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Winter College on Optics and Photonics
7 - 25 February 2000

1218-21

"Field Quantization Non-Classical Light Quantum Entanglement Logic Gates"

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FIELD QUANTIZATION

NON CLASSICAL LIGHT

QUANTUM ENTANGLEMENT

LOGIC GATES

Electromagnetic field quantization

In the free space e.m. field obeys

$$\left\{\begin{array}{l}\vec{\nabla}\cdot\vec{B}=0\\ \vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}\\ \vec{\nabla}\cdot\vec{D}=0\\ \vec{\nabla}\times\vec{H}=\frac{\partial}{\partial t}\vec{D}\end{array}\right.$$

Maxwell Eqns.

Many books — suitable for quantum optics:
- Walls + Milburn "Quantum Optics"
- Louisell "Quant. stat. prop. of Radiation"

⋮

with
$$\vec{B}=\mu_0\vec{H}$$
$$\vec{D}=\varepsilon_0\vec{E}$$

μ_0 is the magnetic permeability
ε_0 is the electric permettivity
$$\mu_0\varepsilon_0=c^{-2}$$

— gauge invariance —

We can introduce the vector potential $\vec{A}$

and choose the gauge

$$\vec{\nabla}\cdot\vec{A}=0$$

the Coulomb gauge.

in this gauge

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

then in M.E.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{A}$$

using the identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = (\vec{\nabla}\cdot\vec{A})\vec{\nabla} - (\vec{\nabla}\cdot\vec{\nabla})\vec{A}$$

we get the wave eqn.

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = 0$$

We can introduce

$$\vec{A} = \vec{A}^{(+)} + \vec{A}^{(-)}$$

$$\vec{A}^{(-)} = \sum_k c_k^* \vec{u}_k^*(\underline{r})\, e^{i\omega_k t}$$

$$\vec{A}^{(+)} = \sum_k c_k \vec{u}_k(\underline{r})\, e^{-i\omega_k t}$$

$$\vec{A}^{(-)} = \left(\vec{A}^{(+)}\right)^*$$

c_k are constants in free space and we consider a finite volume.

Thus in the wave eqn.

$$\nabla^2 \vec{A} = \sum_k c_k \nabla^2 \vec{u}_k(\underline{r})\, e^{-i\omega_k t} + c.c.$$

$$\frac{\partial^2}{\partial t^2} \vec{A} = -\sum_k \omega_k^2 c_k \vec{u}_k(\underline{r})\, e^{-i\omega_k t} + c.c.$$

then the w.eqn. is

$$\sum_k c_k e^{-i\omega_k t} \left(\nabla^2 \vec{u}_k(\underline{r}) + \frac{\omega_k^2}{c^2} \vec{u}_k(\underline{r}) \right) + c.c. = 0$$

Functions $\vec{u}_k(\underline{r})$ are orthogonal mode functions and they satisfy

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2}\right) \vec{u}_k(\underline{r}) = 0$$

Because of $\vec{\nabla} \cdot \vec{A} = 0$

they satisfy the Transversality condition

$$\vec{\nabla} \cdot \vec{u}_k(\underline{r}) = 0$$

they form a complete orthonormal set

$$\int_V \vec{u}_k^*(\underline{r})\, \vec{u}_{k'}(\underline{r})\, d^3r = \delta_{kk'}$$

The mode function $\vec{u}_k(\underline{r})$ depends on the boundary conditions.

in a cubic volume L^3

$$\vec{u}_k(\underline{r}) = \frac{1}{\sqrt{L^3}}\, \hat{\underline{e}}(\lambda)\, e^{i\vec{k}\cdot\vec{r}}$$

with the propagation vector $\underline{k}$

$$k_x = \frac{2\pi}{L} n_x \;,\; k_y = \frac{2\pi}{L} n_y \;,\; k_z = \frac{2\pi}{L} n_z$$

$\hat{\underline{e}}(\lambda)$ is the polarization vector $(\lambda = 1, 2)$

and by the transversality condition must be orthogonal to $\underline{k}$.

i.e $\quad \vec{\nabla}\cdot\vec{u}_k = 0 \Rightarrow \frac{1}{\sqrt{L^3}}\vec{\nabla}\cdot \hat{\underline{e}}(\lambda) e^{i\underline{k}\cdot\underline{r}} = \frac{i}{\sqrt{L^3}} e^{i\underline{k}\cdot\underline{r}}\, \underline{k}\cdot\hat{\underline{e}}(\lambda) = 0$

Finally :

$$\vec{A}(\underline{r},t) = \sum_k \left(\frac{\hbar}{2\epsilon_0\omega_k}\right)^{\frac{1}{2}} \left[a_k \vec{u}_k(\underline{r}) e^{-i\omega_k t} + a_k^* \vec{u}_k^{\,*}(\underline{r}) e^{i\omega_k t}\right]$$

where a_k and a_k^* are complex dimensionless numbers

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} = i\sum_k \left(\frac{\hbar\omega_k}{2\epsilon_0}\right)^{1/2} \left[a_k \vec{u}_k(\underline{r}) e^{-i\omega_k t} - a_k^* \vec{u}_k^{\,*}(\underline{r}) e^{i\omega_k t}\right]$$

The field quantization is obtained considering a_k and a_k^* self-adjoint operators:

i.e.

$$a_k \to \hat{a}_k$$

$$a_k^* \to \hat{a}_k^+$$

they represent the orthonormal modes of e.m. field. Being the e.m. field a photons ensemble satisfying the Bose statistics we must require the commutation relation

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^+, \hat{a}_{k'}^+] = 0$$

$$[\hat{a}_k, \hat{a}_{k'}^+] = \delta_{kk'}$$

The dynamics of the e.m. field can be described by a set of harmonic oscillators the quantum state of each oscillator is the state of one mode of radiation -
It is described independently of ~~the~~ other modes:
i.e. photons in free space do not interacts -

The field total energy, or Hamiltonian, is:

$$\mathcal{H} = \frac{1}{2}\int d^3r \left(\varepsilon_0 E^2 + \mu_0 H^2\right)$$

$$H = \vec{\nabla}\times\vec{A} = \sum_k \left(\frac{\hbar}{2\omega_k\varepsilon_0}\right)^{1/2}\left(\hat{a}_k \vec{\nabla}\times\vec{u}_k(r) + h.c.\right)$$

$$H^2 = \sum_k \left(\frac{\hbar}{2\omega_k\varepsilon_0}\right)^{1/2}\left(\hat{a}_k \vec{\nabla}\times\vec{u}_k + \hat{a}_k^\dagger \vec{\nabla}\times\vec{u}_k^*\right)\sum_{k'}\left(\frac{\hbar}{2\omega_{k'}\varepsilon_0}\right)^{1/2}\times$$
$$\times\left(\hat{a}_{k'}\vec{\nabla}\times\vec{u}_{k'} + \hat{a}_{k'}^\dagger \vec{\nabla}\times\vec{u}_{k'}^*\right)$$

we have terms like

$$(\vec{\nabla}\times\vec{u}_k)\cdot(\vec{\nabla}\times\vec{u}_{k'}) = \nabla^2\vec{u}_k\cdot\vec{u}_{k'} - (\vec{\nabla}\cdot\vec{u}_{k'})(\vec{\nabla}\cdot u_k)$$

$= 0$ from transversality.

we use the wave eqn.

$$\nabla^2\vec{u}_k = -\frac{\omega_k^2}{c^2}\vec{u}_k \quad \text{and} \quad \vec{\nabla}\cdot\vec{u}_k = 0$$

thus considering

$\mu_0\int d^3r\, H^2$ there are terms like

$$\int d^3r\, \nabla^2\vec{u}_k\cdot\vec{u}_{k'} \triangleq \nabla^2\int d^3r\, \vec{u}_k(\underline{r})\cdot\vec{u}_{k'}(r) = 0$$

and others

$$\int d^3r\, \nabla^2(\vec{u}_k\cdot\vec{u}_{k'}^*) = -\delta_{kk'}\frac{\omega_k^2}{c^2}$$

The ground state is $|0\rangle$
and is defined in such a way that

$$\hat{a}_k |0\rangle_k = 0$$

The energy of the total ground state i.e.

$$|0\rangle = |0\rangle_0 |0\rangle_1 |0\rangle_2 \cdots\cdots$$

is given by

$$\langle 0|\hat{\mathcal{H}}|0\rangle = \hbar\omega_0 \, {}_0\langle 0|\hat{a}_0^{+}\hat{a}_0 + 1/2|0\rangle_0 + $$
$$+ \hbar\omega_1 \, {}_1\langle 0|\hat{a}_1^{+}\hat{a}_1 + 1/2|0\rangle_1 + $$
$$\vdots$$
$$+ \hbar\omega_m \, {}_m\langle 0|\hat{a}_m^{+}\hat{a}_m + 1/2|0\rangle_m + \cdots$$

with $\quad {}_i\langle 0|0\rangle_j = \delta_{ij}$

and because of $\hat{a}_k |0\rangle_k = 0$

we get

$$\langle 0|\hat{\mathcal{H}}|0\rangle = \Sigma_k \frac{\hbar\omega_k}{2}$$

this is infinite but physically we are only concerned with variation of energy thus this infinite does not have physical consequences —

doing the same with E^2 we finally get

$$\mathcal{H} = \frac{1}{2} \Sigma_k \hbar\omega_k (\hat{a}_k \hat{a}_k^+ + \hat{a}_k^+ \hat{a}_k)$$

$$= \Sigma_k \hbar\omega_k (\hat{a}_k^+ \hat{a}_k + 1/2)$$

<u>Number states or Fock states</u> –

Let introduce $\hat{n}_k = \hat{a}_k^+ \hat{a}_k$

then

$$\mathcal{H} = \Sigma_k \hbar\omega_k (\hat{n}_k + 1/2) = \Sigma_k \mathcal{H}_k$$

is the Hamiltonian of a set of harmonic oscillators, with the wave vector $\underline{k}$ distinguishing the various modes.

For each oscillator the energy eigenvalue is

$$E_n^k = \hbar\omega_k (n_k + \tfrac{1}{2})$$

$$n_k = 0, 1, 2, \ldots \quad \text{integer}$$

the energy eigenstates are eigenstates of $\hat{n}_k$ with

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle$$

The operators

$$\hat{a}^{+}_{k} \quad , \quad \hat{a}_{k}$$

are called the creation operator $\hat{a}^{+}_{k}$ and the destruction operator $\hat{a}_{k}$ for one photon in the mode $\underline{k}$. (and polarization $\underline{\hat{e}}_{k}(\lambda)$)

they are such that.

$$\hat{a}_{k}|n\rangle_{k} = \sqrt{n}\,|n-1\rangle_{k}$$

$$\hat{a}^{+}_{k}|n\rangle_{k} = \sqrt{n+1}\,|n+1\rangle_{k}$$

with $\quad {}_{k}\langle n|m\rangle_{k} = \delta_{nm}$

the state $|n_k\rangle$ is obtained successively applying the creation operator to the ground state $|0\rangle_{k}$ n times.

$$|n\rangle_{k} = \frac{(a^{+}_{k})^{n}}{\sqrt{n!}}|0\rangle_{k}$$

they are complete

$$\Sigma_{n}\,|n\rangle_{k}\,{}_{k}\langle n| = 1 \qquad \forall k$$

They form a complete set of basis vectors for a Hilbert space.

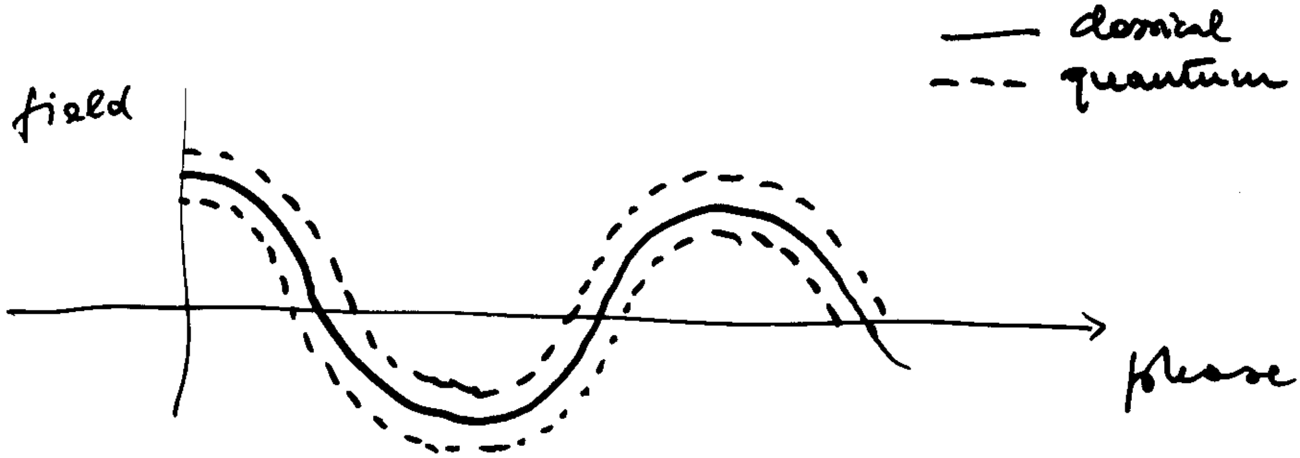
NON CLASSICAL LIGHT

Quantum fields have an inherent quantum indeterminacy also called quantum noise.

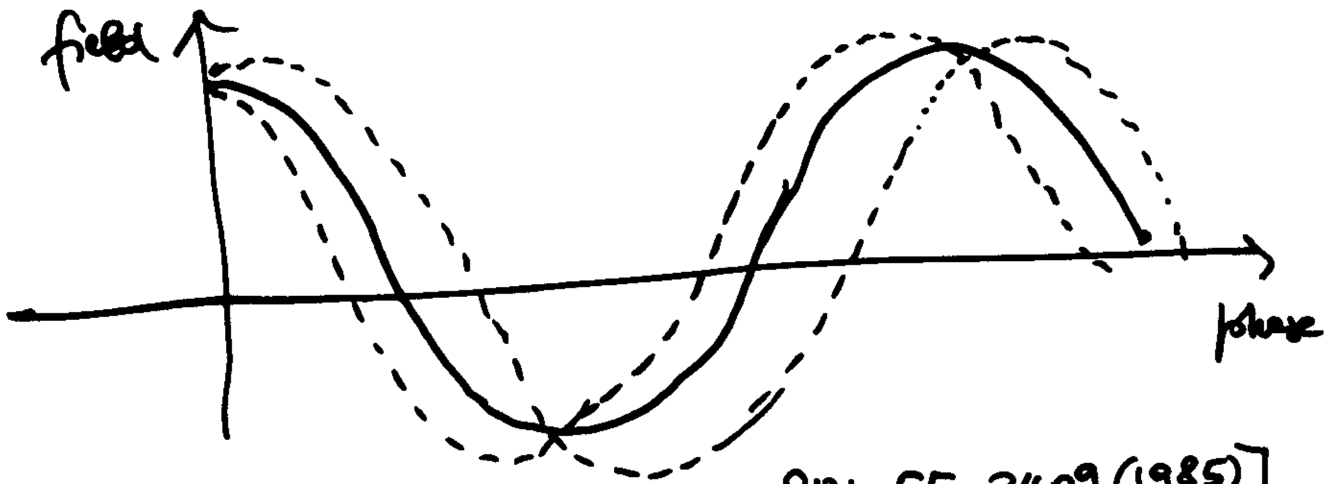
The minimum size of it

$$\mathcal{E}_0 = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2}$$

This noise is present in absence of the field and is connected with the vacuum fluctuations. It is also called the shot noise.



The squeezed state is a new coherent field in which the electric field uncertainty may be reduced below $\mathcal{E}_0$ for given values of the phase angle.



Generated by [Slusher et. al PRL 55, 2409 (1985)]

Various theorists called them:

- pulsating wave packets [Takahasi, Adv. Com. Syst 1, 227 (1965)]
- new coherent states [Lu, Lett. Nuovo Cim. 2, 1241 (1971)]
- Two-photon coherent states [Yuen PRA 13, 2226 (1976)]
- ideal squeezed states [Caves PRD 23, 1693 (1981)]

A single mode of the e.m. field behaves as a simple harmonic oscillator of unit mass

Assuming that the radiation is confined in a one-dimensional cavity and is linearly polarized

$$\hat{E}(z,t)=\left(\frac{2\omega^2}{\varepsilon_0 V}\right)^{1/2}\hat{q}(t)\sin kz$$

$$\hat{H}(z,t)=\left(\frac{2\varepsilon_0 c^2}{V}\right)^{1/2}\hat{p}(t)\cos kz$$

$k=\frac{\omega}{c}$ is the wavevector.

$$[\hat{q}(t),\hat{p}(t)]=i\hbar$$

dimension-less operator.

$$\hat{a}=\left(\frac{1}{2\hbar\omega}\right)^{1/2}(\omega\hat{q}+i\hat{p})$$

$$\hat{a}^{+}=\left(\frac{1}{2\hbar\omega}\right)^{1/2}(\omega\hat{q}-i\hat{p})$$

$$[\hat{a},\hat{a}^{+}]=1$$

then

$$\hat{E}(z,t)=\frac{1}{2}\mathcal{E}\,[\hat{a}(t)+\hat{a}^{+}(t)]$$

$$\mathcal{E}=\sqrt{\frac{4\hbar\omega}{\varepsilon_0 V}}\sin kz$$

$$\hat{a}(t)=\hat{a}(0)\,e^{-i\omega t}$$

$$\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle$$

$$\hat{a}^{+}|n\rangle=\sqrt{n+1}\,|n+1\rangle$$

$$|n\rangle=\frac{(\hat{a}^{+})^n}{\sqrt{n!}}|0\rangle$$

The fluctuations of any observable operator $\hat{O}$ are defined by the variance as

$$\langle\psi|(\Delta\hat{O})^2|\psi\rangle = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$$

The expectation is taken with respect to any state $|\psi\rangle$.

The variances of two observables $\hat{O}_1$ and $\hat{O}_2$ for the same state satisfy the uncertainty relation

$$\langle(\Delta\hat{O}_1)^2\rangle\langle(\Delta\hat{O}_2)^2\rangle \geqslant \frac{1}{4}\left|\langle[\hat{O}_1,\hat{O}_2]\rangle\right|^2$$

= holds, the state $|\psi\rangle$ is called <u>MINIMUM UNCERTAINTY STATE</u>

The ground state $|0\rangle$ of the single-mode is a minimum uncertainty state.

$$\langle 0|\hat{q}|0\rangle = \langle 0|\hat{p}|0\rangle = 0$$

$$\langle 0|(\Delta\hat{q})^2|0\rangle = \langle 0|\hat{q}^2|0\rangle = \langle 0|\frac{\hbar}{2\omega}(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger)|0\rangle$$

$$= \frac{\hbar}{2\omega}$$

$$\langle 0|(\Delta\hat{p})^2|0\rangle = \frac{\hbar\omega}{2}$$

then

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \left(\frac{\hbar}{2}\right)^2$$

The coherent states introduced by Glauber [Phys. Rev. **131**, 2766 (1963)] are minimum uncertainty states. They are the eigenstates of the destruction operator.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

α a complex number - i.e. $\alpha = |\alpha| e^{i\varphi}$

In terms of number states

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_0^\infty {}_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

The photon number variance for a single-mode coherent state is.

$$\langle\alpha|(\Delta\hat{n})^2|\alpha\rangle = \langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2 = |\alpha|^2 \equiv \langle\alpha|\hat{n}|\alpha\rangle$$

$|\alpha|^2$ is the mean number of photons in the coherent state.

<u>Quadrature Operators</u>

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \equiv \left(\frac{\omega}{2\hbar}\right)^{1/2} \hat{q}$$

$$\hat{Y} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \equiv \left(\frac{1}{2\hbar\omega}\right)^{1/2} \hat{p}$$

the electric field can be written

$$\hat{E}(z,t) = \mathcal{E}(\hat{X}\cos\omega t + \hat{Y}\sin\omega t)$$

$$\mathcal{E} = \sqrt{\frac{4\hbar\omega}{\epsilon_0 V}} \sin kz.$$

$$[\hat{X}, \hat{Y}] = \frac{i}{2}$$

the variances

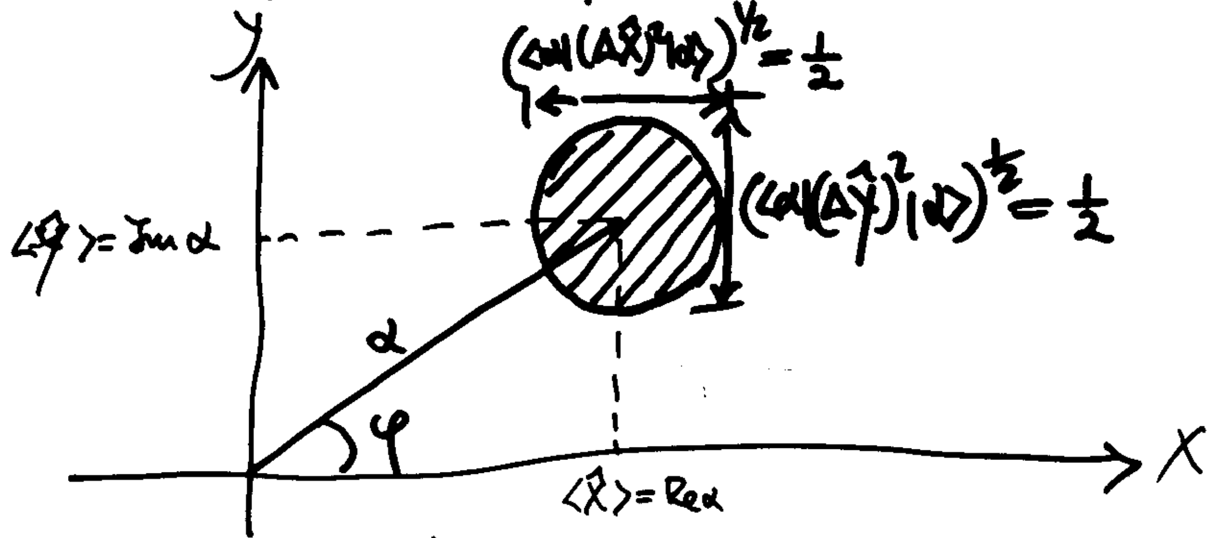
$$\langle (\Delta\hat{X})^2 \rangle \langle (\Delta\hat{Y})^2 \rangle \geq \frac{1}{16}$$

for a coherent state

$$\langle\alpha|(\Delta\hat{X})^2|\alpha\rangle \langle\alpha|(\Delta\hat{Y})^2|\alpha\rangle = \frac{1}{16}$$

with

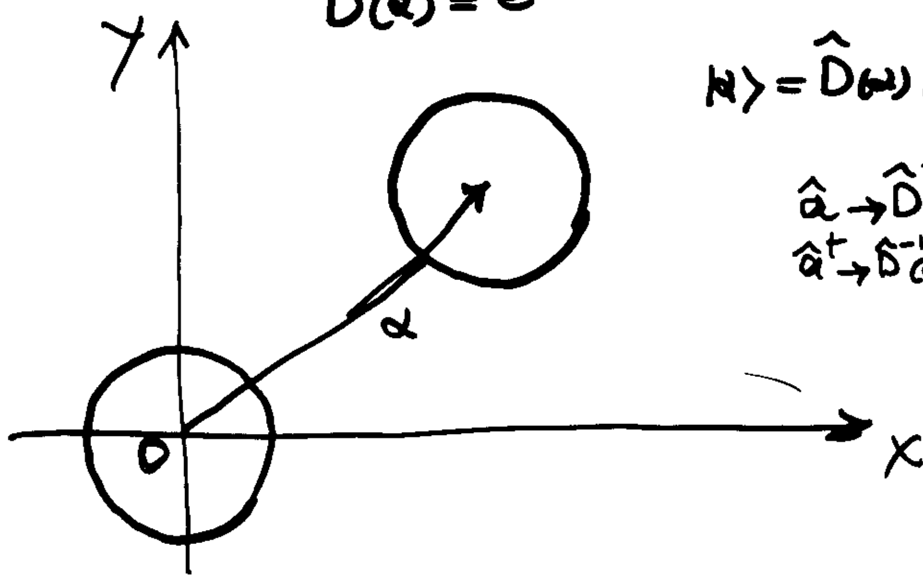
$$\langle\alpha|(\Delta\hat{X})^2|\alpha\rangle = \langle\alpha|(\Delta\hat{Y})^2|\alpha\rangle = \frac{1}{4}$$



the centre is in $\langle\alpha|\hat{X} + i\hat{Y}|\alpha\rangle = \alpha$

Displacement operator

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}$$



$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$\hat{a} \to \hat{D}^{-1}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$

$\hat{a}^{\dagger} \to \hat{D}^{-1}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a}^{\dagger} + \alpha^{*}$

$$\hat{X} \to \hat{D}^{-1}(\alpha)\hat{X}\hat{D}(\alpha) = \hat{X} + \mathrm{Re}\,\alpha$$
$$\hat{Y} \to \hat{D}^{-1}(\alpha)\hat{Y}D(\alpha) = \hat{Y} + \mathrm{Im}\,\alpha$$

Optical correlation functions.

The intensity fluctuations of the optical field are described by the correlation function $G^{(2)}$ which is evaluated by considering the joint absorption of photons at two space-time points.

The electric field $\hat{E}(t)$

$$\hat{E}(t) = \hat{E}^{+}(t) + \hat{E}^{-}(t)$$

wl

$$\hat{E}^{+}(t) = \frac{1}{2}\mathcal{E}\,\hat{a}\,e^{-i\omega t}$$

$$\hat{E}^{-}(t) = (\hat{E}^{+})^{\dagger} = \frac{1}{2}\mathcal{E}\,\hat{a}^{\dagger}e^{i\omega t}$$

When one is interested in purely temporal correlations one needs the correlations at times t and $t+\tau$ measured at the same space point

$$G^{(2)}(t,t+\tau) = \langle \hat{E}^{-}(t)\hat{E}^{-}(t+\tau)\hat{E}^{+}(t+\tau)\hat{E}^{+}(t)\rangle$$

or, normalized

$$g^{(2)}(t,t+\tau) = \frac{\langle \hat{E}^{-}(t)\hat{E}^{-}(t+\tau)\hat{E}^{+}(t+\tau)\hat{E}^{+}(t)\rangle}{\langle \hat{E}^{-}(t)\hat{E}^{+}(t)\rangle^{2}}$$

For a single mode at the same instant one gets

$$g^{(2)}(t,t) \equiv g^{(2)}(0) = \frac{\langle \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle^{2}}$$

$$\hat{a}^{\dagger}\hat{a} = \hat{n}$$

Thus,

$$g^{(2)}(0) = \frac{\langle \hat{a}^{\dagger}\hat{a}\,\hat{a}^{\dagger}\hat{a}\rangle - \langle \hat{a}^{\dagger}\hat{a}\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle^{2}} = \frac{\langle \hat{n}^{2}\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^{2}} = \frac{\langle(\Delta\hat{n})^{2}\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^{2}} + 1$$

For a single-mode coherent state $|\alpha\rangle$

$$\langle\alpha|(\Delta\hat{n})^2|\alpha\rangle = \langle\alpha|\hat{n}|\alpha\rangle$$

The photon number variance is also described by the Mandel Q-parameter

$$Q = \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle} = \langle\hat{n}\rangle(g^{(2)}(0) - 1)$$

Thus, for a <u>coherent state $|\alpha\rangle$</u> the Q parameter is

$$Q = 0$$

This condition defines the Poissonian statistics

when $g^{(2)}(0) < 1$ or $Q < 0$

the field is sub-Poissonian

when $g^{(2)}(0) > 1$ or $Q > 0$

the field is super-Poissonian

<u>Squeezed states</u>

The coherent states have the two quadratures variances equal and are minimum uncertainty states

It might exist a minimum uncertainty state without equal variances.

The modified variances can be written:

$$\langle(\Delta\hat{X})^2\rangle = \frac{1}{4}e^{-2s}$$

$$\langle(\Delta\hat{Y})^2\rangle = \frac{1}{4}e^{+2s}$$

s = squeezing parameter. for $s=0$ one obtains the coherent state –

Squeezing transformation:

$$\hat{X} \to \hat{X}_s = \hat{X} e^{-s}$$
$$\hat{Y} \to \hat{Y}_s = \hat{Y} e^{+s}$$

the corresponding $\hat{a}_s$

$$\hat{a}_s = \hat{X}_s + i\hat{Y}_s = \frac{\hat{a}+\hat{a}^+}{2} e^{-s} + \frac{i}{2i}(\hat{a}-\hat{a}^+) e^{+s}$$
$$= \hat{a}\cosh s - \hat{a}^+ \sinh s$$
$$\hat{a}_s^+ = \hat{a}^+ \cosh s - \hat{a} \sinh s$$
$$[\hat{a}_s, \hat{a}_s^+] = 1 \quad [\hat{X}_s, \hat{Y}_s] = \frac{i}{2}$$

the transformed quantum oscillator Hamiltonian

$$H = \hbar\omega(\hat{a}_s^+ \hat{a}_s + 1/2)$$

defines a pseudo-number state $|n_s\rangle$ eigenvector of

$$\hat{n}_s|n_s\rangle = \hat{a}_s^+ \hat{a}_s |n_s\rangle = n_s |n_s\rangle$$

the new coherent state is defined as

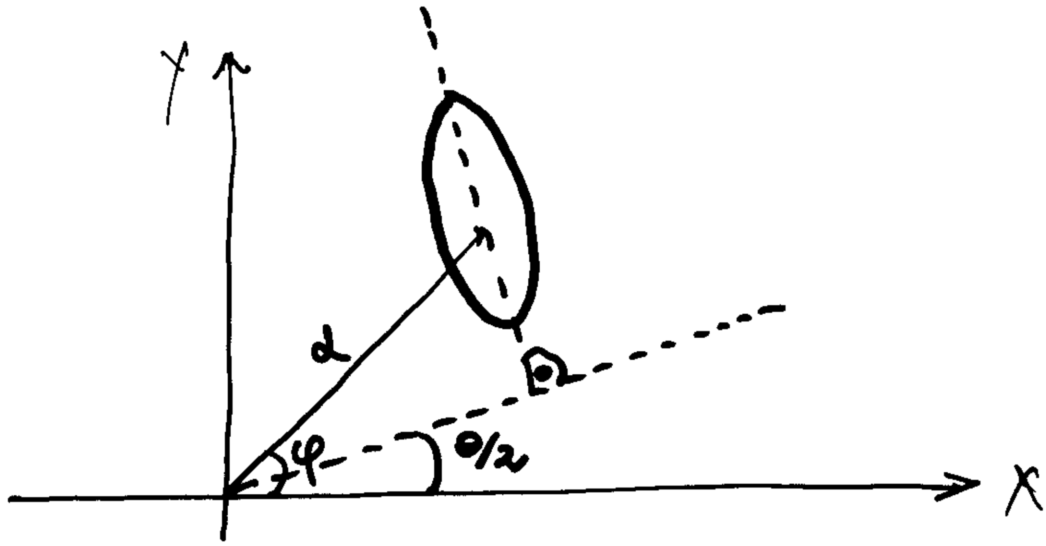
$$\hat{D}_s(\alpha)|0_s\rangle = |\alpha_s\rangle \; ; \; \hat{D}_s(\alpha) = e^{\alpha\hat{a}_s^+ - \alpha^*\hat{a}_s}$$

and

$$\hat{X}_s \xrightarrow{\hat{D}_s} \hat{X}_s + \mathrm{Re}\,\alpha$$
$$\hat{Y}_s \xrightarrow{\hat{D}_s} \hat{Y}_s + \mathrm{Im}\,\alpha$$

In the transformed space the error circle remains a circle because the coherent state defined in this space is a true coherent state.

However, in the original space there is a compression of the $\hat{X}$ quadrature and an expansion of the $\hat{Y}$ quadrature to produce an elliptical error contour.



The transformation is

$$\hat{S}(\zeta) = e^{+\frac{1}{2}\zeta^* \hat{a}^2 - \frac{1}{2}\zeta \hat{a}^{+2}}$$

with $\zeta = s e^{i\theta}$ $\quad 0 \le s \le \infty \; ; \; 0 \le \theta \le 2\pi$

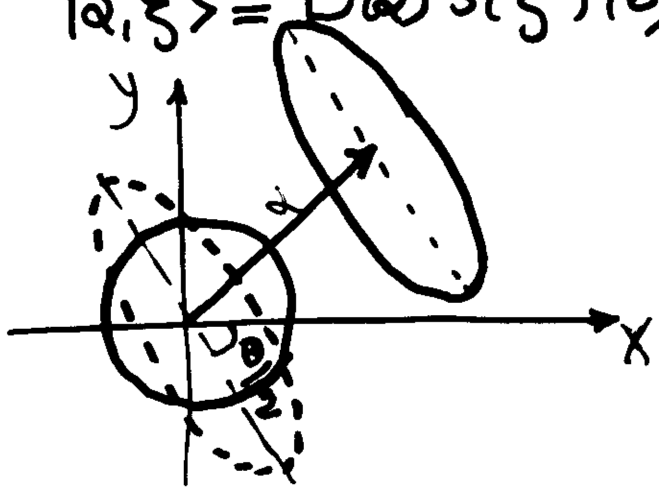
This general transformation gives a compression and an expansion of the canonical variables $\hat{X}$ and $\hat{Y}$ in a direction which forms an angle $\frac{\theta}{2}$ with X and Y axes.

$$\hat{a} \to \hat{S}^{-1}(\zeta)\hat{a}\hat{S}(\zeta) = \hat{a}\cosh s - \hat{a}^{+} e^{i\theta} \sinh s$$

$$\hat{a}^{+} \to \hat{S}^{-1}(\zeta)\hat{a}^{+}\hat{S}(\zeta) = \hat{a}^{+}\cosh s - \hat{a}\, e^{-i\theta} \sinh s$$

Alternative and equivalent way:

$$|\alpha, \zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$$



Expectation values with respect to $|\alpha,\zeta\rangle$.

$$\langle\alpha,\zeta|\hat{a}|\alpha,\zeta\rangle = \langle 0|\hat{S}^{-1}(\zeta)\hat{D}^{-1}(\alpha)\hat{a}\,\hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$$

$$= \langle 0|\hat{S}^{-1}(\zeta)\hat{a}\,\hat{S}(\zeta)|0\rangle + \alpha$$

$$= \langle 0|\hat{a}\cosh s - \hat{a}^{\dagger}e^{i\theta}\sinh s|0\rangle + \alpha$$

$$= \alpha$$

$$\langle\alpha,\zeta|\hat{a}^{\dagger}|\alpha,\zeta\rangle = \alpha^{*}$$

$$\langle\alpha,\zeta|\hat{a}^{\dagger}\hat{a}|\alpha,\zeta\rangle = \langle 0|\hat{S}^{-1}(\zeta)\hat{D}^{-1}(\alpha)\hat{a}^{\dagger}\hat{a}\,\hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$$

$$\mathbb{1} = \hat{D}(\alpha)\hat{D}^{-1}(\alpha)$$

$$= \langle 0|\hat{S}^{-1}(\zeta)\hat{a}^{\dagger}\hat{a}\,\hat{S}(\zeta)|0\rangle + |\alpha|^2$$

$$\hat{S}(\zeta)\hat{S}^{-1}(\zeta) = \mathbb{1}$$

$$= |\alpha|^2 + \sinh^2 s$$

In the same way

$$\langle\alpha,\zeta|\hat{a}\hat{a}|\alpha,\zeta\rangle = \alpha^2 - e^{i\theta}\sinh s\cosh s$$

$$\langle\alpha,\zeta|\hat{a}^{\dagger}\hat{a}^{\dagger}|\alpha,\zeta\rangle = \alpha^{*2} - e^{-i\theta}\sinh s\cosh s$$

and the quadrature variances:

$$\langle\alpha,\zeta|(\Delta\hat{X})^2|\alpha,\zeta\rangle = \frac{1}{4}\left[e^{-2s}\cos^2\frac{\theta}{2} + e^{2s}\sin^2\frac{\theta}{2}\right]$$

$$\langle\alpha,\zeta|(\Delta\hat{Y})^2|\alpha,\zeta\rangle = \frac{1}{4}\left[e^{-2s}\sin^2\frac{\theta}{2} + e^{2s}\cos^2\frac{\theta}{2}\right]$$

obviously $\langle\alpha,\zeta|\hat{X}|\alpha,\zeta\rangle = \frac{1}{2}(\alpha+\alpha^*) = \mathrm{Re}\,\alpha$

$\langle\alpha,\zeta|\hat{Y}|\alpha,\zeta\rangle = \frac{1}{2i}(\alpha-\alpha^*) = \mathrm{Im}\,\alpha$

Uncertainty product:

$$\left(\langle\alpha,\zeta|(\Delta\hat{X})^2|\alpha,\zeta\rangle\right)^{1/2}\left(\langle\alpha,\zeta|(\Delta\hat{Y})^2|\alpha,\zeta\rangle\right)^{1/2}=\frac{1}{4}\left(\cosh^2 2s\,\sin^2\theta+\cos^2\theta\right)^{1/2}$$

takes the minimum value $\frac{1}{4}$ for $\theta=0$ or π

it is maximum for $\theta=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$

and

$$\left(\langle\alpha,\zeta|(\Delta\hat{X})^2|\alpha,\zeta\rangle\right)^{1/2}\left(\langle\alpha,\zeta|(\Delta\hat{Y})^2|\alpha,\zeta\rangle\right)^{1/2}=\frac{1}{4}\cosh 2s$$

Rotated quadrature:

$$\hat{X}_\theta=\frac{e^{-i\frac{\theta}{2}}\hat{a}+e^{i\frac{\theta}{2}}\hat{a}^\dagger}{2}$$

$$\langle\alpha,\zeta|\hat{X}_\theta^2|\alpha,\zeta\rangle=\frac{1}{4}(\alpha e^{-i\theta/2}+\alpha^* e^{i\theta/2})^2+\frac{1}{4}e^{-2s}$$

$$\langle\alpha,\zeta|\hat{X}_\theta|\alpha,\zeta\rangle=\frac{1}{2}(\alpha e^{-i\theta/2}+\alpha^* e^{i\frac{\theta}{2}})$$

and

$$\langle\alpha,\zeta|(\Delta\hat{X}_\theta)^2|\alpha,\zeta\rangle=\frac{1}{4}e^{-2s}$$

$$\langle\alpha,\zeta|(\Delta\hat{Y}_\theta)^2|\alpha,\zeta\rangle=\frac{1}{4}e^{2s}$$

the rotated quadratures have minimum uncertainty product.

The squeezed state becomes a coherent state for $s=0$.

For $s\to\infty$ it becomes the so-called "line state".

For $\theta=0$ it coincides with the $\hat{X}$ or $\hat{Y}$ axes.

The line state is eigenvector of $\hat{X}$ ($\hat{Y}$)

$$\hat{X}|x\rangle=x|x\rangle$$

The coordinate representation is the projection of $|\alpha,\zeta\rangle$ onto $|x\rangle$

$$\langle x|\alpha,\zeta\rangle=\left(\frac{2e^{2s}}{\pi}\right)^{1/4}e^{-(x-\mathrm{Re}\,\alpha)^2e^{2s}+2ix\,\mathrm{Im}\,\alpha-i\,\mathrm{Re}\,\alpha\cdot\mathrm{Im}\,\alpha}$$

it forms a Gaussian wave-packet

$$|\langle x|\alpha,\zeta\rangle|^2=\left(\frac{2e^{2s}}{\pi}\right)^{1/2}e^{-2(x-\mathrm{Re}\,\alpha)^2e^{2s}}$$

Number uncertainty for the squeezed state

$$\langle\alpha,\zeta|(\Delta\hat{n})^2|\alpha,\zeta\rangle = \langle\alpha,\zeta|\hat{n}^2|\alpha,\zeta\rangle - \langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle^2$$

$$= \langle\alpha,\zeta|\hat{a}^{\dagger 2}\hat{a}^2|\alpha,\zeta\rangle + \langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle - \langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle^2$$

$$= |\alpha|^2\left\{e^{-2s}\cos^2\left(\varphi-\frac{\theta}{2}\right) + e^{2s}\sin^2\left(\varphi-\frac{\theta}{2}\right)\right\} + 2\sinh^2 s\cosh^2 s$$

$\alpha = |\alpha|e^{i\varphi}$

Q-parameter:

$$Q = \frac{\langle\alpha,\zeta|(\Delta\hat{n})^2|\alpha,\zeta\rangle - \langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle}{\langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle}$$

for the vacuum - ~~[illegible]~~

$$Q = 2\langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle + 1$$

i.e. the squeezed vacuum shows photon bunching and super-Poissonian statistics.

Squeezed light may have a sub-Poissonian statistics and photon antibunching.

assume: $|\alpha|^2 \gg e^{2s} \Rightarrow \langle\alpha,\zeta|\hat{n}|\alpha,\zeta\rangle \approx |\alpha|^2$

$$\langle\alpha,\zeta|(\Delta\hat{n})^2|\alpha,\zeta\rangle \approx |\alpha|^2\left\{e^{-2s}\cos^2\left(\varphi-\frac{\theta}{2}\right) + e^{2s}\sin^2\left(\varphi-\frac{\theta}{2}\right)\right\}$$

$$Q \cong (e^{-2s}-1)\cos^2\left(\varphi-\frac{\theta}{2}\right) + (e^{2s}-1)\sin^2\left(\varphi-\frac{\theta}{2}\right)$$

antibunching for $\varphi = \frac{\theta}{2}$

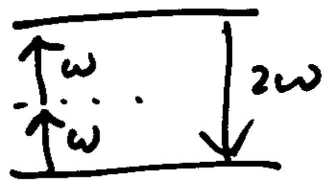
$$Q \cong (e^{-2s}-1) < 0 \quad \Rightarrow Q = \langle\hat{n}\rangle(g^2(0)-1) < 0$$

i.e. $g^2(0) < 1$ —

Generation of squeezed light.

Nonlinearity $\rightarrow$ modify the noise properties of light

eg. two photon absorption



depends quadratically on the instantaneous field intensity.

- It will absorb from the peak of the intensity and reduce the amplitude fluctuations.
- Noise fluctuations may be affected with phase dependent noise amplification or attenuation

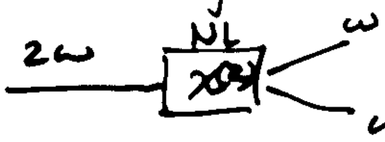
Simplest quadratic interaction:

$$H = \hbar\left(\chi^*(\varepsilon)\hat{a}^2 + \chi(\varepsilon)\hat{a}^{\dagger 2}\right)$$

$\chi(\varepsilon)$ effective non linear susceptibility.

$$\chi(\varepsilon) = \varepsilon\,\chi^{(2)}$$

H represents a degenerate parametric amplifier



down-converted light

A single mode electric field enters in a crystal with $\chi^{(2)}$ the polarization

$$P(t) \simeq \chi^{(1)}E(t) + \chi^{(2)}E^2(t)$$

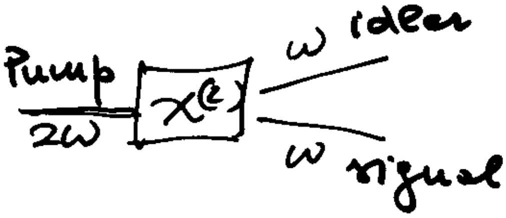
Interaction Hamiltonian

$$H_{int} = -\bar{E}\cdot P \qquad (E = E_p + E_s + E_i)$$

phase-matching and energy conservation.

E_p very intense and undepleted

$$H = \hbar\,(\chi^* \hat{a}^2 + \chi\,\hat{a}^{\dagger 2})$$



State of the signal at time t:

$$|s(t)\rangle = e^{-\frac{i}{\hbar}Ht}|0\rangle$$

$$= e^{-i(\chi^* \hat{a}^2 + \chi\,\hat{a}^{\dagger 2})t}|0\rangle$$

$$\zeta = 2i\chi t$$

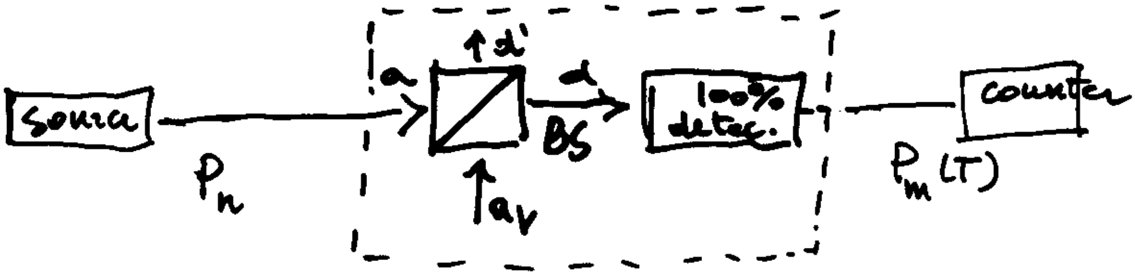
$$|s(t)\rangle = e^{\frac{1}{2}\zeta^* \hat{a}^2 - \frac{1}{2}\zeta\,\hat{a}^{\dagger 2}}|0\rangle \equiv |0,\zeta\rangle$$

In real situations

fluctuations of the pump field and other dissipative losses will act to spoil the squeezing.

Detection of squeezed light

Photon-statistical experiment.



"black box" ≡ real detector

The BS and the perfectly efficient detector model a real detector with quantum efficiency η.

The photocount distribution provides a record of the photon-number distribution distorted by the effect of efficiency η and integration time T.

In the model inefficiency is ascribed to the loss of light at a beam splitter.

only a fraction $\eta^{1/2}$ of the incident light is transmitted

$$\begin{pmatrix}\hat{d}\\ \hat{d}'\end{pmatrix} = \begin{pmatrix}\sqrt{\eta} & e^{i\theta}\sqrt{1-\eta}\\ e^{-i\theta}\sqrt{1-\eta} & \sqrt{\eta}\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{a}_v\end{pmatrix}$$

$$\hat{d} = \sqrt{\eta}\,\hat{a} + i\sqrt{1-\eta}\,\hat{a}_v \qquad \theta = \frac{\pi}{2}$$

The initial state is $|0\rangle_v |\cdots\rangle_s$

$$\langle\cdots|\langle 0|\hat{d}^{+}\hat{d}|0\rangle|\cdots\rangle = \langle \hat{d}^{+}\hat{d}\rangle$$

$$= \eta\langle \hat{a}^{+}\hat{a}\rangle + i\sqrt{\eta(1-\eta)}\cancel{\langle \hat{a}^{+} a_v\rangle} - i\sqrt{\eta(1-\eta)}\cancel{\langle a_v^{+} a\rangle}$$
$$+ (1-\eta)\cancel{\langle a_v^{+} a_v\rangle}$$

$$\langle \hat{d}^{+}\hat{d}^{+}\hat{d}\hat{d}\rangle = \eta^2\langle \hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}\rangle = \eta^2\langle \hat{n}(1-\hat{n})\rangle$$

$\vdots$

The statistic of the signal $\underline{a}$ is taken into account considering the statistics of $\underline{d}$.

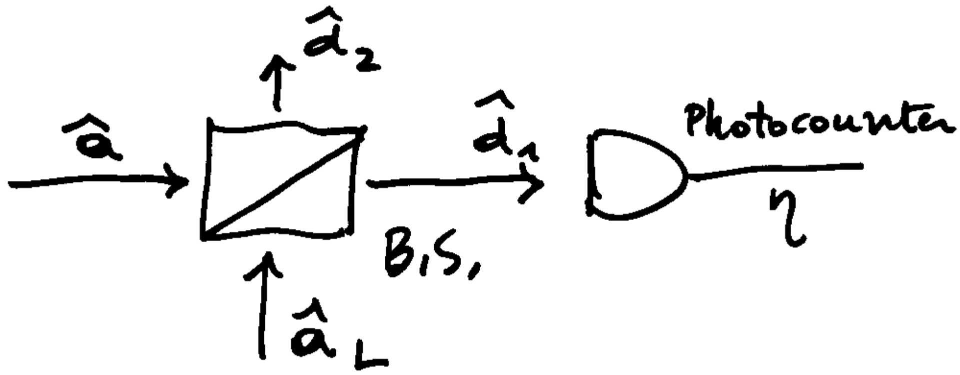
With the direct detection one can observe the possible antibunching but it is not sensitive to the phase of the detected field

Not suitable to detect the squeezing -

<u>We need a phase dependent detection</u>

Homodyne detection

(Yuen + Shapiro IEEE Trans. Inf. Th. 24, 657 (1978))



$\hat a_L \equiv$ local oscillator $\quad \hat\alpha_L = |\alpha_L| e^{i\varphi_L}$

strong field with the <u>same frequency</u> of the signal

$r \equiv$ reflection
$t \equiv$ transmission

$$\begin{pmatrix} \hat d_1 \\ \hat d_2 \end{pmatrix} = \underbrace{\begin{pmatrix} r & t \\ t & r \end{pmatrix}}_{T} \begin{pmatrix} \hat a_L \\ \hat a \end{pmatrix}$$

unitarity requires $\quad T T^\dagger = \mathbb{1}$

i.e. $\quad |r|^2 + |t|^2 = 1$

$\quad\quad r^* t + r t^* = 0$

Ordinary homodyne detection requires

$$|r| \ll |t|$$

$$\hat d_1 = r \hat a_L + t \hat a \quad ; \quad \hat d_1^\dagger = r^* \hat a_L^\dagger + t^* \hat a^\dagger$$

$$\hat d_1^\dagger \hat d_1 = |r|^2 \hat a_L^\dagger \hat a_L + r t^* \hat a^\dagger \hat a_L + r^* t \hat a \hat a_L^\dagger + |t|^2 \hat a^\dagger \hat a$$

$$\langle \hat{a}_1^{+}\hat{a}_1\rangle = |r|^2|\alpha_L|^2 + |t|^2\langle \hat{a}^{+}\hat{a}\rangle + tr^*|\alpha_L|e^{-i\varphi_L}\langle \hat{a}\rangle$$
$$+ t^* r|\alpha_L|e^{i\varphi_L}\langle \hat{a}^{+}\rangle$$

$$x = \arg r - \arg t + \varphi_L$$

define:
$$\hat{X}_x = \frac{\hat{a}e^{ix} + \hat{a}^{+}e^{-ix}}{2}$$

$$\langle \hat{a}_1^{+}\hat{a}_1\rangle = |r|^2|\alpha_L|^2 + |t|^2\langle \hat{a}^{+}\hat{a}\rangle + 2|r||t|\langle \hat{X}_x\rangle$$

ordinary homodyne detection
$$|r| \ll |t|$$
$$|r||\alpha_L| \gg |t||\alpha|$$

Subtracting the known contribution the mean photocurrent at the detector
$$\langle m_1\rangle = \eta\left(\langle \hat{a}_1^{+}\hat{a}_1\rangle - |r|^2|\alpha_L|^2\right) = 2\eta|r||t|\langle \hat{X}_x\rangle$$

choosing $\arg r = \arg t$.

the signal quadrature is governed by the local oscillator phase
$$\tilde{X}_x \equiv \hat{X}_\varphi = \frac{\hat{a}e^{i\varphi_L} + \hat{a}^{+}e^{-i\varphi_L}}{2}$$

Photocurrent variance
$$\langle(\Delta m_1)^2\rangle = \eta|r|^2|\alpha_L|^2\left\{1 + 4\eta|t|^2\left(\langle(\Delta\hat{X}_{\varphi_L})^2\rangle - \frac{1}{4}\right)\right\}$$

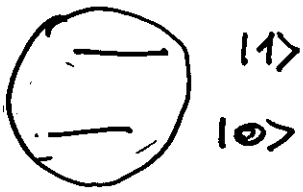
if input $|\alpha\rangle \rightarrow \langle(\Delta\hat{X}_\varphi)^2\rangle = 1/4$

if input $|\alpha,\zeta\rangle \rightarrow \langle(\Delta\hat{X}_\varphi)^2\rangle < 1/4$

QUANTUM ENTANGLEMENT

Basics.

es: - trapped electron
- two-level atom
- ~~single~~ mode cavity
- trapped ion.
⋮



2-dimensional Hilbert space.

$$\langle 1|0\rangle = 0 \qquad \langle 0|0\rangle = \langle 1|1\rangle = 1$$

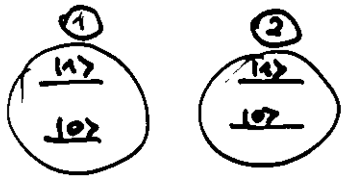
any state:
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

qubit quantum information

classical bit $\begin{cases} \text{or } |0\rangle \\ \text{or } |1\rangle \end{cases}$

quantum bit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
both states are contemporarily present. !!

TWO QUBITS



Register —

classically only one of the states:

$$\overset{1}{|0\rangle_1|0\rangle_2} ; \overset{2}{|0\rangle_1|1\rangle_2} ; \overset{3}{|1\rangle_1|0\rangle_2} ; \overset{4}{|1\rangle_1|1\rangle_2}$$

is present

quantum mechanically:

$$|\psi\rangle = \frac{1}{2}\left(|0\rangle_1|0\rangle_2 + |0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right)$$

all 4 numbers are present in the register.

we'll write

$$|nm\rangle = |n\rangle_1 |m\rangle_2 \,.$$

Assume there are two states in the register with 2 qubits.

$$|\psi_1\rangle = \alpha|00\rangle + \beta|01\rangle$$

$$|\psi_2\rangle = \alpha|00\rangle + \beta|11\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

We may ask what is the quantum state of each qubit in the previous superpositions?

$$|\psi_1\rangle = |0\rangle_1\left(\alpha|0\rangle_2 + \beta|1\rangle_2\right)$$

$$|\psi_2\rangle = \alpha|0\rangle_1|0\rangle_2 + \beta|1\rangle_1|1\rangle_2$$

We define the state which does not allow decomposition **entangled state**.

The prototype of entangled states in 4-dimensional Hilbert space are the so called Bell's states

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

With entangled states one needs the concept of

density operator $\hat{\varrho}$

also called density matrix.

Properties:

$$\mathrm{Tr}\,\hat{\varrho} = 1$$

$$\hat{\varrho} \geqslant 0 \quad ; \quad \hat{\varrho} = \hat{\varrho}^{\dagger}$$

For a pure state $|\psi\rangle$

$$\hat{\varrho} = |\psi\rangle\langle\psi|$$

i.e. the projector onto the Hilbert space state vector $|\psi\rangle$

If

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\hat{\varrho} = (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|)$$

$$= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|$$

In the basis $(|0\rangle, |1\rangle)$ of H-space we can construct:

$$\langle 0|\hat{\varrho}|0\rangle \; ; \; \langle 0|\hat{\varrho}|1\rangle \; ; \; \langle 1|\hat{\varrho}|0\rangle \; ; \; \langle 1|\varrho|1\rangle$$

then the matrix

$$\hat{\varrho} = \begin{array}{cc} |0\rangle & |1\rangle \\ \left(\begin{array}{cc} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{array}\right) & \begin{array}{c} \langle 0| \\ \langle 1| \end{array} \end{array}$$

$$\langle 0|\hat{\varrho}|0\rangle = \langle 0|\{|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \alpha\beta^*|0\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0|\}|0\rangle$$

$$= |\alpha|^2$$

$$\langle 0|\hat{\varrho}|1\rangle = \alpha\beta^*$$

$$\langle 1|\hat{\varrho}|0\rangle = \beta\alpha^*$$

$$\langle 1|\hat{\varrho}|1\rangle = |\beta|^2$$

Positivity of $\hat{\varrho}$ means that $\forall\ |\phi\rangle$ in H-space

$$\langle\phi|\hat{\varrho}|\phi\rangle \geq 0$$

Expectation values with $\hat{\varrho}$ -

we know

$$\langle\hat{A}\rangle = \langle\varphi|\hat{A}|\varphi\rangle$$

is the expectation value of $\hat{A}$ on the state $|\varphi\rangle$

we define also:

$$\langle\hat{A}\rangle = \mathrm{Tr}\{\hat{\varrho}\hat{A}\} = \mathrm{Tr}\{|\varphi\rangle\langle\varphi|\hat{A}\}$$

for a pure state $\hat{\varrho} = |\varphi\rangle\langle\varphi|$ -

The trace can be performed with respect to any orthogonal basis; thus:

$$\mathrm{Tr}\{|\varphi\rangle\langle\varphi|\hat{A}\} = \langle\varphi|\hat{A}|\varphi\rangle$$

What is the matrix for $|\Phi^+\rangle$?

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2 \right)$$

the H-space has 4 dimensions.

$$\hat{\rho} = |\Phi^+\rangle\langle\Phi^+|$$

$$= \frac{1}{2}\Big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \Big)$$

We choose as a basis in H-space the 4 vectors

$$|00\rangle, \ |01\rangle, \ |10\rangle, \ |11\rangle$$

then:

$\langle 00|\hat{\rho}|00\rangle = \frac{1}{2}$ $\quad \langle 01|\hat{\rho}|00\rangle = 0$

$\langle 00|\hat{\rho}|01\rangle = 0$ $\quad \langle 01|\hat{\rho}|01\rangle = 0$

$\langle 00|\hat{\rho}|10\rangle = 0$ $\quad \langle 01|\hat{\rho}|10\rangle = 0$

$\langle 00|\hat{\rho}|11\rangle = \frac{1}{2}$ $\quad \langle 01|\hat{\rho}|11\rangle = 0$

$\langle 10|\hat{\rho}|00\rangle = 0$ $\quad \langle 11|\hat{\rho}|00\rangle = \frac{1}{2}$

$\langle 10|\hat{\rho}|01\rangle = 0$ $\quad \langle 11|\hat{\rho}|01\rangle = 0$

$\langle 10|\hat{\rho}|10\rangle = 0$ $\quad \langle 11|\hat{\rho}|10\rangle = 0$

$\langle 10|\hat{\rho}|11\rangle = 0$ $\quad \langle 11|\hat{\rho}|11\rangle = \frac{1}{2}$

$$\hat{\rho} = |\Phi^+\rangle\langle\Phi^+| \equiv \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} \\ \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array}$$

The most general state in 4-d H-space is represented by the density matrix:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

$$\hat{\rho} = |\psi\rangle\langle\psi| \equiv \begin{array}{cc} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} & \\ \left(\begin{array}{cc|cc} |\alpha|^2 & \alpha\beta^* & \alpha\gamma^* & \alpha\delta^* \\ \alpha^*\beta & |\beta|^2 & \beta\gamma^* & \beta\delta^* \\ \hline \alpha^*\gamma & \beta^*\gamma & |\gamma|^2 & \gamma\delta^* \\ \alpha^*\delta & \beta^*\delta & \gamma^*\delta & |\delta|^2 \end{array}\right) & \begin{array}{c} \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array} \end{array}$$

The off-diagonal elements are complex conjugate thus $\hat{\rho} = \hat{\rho}^\dagger$

These off-diagonal terms are called the coherence terms: any phenomenon which destroys them is called de-coherence

The partial trace is, by definition, the trace with respect only one of the two qubits.

We get the so called reduced density matrix

Consider $\hat{\rho} = |\Phi^+\rangle\langle\Phi^+|$

the trace with respect to qubit 1 gives

$$\hat{\rho}_2 = \frac{1}{2}\left(|0\rangle_2\, {}_2\langle 0| + |1\rangle_2\, {}_2\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the trace with respect to qubit 2 gives

$$\hat{\rho}_1 = \frac{1}{2}\left(|0\rangle_1\, {}_1\langle 0| + |1\rangle_1\, {}_1\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The state is entangled when the difference between the density matrix and the tensor product of the reduced density matrix is different from zero.

$$\hat{\rho} - \hat{\rho}_1 \otimes \hat{\rho}_2 \neq 0 \qquad \text{entangled}$$

$$\hat{\rho}_1 \otimes \hat{\rho}_2 = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\hat{\rho} - \hat{\rho}_1 \otimes \hat{\rho}_2 = \frac{1}{4} \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

An entangled state can be a mixture of disentangled states

$$\hat{\rho} = \Sigma_i \, p_i \, \hat{\rho}_1^{(i)} \otimes \hat{\rho}_2^{(i)}$$

$$\text{with} \quad p_i > 0 \qquad \Sigma_i \, p_i = 1$$

In this case an entangled state is called separable.

The correlations associated with an entangled but separable state are not of quantum nature and can be understood classically.

Any two-qubit state $\hat{\varrho}$ can be written as

$$\hat{\varrho} = \lambda \hat{\varrho}_{sep} + (1-\lambda)\hat{\varrho}_{pure} \quad \left(\begin{matrix}\text{not}\\ \text{unique}\end{matrix}\right)$$

$$(\hat{\varrho}_{pure} = \hat{\varrho}_{pure}^2)$$

the maximum value λ_{max} define the degree of separability.

$$\hat{\varrho} = \lambda_{max} \hat{\varrho}_{sep} + (1-\lambda_{max})\hat{\varrho}_{pure}$$

is unique.

λ_{max} measure to which extent the correlation associated with $\hat{\varrho}$ are classical.

For quantum information purposes a state $\hat{\varrho}$ is the more useful the smaller is λ_{max}, i.e. its degree of separability.

Entanglement and quantum logic with atoms and cavities

The model

- The qubits are identical **microwave cavity operating at low-order modes [1]:**

$$\omega \sim 10 - 100 \ \mathbf{GHz}$$

- The Fock state with one photon and the vacuum state are the logical states:

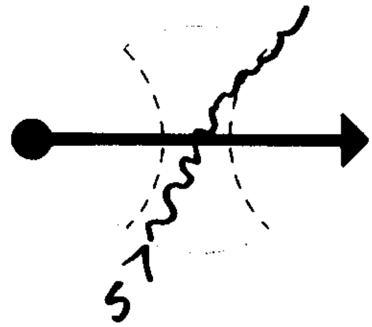
$$|0\rangle \equiv |0\rangle_L \qquad\qquad |1\rangle \equiv |1\rangle_L$$

- Mutual interaction between separated qubits are implemented by means of **two-level circular Rydberg atoms slightly detuned from the cavities frequency:**

$$\left.\begin{matrix} |e\rangle \\ \\ |g\rangle \end{matrix}\right\} \hbar\omega_{eg}$$

Domokos et al. PRA 52, 3554 (1995)

Two level Rydberg atom entering in a cavity (off-resonant case)



The Hamiltonian is time dependent, [3]:

$$\mathcal{H}(t) = \frac{\hbar\omega}{2}\left[b^\dagger b + b b^\dagger\right] + \frac{\hbar\omega_{eg}}{2}\left[|e\rangle\langle e| - |g\rangle\langle g|\right]$$

$$+\hbar\Omega(t)\left[|e\rangle\langle g| b + |g\rangle\langle e| b^\dagger\right]$$

with

$$\Omega(t) = \Omega_0 e^{-\left(\frac{t}{\tau}\right)^2}$$

the instantaneous eigenvectors are the so called dressed states

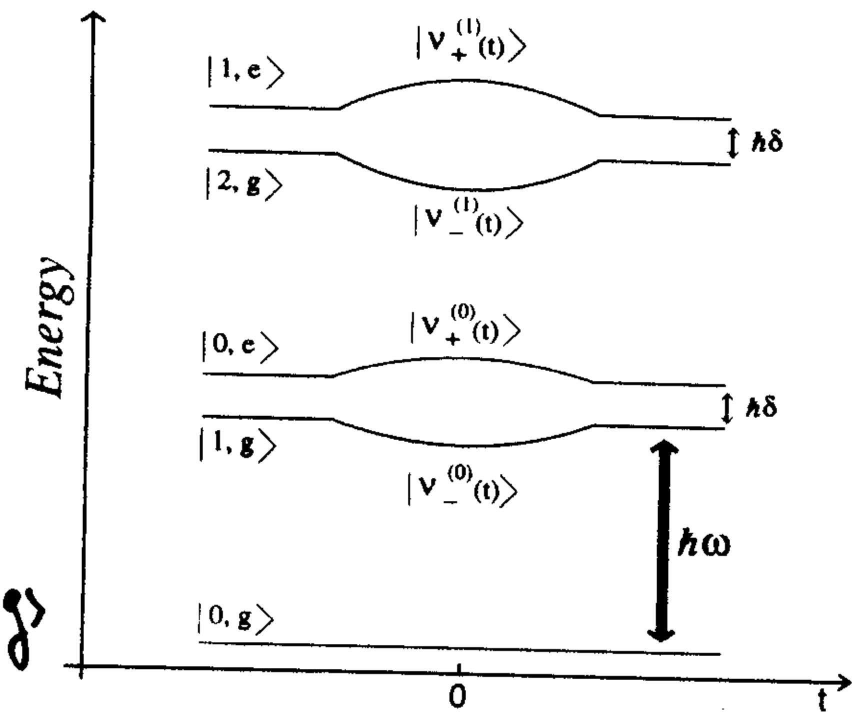
$$|\mathcal{V}_\pm^{(n)}(t)\rangle = \frac{(\delta/2 \pm \sqrt{(\delta/2)^2 + \Omega^2(t)(n+1)})|e,n\rangle + \Omega(t)\sqrt{n+1}|g,n+1\rangle}{\sqrt{\delta^2/2 + 2\Omega^2(t)(n+1) \pm \delta\sqrt{(\delta/2)^2 + \Omega^2(t)(n+1)}}}$$

while the eigenvalues

$$E_\pm^{(n)}(t) = \hbar\omega(n+1) \pm \hbar\sqrt{(\delta/2)^2 + \Omega^2(t)(n+1)}$$

Position of the atom-cavity energy levels

$(\delta = \omega_{eg} - \omega > 0)$



$|ng\rangle = |n\rangle|g\rangle$

- adiabatic condition

$$\frac{\dot{\Omega}(t)\,\delta\,\sqrt{n}}{4\left[(\delta/2)^2 + \Omega^2(t)n\right]^{3/2}} \ll 1$$

- coupling with classical field

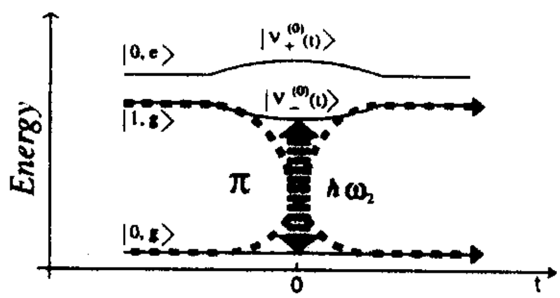
$$\mathcal{H}_S(t) = -\hbar\Xi_0 \cos(\omega_S t + \varphi_S) e^{-\left(\frac{t}{\tau_S}\right)^2} \left[|e\rangle\langle g| + |g\rangle\langle e|\right]$$

One qubit operation

One qubit operations are implemented with just an atom in $|g\rangle$ and the classical field tuned at ω_L

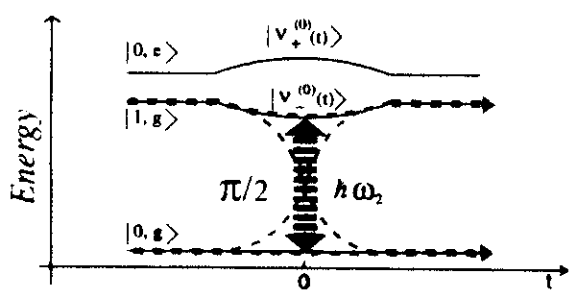
- if the atom undergoes to a π pulse we have a NOT gate on the cavity

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \beta|0\rangle + \alpha|1\rangle$$



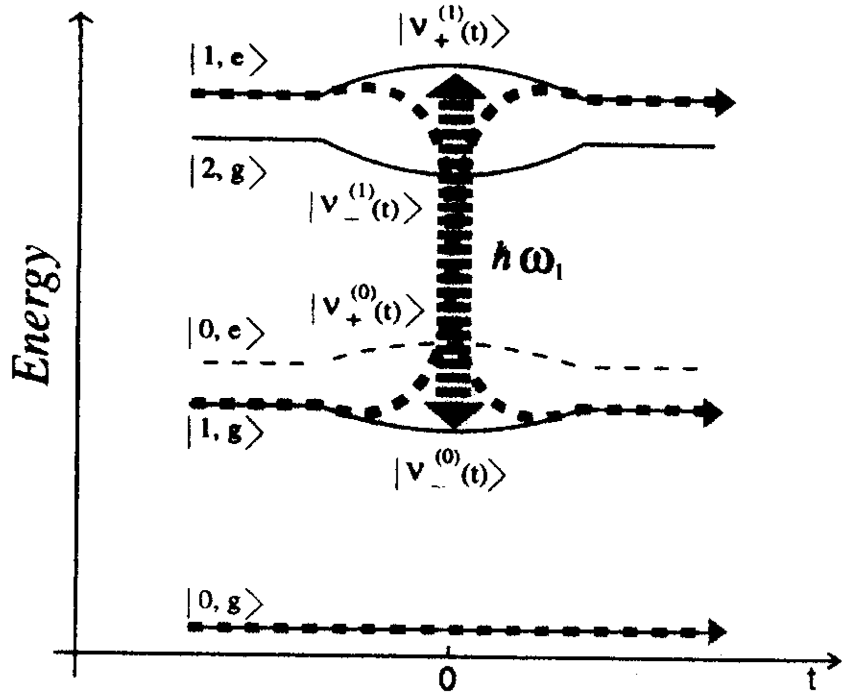
- if the atom undergoes to a $\pi/2$ pulse then we obtain an Hadamard gate

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$



- It is possible to vary also the relative phase of the qubit components acting on the phase φ_S of the classical field.

C-NOT cavity $\longrightarrow$ atom



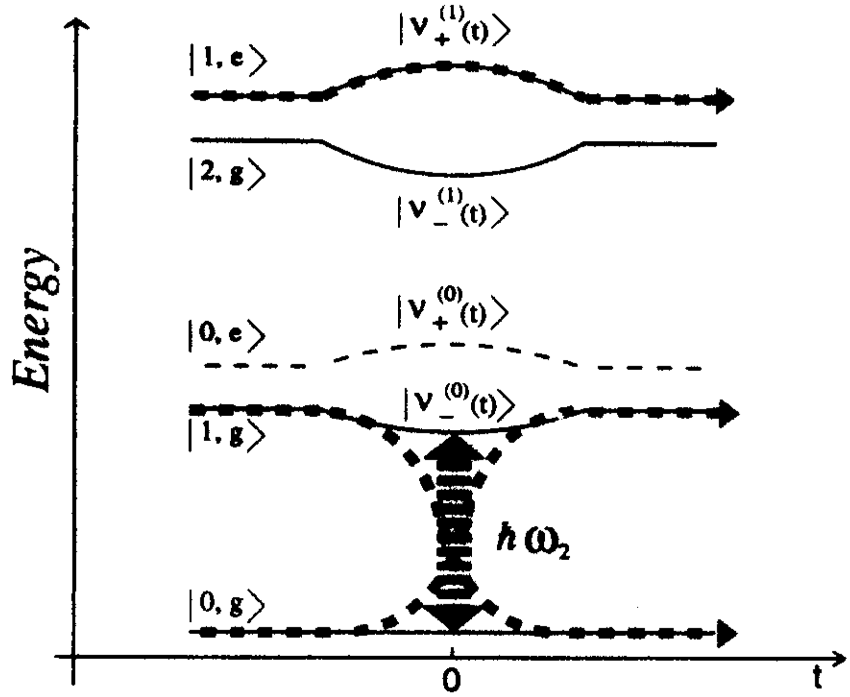
An atom entering in the cavity undergoes a π pulse from a classical field source of frequency ω_1

$$(a, b, c, d) \longrightarrow (a, d, c, b)$$

this is a C-NOT in which the cavity is the control qubit and the atom is the target qubit, (Ref. [2]).

$$\boxed{(a, b, c, d) \equiv a|g, 0\rangle + b|g, 1\rangle + c|e, 0\rangle + d|e, 1\rangle}$$

C-NOT atom $\longrightarrow$ cavity



The atom entering in the cavity now undergoes a π pulse from a classical field source of frequency ω_2

$$(a, b, c, d) \longrightarrow (b, a, c, d)$$

in this case the transformation is a C-NOT in which the atom is the control qubit.

$$(a,b,c,d) \equiv a|g0\rangle + b|g1\rangle + c|e0\rangle + d|e1\rangle$$

Parameters

atom-cavity coupling

$$\Omega_0 = 420 \text{ KHz}$$

atom-cavity detuning

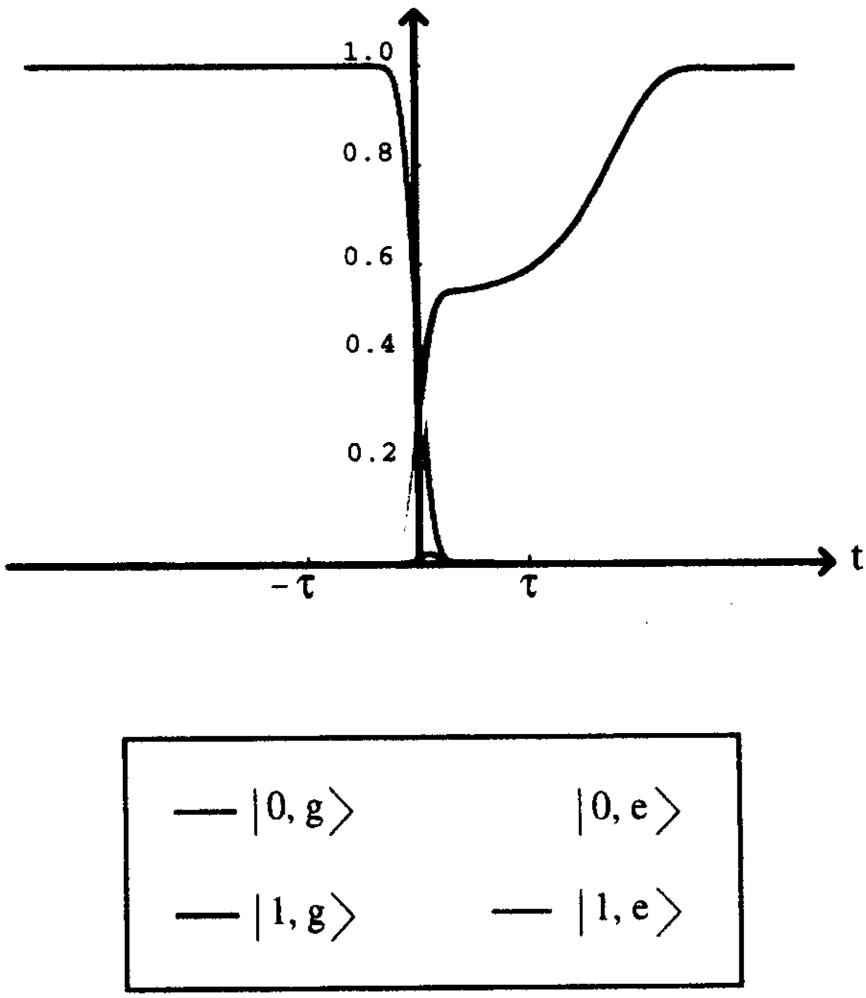
$$\delta = 0.18\,\Omega_0$$

typical cavity crossing time

$$\tau \sim 100\ \mu s$$

- C-NOT cavity $\longrightarrow$ atom
  - atom-classical field coupling: $\Xi_0 = 240$ KHz
  - characteristic time of classical pulse: $\tau_S = 14\ \mu s$
- C-NOT atom $\longrightarrow$ cavity
  - atom-classical field coupling: $\Xi_0 = 141.5$ KHz
  - characteristic time of classical pulse: $\tau_S = 19\ \mu s$

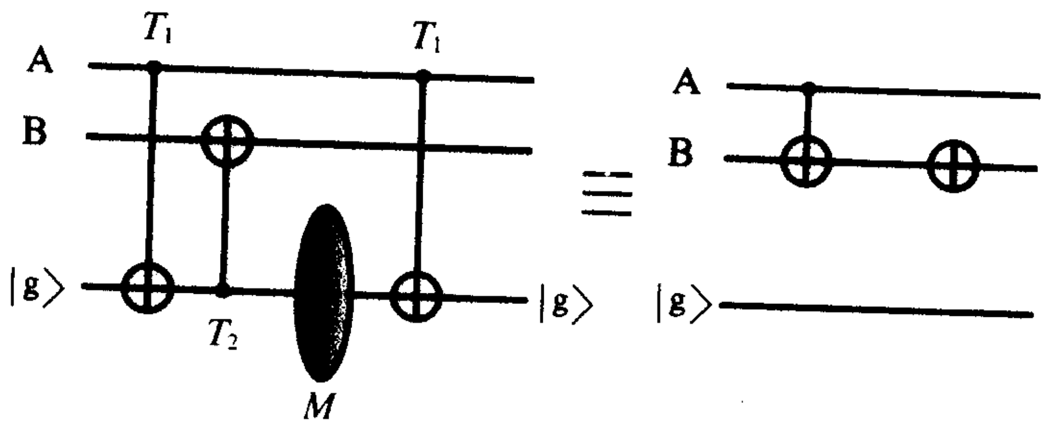
An example of dynamical evolution of the populations for the case of the C-NOT atom $\longrightarrow$ cavity with the system initially prepared in the state $|g, 0\rangle$



- Parasitic phases terms can be adjusted acting on the phase φ_S of the classical field.

C-NOT cavity $A \longrightarrow$ cavity B

It is possible to implement a C-NOT gate in which a cavity A controls another cavity B of the same type using this logical scheme



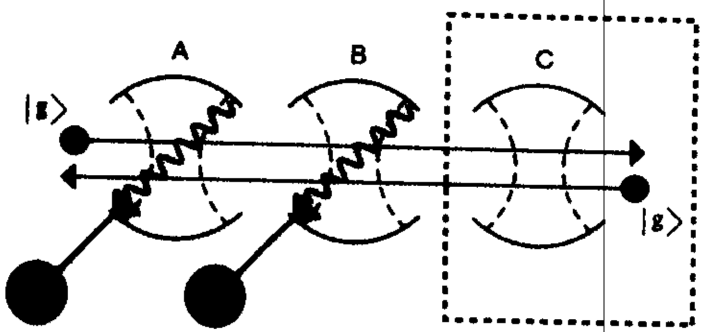
where

- T_1 is the C-NOT between atom and cavity in which the cavity A is the control qubit;

- T_2 is the C-NOT between atom and cavity in which the atom is the control qubit and cavity B is the target qubit;

- M is a quantum mirror which permits to *reflect back* the atom in the first cavity

Physical implementation



The dashed line box includes the quantum mirror:

- two atoms are injected in the apparatus, both in the ground state: first a_1 from left to right, then a_2 in the opposite direction;
- cavity C is resonant with the atoms and it is prepared in the vacuum state;
- when a_1 pass through C its excited component relases one photon via resonant interaction. The same photon is then absorbed by a_2 when it enters in C;

This realizes a quantum information transfer from a_1 to a_2. The overall transformation is equivalent to reflect back the first atom.

All the passages step by step

The initial condition is:

$$|\phi\rangle_A \otimes |\psi\rangle_B \otimes |g\rangle_{a_1} \otimes |0\rangle_C$$

- $T_1 \longrightarrow$
  $$\Big[\,\alpha_A|0\rangle_A \otimes |g\rangle_{a_1} + \beta_A|1\rangle_A \otimes |e\rangle_{a_1}\,\Big] \otimes |\psi\rangle_B \otimes |0\rangle_C$$
- $T_2 \longrightarrow$
  $$\Big[\,\alpha_A|0\rangle_A \otimes \overline{|\psi\rangle}_B \otimes |g\rangle_{a_1} + \beta_A|1\rangle_A \otimes |\psi\rangle_B \otimes |e\rangle_{a_1}\,\Big] \otimes |0\rangle_C$$
- **a_1 in C** $\longrightarrow$
  $$\Big[\,\alpha_A|0\rangle_A \otimes \overline{|\psi\rangle}_B \otimes |0\rangle_C + \beta_A|1\rangle_A \otimes |\psi\rangle_B \otimes |1\rangle_C\,\Big] \cancel{\otimes |g\rangle_{a_1}}$$
- **a_2 in C** $\longrightarrow$
  $$\Big[\,\alpha_A|0\rangle_A \otimes \overline{|\psi\rangle}_B \otimes |g\rangle_{a_2} + \beta_A|1\rangle_A \otimes |\psi\rangle_B \otimes |e\rangle_{a_2}\,\Big] \cancel{\otimes |0\rangle_C}$$
- $T_1 \longrightarrow$
  $$\Big[\,\alpha_A|0\rangle_A \otimes \overline{|\psi\rangle}_B + \beta_A|1\rangle_A \otimes |\psi\rangle_B\,\Big] \cancel{\otimes |g\rangle_{a_2}}$$

$$\boxed{\begin{aligned} |\phi\rangle_A &= \alpha_A|0\rangle_A + \beta_A|1\rangle_A \\ |\psi\rangle_B &= \alpha_B|0\rangle_B + \beta_B|1\rangle_B \\ \overline{|\psi\rangle}_B &= \beta_B|0\rangle_B + \alpha_B|1\rangle_B \end{aligned}}$$