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1. INTRODUCTION

Since the beginning of nuclear reactor physics studies, perturbation theory has played an important role. As well known, it was first proposed by Wigner [1] as early as 1945 to study fundamental quantities such as the reactivity worths of different materials in the reactor core. It is also well known that this first formulation, today widely used by reactor analysts, makes a consistent use of the adjoint flux concept.

The advantage of using perturbation theory lies in the fact that instead of making a new, often lengthy direct calculation of the eigenvalue (and then of the real flux) for every perturbed system configuration, a simple integration operation is required in terms of unperturbed quantities.

It is interesting that as early as 1948 Soodak [2] associated to the adjoint flux the concept of importance, viewing it as proportional to the contribution of a neutron, inserted in a given point of a critical system, to the asymptotic power.

Along with the introduction of the concept of importance and, parallel to it, along with the development of calculational methods and machines, from the early 60' a flourishing of perturbation methods, at first in the linear domain and then in the nonlinear one, have been proposed for the analysis of reactor core, shielding, nuclide evolution, thermohydraulics, as well as other fields.

The perturbation formulations proposed by various authors may be subdivided into three main categories, according to the approach followed in their derivation:

1. The heuristic approach, making exclusive use of importance conservation concepts, adopted first by Usachev [3] and then extensively developed by Gandini [4-7]. It will be referred to, in the following, as heuristic generalized perturbation theory (HGPT) method.
2. The variational approach adopted, in particular, by Lewins [8], Pomraning [9], Stacey [10], Harris and Becker [11] and Williams [12].
3. The differential method, proposed by Oblow [13] and extensively developed by Cacuci [14], based on a formal differentiation of the response considered.

Each of the above methods has its own merit, although all of them can be shown equivalent to each other [15].

Here the heuristically based, generalized perturbation theory (HGPT) [3-7] methodology is considered for analysis of subcritical, source-driven systems, in particular the so called ADS (Accelerator Driven Systems), now considered with increasing interest in many laboratories for their potential use as nuclear waste burners and/or safer energy producers. A formulation for time dependent, as well as for stationary conditions, is correspondingly derived, clarifying some peculiar physical aspects of subcritical systems with respect to their critical counterparts.

2. THE HGPT METHOD

In the HGPT method the importance function is uniquely defined in relation to a given system response, for example, a neutron dose, the quantity of plutonium in the core at end of cycle, the temperature of the outlet coolant.

The HGPT method was first derived in relation to the linear neutron density field. Then it was extended to other linear ones. For all these fields the equation governing the importance function was obtained directly by imposing that on average the contribution to the chosen response from a particle [a neutron, or a nuclide, or an energy carrier] introduced at a given time in a given phase space point of the system is conserved through time (importance conservation principle). Obviously such importance will result generally dependent on the time, position, and, when the case, energy and direction, of the inserted particle.

Consider a linear particle field density represented by vector $\mathbf{f}$ (e.g., the multigroup neutron density field) and a response Q of the type

$$Q = \int_{t_o}^{t_F} < \mathbf{s}^+, \mathbf{f} > dt \equiv << \mathbf{s}^+, \mathbf{f} >> , \qquad (2.1)$$

where $\mathbf{s}^+$ is an assigned vector function and where < > indicate integration over the phase space. Weighting all the particles inserted into the system, let's assume a source $\mathbf{s}$, with the corresponding importance ($\mathbf{f}^*$) will obviously give the response itself, i.e.,

$$<<\mathbf{f}^*, s>> = Q = <<\mathbf{s}^+, \mathbf{f}>> , \qquad (2.2)$$

which represents an important reciprocity relationship.

From the first derivations mentioned above the rules for determining the equation governing the importance function $\mathbf{f}^*$ were learned. They imply, in relation to the equation governing $\mathbf{f}^*$:

- change of sign of the odd derivatives,
- transposing matrix elements,
- reversing the order of operators,
- substitution of the source $\mathbf{s}$ with $\mathbf{s}^+$.

The first three rules will be generally called "operator reversal" rules.

The HGPT method was then extended to any field governed by linear operators for which the rules for their reversal were known. In particular, it was extended to the derivative fields, obtained from expanding to first order, around a given starting solution, a number of important nonlinear equations as those governing:

- the coupled neutron/nuclide field, relevant to core evolution and control problems,
- the temperature field, relevant to thermohydraulics.

2.1 General Formulation.

Consider a generic physical model defined by a number of parameters p_j $(j=1,2,\ldots,J)$ and described by an N-component vector field $\mathbf{f}$ obeying equation

$$\mathbf{m}(\mathbf{f}|\mathbf{p}) = \mathbf{0} \quad . \tag{2.3}$$

Vector $\mathbf{f}(\mathbf{q},t)$ generally depends on the phase space coordinates $\mathbf{q}$ and time t. Vector $\mathbf{p}$ represents the set of independent parameters p_j $(j=1,2,\ldots)$ fully describing the system and entering into Eq.(2.3). Their value generally determines physical constants, initial conditions, source terms, etc. Equation (2.3) can be viewed as an equation comprising linear, as well as nonlinear, operators and is assumed to be derivable with respect to parameters p_j and (in the Frechet sense) component functions f_n $(n=1,2,\ldots,N)$.

Consider now a response of interest, or functional Q given by Eq.(2.1). In the following, we shall look for an expression giving perturbatively the change δQ of the response Q in terms of perturbations δp_j of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to each parameter p_j will be obtained.

Expanding equation (2.3) around a reference solution gives setting $\mathbf{f}_{/j} \equiv \frac{d\mathbf{f}}{dp_j}$,

$$\sum_{j=1}^{J} \delta p_j (H\mathbf{f}_{/j} + \mathbf{m}_{/j}) + O_2 = \mathbf{0} \ , \tag{2.4}$$

where O_2 is a second, or higher order term, and where $\mathbf{m}_{/j} = \frac{\partial \mathbf{m}}{\partial p_j}$.

The (Jacobian) operator H is given by the expression

$$H = \begin{pmatrix} \frac{\bar{\partial} m_1}{\partial f_1} & \frac{\bar{\partial} m_1}{\partial f_2} & \cdots & \frac{\bar{\partial} m_1}{\partial f_N} \\ \frac{\bar{\partial} m_2}{\partial f_1} & \frac{\bar{\partial} m_2}{\partial f_2} & \cdots & \frac{\bar{\partial} m_2}{\partial f_N} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\bar{\partial} m_N}{\partial f_1} & \frac{\bar{\partial} m_N}{\partial f_2} & \cdots & \frac{\bar{\partial} m_N}{\partial f_N} \end{pmatrix} \tag{2.5}$$

where by $\frac{\bar{\partial}}{\partial f_n}$ we have indicated a Frechet derivative.

Since parameters p_j, and then their changes δp_j, have been assumed to be independent from each other, it must follow

$$H\mathbf{f}_{/j} + \mathbf{m}_{/j} = \mathbf{0} \quad , \tag{2.6}$$

which represents the (linear) equation governing the derivative functions $\mathbf{f}_{/j}$. The source term $\mathbf{m}_{/j}$ is here intended to account also, via appropriate delta functions, for the initial and, if the case, boundary conditions.

Consider now functional

$$Q_j = \langle\langle \mathbf{h}^+, \mathbf{f}_{/j} \rangle\rangle \quad . \tag{2.7}$$

Introducing the importance ($\mathbf{f}^*$) associated with field $\mathbf{f}_{/j}$, if we use it as weight of the source term $\mathbf{m}_{/j}$, and integrate space-and time-wise, according to the source reciprocity relationship, Eq.(2.2), the resulting quantity will be equivalent to functional Q_j, i.e.,

$$Q_j = \langle\langle \mathbf{f}^*, \mathbf{m}_{/j} \rangle\rangle \quad , \tag{2.8}$$

where the importance $\mathbf{f}^*$ obeys the (index-independent) equation

$$H^*\mathbf{f}^* + \mathbf{h}^+ = \mathbf{0} \quad , \tag{2.9}$$

H^* being obtained by reversing operator H. As said above, this implies transposing matrix elements, changing sign of the odd derivatives, inverting the order of operators.

We can easily see that the sensitivities s_j (j=1,2,...,J) of system parameters can be written

$$s_j \equiv \frac{dQ}{dp_j} = \langle\langle \frac{\partial \mathbf{h}^+}{\partial p_j} , \mathbf{f} \rangle\rangle + \langle\langle \mathbf{f}^*, \frac{\partial \mathbf{m}}{\partial p_j} \rangle\rangle \quad , \tag{2.10}$$

where the first term at the right-hand side represents the so called, easy to calculate, direct term.

The overall change δQ due to perturbations δp_j (j=1,2,...,J) of system parameters can be written, at first order,

$$\delta Q = \sum_{j=1}^{J} \delta p_j \left[\langle\langle \frac{\partial \mathbf{h}^+}{\partial p_j} , \mathbf{f} \rangle\rangle + \langle\langle \mathbf{f}^*, \frac{\partial \mathbf{m}}{\partial p_j} \rangle\rangle \right] . \tag{2.11}$$

Higher order expression may be obtained making explicit use of the derivative functions described above.

3. NUCLIDE/NEUTRON FIELD

The HGPT methodology may be applied to the core cycle analysis during burnup, in which optimal design parameters and fuel shuffling strategies are sought. Before considering these problems in subcritical systems, HGPT methods developed in the past for analogous problems in critical systems are first reviewed. These same methods can be in fact easily extended to cope with such new problems.

An HGPT related perturbation methodology relevant to the (heavy) nuclide density evolution has been first developed in 1975 [17]. Kallfeltz et al. [18] coupled it with the HGPT methodology relevant to the neutron field to account for nonlinear effects inherent to burnup problems. Other efforts in the nonlinear domain have been made by Harris and Becker [11], who arrived at a still crude formulation, and, successively, by Williams [12] and Gandini [6,7].

Williams used variational techniques starting from the time-wise discretized neutron and nuclide density equations, along with the quasi-static approximation.

Gandini used the heuristically based HGPT method after having formally extended the neutron and nuclide densities to a control (intensive) variable. The equations obtained governing the corresponding (time-wise continuous) importance functions are relevant to the physical solution. Different schemes of integration can then be defined [19] .

Typical quantities which can be analysed with this methodology are:

- the amount of a material specified in a given region at the end of the reactor life cycle;
- the d.p.a. of a specific material and at a given position;
- the residual reactivity at the end of the reactor life cycle. The analysis of this quantity may be of particular interest in studies aiming at extending the reactor life cycle.

3.1. Source driven systems

The methodology mentioned above for long term nuclide/neutron core cycle evolution analysis may be very well applied to source driven, subcritical systems.

One of the advantages often claimed for the subcritical source driven power systems is associated to the fact that the power level may be basically controlled by the source strength (via the regulation of the accelerator current). So, no control, or regulating elements would be necessary, if a sufficient breeding is available (and/or an appropriate core burnable poison distribution is provided at the beginning of cycle) in the core for compensating the reactivity loss during burnup. To represent this, we shall write the equations relevant to the neutron and fuel nuclide densities $\mathbf{n}$ and $\mathbf{c}_f$, respectively, and to the control (intensive) variable ρ (the time behaviour of this control variable will be such to maintain the assigned power history W):

$$\mathbf{m}_{(n)}(\mathbf{n}, \mathbf{c}_f, \rho | \mathbf{p}) \equiv -\frac{\partial \mathbf{n}}{\partial t} + B\mathbf{n} + \rho \mathbf{s}_n = 0 \qquad (3.1)$$

$$\mathbf{m}_{(c)}(\mathbf{n}, \mathbf{c}_f | \mathbf{p}) \equiv -\frac{\partial \mathbf{c}}{\partial t} + E\mathbf{c}_f + \mathbf{s}_c = 0 \qquad (3.2)$$

$$m_{(\rho)}(\mathbf{n}, \mathbf{c}_f | \mathbf{p}) \equiv \langle \mathbf{c}_f, S\mathbf{n} \rangle - W = 0 \quad , \qquad (3.3)$$

where B is the neutron diffusion, or transport, matrix operator (depending on $\mathbf{c}_f$ and ρ), E the nuclide evolution matrix (depending on $\mathbf{n}$), $\mathbf{s}_n$ and $\mathbf{s}_c$ are given source terms, while

$$S = \gamma \begin{vmatrix} \sigma_{f1}^1 & \sigma_{f2}^1 & \dots & \sigma_{fG}^1 \\ \sigma_{f1}^2 & \sigma_{f2}^2 & \dots & \sigma_{fG}^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sigma_{f1}^M & \sigma_{f2}^M & \dots & \sigma_{fG}^M \end{vmatrix} V \quad ,$$

γ being the amount of energy per fission, and σ_{fg}^m the microscopic g'th group fission cross-section of the m'th heavy isotope. V is the diagonal neutron velocity matrix. Quantities γ, V, W and σ_{fg}^m may be considered generally represented by (or function of) system parameters p_j. Source terms $\mathbf{s}_n$ and $\mathbf{s}_c$ are also parameter dependent.

Since we generally consider systems at quasi-static, i.e., stationary conditions, the time derivative at second member of Eq.(3.1) may be neglected in the course of the integration process.

Any response, functional of variables $\mathbf{n}$, $\mathbf{c}_f$, and ρ, could be considered for analysis. We think instructive to limit consideration to the response defined by the expression

$$Q = \rho(t_F) \equiv \int_{t_o}^{t_F} \delta(t-t_F)\ \rho(t)\, dt \quad , \tag{3.4}$$

which corresponds to the relative source strength required at t_F to assure the power level imposed. We may assume that, at unperturbed conditions, $\rho(t)=1$ in the interval (t_o, t_F). If some system parameter (for instance, the initial enrichment, or some other material density) is altered, as in an optimization search analysis, it may be of interest to evaluate the corresponding change of ρ at the end of cycle, to make sure that given upper limit specifications of the source strength are non exceeded.

Along with the HGPT methodology, the equations for the corresponding importance functions result

$$-\frac{\partial \mathbf{n}^*}{\partial t} = B^*\mathbf{n}^* + \Omega_c^*\mathbf{c}_f^* + S^T\mathbf{c}_f\rho^* \tag{3.5}$$

$$-\frac{\partial \mathbf{c}_f^*}{\partial t} = E^*\mathbf{c}_f^* + \Omega_n^*\mathbf{n}^* + S\mathbf{n}\rho^* \tag{3.6}$$

$$\langle \mathbf{n}^*, \mathbf{s}_n \rangle + \delta(t-t_F) = 0 \tag{3.7}$$

Ω_c^* and Ω_n^* being operators adjoint of Ω_c $[\equiv \frac{\bar{\partial}(E\mathbf{c}_f)}{\partial \mathbf{n}}]$ and Ω_n $[\equiv \frac{\bar{\partial}(B\mathbf{n})}{\partial \mathbf{c}_f}]$, respectively.

Eq.(3.7) corresponds to an orthonormal condition for $\mathbf{n}^*$.

In order to determine the 'final' value $\mathbf{n}^*(t_F)$ required for starting the integration of Eq.(3.5), in consideration of the nature of the above governing equations, we shall first write $\mathbf{n}^*$ and ρ in the form+

$$\mathbf{n}^*(\mathbf{r},t) = \mathbf{n}_F^*(\mathbf{r})\ \delta(t-t_F) + \tilde{\mathbf{n}}^*(\mathbf{r},t) \qquad (3.8)$$

$$\rho^*(t) = \rho_F^*\ \delta(t-t_F) + \tilde{\rho}^*(t) \qquad (3.9)$$

with $\tilde{\mathbf{n}}^*(\mathbf{r},t)$ and $\tilde{\rho}^*(t)$ being finite functions, vanishing at t_F.

Replacing into Eq.(3.5), integrating in the interval $(t_F-\varepsilon, t_F+\varepsilon)$, and then making $\varepsilon\to 0$, we obtain the equation

$$B\mathbf{n}_F^* + S^T\mathbf{c}_f(t_F)\rho_F^* = 0 \qquad (3.10)$$

Let us now define $\hat{\mathbf{n}}_F^*$ as obeying equation

$$B^*\hat{\mathbf{n}}_F^* + S^T\mathbf{c}_f(t_F) = 0\ , \qquad (3.11)$$

We note that $\hat{\mathbf{n}}_F^*$ corresponds to the importance relevant to functional $\langle \mathbf{c}_f(t_F), S\mathbf{n}(t_F)\rangle$, i.e., to the system power W. From the source reciprocity relationship, we may write

$$\langle \mathbf{c}_f(t_F), S\mathbf{n}(t_F)\rangle = \langle \hat{\mathbf{n}}_F^*, \mathbf{s}_n\rangle \equiv W\ . \qquad (3.12)$$

From constraint, Eq.(3.7), we easily obtain

$$\rho_F^* = -\frac{1}{\langle \hat{\mathbf{n}}_F^*, \mathbf{s}_n\rangle} \equiv -\frac{1}{W} \qquad (3.13)$$

and then

$$\mathbf{n}_F^* \equiv \hat{\mathbf{n}}_F^*\ \rho_F^* = -\frac{\hat{\mathbf{n}}_F^*}{W}\ . \qquad (3.14)$$

+ The diverging of $\mathbf{n}^*(\mathbf{r},t)$ at t_F may be explained on physical grounds recalling the meaning of importance (in this case, the contribution to the given response by a neutron with the same space/time coordinates) and considering that the response here is $\rho(t_F)$, i.e., the control assumed to maintain the power at a prefixed level. A neutron introduced at t_F into the system would in fact produce a (delta-like) impulse of control ρ to balance its effect on the power level. Then, the importance associated to such neutron would be characterized by a similar delta-like behaviour. A quite similar reasoning applies in relation to the diverging of importance $\rho^*(t)$ at t_F, considering that its physical meaning corresponds to the contribution to the response $[\rho(t_F)]$ due to a unit energy insertion at t_F or, which is the same, to an overall power pulse $\delta(t-t_F)$.

From this 'final' value, a recurrent calculational scheme may be defined starting from t_F and proceeding backward.

The sensitivity coefficient relevant to the j'th parameter p_j can then be defined as

$$\frac{d\rho(t_F)}{dp_j} = \rho_F^* \left[\langle \hat{\mathbf{n}}_F^*, \frac{\partial}{\partial p_j} (B\mathbf{n} + \mathbf{s}_n) \rangle + \frac{\partial}{\partial p_j} (\langle \mathbf{c}_f, S\mathbf{n} \rangle - W) \right]_{t_F}$$

$$+ \int_{t_o}^{t_F} \left[\langle \tilde{\mathbf{n}}^*, \frac{\partial}{\partial p_j} (B\mathbf{n} + \mathbf{s}_n) \rangle + \langle \mathbf{c}^*, \frac{\partial E}{\partial p_j} \mathbf{c}_f \rangle + \tilde{\rho}^* \frac{\partial}{\partial p_j} (\langle \mathbf{c}_f, S\mathbf{n} \rangle - W) \right] dt. \tag{3.15}$$

with ρ_F^* given by Eq.(3.13). The first term at right side accounts for effects on $\rho(t_F)$ due to parameter changes at t_F. The second, integral term accounts for analogous effects on $\rho(t_F)$ produced by parameter changes at times $t<t_F$.

Rather than on the source term, a control on the neutron absorption in the multiplying region could be of interest. In this case, the (intensive) control variable ρ would represent the average penetration of the neutron elements, or the average density of the soluble boron in the coolant, and then would enter into the (transport, or diffusion) operator B. The orthonormal condition for the neutron importance $\mathbf{n}^*$ would now be, rather than Eq. (3.7),

$$< \mathbf{n}^*, \frac{\partial B}{\partial \rho} \mathbf{n} > + \delta(t-t_F) = 0 \ . \qquad (3.16)$$

In this case, the sensitivity coefficient with respect to a given parameter p_j would always be given by Eq. (3.15), with $\hat{\mathbf{n}}_F^*$ obeying Eq. (3.11), but with

$$\rho_F^* = - \frac{1}{<\hat{\mathbf{n}}_F^*, \frac{\partial B}{\partial \rho} \mathbf{n}>} \ . \qquad (3.17)$$

In general, a control strategy, by which an automatic resetting of the imposed overall power is actuated, might imply a control intervention on both the neutron source strength and the absorbing elements within the multiplying region. In this case, ρ (which remains a unique, intensive control variable) would affect both operator B and the neutron source [in this latter case, via an appropriate ρ- and parameter dependent coefficient $\alpha(\rho|\mathbf{p})$, assumed unity at unperturbed conditions]. The distribution between these two control mechanisms could be described by appropriate parameters (subject to perturbation analysis). The sensitivity coefficient, in this case, with respect to a given parameter p_j would always be given by Eq. (3.15), with $\hat{\mathbf{n}}_F^*$ obeying Eq. (3.11), but with

$$\rho_F^* = - \frac{1}{<\hat{\mathbf{n}}_F^*, (\frac{\partial B}{\partial \rho} \mathbf{n} + \frac{\partial \alpha}{\partial \rho} \mathbf{s}_n)>} \ . \qquad (3.18)$$

Stationary Case

To study a given subcritical system at the beginning of its cycle life, we may consider the corresponding stationary case, i.e., that same system in which the neutron source and the nuclide density are assumed time-independent during an arbitrary time interval (t_o,t_B). We assume that at t_o the neutron density $(\mathbf{n}_o)$, as well as the control (ρ_o) have already reached stationary conditions. So, also these two quantities are time-independent in the same time interval. Their governing equations can then be written, in case the power level is controlled by the source strength,

$$B\mathbf{n}_o + \rho_o \mathbf{s}_{no} = 0 \tag{3.19}$$

$$\langle \mathbf{c}_{fo}, S\mathbf{n}_o \rangle - W_o = 0 \ . \tag{3.20}$$

Also here we shall assume that at unperturbed conditions $\rho_o=1$.

The same equations derived previously are applicable to this case, with the advertence of replacing t_F with with t_B and setting the coupling operators Ω_c^* and Ω_n^* appearing in Eqs.(3.5) and (3.6) equal to zero. The sensitivity coefficient of the response $\rho(t_B)$ $[=\rho(t)\equiv\rho_o$, i.e., constant in the whole interval $(t_o,t_B)]$ relevant to the j'th parameter p_j can then be obtained. Since in this case $\mathbf{c}^*$, as well as $\tilde{\mathbf{n}}^*$ and $\tilde{\rho}^*$, vanish, recalling Eq.(3.15), we obtain

$$\frac{d\rho_o}{dp_j} = \rho_o^* \left[\langle \mathbf{n}_o^*, \frac{\partial}{\partial p_j} (B\mathbf{n}_o + \mathbf{s}_{no}) \rangle + \frac{\partial}{\partial p_j} (\langle \mathbf{c}_{fo}, S\mathbf{n}_o \rangle - W_o) \right] \tag{3.21}$$

where

$$\rho_o^* = -\frac{1}{W_o} \tag{3.22}$$

and $\mathbf{n}_o^*$ obeys equation

$$B^*\mathbf{n}_o^* + S^T\mathbf{c}_{fo} = 0 \ . \tag{3.23}$$

If, rather than via the source strength, the power level reset control is assumed to be regulated via neutron absorption, so that the control ρ_o would enter into operator B, the sensitivity coefficient would be given always by Eq.(3.21), but with

$$\rho_o^* = - \frac{1}{\langle \mathbf{n}_o^*, \frac{\partial B}{\partial \rho_o} \mathbf{n}_o \rangle} \quad . \tag{3.24}$$

We might as well consider a (fictitious) control mechanism affecting the fission source, rather than the neutron absorption, i.e., we might choose as control a coefficient multiplying the fission matrix (F) and, therefore, entering into the Boltzmann, or diffusion, operator B ($\equiv A+\rho_o F$). The sensitivity coefficient would be given again by Eq.(3.21), but with

$$\rho_o^* = - \frac{1}{\langle \mathbf{n}_o^*, F\, \mathbf{n}_o \rangle} \quad . \tag{3.25}$$

Reactivity of Subcritical Systems

For resetting the power level, we have considered above different control mechanisms to which the following types of equations governing the neutron density may be associated:

$$B(\mathbf{p})\,\mathbf{n}_o + \rho_o \mathbf{s}_{no}(\mathbf{p}) = 0 \qquad \text{(source control)} \tag{3.26}$$

$$B(\rho_o|\mathbf{p})\,\mathbf{n}_o + \mathbf{s}_{no}(\mathbf{p}) = 0 \qquad \text{(neutron absorption, or fission control)} \tag{3.27}$$

$$B(\rho_o|\mathbf{p})\,\mathbf{n}_o + \alpha(\rho_o|\mathbf{p})\,\mathbf{s}_{no}(\mathbf{p}) = 0 \qquad \text{(mixed control)}^{+} \tag{3.28}$$

where the control and parameter dependence is indicated. Coefficient α is given and reflects the mixed strategy chosen. Eqs. (3.26), (3.27) and (3.28) may be generally represented by equation

$$\mathbf{m}_{(n,o)}(\mathbf{n}_o, \rho_o|\mathbf{p}) = 0 \,. \tag{3.29}$$

The sensitivity expression (3.21) may be generalized so that

$$\frac{d\rho_o}{dp_j} = -\frac{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial p_j}\rangle + \dfrac{\partial}{\partial p_j}\left(\langle \mathbf{c}_{fo}, S\mathbf{n}_o\rangle - W_o\right)}{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o}\rangle} \,, \tag{3.30}$$

with $\mathbf{n}_o^*$ obeying Eq. (3.23).

A corresponding perturbation expression may now be obtained. Assuming that the power W_o appearing in Eq. (3.30) is not subject to perturbation, we may write:

$$\delta\rho_o = -\frac{\langle \mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)}\rangle + \langle \mathbf{n}_o, \delta(S^T\mathbf{c}_{fo})\rangle}{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o}\rangle} \,, \tag{3.31}$$

where $\delta\mathbf{m}_{(n,o)} = \sum_j \delta p_j \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial p_j}$ and $\delta(S^T\mathbf{c}_{fo}) = \sum_j \delta p_j \dfrac{\partial (S^T\mathbf{c}_{fo})}{\partial p_j}$.

+ A mixed control strategy may be considered also using Eq. (3.26), or Eq. (3.27). Adopting, for instance, Eq. (3.26), relevant to the neutron source control, part of the power level would be taken care of parametrically (e.g., by properly changing the control rod position, or the soluble boron density). The remaining reset would be taken care of intrinsically, by the ρ-control chosen.

As said previously, $\delta\rho_o$ corresponds to the control change necessary to reestablish the power level existing before the perturbation $\delta\mathbf{m}_{(n,o)}$. We may as well say that the perturbation $\delta\mathbf{m}_{(n,o)}$ [and $\delta(S^T\mathbf{c}_{fo})$] would produce a power level change equivalent to that produced by a control change δK_ρ given by the equation

$$\delta K_\rho = \frac{\langle \mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)} \rangle + \langle \mathbf{n}_o, \delta(S^T\mathbf{c}_{fo}) \rangle}{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} \rangle} \quad . \tag{3.32}$$

In the case of the (fictitious) control on the neutron fission, setting λ in place of ρ to distinguish this peculiar case, we may explicitly write

$$\delta K_\lambda = \frac{\langle \mathbf{n}_o^*, \delta B\, \mathbf{n}_o \rangle}{\langle \mathbf{n}_o^*, F\mathbf{n}_o \rangle} + \frac{\langle \mathbf{n}_o^*, \delta\mathbf{s}_{no} \rangle}{\langle \mathbf{n}_o^*, F\mathbf{n}_o \rangle} + \frac{\langle \mathbf{n}_o, \delta(S^T\mathbf{c}_{fo}) \rangle}{\langle \mathbf{n}_o^*, F\mathbf{n}_o \rangle} \quad . \tag{3.33}$$

The first term at the right side closely resembles the reactivity expression for critical systems[+] . So, we shall call a quantity δK_λ as given by expression (3.33) a 'generalized reactivity'. To account for a generic ρ-mode control mechanism, we shall extend this definition to δK_ρ, similarly defined by Eq.(3.32), i.e.,

$$\delta K_\rho = \frac{\langle \mathbf{n}_o^*, \delta B\, \mathbf{n}_o \rangle}{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} \rangle} + \frac{\langle \mathbf{n}_o^*, \delta\mathbf{s}_{no} \rangle}{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} \rangle} + \frac{\langle \mathbf{n}_o, \delta(S^T\mathbf{c}_{fo}) \rangle}{\langle \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} \rangle} \quad . \tag{3.34}$$

and call it generalized ρ-mode reactivity[++] .

[+] The first term at right hand side of Eq.(3.33) can be demonstrated to formally approach the standard reactivity expression as the (reference) system considered gets close to criticality conditions (See Appendix B).

[++] In the following, if no ambiguity occurs, we shall call it simply 'reactivity'.

We may as well define a (generalized) reactivity coefficient, as given by the expression

$$\frac{\partial K_\rho}{\partial p_j} = \frac{\langle \mathbf{n}^*_o, \frac{\partial B}{\partial p_j}\mathbf{n}_o\rangle}{\langle \mathbf{n}^*_o, \frac{\partial B}{\partial \rho_o}\mathbf{n}_o\rangle} + \frac{\langle \mathbf{n}^*_o, \frac{\partial \mathbf{s}_{no}}{\partial p_j}\rangle}{\langle \mathbf{n}^*_o, \frac{\partial B}{\partial \rho_o}\mathbf{n}_o\rangle} + \frac{\langle \mathbf{n}_o, \frac{\partial (S^T \mathbf{c}_{fo})}{\partial p_j}\rangle}{\langle \mathbf{n}^*_o, \frac{\partial B}{\partial \rho_o}\mathbf{n}_o\rangle} \quad . \qquad (3.35)$$

Expressions (3.34) and (3.35) can be useful in the analysis and exploitation of measurements on subcritical experimental facilities, as well as for analytical studies of power source driven reactors (for example, for optimal configuration searches).

In relation to the application of above formulations to experimental facility analysis, a (measurable) change of the flux level produced by a perturbation of a parameter (such as a material density, the neutron source intensity, etc.) would be associated to the corresponding 'reactivity' δK_ρ^{ex}, where apex "ex" indicates that it would correspond to a measured quantity. The determination of δK_ρ^{ex} could be effected either directly, by resetting the initial flux level conditions via the specific ρ-mode control chosen (for instance, as is the case for an experimental facility, through a regulatory rod movement), or indirectly, through its previous calibration vs. neutron flux level.

Calculating value δK_ρ^{cal} of the same 'reactivity' from Eq.(3.34), would enable a comparison between experimental and calculational results, in view, for instance, of data adjustments exercises via statistical fitting methods [20]. The data to be adjusted could be differential quantities (e.g., cross-sections), as well as neutron source parameters (e.g., related to the energy distribution and intensity) to which 'a priori' uncertainties have been associated.

A measurement of the power level change consequent to a perturbation of system parameters could be also used directly in an 'unconstrained' system, i.e., in a system in which no reset mechanism is considered (which may be the case for an experimental subcritical facility). In this case, the neutron density $\mathbf{n}_o$ at unperturbed conditions would obey equation

$$B\mathbf{n}_o + \mathbf{s}_{no} = 0 \ . \qquad (3.36)$$

Considering the importance $\mathbf{n}_o^*$ governed by Eq.(3.23), relevant to the system power $W_o = \langle \mathbf{c}_{fo}, S\mathbf{n}_o \rangle$, along with the HGPT methodology [7] we would obtain the perturbation expression (inclusive also of the so called 'direct effect' term)

$$\delta W_o = \langle \mathbf{n}_o^*, \delta B\, \mathbf{n}_o \rangle + \langle \mathbf{n}_o^*, \delta \mathbf{s}_{no} \rangle + \langle \mathbf{n}_o, \delta(S^T \mathbf{c}_{fo}) \rangle \ . \qquad (3.37)$$

This expression could also be used for experimental data analysis.

In certain circumstances, Eq.(3.37) could as well be adopted for system analysis and optimization searches, even though in this case no direct appreciation would be obtained on the 'reactivity' δK_ρ associated with the control mode selected.

APPENDIX B. On the 'generalized reactivity'

At previous section an expression was obtained relevant to the so called 'generalized reactivity' dK_λ, i.e.,

$$\delta K_\lambda = \frac{\langle \mathbf{n}_o^*, \delta B \mathbf{n}_o \rangle}{\langle \mathbf{n}_o^*, F \mathbf{n}_o \rangle} + \frac{\langle \mathbf{n}_o^*, \delta \mathbf{s}_{no} \rangle}{\langle \mathbf{n}_o^*, F \mathbf{n}_o \rangle} + \frac{\langle \mathbf{n}_o, \delta(S^T \mathbf{c}_o) \rangle}{\langle \mathbf{n}_o^*, F \mathbf{n}_o \rangle} . \qquad \text{(B.1)}$$

It was evidenced how the first term at the right side closely resembles the reactivity expression for critical systems. This term can be demonstrated to formally approach the standard reactivity expression as the (reference) system considered gets close to criticality conditions, the λ control coefficient (at reference conditions) approaching its critical value λ_c [for example, consequent to assuming the (reference) neutron source term, defined as $\mathbf{s}_{no} \equiv \zeta \hat{\mathbf{s}}_{no}$, with (positive) coefficient ζ approaching zero, while maintaining unaltered the power level W_o]. To show this, let us consider (arbitrarily normalized) functions $\bar{\mathbf{n}}_o$ and ϕ_o^* relevant to the corresponding critical system, obeying equations

$$A \bar{\mathbf{n}}_o + \lambda_c F \bar{\mathbf{n}}_o = 0; \qquad A^* \phi_o^* + \lambda_c F^* \phi_o^* = 0 . \qquad \text{(B.2-3)}$$

Clearly, functions $\mathbf{n}_o$ and $\mathbf{n}_o^*$ obeying heterogeneous equations (3.27) (with source term $\mathbf{s}_{no} \equiv \zeta \hat{\mathbf{s}}_{no}$, and with $\rho \equiv \lambda$) and (3.23), respectively, for $\lambda \to \lambda_c$ (corresponding to $\zeta \to 0$) approach limiting values, i.e.,

$$\mathbf{n}_o \to \alpha_1 \bar{\mathbf{n}}_o; \quad \mathbf{n}_o^* \to \alpha_2 \phi_o^* , \qquad \text{(B.4-5)}$$

where α_1 is a finite (positive) coefficient while α_2 diverges. Correspondingly, the third term at the right side of Eq.(B.1) tends to vanish. Eq.(B.1) then approaches the asymptotic expression,

$$\delta K_\lambda^{as} = \frac{\langle \phi_o^*, \delta B \bar{\mathbf{n}}_o \rangle}{\langle \phi_o^*, F \bar{\mathbf{n}}_o \rangle} + \frac{\langle \phi_o^*, \delta \mathbf{s}_{no} \rangle}{\langle \phi_o^*, F \bar{\mathbf{n}}_o \rangle} . \qquad \text{(B.6)}$$

The first term at right side formally coincides with the reactivity expression for critical systems. The sum of the first and second term may be viewed as a generalization of the traditional reactivity expression. The second term would allow to account for the possibility of introducing into a critical system a neutron source $\mathbf{s}'_{no} (\equiv \delta \mathbf{s}_{no})$, viewed as a perturbation. The quantity $-\delta K_\lambda^{as}$ would in this latter case correspond to the control (λ) change associated with $\delta \mathbf{s}_{no}$, so that in the altered system (subcritical, after the control reset) the previous power level is maintained. This possibility so far has not been of particular interest in critical system studies, so that reactivity expressions containing only the first term at right side have been so far generally considered.

Source Reactivity

Limiting consideration to the source term perturbation, Eq.(B.6) reduces to

$$\delta K_{\lambda}^{as} = \frac{\langle \phi_o^*, \delta s_{no} \rangle}{\langle \phi_o^*, F \bar{n}_o \rangle} \quad . \tag{B.7}$$

where $-\delta K_{\lambda}^{as}$ corresponds (to first order) to the control (λ) change associated with δs_{no}, so that in the altered system (subcritical, after the control reset) the previous power level is maintained. So, if the source term perturbation is sufficiently small, the fission source distribution $F\bar{n}_o$ will not differ significantly from that, Fn_o, relevant to the corresponding subcritical system with a multiplication coefficient $K_o = 1 - \delta K_{\lambda}^{as}$. Replacing δK_{λ}^{as} with $(1-K_o)$, and $F\bar{n}_o$ with Fn_o, Eq.(B.7) may be rewritten in the form

$$1-K_o = \frac{\langle \phi_o^*, \delta s_{no} \rangle}{\langle \phi_o^*, F n_o \rangle} , \tag{B.8}$$

A similar relationship may be easily obtained in relation to any degree of subcriticality. Consider a subcritical system described by the equation

$$(A + F)n_o + s_{no} = 0 \tag{B.9}$$

and corresponding to a given multiplication factor K_o.

Consider then function ϕ_o^* relevant to the corresponding critical system, governed by Eq.(B.3) (with $\frac{1}{K_o}$ in place of λ_c), i.e.,

$$(A^* + \frac{1}{K_o} F^*)\phi_o^* = 0 \quad . \tag{B.10}$$

As well known ϕ_o^* corresponds to the adjoint of the fundamental mode $\bar{n}_o$ [governed by Eq.(B.2), with $\frac{1}{K_o}$ in place of λ_c] and, according to Soodak (1948), may be viewed as proportional to the contribution to the asymptotic power by a neutron inserted in a given point of the corresponding critical system.

Adding and subtracting $\frac{1}{K_o}F\mathbf{n}_o$ at the left term of Eq.(B.9), multiplying on the left by ϕ_o^* and integrating, we easily obtain the relationship

$$\frac{1-K_o}{K_o} = \frac{\langle \phi_o^*, s_{no} \rangle}{\langle \phi_o^*, F\mathbf{n}_o \rangle} \tag{B.11}$$

which corresponds to the source multiplication reactivity defined by Greenspan (1976). We can also rewrite it in the form

$$\langle \phi_o^*, F\mathbf{n}_o \rangle = K_o \frac{\langle \phi_o^*, s_{no} \rangle}{1-K_o} \tag{B.12}$$

which evidences the source amplification effect on the integrated (weighted) fission source. Along with Soodak (1948), quantity $\langle \phi_o^*, F\mathbf{n}_o \rangle$ could be properly interpreted as proportional to the asymptotic power produced in the corresponding K-mode+ critical system (i.e., with the number of secondary neutrons per fission, ν, replacing ν/K_o) by a neutron source distruibuted as $F\mathbf{n}_o$, introduced for a given limited time interval, propagating into it and so giving rise to a persistent, steady state distribution.

+ i.e., relevant to replacing the fission source $F\mathbf{n}_o$ with $\frac{1}{K_o}F\mathbf{n}_o$. One could as well consider different mo for instance the so called ω-mode (Gandini, 1981b). In this case, the equations governing the real adjoint fundamental eigenfunctions would be:

$$\omega_o \mathbf{n}_o^{(\omega)} + (A+F)\mathbf{n}_o^{(\omega)} = 0\,; \qquad \omega_o \phi_o^{*(\omega)} + (A^*+F^*)\phi_o^{*(\omega)} = 0\ ,$$

ω_o being the corresponding eigenvalue. Rather than expression (B.12), we would have then obtained

$$\langle \phi_o^{*(\omega)}, \mathbf{n}_o \rangle = \frac{\langle \phi_o^{*(\omega)}, s_{no} \rangle}{\omega_o}\ ,$$

where $\omega_o \to 0$ for the system approaching criticality conditions. Dividing both sides by $\langle \phi_o^{*(\omega)}, Fr$ rearranging and defining the "effective neutron lifetime" $l_{eff} = \frac{\langle \phi_o^{*(\omega)}, \mathbf{n}_o \rangle}{\langle \phi_o^{*(\omega)}, F\mathbf{n}_o \rangle}$, we may also write

$$\langle \phi_o^{*(\omega)}, F\mathbf{n}_o \rangle = \frac{\langle \phi_o^{*(\omega)}, s_{no} \rangle}{l_{eff}\omega_o}\ .$$

The above relationship may be rearranged so that

$$K_o = \frac{\langle \phi_o^*, F\mathbf{n}_o \rangle}{\langle \phi_o^*, F\mathbf{n}_o \rangle + \langle \phi_o^*, \mathbf{s}_{no} \rangle} \equiv \frac{1}{1 + \dfrac{\langle \phi_o^*, \mathbf{s}_{no} \rangle}{\langle \phi_o^*, F\mathbf{n}_o \rangle}} , \qquad (B.13)$$

which defines, the multiplication coefficient as the ratio between the weighted integrated fission source $\langle \phi_o^*, F\mathbf{n}_o \rangle$ and the sum $(\langle \phi_o^*, F\mathbf{n}_o \rangle + \langle \phi_o^*, \mathbf{s}_{no} \rangle)$ of this same quantity and the weighted integrated external source. From the definition of ϕ_o^*, all these source terms are weighted so that only their contribution to the fundamental mode (or, better, to the asymptotic power in the corresponding critical system) is kept. Then, the larger the (average) importance of the external neutron source with respect to the inportance of the fission one, the smaller the multiplication coefficient K_o required to maintain the prescribed power level with that same neutron source.

On the other hand, if, rather than the adjoint flux, we adopt a unit weight vector $\mathbf{u}$ we would obtain the 'actual' multiplication factor K_{act}, i.e.,

$$K_{act} = \frac{\langle \mathbf{u}, F\mathbf{n}_o \rangle}{\langle \mathbf{u}, F\mathbf{n}_o \rangle + \langle \mathbf{u}, \mathbf{s}_{no} \rangle} \equiv \frac{1}{1 + \dfrac{\langle \mathbf{u}, \mathbf{s}_{no} \rangle}{\langle \mathbf{u}, F\mathbf{n}_o \rangle}} . \qquad (B.14)$$

It is quite evident that $K_{act}=K_o$ in case the external source has the distribution of the fission source, i.e., if $\mathbf{s}_{no}=\alpha F\mathbf{n}_o$, α being a given (positive) coefficient.

From Eq.(B.14) we could as well obtain the expression

$$\langle \mathbf{u}, F\mathbf{n}_o \rangle = K_{act} \frac{\langle \mathbf{u}, \mathbf{s}_{no} \rangle}{1-K_{act}} \qquad (B.15)$$

which defines the amplification in terms of the 'actual' source multiplication. Quantity $\langle \mathbf{u}, F\mathbf{n}_o \rangle$ multiplied by $\gamma/\bar{\nu}$, $\bar{\nu}$ being the (properly weighted) average number of secondary neutrons and γ that of energy units per fission, corresponds to the subcritical system power.

In Eqs.(B.12) and (B.15) both left hand side terms are proportional to the fission rate and, then, to the power level. Two different amplification factors, $K_o/(1-K_o)$ and $K_{act}/(1-K_{act})$, appear at the right hand side of these same equations. There is no inconsistency in this fact, since these factors are applied to different integrated source terms $\langle \boldsymbol{\phi}_o^*, \boldsymbol{s}_{no} \rangle$ and $\langle \mathbf{u}, \boldsymbol{s}_{no} \rangle$, respectively. So, if the neutron source is located in a 'high weight' position, relatively to the average fission neutron,[+] the corresponding amplification factor $K_o/(1-K_o)$ would result significantly smaller than $K_{act}/(1-K_{act})$ (and, consequently, K_o would result significantly smaller than K_{act}).

A few comments on the concept of subcriticality are appropriate. When we say that a system is subcritical, usually we implicitly refer to the fundamental mode since it is with this mode that the eigenvalue K_o is associated. As well known (Glasstone and Edlund, 1952) higher modes may be excited during a transient in a subcritical system but, in absence of an external neutron source, these modes, being associated with smaller eigenvalues, would vanish more rapidly than the fundamental one.[++] In a reactor at subcritical conditions it is then the value of K_o, rather than that of K_{act}, which identifies the degree of subcriticality, i.e., the 'distance' from reactor conditions such that this same quantity K_o reaches a unit value.

The notion of the coefficient K_{act} may be otherwise useful, for instance as a 'measure' of the efficiency of the neutron source, as far as the overall 'actual' power is concerned, in a search for its optimal location [which could as well be identified by considering, for different source positions, the quantity $\langle \boldsymbol{\phi}_o^*, \boldsymbol{s}_{no} \rangle$ or, better, $\langle \mathbf{n}_o^*, \boldsymbol{s}_{no} \rangle$, with the importance $\mathbf{n}_o^*$ obeying Eq.(3.23) and directly associated with the subcritical system power].

[+]i.e., with a relatively high value $\frac{\langle \boldsymbol{\phi}_o^*, \boldsymbol{s}_{no} \rangle}{\langle \boldsymbol{\phi}_o^*, F\mathbf{n}_o \rangle}$, or, more precisely, with ratio $\frac{\langle \boldsymbol{\phi}_o^*, \boldsymbol{s}_{no} \rangle}{\langle \mathbf{u}, \boldsymbol{s}_{no} \rangle}$ significantly larger t $\frac{\langle \boldsymbol{\phi}_o^*, F\mathbf{n}_o \rangle}{\langle \mathbf{u}, F\mathbf{n}_o \rangle}$.

[++]Experimentally, this means that, removing the neutron source from a subcritical reactor, the sin determination of the fundamental mode period after the higher modes have vanished would allow, using inhour formula (Glasstone and Edlund, 1952), the evaluation of K_o.

ADJUSTMENT METHODS

C_ℓ ($\ell=1,2,\ldots,L$) — Integral quantities, or responses, relevant to one or more physical systems

p_j ($j=1,2,\ldots,J$) — Systems parameters.

C_ℓ^{ex} ($\ell=1,2,\ldots,L$) and p_j^{ex} — Experimental values

Theoretical model

$$C_\ell = Q_\ell(p_1,\ldots,p_J) . \quad (\ell=1,2,\ldots,L) \tag{1}$$

Assume a given set of values $p_{o,j}$ close enough to the true ones p_j, we may expand Eq.(1) disregarding second and higher order terms and obtain

$$Q_\ell = Q_\ell^{cal} + \sum_{j=1}^{J} \left.\frac{\partial Q_\ell}{\partial p_j}\right|_{p_o} (p_j - p_{o,j}). \tag{2}$$

$$C_\ell^{cal} = Q_\ell(p_{o,1},\ldots,p_{o,J}) . \tag{3}$$

Define (assuming p and Q different from zero)

$$y_{p,j} = \frac{p_j - p_{o,j}}{p_{o,j}} \tag{4}$$

$$y_{Q,\ell} = \frac{Q_\ell - Q_\ell^{cal}}{Q_\ell^{cal}} \quad (\ell=1,\ldots,L) \tag{5}$$

$$s_{\ell,j} = \frac{p_{o,j}}{Q_\ell^{cal}} \left.\frac{\partial Q_\ell}{\partial p_j}\right|_{p_o} \quad \text{(sensitivity coefficients)} \tag{6}$$

Introducing vectors

$$y_P = \begin{vmatrix} y_{P.1} \\ \cdot \\ \cdot \\ \cdot \\ y_{P.J} \end{vmatrix}, \quad y_Q = \begin{vmatrix} y_{Q.1} \\ \cdot \\ \cdot \\ \cdot \\ y_{Q.L} \end{vmatrix} \tag{7}$$

and the sensitivity matrix

$$S = \begin{vmatrix} s_{11} & s_{12} & \cdot & \cdot & s_{1J} \\ s_{21} & s_{22} & \cdot & \cdot & s_{2J} \\ \vdots & \vdots & : & : & \vdots \\ s_{L1} & s_{L2} & \cdot & \cdot & s_{LJ} \end{vmatrix}, \tag{8}$$

Eq.(2) may be written, in vector form,

$$y_Q - S y_P = 0. \tag{9}$$

Define

$$y_{P.j}^{ex} = \frac{p_j^{ex} - p_{o.j}}{p_{o.j}} \tag{10}$$

$$y_{Q.j}^{ex} = \frac{Q_\ell^{ex} - Q_\ell^{cal}}{Q_\ell^{cal}} \tag{11}$$

Assume that the that measured quantities p_j^{ex} and Q_ℓ^{ex} (and, therefore, $y_{P.j}^{ex}$ and $y_{Q.j}^{ex}$) are normally distributed with dispersion matrices C_P and C_Q, respectively.

The likelihood function relevant to vectors y_Q and y_P results

$$L = a \exp\left\{-\frac{1}{2}\left[(y_Q^{ex}-y_Q)^T C_Q^{-1}(y_Q^{ex}-y_Q) + (y_P^{ex}-y_P)^T C_P^{-1}(y_P^{ex}-y_P)\right]\right\}, \tag{14}$$

An estimator $\tilde{y}_P$ of vector y_P [and, consequently, via relationship (9), an estimator $\tilde{y}_Q$ of y_Q] is searched such that the likelihood function L is maximized. This is obtained by solving equation

$$(y_Q^{ex}-\tilde{y}_Q)^T C_Q^{-1}(y_Q^{ex}-\tilde{y}_Q) + (y_P^{ex}-\tilde{y}_P))^T C_P^{-1}(y_P^{ex}-\tilde{y}_P)) = \min \tag{15}$$

with the constraints

$$\tilde{y}_Q - S\tilde{y}_P = 0. \tag{16}$$

Having assumed, for simplicity, that (numerically) $p_j^{ex} = p_{o,j}$ (or, which is the same, that $y_{p,j}^{ex} = 0$), two equivalent solutions may be obtained, i.e.,

$$\tilde{y}_p = C_p S^T (C_Q + S C_p S^T)^{-1} y_Q^{ex} \qquad (17)$$

$$\tilde{y}_p = (C_p^{-1} + S^T C_Q^{-1} S)^{-1} S^T C_Q^{-1} y_Q^{ex} \; . \qquad (18)$$

The first solution (implying inverting a matrix of the order of the number L of experimental data Q_ℓ^{ex}) corresponds to the method of reduction by the Lagrange multipliers, the second one (implying inverting a matrix of the order of the number J of experimental data p_j^{ex}) to the so called method of reduction by elements.

Corresponding dispersion matries $\tilde{C}_p$

$$\tilde{C}_p = C_p - C_p S^T (C_Q + S C_p S^T)^{-1} S C_p \qquad (19)$$

$$\tilde{C}_p^{-1} = C_p^{-1} + S^T C_Q^{-1} S \; . \qquad (20)$$

Pieces of different, independent integral information can be added subsequently, adopting at each stage the latest up-dated estimates $\tilde{y}_p$ with dispersion matrix $\tilde{C}_p$.

Criteria for establishing the degree of confidence can be adopted, for example χ^2 tests, since it results that the residual quantity

$$\tilde{R} = y_Q^{ex.T} (S C_p S^T + C_Q)^{-1} y_Q^{ex} \qquad (21)$$

is distributed as χ^2, i.e., with L degrees of freedom. Its expected value is, therefore, equal to L.

Ref. in Uncertainty Analysis, CRC Press, Boca Raton, FL (1988)
(Y. Ronen Editor)