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"Geometrical Optics"

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GEOMETRICAL OPTICS

Approximation according to which electromagnetic propagation takes place along paths, rays, which are straight lines in free space.

In case of incoherent radiation, such as light from thermal sources, rays describe energy propagation;

In the case of coherent radiation, such as light from a laser source, rays are the normals to the wavefronts or conversely wavefronts can be obtained from rays.

The ray approximation, although originally used in Optics, is valid at all frequencies, provided suitable conditions are satisfied.

Examples:

-Free propagation at all frequencies,
-Reflection of radio waves, in the Ionosphere,
-Reflection of microwaves from metal surfaces.

COHERENT RADIATION

Let us start from waves:

Light: electromagnetic waves

Scalar approximation: one Cartesian component of the e. m. field V(P,t) is representative of entire field

Energy: proportional to the square of this component (power flux α Poynting vector)

Complex form (coherent monochromatic)

1.1) $$V(P,t)=u(P)e^{-i\omega t}$$

time dependence:

oscillation with frequency $\nu = \omega / 2\pi$

$\omega \rightarrow$ source

u(P) complex amplitude

1.2) $$u(P) = A(P)\, e^{i\varphi}$$

A amplitude φ phase

wavefronts: surfaces φ = constant

Wave equation u=u(P)

1.3) $$\nabla^2 u + n^2 k_o^2 u = 0$$

$k_o = 2\pi/\lambda_o$ λ_o wavelength, empty space

$n = n(P) = c/v(P)$ $v(P)$ velocity in the medium

and $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ Laplacian

Eq 3 valid also for inhomogeneous media, if n(P) varies slowly over distances of order of wavelength.

Introduction of Eq 2 into Eq.3, on account that

$$\nabla^2 u = \nabla \cdot \text{grad } u$$

and

$$\text{grad } e^{i\varphi} = i\, e^{i\varphi} \text{grad } \varphi$$

gives:

$$2i \,\text{grad } A \cdot \text{grad } \varphi - A|\text{grad } \varphi|^2 + Ai\nabla^2\varphi + \nabla^2 A + n^2 k_o^2 A = 0$$

Separation of real and imaginary parts gives:

1.4) $$\nabla^2 A + n^2 k_o^2 A - A|\text{grad } \varphi|^2 = 0$$

and

1.5) $$2 \text{ grad } A \cdot \text{grad } \varphi + A \nabla^2 \varphi = 0$$

ASSUMPTION:

1.6) $$\nabla^2 A << n^2 k_o^2 A$$

In Eq. 1.4) first term negligible

Eq. 1.6 becomes

1.7) $$n^2 k_o^2 - |\text{grad } \varphi|^2 = 0$$

that is

1.8) $$\text{grad } \varphi = n k_o \underline{s}$$

$\underline{s}$ unit vector in the direction of grad φ.

Phase independent of amplitude; Eq 1.8 basic Equation of Geometrical Optics.

From this Equation:

-Fermat Principle can be found

-Many applications to inhomogeneous media

Meaning of condition Eq.1.6:

1.9) $$\frac{\nabla^2 A}{A} = \frac{d^2 A}{A\ dP^2} << \left[\frac{4\pi}{\lambda}\right]^2$$

where dP is a small displacement. If one chooses

$$dP = \lambda / 2\pi$$

condition reads

1.10) $$\frac{d^2 A}{A} << 1$$

Therefore the first term of Eq 1.4 is negligible if the second variation of the amplitude on a space of the order of the wavelength is negligible. In other words the amplitude has no abrupt changes.

In conclusion:

-Geometrical Optics (G.O) is valid when there are no abrupt changes of amplitude.

-In G.O. phase is independent of amplitude.

The unit vector $\underline{s}$ is the normal to the wavefront and is, at each point, the tangent to the ray. Therefore a ray (in the physics meaning, not just mathematical) in geometrical optics is a tube of flux of the phase. Energy propagates along these tubes of flux.

Integration of Eq 1.8 along a ray gives a simple formula to obtain the phase difference "from geometrical optics"

1.11) $$\varphi(P) - \varphi(P') = k_0 \int_{ray} n \, ds$$

Useful to understand formation of images with lenses. Point P' is an Image of point P if along rays from P to P' the phase difference is the same.

Quantity

1.12) $$S = n / k_0$$

is called eikonal (from Greek εικων ,image) and Eq. 1.8 becomes

1.13) $$\text{grad}\, S = n \underline{s}$$ Eikonal Equation

where in general n=n(P).

As rot grad S=0, (rot=curl) one has

1.14) $$\text{rot}\, n \underline{s} = 0$$

It can be also shown (Born and Wolf) that in an inhomogeneous medium a ray "bends" toward the region of higher refractive index.

Important cases for application: propagation in graded index materials (optical fibers).

IMAGES

Geometrical Optics allows evaluation of images formed by:

Mirrors

Lenses
- traditional
- others such as aspherical or graded index

Systems, more or less sophisticated

<u>Advantages:</u>

1 - Geometrical optics:

- -simple
- -rays
- -allows accounting for aberrations
- -neglects diffraction, but final test of images also requires knowledge of the limits set by diffraction

2 - Wave optics:

- -more general
- -allows accounting for:
  - aberrations
  - diffraction

a) - Free space :

rectilinear propagation (straight rays), velocity c.

b) - Homogeneous transparent medium

rectilinear propagation, velocity $v<c$

The ratio between c and the velocity in the medium is the refractive index of the medium

2.1) $$n = c/v$$

Important: $n \geq 1$

DISPERSION:

refractive index of materials n depends on wavelength λ

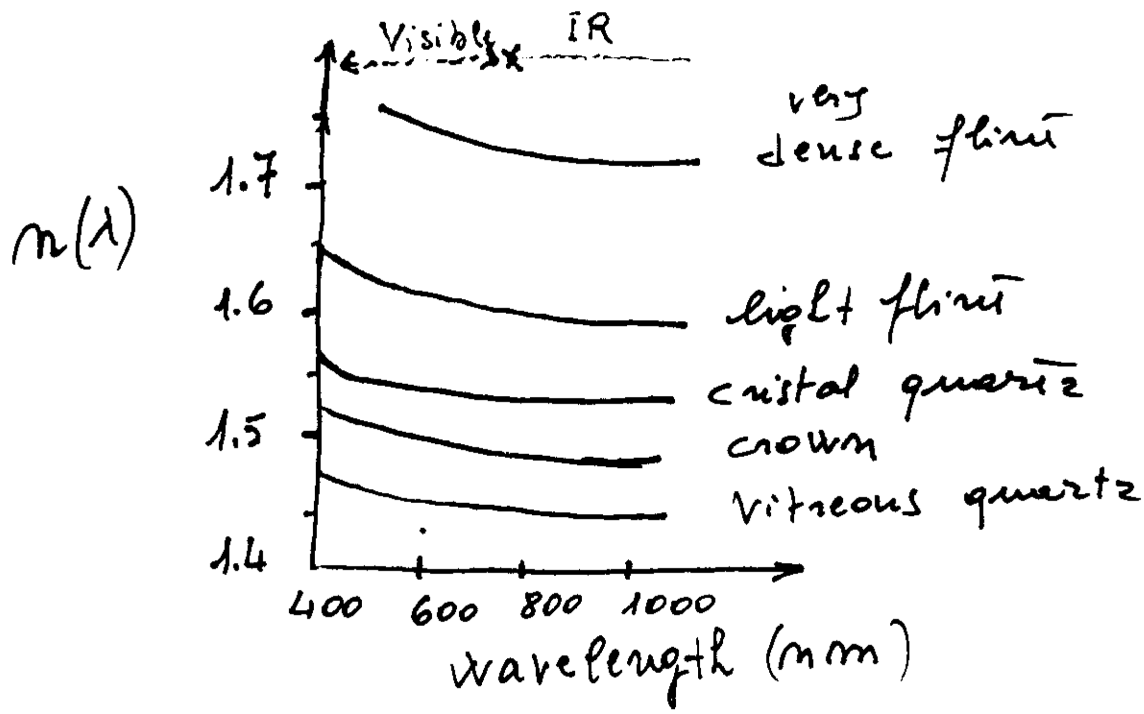
$$n = n(\lambda)$$

An important quantity in dispersion is the so called Abbe's Number, generally denoted as ν, defined as:

2.2) $$\nu = \frac{n_D - 1}{n_F - n_C}$$

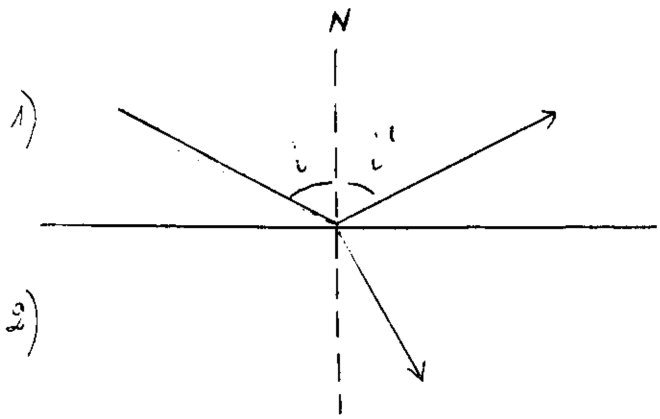
where C, D, F, denote the three standard reference lines in optics, known as Fraunhofer lines:

C, red	λ = 656.28 nm	(hydrogen)
D, yellow	λ = 587.56 nm	(sodium)
F, blue	λ = 486.13 nm	(hydrogen)



Plane surface between two media:

ray from medium 1 impinges on the surface between medium 1 and 2.
Energy splits into two parts one reflected and one transmitted.



REFLECTION LAWS:

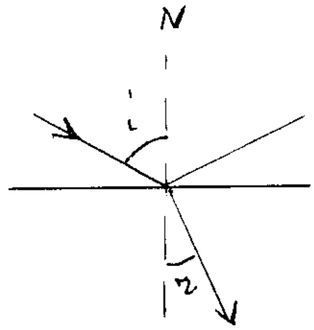
Ray impinges on the reflecting surface at point P

1 - The phenomenon takes place in a plane set by the ray and the normal to the surface at P

2 - Incidence, i, and reflection, i', angles are equal

2.3) $$i = i'$$

REFRACTION LAWS



Ray from medium 1 impinges on the refracting surface at point P

1 - The phenomenon takes place in the plane, set by the ray and the normal to the surface at P

2 - The ratio of the sines of the incidence, *i*, and refraction, *r*, angles is constant. The constant, denoted by n_{21}, is called the refractive index of the second medium with respect to the first one: it is the ratio of the refractive index n_2 of the second medium to that n_1 of the first medium. In formulas

2.4) $$\frac{\sin i}{\sin r} = n_{2,1} = \frac{n_2}{n_1} = \frac{v_1}{v_2}$$

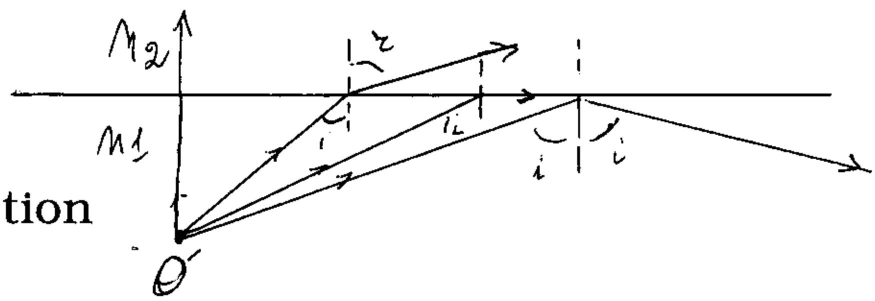
In other words each sine is proportional to the corresponding velocity.

Note that "constant" in a law, means constant with respect to the other quantities appearing in the law. On its turn the "constant" may depend on some other quantity. In our case the refractive index depends on the wavelength.

A ray passing from a medium to another having higher refractive index approaches the normal.

Consequence:
- The higher the velocity, the larger the corresponding angle.
- Quantity n_{21} can be larger or lower than 1. The phenomenon of total internal reflection takes place when light passes from a medium to another with lower refractive index.

FIGURe total reflection



$\sin\theta_L = \frac{n_2}{n_1}$ $\qquad$ $n_1 > n_2$

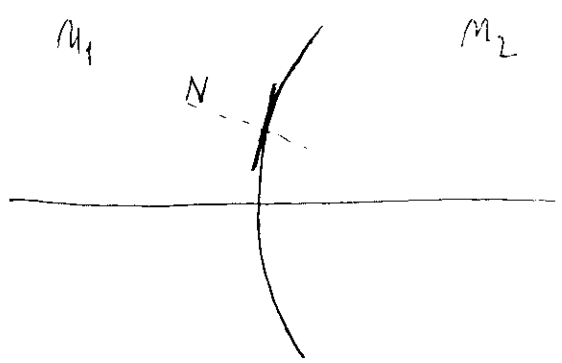
$i_L = \arcsin\frac{n_2}{n_1}$ limit angle

This effect finds important application in guided optical propagation (plane, or integrated optics) and optical fibers and prisms. In guided propagation it is utilized both for guiding of radiation and for coupling between fibers and other devices.

CURVED SURFACES

Reflection and refraction laws can be applied to curved surfaces, provided that the curvature is weeck, or in other words the radii of curvature are very large (with respect to wavelength). Application to curved surfaces constitutes the basis of all treatment of images by lenses and systems. After a curved surfaces parallel rays change their behaviour, eg converge or diverge. The simplest case is that of spherical surfaces, but also aspherical surfaces are important.

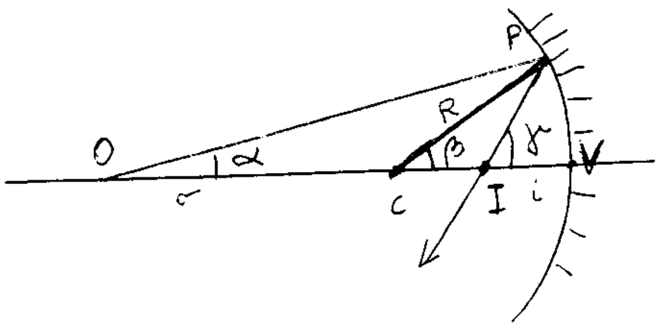
Simple formulas in the case of "paraxial" or "gaussian" approximation.



Locally: the spere is replaced by its tangent plane

EXAMPLE. In order to have an idea of methods and approximations involved let us start with a simple case: the evaluation of the conjugate points formula for a spherical concave mirror in paraxial approximation.

Spherical mirror: center C, radius R.
Point source, O, on axis at a distance o from the surface vertex V. A meridional ray (in a plane though the axis, meridional plane) from the source impinges on the mirror surface at P, and after reflection crosses the axis at I, distant i from V.



Let α β and γ the three angles, with respect to the axis, under which P is seen from O, C and I respectively. Note that β measured in radiants is given by γ = (arch PV)/R.
Relationships between external and internal angles in the two triangles and reflection laws give:

2.5) $\gamma = i + \alpha$
2.6) $\beta = 2i + \alpha$

This is the basic couple of equations which tells the path of any ray leaving source O.
Eliminating i from these two equations gives

$$\beta + \alpha = 2\gamma \quad (2.7)$$

This is an exact relation between the three angles.
Now the paraxial approximation is made:

"α is a small angle".

From now on it can be replaced by some approximate expression such as $\alpha \approx$ (arch PV)/o, This corresponds to consider the (arch PV) as belonging to a sphere with radius o. The same position is also made for β which is replaced by (arch PV)/i. However, in making this approximation, one must observe that smallness of α does not automatically guarantee the smallness of β. The approximation on β requires also that point P must be sufficiently near to the axis, and this also depends on the position of the source. The need of an entrance pupil appears.
Within these approximations one gets the well known formula relating object and image positions

$$\frac{1}{o} + \frac{1}{i} = \frac{2}{R} \quad (2.8)$$

This simple result helps us in remembering some basic concepts, notations and nomenclature.
When the source goes very far from the mirror to infinity, o -> ∞, the image position becomes the focus i -> f =R/2). In other words, the focus is the point where the mirror makes a beam of parallel rays converge.
From this case also the concept of real and virtual images (when i becomes negative) is apparent.
In addition the result can be extended to the case of convex mirror provided that R is made negative.

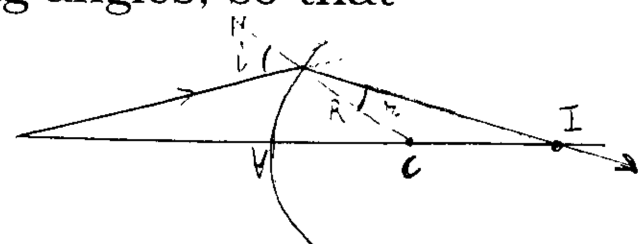
SPERICAL REFRACTING SURFACES

As an exercise it is not difficult to evaluate the analogous expression in the case of refraction at a spherical surface separating two media: medium 1, with refractive index n_1 and medium 2, refractive index n_2.
Point source, O, is on the axis at a distance o from the surface vertex, V. A meridional ray from the source is refracted at the surface in P, then crosses the axis at I, at distance i from the surface.
Sign conventions:

- o>0 if the source is on left hand side with respect to the surface
- i>0 if I on the right hand side
- Surface radius R>0 if center on right side

In addition to the approximations made in the previous case, here there is also the need to approximate the refraction formula by replacing the sines with the corresponding angles, so that

2.9) $$\frac{i}{r} \approx \frac{n_2}{n_1}$$



Substitution of sines with angles is not the most stringent one and can be a good approximation for angles up to 10° and more, depending on the required accuracy. The final result is

2.10) $$\frac{n_1}{o} + \frac{n_2}{i} = \frac{n_2 - n_1}{R}$$

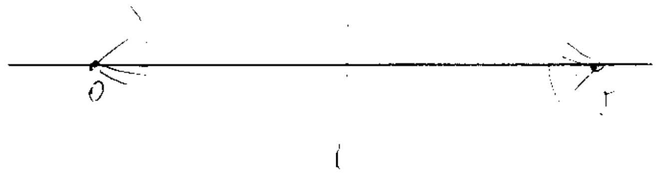
from which one immediately sees that the position of point I does not depend on the particular ray chosen from the source. All rays from the object which reach the refracting surface (and fit the paraxial approximation) are refracted in such a way that they all cross I. Point I is called the image of the point source O.

The previous result is the first step to evaluate a final result after a number of refractions at different subsequent surfaces.
If, there is another spherical surface, separating medium 2 from another medium, point I can be utilized as source for this second surface and the equation applied to obtain the image after this second refraction. A general formula can be obtained for a thick lens, whose limit gives the thin lens law.

THIN LENSES

A thin lens is a lens whose thickness is negligible with respect to the other dimensions involved. It is consists of material of refractive index n, limited by two spherical surfaces not necessarily of the same radii.
The relationship between object and image positions is called the equation of the conjugate points. In the paraxial approximation, gaussian optics, and in air ($n_{air} = 1$) one gets

3.1) $$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}$$



where the focal length, f, is given by

3.2) $$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

This equation is known as "lens maker's formula".

Sign conventions

- $o>0$ if object on left lens side (object space)
- $i>0$ if image on right side (image space)
- Lens radius >0 if center on right side
  in fig $R_1>0$, $R_2<0$
- $f>0$ converging lens
- $f<0$ diverging lens

When $o \rightarrow \infty$, i=Focus. Perfect lens makes parallel rays converge to (or diverge from) focus.
Ex: $f>0$: the lens makes rays converge. It transforms plane wave into spherical converging wave.

In a system one defines two focal points: front focus, before the lens, and back focus, after it. If the positions of object and image are counted from the two foci, $o=f+x$ and $i=f+x'$, respectively, from 2.1 one immediately gets Newton's formula :

3.3) $$x\,x' = f^2$$

Images: Real or Imaginary.

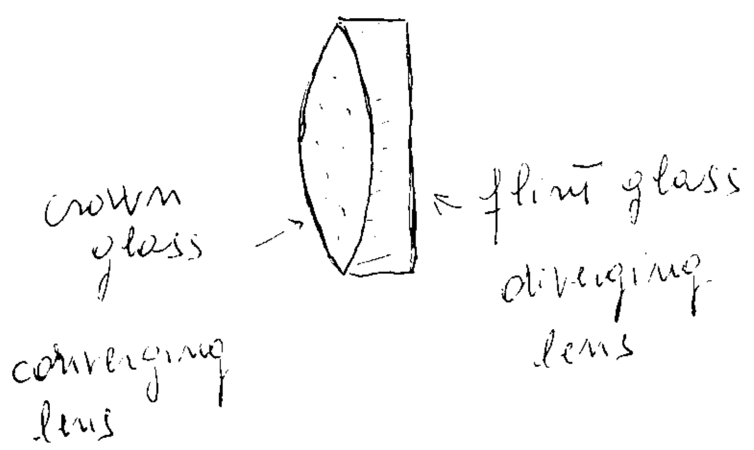
In general, for lenses or systems of elements:

From source to image optical path along each ray the same (Fermat principle). Phase along each ray the same; at image point positive interference.

CHROMATIC ABERRATION

In the paraxial approximation one image point corresponds to a source point, only in the case of monochromatic (or quasi monochromatic) radiation. Due to dispersion, the focus is different for the different colors, and the rays from a source do not have one common cross point if they are of different "color" (wavelength). This effect is known as chromatic aberration and is the only aberration present also in the paraxial approximation. To avoid it double lenses "doublets" were first invented, where two different glasses (flint and crown) were used.

example: an achromatic doublet



EXTENDED OBJECT

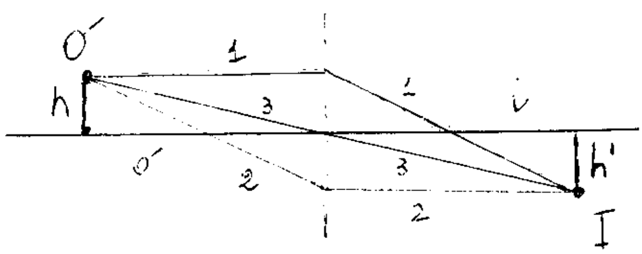
For extended objects one can use a method based on rays; the object is divided in "small elements" and each element is considered a point-like source.

Consider : a point source at height h above the axis.

There are particular rays from this point (figure):
1) The ray reaching the lens parallel to the axis, which passes through the back focal point,
2) The ray reaching the lens after passing through the front focus, which emerges parallel to the axis,
3) The ray crossing the lens through the optical center (central point of the lens on the axis), which propagates straight, without deviation.

Position of the image is obtained where all these rays cross, and is described by i (along axis) and h' normal to axis:

3.4) $h' = h\, i/o$



Inverted image
Transversal magnification.

3.5) $M = -h'/h$

The sign means an inverted image.

LENSES IN SEQUENCE

Two lenses in sequence at distance d are equivalent to a new lens of which one can evaluate the focus: one obtains (obvious symbols)

3.6) front focal length $= \dfrac{f_1\,(d - f_2)}{d - (f_1 + f_2)}$

3.7) back focal length $= \dfrac{f_2\,(d - f_1)}{d - (f_1 + f_2)}$

from which the role of d is clear.

If d=0, lenses in contact. Multiplets of N lenses in contact are equivalent to a lens with focus

3.8) $$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \ldots + \frac{1}{f_N}$$

Inverse of focus: power. Summation of powers.

MATRIX METHOD for RAY TRACING

One ray impinges at a given height on the surface of a lens. Finding the position and angle after a given path can be done by the matrix method (called ABCD).

For instance, in the paraxial approximation, refraction at a surface betwee two media can be described by a matrix of refraction

$$\begin{vmatrix} 1 & 0 \\ (n_1-n_2)/R & 1 \end{vmatrix}$$

This matrix, applied to a one column matrix with elements height h and "angle" $n_1\ \theta_1$ (note angle multiplied by the refractive index of the medium) of the ray impinging on the surface gives as a result height h and "angle" $n_2\ \theta_2$ of the output ray. This can be easily verified by the student.

From this matrix, the application to two surfaces allows one to obtain a matrix for thin a lens, a matrix a for a thick lens and so on.

This simple procedure allows one to trace rays in more complicated systems, in the paraxial approximation.

Going further here is beyond the scope of the present lectures.

THICK LENSES

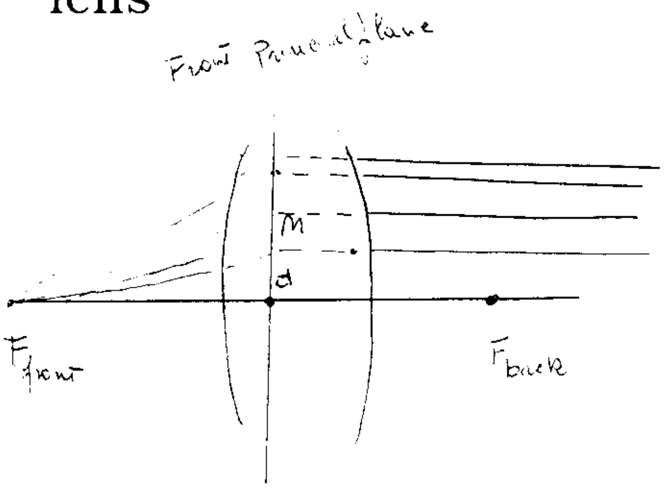
Thick lenses: Thickness non negligible.

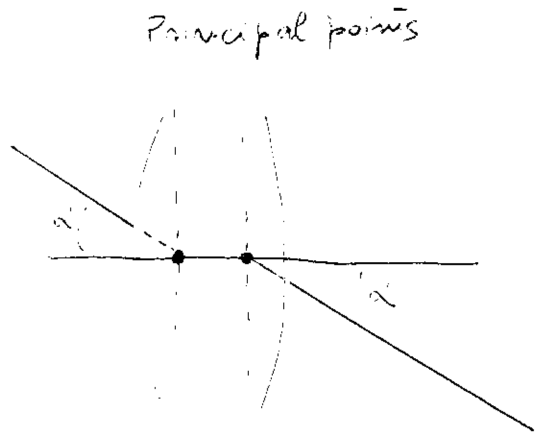
The focus positions of the lens (front and back) are measured from the two Vertices on the axis.

One defines two “principal planes” (primary and secondary) and two “principal points”. The primary principal plane is a plane, internal or even external to the lens, such that a ray which impinges at a point of it from the focus goes out of the lens parallel to the axis. For secondary ray it “converges” to the focus if it arives parallel to the axis.

The two principal points are the crossing points of the planes with the axis and are such that a ray reaching the primary principal plane at a given angle goes out from the lens as it were coming with the same angle from the secondary principal point (generalization of what happens in thin lenses).

If one measures the distances from these planes one can write an equation formally equal to that of the thin lens





QUALITY OF IMAGES

In general rays do not all cross at the same point; aberrations.

In addition there are diffraction effects. Images by perfect systems, that is systems without aberrations, are

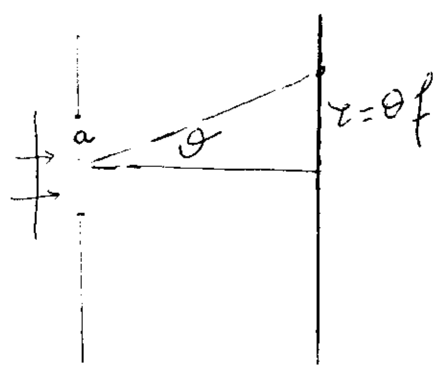
diffraction limited.

The best one can obtain is a "diffraction limited image"

As is well known due to diffraction, the image of a point given by, say, a lens is not a point, but is a "spot" called diffraction pattern. It originates by the fact that the lens has a border (it is not infinitely extended), at the border there is an abrupt change of the amplitude (see condition 1.6) and diffraction takes place.

Consider a lens of radius "a" illuminated by a plane wave, originating from a monochromatic source of wavenumber $k=2\pi/\lambda$ infinitely distant on axis (parallel incident rays). The theory of diffraction allows us to know the energy (intensity) in direction θ. Apart from unessential constant

$$I = u\,u^{*} \propto \frac{J_1^2(ka\sin\theta)}{(ka\sin\theta)^2}$$



Now we take into account that there is the lens; in the lens focus, one has the diffraction pattern. The energy diffracted in direction θ is seen at point $r=\theta f$ of the focal plane: a spot “Airy function” taht is a central circle with surroundings wings.

Analyze the behaviour of I as function of θ, maxima and minima

when $\sin\theta = 0$ central maximum, main direction

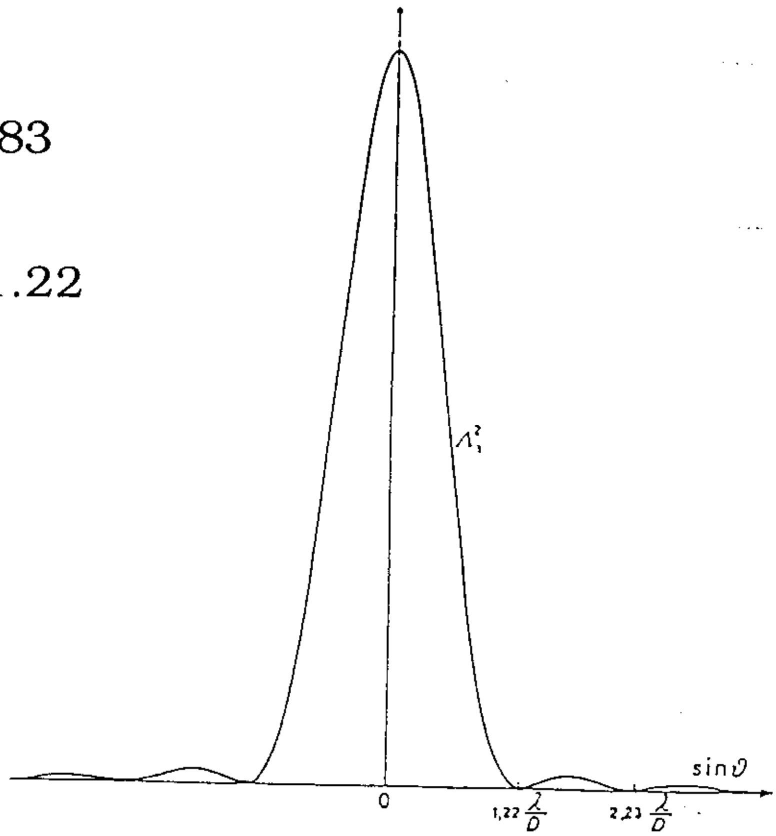
first four zeros of $J_1(x)$

$x = 3.83\,;\ 7.02\,;\ 10.17\,;\ 13.23$

FIRST ZERO

when $ka\sin\theta = 3.83$

$$2\frac{a}{\lambda}\sin\theta = \frac{3.83}{3.14} = 1.22$$



Values of subsequent maxima, with respect to the central one

central	1
first	0.0175
second	0.0042
third	0.0016

It can be shown (Rayleigh 1899) that the energy flux the i-th ring is

$$\Phi_i = J_0^2(x_i) - J_0^2(x_{i+1})$$

Through the central disc and subsequent rings Energy flux (total flux = 1)

central disc	0.8378
first ring	0.0722
second "	0.0276
third "	0.0147
and so on	

The energy in the central disc of the pattern is ~ 84% of the total.

Energy is mostly concentrated in the central ring, whose total angular with is (ka sin θ = 3.83)

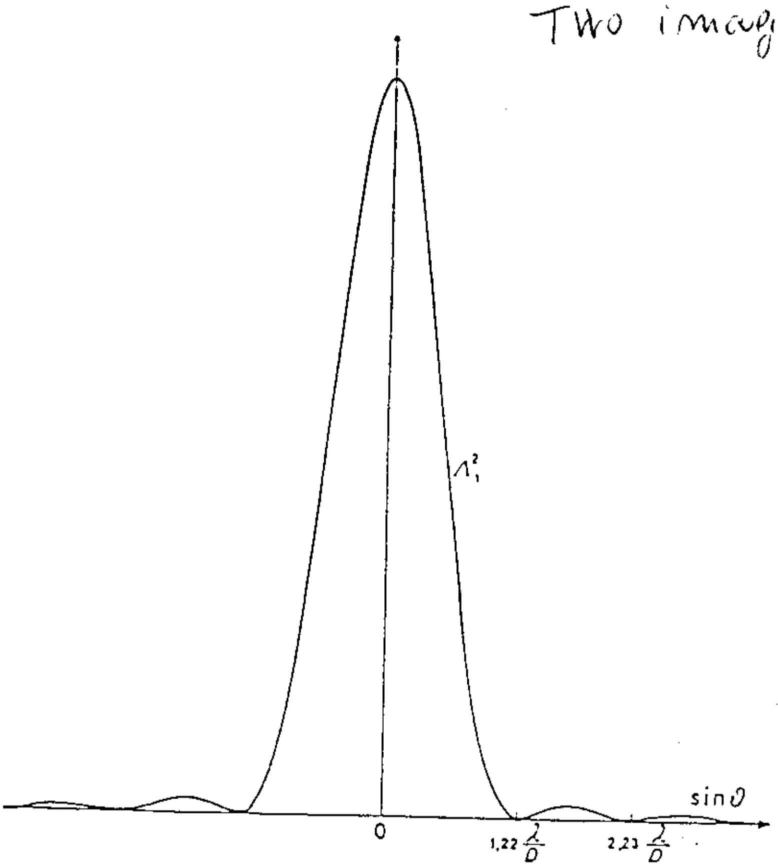
$$2a/\lambda \ \sin\theta = \frac{3.83}{3.14} = 1.22$$

2a = D diameter

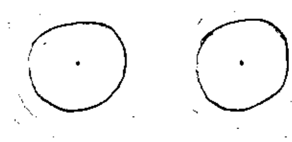
$$\sin\theta = 1.22\frac{\lambda}{D}$$

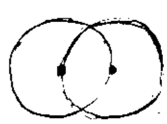
effect on Resolving power of instruments.

Two images can be distinguished if Rayleigh criterion



$$\theta = \frac{\lambda}{D}$$





Rayleigh

- RESOLVING POWER

1- Strehl ratio

2- Rayleigh criterion (v diffraction)

2- OTF or MTF half width