n_s^2 + 2n_s\Delta n f(x)g(y) & y > 0 \\ n_c^2 & y < 0 \end{cases} \tag{1}$$

where n_s= substrate index; n_c= cover (air) index; Δn= the maximum index change from substrate to the guiding region. $n_0 = \sqrt{(n_s^2 + 2n_s\Delta n)} \approx n_s + \Delta n$= the maximum index of the film, generally at the central point ($x = 0$) on the top surface ($y = 0$) of the waveguide film.

- The wave equation for such a guiding structure is given by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + [k_0^2 n^2(x,y) - \beta^2]\psi(x,y) = 0 \tag{2}$$

- In a channel waveguide, the refractive-index distribution is symmetric along the surface of the waveguide (along the x-direction) and the commonly used functions to model the index variation are

$$f(x) = \begin{cases} \exp(-x^2/W^2) & \text{Gaussian} \\ \left[\text{erf}\left\{\frac{x+W}{D}\right\} - \text{erf}\left\{\frac{x-W}{D}\right\}\right] / \left[2\text{erf}(W/D)\right] & \text{error function} \end{cases} \tag{3}$$

On the other hand, the refractive-index distribution along the depth (the y-direction) is highly asymmetric and the commonly used funtional forms for $g(y)$ are

$$g(y) = \begin{cases} \exp(-y/D) & \text{exponential} \\ \exp(-y^2/D^2) & \text{Gaussian} \\ \text{erfc}(y/D) & \text{complementary error function} \end{cases} \tag{4}$$

OPTIMAL VARIATIONAL METHOD FOR CHANNEL WAVEGUIDES

- In the variational methods describe earlier the accuracy is limited on account of
  - Assumption of separability of $\psi(x, y)$:
  $$\psi(x,y) = \chi(x)\ \phi(y)$$
  - Assumption of specific field forms for $\chi(x)$ and $\phi(y)$.
- In the Optimal Variational (VOPT) method, any specific forms for $\chi(x)$ and $\phi(y)$ are not assumed and these are automatically generated by the variational method in the process of optimization. However, the separability is still assumed.
- Thus, under the assumption of separability, this method generates an *optimal* trial field and gives the *best* accuracy for the propagation constant.

THE OPTIMAL PROCEDURE

- **With $\psi_t(x,y) = \chi(x)\ \phi(y)$, the variational expression becomes**

$$\beta^2 = \int\int k_0^2 n^2(x,y)|\chi|^2\ |\phi|^2 dxdy - \int\left|\frac{d\chi}{dx}\right|^2 dx - \int\left|\frac{d\phi}{dy}\right|^2 dy$$

Normailsation is assumed: $\int|\chi|^2 dx = 1 = \int|\phi|^2 dy$.

- **Consider a planar index distribution $n_x^2(x)$ and rewrite the variational expression as**

$$\begin{aligned}\beta^2 &= \int k_0^2 n_x^2(x)|\chi(x)|^2 dx - \int\left|\tfrac{d\chi}{dx}\right|^2 dx \\ &+ \int k_0^2|\phi(y)|^2\ \left[\int\{n^2(x,y) - n_x^2(x)\}|\chi(x)|^2 dx\right] dy - \int\left|\tfrac{d\phi}{dy}\right|^2 dy\end{aligned}$$

or,

$\beta^2 =$ **Term#1 + Term#2**

- **Term#1** $=\int k_0^2 n_x^2|\chi|^2 dx - \int |d\chi/dx|^2 dx$ **is simply the variational expression for the profile $n_x^2(x)$ and has a maximum value, say, β_x^2.**
- **This β_x^2 is the propagation constant of the mode of the waveguide defined by $n_x^2(x)$ and $\chi(x)$ is the corresponding modal field. These can be evaluated by any standard numerical method. $\chi(x)$ is then normalised.**

Term#2

$$\int k_0^2|\phi|^2 \underbrace{\left[\int \{n^2(x,y) - n_x^2(x)\}|\chi(x)|^2 dx\right]}_{n_y^2(y)} dy - \int \left|\frac{d\phi}{dy}\right|^2 dy$$

- This term is also the variational expression for a planar index profile defined by

$$n_y^2(y) = \int \{n^2(x,y) - n_x^2(x)\}|\chi(x)|^2 dx$$

- The maximum value of Term#2, therefore, is β_y^2, the propagation constant of the mode of the waveguide defined by $n_y^2(y)$ with $\phi(y)$ being the corresponding modal field.

- These can again be evaluated by a standard numerical method and $\phi(y)$ can be normalised.

- In the evaluation of $n_y^2(y)$, use is made of $n_x^2(x)$ and $\chi(x)$ of the Term#1.

- Next, using $n_y^2(y)$ obtained in Term#2, we can rewrite the variational expression as

$$\beta^2 = \int k_0^2 n_y^2(y)|\phi(y)|^2 dy - \int \left|\frac{d\phi}{dy}\right|^2 dy$$
$$+\int k_0^2 |\chi(x)|^2 \left[\int \{n^2(x,y) - n_y^2(y)\}|\phi(y)|^2 dy\right] dx - \int \left|\frac{d\chi}{dx}\right|^2 dx$$

or, $\beta^2 =$ Term#2 + Term#3

- The maximum value of Term#2 has already been obtained as β_y^2.
- The maximum value of Term#3 is β_x^2 which is now the propagation constant of the mode of the waveguide defined by

$$n_x^2(x) = \int \{n^2(x,y) - n_y^2(y)\}|\phi(y)|^2 dy$$

- $\chi(x)$ is the corresponding modal field which, alongwith β_x^2, can be evaluated by any standard numerical method. $\chi(x)$ is then normalised.
- Thus, $\beta_x^2 + \beta_y^2$ gives an estimate for β^2 and $\chi(x)\phi(y)$ is an approximation for $\psi(x,y)$.
- This completes one cycle of iteration.
- The $n_x^2(x)$ of this cycle is used for the next cycle and the iterations are continued till a desired convergence is obtained for $\beta_x^2 + \beta_y^2$.

IMPLEMENTATION PROCEDURE

STEP 1: Choose an $n_x^2(x)$. A good choice is $n_x^2(x) = n^2(x, y = 0)$.

STEP 2: Obtain β_x^2 and $\chi(x)$ numerically. Normalize $\chi(x)$.

STEP 3: Obtain $n_y^2(y)$ using

$$n_y^2(y) = \int \{n^2(x,y) - n_x^2(x)\} |\chi(x)|^2 dx$$

STEP 4: Obtain β_y^2 and $\phi(y)$ numerically. Normalize $\phi(y)$.

STEP 5: Obtain $n_x^2(x)$ using

$$n_x^2(x) = \int \{n^2(x,y) - n_y^2(y)\} |\phi(y)|^2 dy$$

STEP 6: Obtain β_x^2 and $\chi(x)$ numerically. Normalize $\chi(x)$.

STEP 7: Compute $\beta_t^2 = \beta_x^2 + \beta_y^2$. Check for convergence in β_t^2.
IF Converged, GOTO STEP 8
OTHERWISE, GOTO STEP 3

STEP 8: β_t^2 and $\psi_t(x,y) = \chi(x)\phi(y)$ are the required propagation constant and the modal field.

RESULTS FOR CHANNEL WAVEGUIDES

Profile

$$f(x) = \frac{\mathrm{erf}\left\{\frac{x+W}{D}\right\} - \mathrm{erf}\left\{\frac{x-W}{D}\right\}}{2\mathrm{erf}(W/D)}$$

$$g(y) = \exp(-y^2/D^2)$$

Values of $B = [(\beta/k_0)^2 - n_s^2]/2n_s\Delta n$

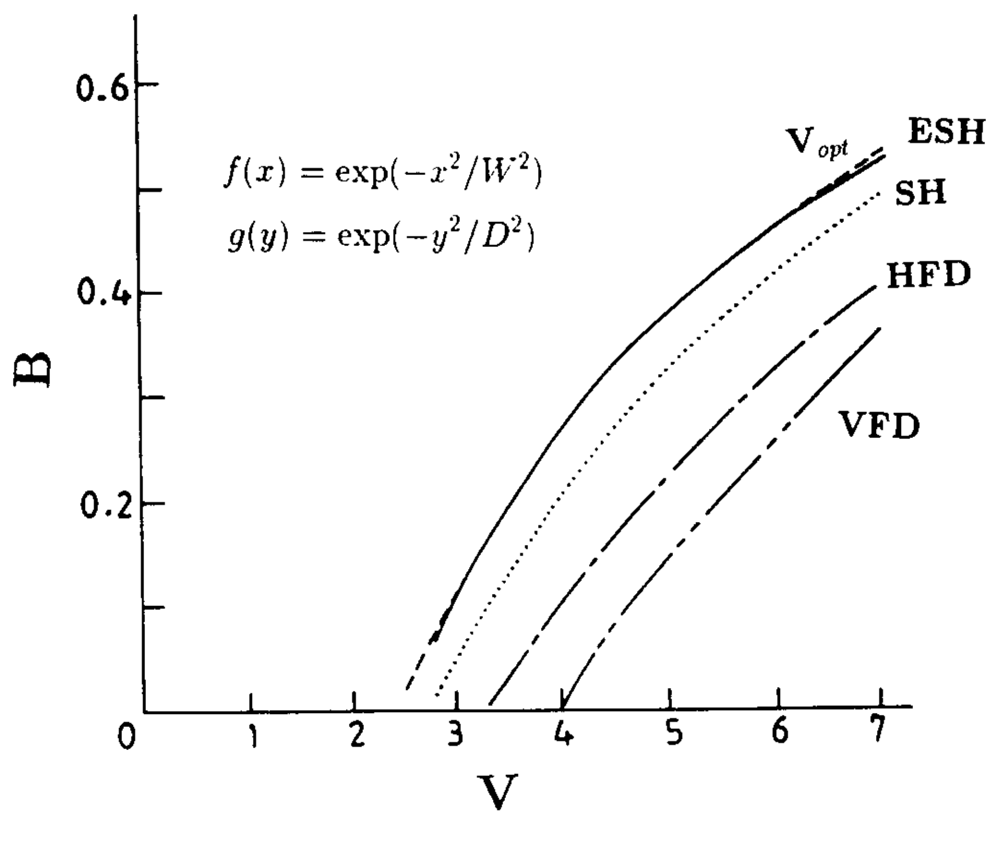
Table 1: W=3.0 μm, D=3.35 μm, (thickness, τ of Ti film varies)

St	V	VOPT	HG	EHG	SH	ESH	CE
20	2.12	0.133	0.112	0.125	0.118	0.130	0.123
30	2.59	0.248	0.233	0.247	0.231	0.245	0.234
40	3.00	0.329	0.313	0.328	0.313	0.327	0.318

Table 2: τ=720 A, D=5.08 μm (width, W, of Ti film varies)

St	V	VOPT	HG	EHG	SH	ESH
20	1.06	0.143	0.126	0.135	0.125	0.133
30	1.91	0.254	0.240	0.250	0.240	0.249
40	2.86	0.328	0.315	0.324	0.315	0.324

CHANNEL WAVEGUIDE



EQUIVALENT 1-D WAVEGUIDES

- The index profiles $n_x^2(x)$ or $n_y^2(y)$, generated in the numerical method, can be used to obtain 1-D equivalent profiles for the given channel waveguide, $n^2(x, y)$:

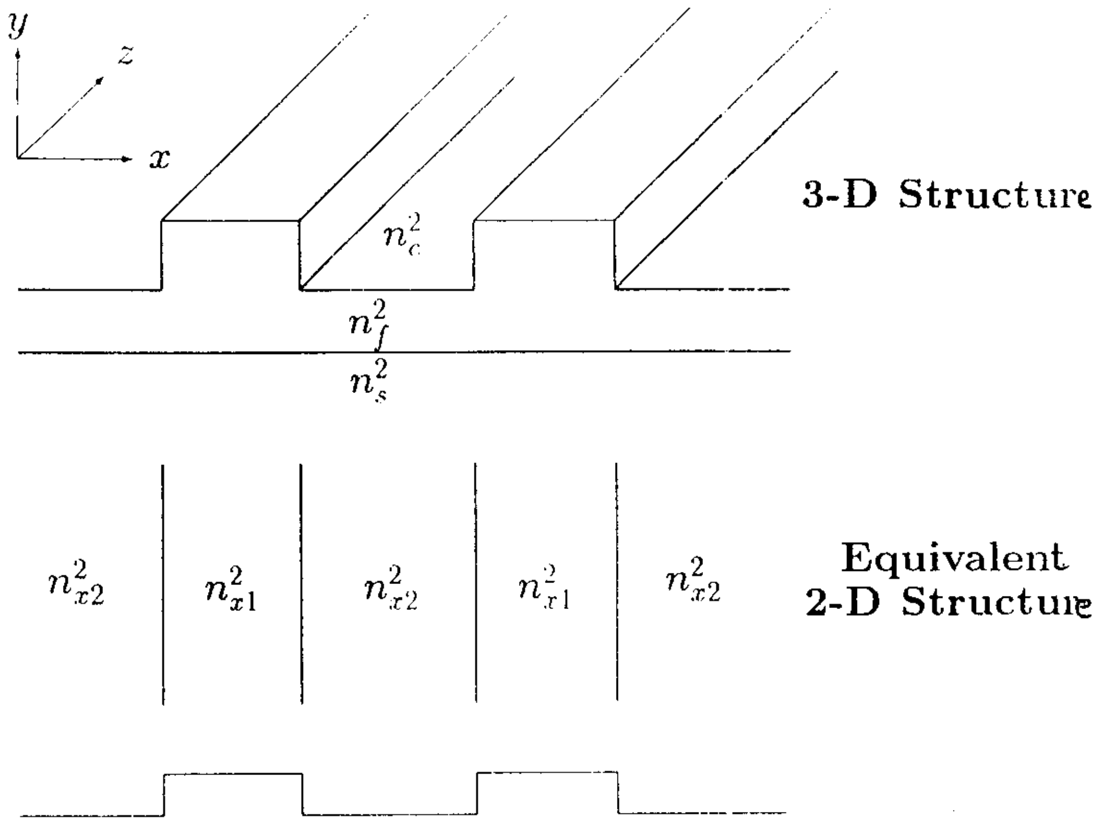
$$n_{xeq}^2(x) = \int \{n^2(x, y) - n_y^2(y)\} |\phi(y)|^2 dy + \beta^2 / k_0^2$$

$$n_{yeq}^2(y) = \int \{n^2(x, y) - n_x^2(x)\} |\chi(x)|^2 dx + \beta^2 / k_0^2$$

- $n_{xeq}^2(x)$ represents a 1-D waveguide the mode of which has the same propagation constant and the x-variation of the field as those of the mode of the given waveguide $n^2(x, y)$. In other words, its mode is a projection of the actual propagating modal field $\psi(x, y)e^{-i\beta z}$ on the x-z plane.

- This equivalent waveguide $n_{xeq}^2(x)$ [or, $n_{yeq}^2(y)$] can be used to simulate various effects and interactions in the x-z [or, y-z] plane.

Rib Waveguide Directional Coupler



- A full 3-D analysis is very involved and time consuming.
- An equivalent 2-D model can be used with good accuracy for a majority of waveguiding structures.
- This requires an equivalent planar waveguide for a 2-D waveguiding structure (cross-section).

Methods for Obtaining Equivalent Waveguides

- EFFECTIVE-INDEX METHOD
  - Most commonly used method
  - Usually overestimates the propagation constant
  - Can be time consuming
- VARIATIONAL METHODS
  - Limited by the choice of the form of the trial field
  - More accurate than the effective-index method
- PRESENT METHOD
  - Based on the variational principle
  - Does not require *a priori* ansatz for the form of the field
  - The only assumption is $\psi(x, y) \approx \psi_x(x)\ \psi_y(y)$
  - The fields $\psi_x(x)$ and $\psi_y(y)$ are automatically generated and hence are optimal approximation for the modal field

Principle of the Present Method

- Using the variational principle and the assumption $\psi(x,y) = \psi_x(x)\psi_y(y)$, we obain the expressions for equivalent index profiles [Sharma and Bindal, *Opt. Quantum Electron.* 24, 1359 (1992)]

$$n_x^2(x) = \int \{n^2(x,y) - n_y^2(y)\} |\psi_y(y)|^2 dy + \beta_y^2/k_0^2 \qquad (1)$$

$$n_y^2(y) = \int \{n^2(x,y) - n_x^2(x)\} |\psi_x(x)|^2 dx + \beta_x^2/k_0^2 \qquad (2)$$

Algorithm

1. Choose a starting index profile $n_x^2(x)$.
2. Obtain the modal field, $\psi_x(x)$ and the propagation constant, β_x^2 for $n_x^2(x)$. Normalize $\psi_x(x)$.
3. Compute $n_y^2(y)$ using Eq.(2) above.
4. Obtain the modal field, $\psi_y(y)$ and the propagation constant, β_y^2 for $n_y^2(y)$. Normalize $\psi_y(y)$.
5. Compute $n_x^2(x)$ using Eq.(1) above.
6. Repeat steps 2-5 till convergence in β_x^2 and β_y^2 is obtained.
7. In the converged state, $n_x^2(x)$ defines the equivalent x-slab and $n_y^2(y)$ defines the equivalent y-slab.

Typically 3-4 iterations are sufficient.

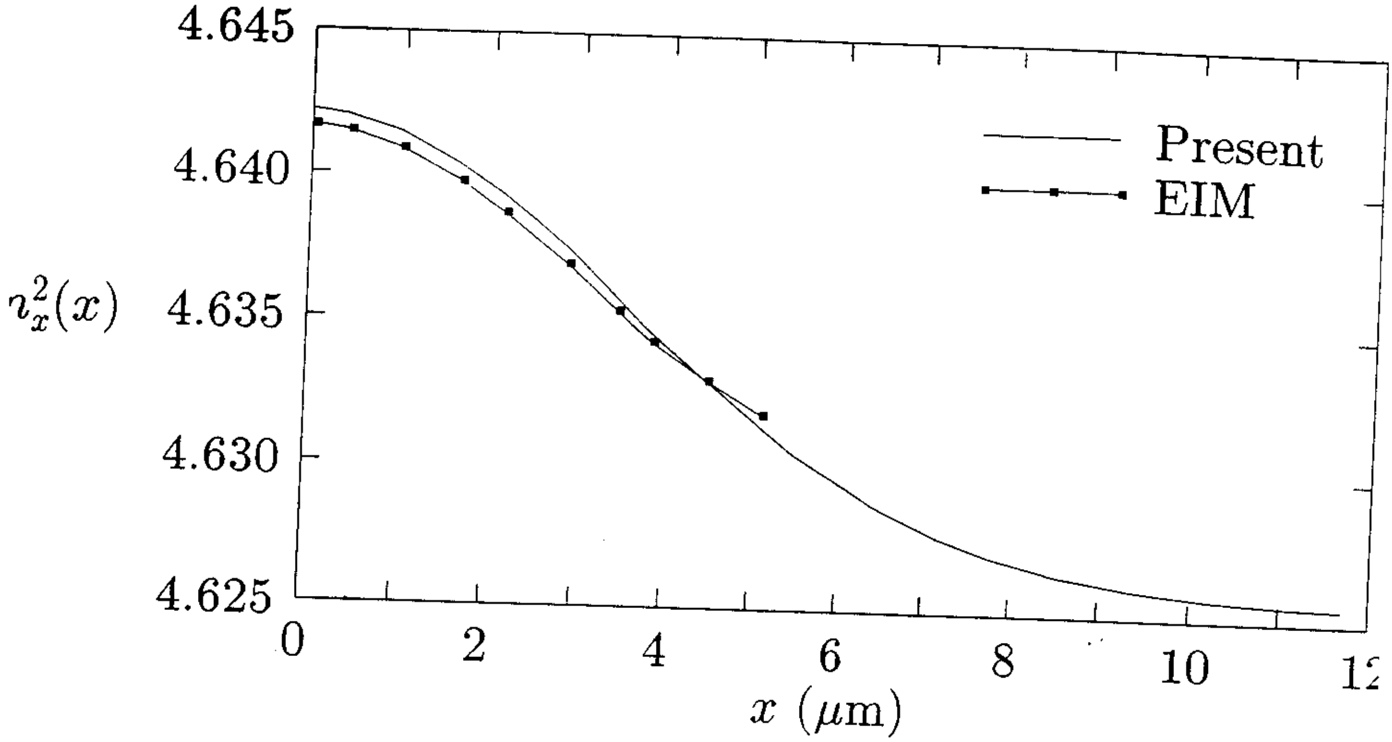
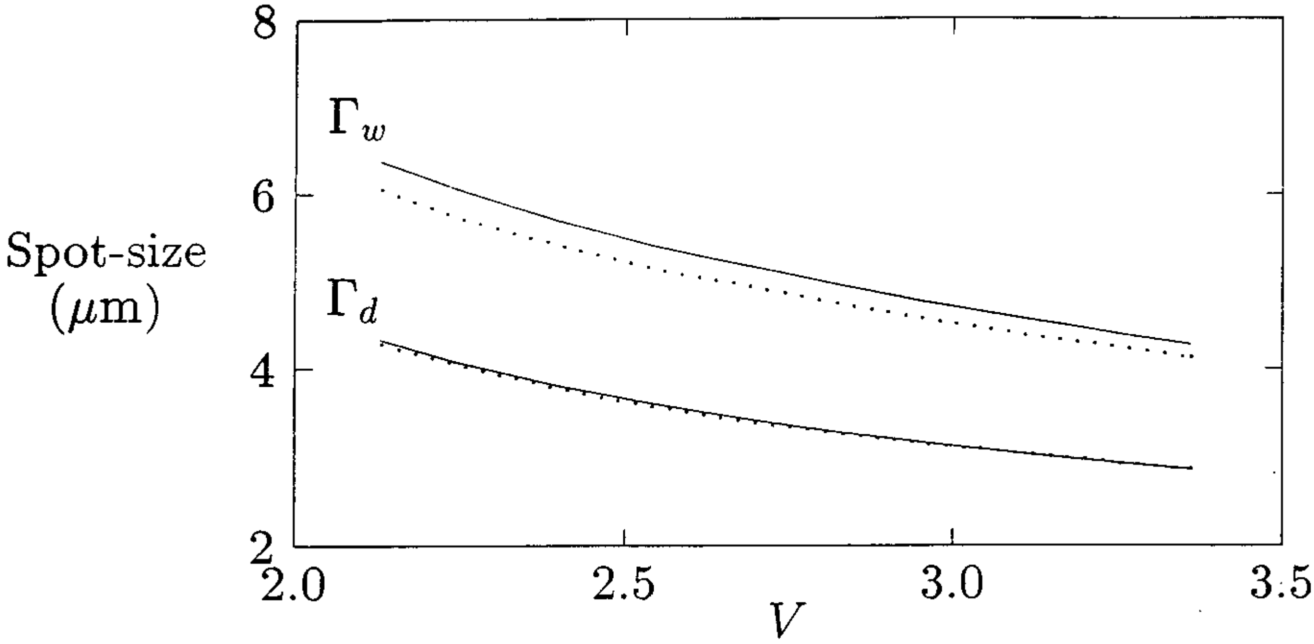
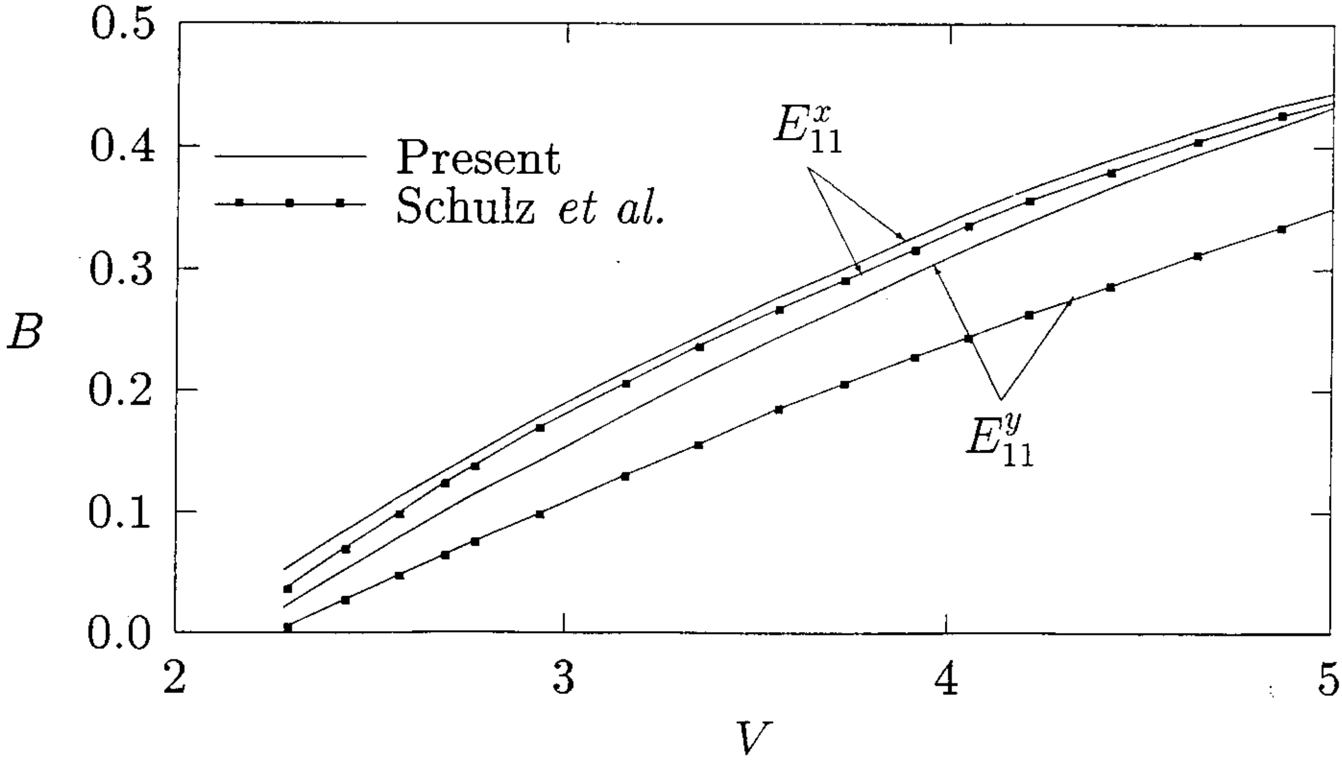


Table-I
Coupling Length (mm) of a Directional Coupler

Parameters	Present	EIM	Experiments[1
Δn=0.006, s=6.08μm	5.8019	7.9054	5.2850
Δn=0.006, s=6.58μm	10.536	11.687	10.600
Δn=0.004, s=6.08μm	3.6892	4.5916	3.4000
Δn=0.004, s=6.56μm	5.5463	5.8601	5.2850



Spot-size
(μm)
8
6
4
2
2.0
2.5
3.0
3.5
V
Γw
Γd



0.5
0.4
0.3
0.2
0.1
0.0
B
2
3
4
5
V
Present
Schulz et al.
E11x
E11y

DIRECTIONAL COUPLERS

- Index profile:

$$n^2(x,y) = \begin{cases} n_s^2 + 2n_s\Delta n g(y)[f(x-s) + f(x+s)] & y > 0 \\ n_c^2 & y < 0 \end{cases}$$

- An equivalent 1-D directional coupler profile would be

$$n_{eq}^2(x) = n_{xeq}^2(x-s) + n_{xeq}^2(x+s) - n_{seq}^2$$

- This 1-D directional coupler is much easier to analyse in comparison to the actual 2-D directional coupler.

RESULTS FOR DIRECTIONAL COUPLERS

Profile

$$n^2(x,y) = \begin{cases} n_s^2 + 2n_s\Delta n \; g(y)[f(x-s) + f(x+s)] & y > 0 \\ n_c^2 & y < 0 \end{cases}$$

$$f(x) = \exp(-x^2/W^2) \qquad g(y) = \exp(-y^2/D^2)$$

$$D = 5.0\mu m, \quad n_s = 2.152, \quad W = 8.0 \quad \mu m, \quad n_c = 1.0$$

Values of Coupling Length, l_c in mm

Process Parameters	2-D Analysis	1-D Analysis
Δn=0.006 s=6.08 μm	5.7965	5.8016
Δn=0.006 s=6.56 μm	10.5211	10.5360
Δn=0.004 s=6.08 μm	3.6956	3.6892
Δn=0.004 s=6.56 μm	5.5469	5.5463