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Reactor Dynamics

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PART I

Fundamentals of the Pseudopotential Theory

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THE KINETIC PSEUDO-POTENTIALS

- Constitute an algorithm suitable for deriving rigorous analytical solutions to the reactor kinetic equations;
- Are helpful in providing a physical insight into reactor physics;
- Can generate useful benchmarks for code testing.

THE STANDARD TWO GROUP REACTOR DYNAMICS

$$\begin{cases} \frac{1}{v_F} \cdot \frac{\partial \Phi_F(r,t)}{\partial t} = D_F \cdot \nabla^2 \Phi_F(r,t) + \\ -\Sigma_{r,F} \cdot \Phi_F(r,t) + (1-\beta) \cdot \chi_F \cdot \{(\nu\Sigma_f)_F \cdot \Phi_F(r,t) + (\nu\Sigma_f)_T \cdot \Phi_T(r,t)\} + \\ + \sum_{i=1}^{6} \chi_{i,F} \cdot \lambda_i \cdot C_i(r,t) + S_{F;ext}(r,t); \\ \\ \frac{1}{v_T} \cdot \frac{\partial \Phi_T(r,t)}{\partial t} = \\ = D_T \cdot \nabla^2 \Phi_T(r,t) + (1-\beta) \cdot \chi_T \cdot \{(\nu\Sigma_f)_F \cdot \Phi_F(r,t) + (\nu\Sigma_f)_T \cdot \Phi_T(r,t)\} + \\ + \Sigma_{F \to T} \cdot \Phi_F(r,t) - \Sigma_{a,T} \cdot \Phi_T(r,t) + \sum_{i=1}^{6} \chi_{i,T} \cdot \lambda_i \cdot C_i(r,t) + S_{T;ext}(r,t); \\ \\ \frac{\partial}{\partial t} C_i(r,t) = \beta_i \cdot \{(\nu\Sigma_f)_F \cdot \Phi_F(r,t) + (\nu\Sigma_f)_T \cdot \Phi_T(r,t)\} - \lambda_i \cdot C_i(r,t); \\ (i = 1, ..., 6). \end{cases} \tag{1}$$

where:

$$\chi_F + \chi_T = 1;\ \chi_{i,F} + \chi_{i,T} = 1,\ (i = 1, ..., 6).$$

Boundary and initial conditions: to be specified.

Formal integration of the precursor equations:

$$C_i(r,t) = C_i^0(r) \cdot e^{-\lambda_i \cdot t} +$$
$$+\beta_i \cdot \int_0^t \{(\nu\Sigma_f)_F \cdot \Phi_F(r,t') + (\nu\Sigma_f)_T \cdot \Phi_T(r,t')\} \cdot e^{-\lambda_i \cdot (t-t')} dt' \ , (i = 1, ..., 6). \tag{2}$$

The substitution of this result into the first two equations displays the "historical memory" of the fission reactor.

THE $\mathcal{L}$-TRANSFORM OF THE MATHEMATICAL MODEL

$$\begin{cases} \nabla^2 \Phi_{F;\mathcal{L}}(r,p) - \left(\dfrac{1}{L_F^2} + \dfrac{p}{v_F \cdot D_F}\right) \cdot \Phi_{F;\mathcal{L}}(r,p) = \\ = -\dfrac{1}{D_F} \cdot (1-\beta) \cdot \chi_F \cdot \{(\nu\Sigma_f)_F \cdot \Phi_{F;\mathcal{L}}(r,p) + (\nu\Sigma_f)_T \cdot \Phi_{T;\mathcal{L}}(r,p)\} - \\ -\dfrac{1}{D_F} \left\{ \sum\limits_{i=1}^{6} \chi_{i,F} \cdot \lambda_i \cdot C_{i,\mathcal{L}}(r,p) + S_{T;\mathcal{L}}(r,p) + \dfrac{1}{v_F} \Phi_F^0(r) \right\}; \\ \\ \nabla^2 \Phi_{T;\mathcal{L}}(r,p) - \left(\dfrac{1}{L_T^2} + \dfrac{p}{v_T \cdot D_T}\right) \cdot \Phi_{T;\mathcal{L}}(r,p) = \\ = -\dfrac{1}{D_T} \cdot (1-\beta) \cdot \chi_T \cdot \{(\nu\Sigma_f)_F \cdot \Phi_{F;\mathcal{L}}(r,p) + (\nu\Sigma_f)_T \cdot \Phi_{T;\mathcal{L}}(r,p)\} - \\ -\dfrac{1}{D_T} \cdot \left\{ \Sigma_{F\to T} \cdot \Phi_{F;\mathcal{L}}(r,p) + \sum\limits_{i=1}^{6} \chi_{i,T} \cdot \lambda_i \cdot C_{i,\mathcal{L}}(r,p) + S_{T;\mathcal{L}}(r,p) + \dfrac{1}{v_T} \cdot \Phi_T^0(r) \right\}; \\ \\ C_{i,\mathcal{L}}(r,p) = \dfrac{1}{p+\lambda_i} \cdot C_i^0(r) + \dfrac{\beta_i}{p+\lambda_i} \{(\nu\Sigma_f)_F \cdot \Phi_{F;\mathcal{L}}(r,p) + (\nu\Sigma_f)_T \cdot \Phi_{T;\mathcal{L}}(r,p)\}; \\ (i = 1, ..., 6\ ). \end{cases} \tag{3}$$

As a consequence of the $\mathcal{L}$-transform algorithm all initial values are automatically included in the above equations.

Definitions:

- the unknown state vector:

$$|\Phi_{\mathcal{L}}(r,p)\rangle \doteq \left| \begin{array}{c} \Phi_{F;\mathcal{L}}(r,p) \\ \Phi_{T;\,\mathcal{L}}(r,p) \end{array} \right\rangle; \tag{4}$$

- the generalized sources for each group:

$$\begin{aligned} \overline{S}_F\,(r,p) &= S_{F;\mathcal{L}}(r,p) + \sum_{i=1}^{6} \chi_{i,F} \cdot \frac{\lambda_i}{p+\lambda_i} \cdot C_i^0(r) + \frac{1}{v_F}\Phi_F^0(r); \\ \overline{S}_T(r,p) &= S_{T;\mathcal{L}}(r,p) + \sum_{i=1}^{6} \chi_{i,T} \cdot \frac{\lambda_i}{p+\lambda_i} \cdot C_i^0(r) + \frac{1}{v_T}\Phi_T^0(r); \end{aligned} \tag{5}$$

- the known complex source vector:

$$\left|\overline{S}\,(r,p)\right\rangle \doteq \left| \begin{array}{c} -\dfrac{1}{D_F} \cdot \overline{S}_F\,(r,p) \\ -\dfrac{1}{D_T} \cdot \overline{S}_T(r,p) \end{array} \right\rangle; \tag{6}$$

- the inverse of the generalized, complex diffusion areas:

$$\begin{aligned} \frac{1}{\Lambda_F^2(p)} &\doteq \left(\frac{1}{L_F^2} + \frac{p}{v_F \cdot D_F}\right); \\ \frac{1}{\Lambda_T^2(p)} &\doteq \left(\frac{1}{L_T^2} + \frac{p}{v_T \cdot D_T}\right); \end{aligned} \tag{7}$$

- the complex matrix of material properties:

$$\{A(p)\} \doteq \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \tag{8}$$

$$\begin{aligned} a_{11} &\doteq -\frac{1}{\Lambda_F^2(p)} + \frac{(\nu\Sigma_f)_F}{D_F} \cdot \left\{\chi_F \cdot (1-\beta) + \sum_{i=1}^{6} \beta_i \cdot \chi_{i,F} \cdot \frac{\lambda_i}{p+\lambda_i}\right\}, \\ a_{12} &\doteq \frac{(\nu\Sigma_f)_T}{D_F} \cdot \left\{\chi_F \cdot (1-\beta) + \sum_{i=1}^{6} \beta_i \cdot \chi_{i,F} \cdot \frac{\lambda_i}{p+\lambda_i}\right\}, \\ a_{21} &\doteq \frac{1}{D_T} \cdot \left\{(\nu\Sigma_f)_F \cdot \left[\chi_T \cdot (1-\beta) + \sum_{i=1}^{6} \beta_i \cdot \chi_{i,T} \cdot \frac{\lambda_i}{p+\lambda_i}\right] + \Sigma_{F\to T}\right\}, \\ a_{22} &\doteq -\frac{1}{\Lambda_T^2(p)} + \frac{(\nu\Sigma_f)_T}{D_T} \cdot \left\{\chi_T \cdot (1-\beta) + \sum_{i=1}^{6} \beta_i \cdot \chi_{i,T} \cdot \frac{\lambda_i}{p+\lambda_i}\right\}. \end{aligned}$$

$\{A(p)\}$ is assumed to be **regionwise constant** with respect to the space variable.

The matrix formulation of the $\mathcal{L}$–transform dynamic equations.

By mean of the previous definition, we can write the system of transformed PDS's in matrix form:

$$\boxed{\boxed{\nabla^2 I_{2\times 2} + \{A(p)\}} \cdot |\Phi_{\mathcal{L}}(r,p)\rangle = \boxed{\overline{S}(r,p)\rangle}} \tag{9}$$

where:

$$\nabla^2 I_{2\times 2} \doteq \begin{pmatrix} \nabla^2 * & 0 \\ 0 & \nabla^2 * \end{pmatrix}$$

A preliminary solution of the fundamental auxiliary problem

$$\{A(p)\} \cdot |\psi_k(p)\rangle = \gamma_k(p) \cdot |\psi_k(p)\rangle \tag{10}$$

and its adjoint $\left(\{A^*(p)\}\right.$ represents the matrix complex conjugate of $\left.\{A(p)\}\right)$:

$$\langle\psi_h(p)| \cdot \{A^*(p)\} \cdot = \eta_h(p) \cdot \langle\psi_h(p)| \tag{11}$$

will be useful.

The secular equation, related to the eigenvalue problem for $\{A(p)\}$, is:

$$\begin{vmatrix} (a_{11} - \gamma) & a_{12} \\ a_{21} & (a_{22} - \gamma) \end{vmatrix} = 0. \tag{12}$$

Two **distinct** eigenvalues are generated:

$$\begin{aligned} \gamma_1(p) &= \frac{(a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4 \cdot a_{12} \cdot a_{21}}}{2} \doteq \mu^2(p), \\ &(\text{usually } \mu^2(p) \geq 0, \text{ for } p \equiv 0); \\ \gamma_2(p) &= \frac{(a_{11} + a_{22}) - \sqrt{(a_{11} - a_{22})^2 + 4 \cdot a_{12} \cdot a_{21}}}{2} \doteq -\nu^2(p), \\ &(\text{usually } -\nu^2(p) << 0, \text{ for } p \equiv 0). \end{aligned} \tag{13}$$

The associated linearly independent eigenvectors for the direct problem can be given the column form:

FUNDAMENTAL THEOREM:

in each homogeneous multiplying region the general integral of the two group, $\mathcal{L}$-transformed, dynamic equations can be given the form of a linear combination of solutions of the decoupled problems (24), according to:

$$\begin{aligned} \Phi_{F;\mathcal{L}}(r,p) &= f_1(r,p) + f_2(r,p) \\ \Phi_{T;\mathcal{L}}(r,p) &= \psi_1^{(2)}(p) \cdot f_1(r,p) + \psi_2^{(2)}(p) \cdot f_2(r,p) \end{aligned} \tag{25}$$

The problem has thus been shifted to that of solving eqs (24), account being taken of the interface and spatial boundary conditions. This procedure involves, as a rule, the transformed fluxes and **not** the pseudopotentials f_1 and f_2 independently of one another: a situation that actually **amounts to recoupling** the unknown potential with each other, as it is to be expected on physical ground. One gets finally:

$$|\Phi(r,t)\rangle = \left| \begin{array}{l} \mathcal{L}^{-1}_{p\to t}\{f_1(r,p)\} + \mathcal{L}^{-1}_{p\to t}\{f_2(r,p)\} \\ \mathcal{L}^{-1}_{p\to t}\left\{\psi_1^{(2)}(p) \cdot f_1(r,p)\right\} + \mathcal{L}^{-1}_{p\to t}\left\{\psi_2^{(2)}(p) \cdot f_2(r,p)\right\} \end{array} \right\rangle \tag{26}$$

as the particular integral of eqs. (1). It is to be considered the as **most appropriate** form of the analytical solution to our problem.

A comment on the formal decoupling of the two group diffusion eqs. system.

Actual recoupling occurs through:

1. regularity, interface and boundary conditions: to be imposed on the fluxes and not, separately, on the potentials;

2. the source: actually each of the $s_k(r,p)'s$ involves the generalized sources of both groups.

A parallel situation arises when one attempts to the classical electromagnetic equations, describing the time-dependent vector ($\mathbf{A}$) and scalar (φ) **electromagnetic potentials**.

Actually the Lorentz's condition:

$$div\mathbf{A}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial\varphi(\mathbf{r},t)}{\partial t} \qquad (27)$$

together with the request of imposing boundary and interface conditions on the field component and not on the potentials prevent this decoupling from being formal.

However the usefulness of e.m. potential algorithm is well recognized.

PART II

A First Application of the Theory

The Application of the Pseudopotential Theory to the cylindrical Reactor

A contribution of

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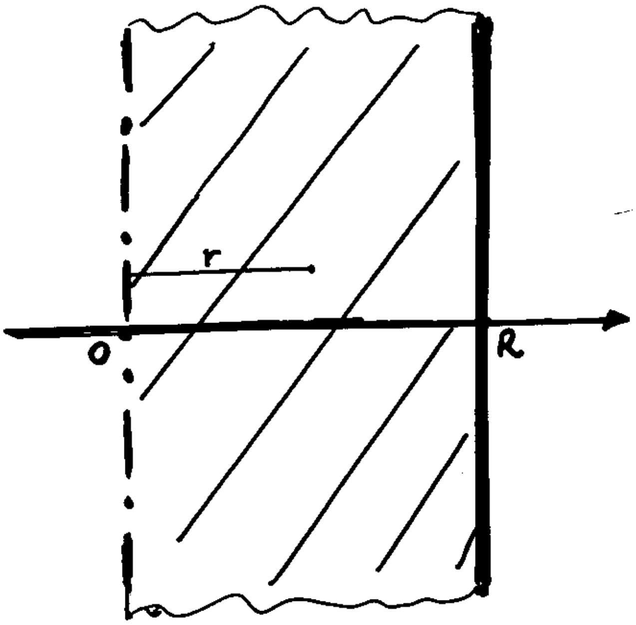
II - TWO GROUPS DYNAMICS OF THE BARE HOMOGENEOUS CYLINDER.



Despite our simplifying assumptions :

- No feedback effects on reactivity ;
- Radial simmetry of the excitation ;
- (Axially infinite reactor);
- (One delayed neutron family);
- Unique extrapolated radius R, for <u>both</u> fluxes,

this problem still appears as an UNSOLVED one : at least in its rigorous and general form developped in this paragraph.

In the present situation : (15)

$$\left.\begin{aligned}\gamma_1(p) &= \mu^2(p)\\ \gamma_2(p) &= -\nu^2(p)\end{aligned}\right\} = -\frac{1}{2}\left[\left(\chi_1^2+\frac{p}{V_1D_1}\right)+\left(\chi_2^2+\frac{p}{V_2D_2}\right)\pm\sqrt{\Omega}\right]$$

where

$$\sqrt{\Omega(p)} = \sqrt{\left[\left(\chi_1^2+\frac{p}{V_1D_1}\right)-\left(\chi_2^2+\frac{p}{V_2D_2}\right)\right]^2+4K_\infty\left(1-\beta\frac{p}{\lambda+p}\right)\chi_1^2\chi_2^2}$$

Furthermore, according to the equivalence theorem,

$$\forall p \in \mathcal{I} \equiv \left\{ p \in \mathbb{C} \;:\; \mathrm{Re}(p) \in [P_1', P_2'],\ \mathrm{Im}(p) \equiv 0 \right\}$$

one gets <u>real eigenvalues of opposite sign</u> :

$$\gamma_1(p_1) \equiv \mu^2(p_1) > 0 \qquad \gamma_2(P_1) \equiv -\nu^2(p_1) < 0$$

(16)

According to the general theory:

$$|\phi_\ell(z,p)\rangle = f_1(z,p)\cdot\left|\begin{matrix}1\\ \\ \psi_1^{(2)}(p)\end{matrix}\right\rangle + f_2(z,p)\left|\begin{matrix}1\\ \\ \psi_2^{(2)}(p)\end{matrix}\right\rangle, \quad (17)$$

where $f_1(z,p)$ and $f_2(z,p)$ come from solving:

$$\frac{d^2 f_1}{dz^2} + \frac{1}{z}\frac{df_1}{dz} + \mu^2(p) f_1 = s_1(z,p) \quad (18a)$$

(pseudo-multiplying medium) ↓ $\forall p \in I$

$$\frac{d^2 f_2}{dz^2} + \frac{1}{z}\frac{df_2}{dz} - \nu^2(p) f_2 = s_2(z,p) \quad (18b)$$

↑ (pseudo-absorbing medium)

and:

$$s_k(z,p) \equiv \langle\psi_k | S(z,p)\rangle \equiv \psi_k^{(1)+}(p)\, S_1^*(z,p) + \psi_k^{(2)+}(p)\, S_2^*(z,p). \quad (19)$$

To solve (18):

$$f_k(z,p) = f_k^{h}(z,p) + f_k^{nh}(z,p) \quad (20)$$

where:

$$f_1^{h}(z,p) = A_1(p)\cdot J_0(\mu z) + B_1(p)\cdot Y_0(\mu z) \quad (21a)$$

$$f_1^{h}(z,p) = A_2(p)\cdot I_0(\nu z) + B_2(p)\cdot K_0(\nu z) \quad (21b)$$

and:

$$f_1^{nh}(z,p) = -\frac{1}{4}\left\{ J_0(\mu z)\int_{\varepsilon\to 0^+}^{z} s_1(z',p)\,Y_0(\mu z')\,2\pi z'\,dz' + \right.$$
$$\left. + Y_0(\mu z)\int_{z}^{R} s_1(z',p)\,J_0(\mu z')\,dz' \right\} \tag{22a}$$

$$f_2^{nh}(z,p) = I_0(\nu z)\int_{\varepsilon\to 0^+}^{z} s_2(z',p)\,K_0(\nu z')\,z'\,dz' +$$
$$+ K_0(\nu z)\int_{z}^{R} s_2(z',p)\,I_0(\nu z')\,z'\,dz' \tag{22b}$$

Our bare reactor is a <u>highly simplified case</u>; the Boundary Conditions to determine $A_K(p)$ and $B_K(p)$ are:

$$\begin{cases} \tilde{\Phi}_1(R,p) = f_1(R,p) + f_2(R,p) = 0 \\ \tilde{\Phi}_2(R,p) = \psi_1^{(2)} f_1(R,p) + \psi_2^{(2)} f_2(R,p) = 0 \end{cases} \tag{23}$$

Due to the linear independence of the $|\psi_K\rangle$'s

$$\begin{Vmatrix} 1 & 1 \\ \psi_1^{(2)}(p) & \psi_2^{(2)}(p) \end{Vmatrix} \neq 0 . \tag{24}$$

As a consequence, (23) cannot be satisfied unless

$$\text{and}\begin{cases} f_1(R,p) = 0 \\ f_2(R,p) = 0 \end{cases} \quad \text{simultaneously} \tag{25}$$

Thus the B_k's, for instance, can be determined each one as a function of the corresponding A_k. The f_k's become:

$$f_1(z,p) = A_1\left[J_0(\mu z) - \frac{J_0(\mu R)}{Y_0(\mu R)}\, Y_0(\mu z)\right] + Y_0(\mu z)\frac{J_0(\mu R)}{Y_0(\mu R)}\,\frac{\pi}{2}\,\cdot$$

$$\cdot\int_0^R s_1 \cdot Y_0(\mu z')\, z'dz' - \frac{\pi}{2}\, J_0(\mu z)\int_0^z s_1 \cdot Y_0(\mu z')\, z'dz' + \qquad (26a)$$

$$+ \frac{\pi}{2}\, Y_0(\mu z)\int_z^R s_1 \cdot J_0(\mu z')\, z'dz' .$$

$$f_2(z,p) = A_2\left[I_0(\nu z) - \frac{I_0(\nu R)}{K_0(\nu R)}\, K_0(\nu z)\right] - K_0(\nu z)\,\frac{I_0(\nu R)}{K_0(\nu R)}\,\cdot$$

$$\cdot\int_0^R s_2 \cdot K_0(\nu z')\, z'dz' + I_0(\nu z)\int_0^z s_2 \cdot K_0(\nu z')\, z'dz' + \qquad (26b)$$

$$+ K_0(\nu z)\int_z^R s_2 \cdot I_0(\nu z')\, z'dz' .$$

The two further "constants" $A_1(p)$ and $A_2(p)$ can now be determined by requiring the continuity (it implies boundedness !!) of both fluxes, when $z \to 0$.
The final result can be achieved by choosing zero values for all coefficients of the (logaritmically) singular terms, and still

taking into account the linear indipendence of the $|\psi_K\rangle$'s.

We write $|\phi\rangle$ in the form :

$$\tilde{\phi}_1(z,p) = \frac{1}{J_0(\mu R)}\left[\left[\mathcal{J}_1(p)\frac{Y_0(\mu R)}{4} J_0(\mu z) + J_0(\mu R)\cdot F_1(z,p)\right]\right] + \quad (27a)$$

$$+ \frac{1}{I_0(\nu R)}\left\{\left\{ \mathcal{J}_2(p)\frac{K_0(\nu R)}{2\pi} I_0(\nu z) + I_0(\nu R)\cdot F_2(z,p)\right\}\right\}$$

$$\tilde{\phi}_2(z,p) = \frac{\psi_1^{(2)}(p)}{J_0(\mu R)}\left[\left[\cdots\cdots \right]\right] + \frac{\psi_2^{(2)}(p)}{I_0(\nu R)}\left\{\left\{ \cdots\cdots \right\}\right\} \quad (27b)$$

where $\mathcal{J}_k(p)$'s represent the <u>excitation integrals</u>:

$$\mathcal{J}_1(p) \equiv -\int_0^R s_1(z',p)\cdot J_0(\mu(p) z')\cdot 2\pi z'\, dz' \;; \quad (28a)$$

$$\mathcal{J}_2(p) \equiv \int_0^R s_2(z',p)\cdot I_0(\nu(p) z')\cdot 2\pi z'\, dz' \;, \quad (28b)$$

and the F_k's are defined by :

$$F_1(z,p) = \frac{1}{4}\Big[Y_0(\mu z)\cdot\int_0^z s_1(z',p)\cdot J_0(\mu z')\, 2\pi z'\, dz' + \quad (29a)$$

$$+ J_0(\mu z)\cdot\int_z^R s_1(z',p)\cdot Y_0(\mu z')\, 2\pi z'\, dz' \Big]$$

$$F_2(r,p) = -\frac{1}{2\pi}\left[K_0(\nu r)\int_0^r s_2(r',p)\, I_0(\nu r')\, 2\pi r'\, dr' + \right.$$

$$\left. + I_0(\nu r)\int_r^R s_2(r';p)\, K_0(\nu r')\, 2\pi r'\, dr'\right] \quad (29b)$$

The F_k's, for non-singular behaviour of the generalized sources, are continous, and hence limited, function of r, $\forall r \in [0, R]$, and not only $\forall p \in I$, but almost everywhere on the complex p-plane: and, in particular, at all values of p where the poles of the (27)'s are located

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WE REMIND NOW THAT:

NO DEEP UNDERSTANDING OF REACTOR DYNAMICS CAN BE ACHIEVED UNLESS STATICS OF THE INJECTED SUBCRITICAL STRUCTURES HAS BEEN FULLY CLARIFIED

So let's go back to the discussion of the source-multiplying media interaction and the experimental approach to criticality, within the two group theory.

Formulae (27), with $p \equiv 0$, are appropriate for studying the interaction between arbitrary time indipendent neutron sources and a cylindrical multiplying structure, whenever their k_{eff} is less than one.

COMMENTS ON (27a) and (27b) for the stationary sub-critical problem:

a) The "pure source transients" and the role of ν-type terms. Local dependence of the energy spectrum.

b) The unique space independent source spectrum that causes the ν contribution to become vanishingly small:

$$\boxed{\frac{S_1(r)}{D_2} : \frac{S_2(r)}{D_1} = 1 : \psi_1^{(2)}} \Rightarrow \mathcal{S}_2^{stat}(r) \equiv 0 \qquad (30)$$

c) The injection of source neutrons with the above energy spectrum, no matter how spatially distributed, causes a neutron response spectrum
(i) independent of the space coordinates and
(ii) similar to that of the critical structure, having the same material constants ($k_\infty > 1$ required).

d) No possibility exists of exciting a pure ν-type, space independent neutron spectrum. This would imply the physical existence of sources of "positive" and "negative" neutrons altogether.

e) The approach to critical : $\mu \to (\mu_{crit} - \varepsilon) \equiv \left(\frac{2.405}{R} - \varepsilon\right)$
Progressive dominance of the critical, space independent spectrum and critical (fundamental) eigenfunction.

THE TIME-DEPENDENT PROBLEM : it looks like the equivalent of infinitely many experiments of "approach to criticality", one for each eigenvalue.

The poles of the (27)'s are located at the roots of

$$\boxed{J_0(\mu(p)R) = 0} \Rightarrow \mu(p) = \frac{j_{0n}}{R} \Rightarrow p = p_j^{(n)},\ j = \genfrac{}{}{0pt}{}{1}{2},\ \forall n$$

$$\Rightarrow J_0\left(\frac{j_{0n}}{R} r\right)$$

and at

$$\boxed{I_0(\nu(p)R) = 0} \Rightarrow i\nu(p_3^{(n)}) = \frac{j_{0n}}{R} \Rightarrow p = p_{j=3}^{(n)},\ j = 3,\ \forall n$$

$$\Rightarrow I_0(i\nu(p_3^{(n)})\, r) = J_0\left(\frac{j_{0n}}{R} r\right) : \text{same type of eigenfunction}$$

and the inverse Laplace transform can be performed by means of residues theorem.

The case of bare homogeneous structures is a quite particular (degenerate) one:

- There are "<u>clusters</u>", each constituted by three (G+R), eigenfunctions having the same spatial shape $J_0(\frac{j_{0n}}{R}r)$ belonging to different time eigenvalues.
  But even inside each cluster the "energy spectra" are quite different from each other, as well as the ratios between neutrons and the nuclei of delayed neutron precursors;
- On the contrary, the excitation integrals are just the same only for the first two time modes inside each cluster;
- The CONCEPT OF DYNAMIC EIGENSTATE OF THE SYSTEM.
- A <u>Vector state</u> approach to the same problem.

$$\star \quad \Phi_1(r,t) = \frac{1}{4R}\sum_{n=1}^{\infty} \frac{Y_0(j_{0n})}{J_1(j_{0n})}\left\{ -\mathcal{J}_1^{(n)} \sum_{j=1}^{2} \frac{1}{\left.\frac{d\mu}{dp}\right|_{p=p_j^{(n)}}} e^{p_j^{(n)} t} + \frac{\mathcal{J}_2^{(n)}}{\left.\frac{d\nu}{dp}\right|_{p=p_3^{(n)}}} e^{p_3^{(n)} t} \right\} J_0\left(\frac{j_{0n}}{R} r\right) \tag{33a}$$

$$\star \quad \Phi_2(r,t) = \frac{1}{4R}\sum_{n=1}^{\infty} \frac{Y_0(j_{0n})}{J_1(j_{0n})}\left\{ -\mathcal{J}_1^{(n)} \sum_{j=1}^{2} \frac{1}{\left.\frac{d\mu}{dp}\right|_{p=p_j^{(n)}}} e^{p_j^{(n)}} \psi_1^{(2)}(p_j^{(n)}) + \frac{\mathcal{J}_2^{(n)}}{\left.\frac{d\nu}{dp}\right|_{p=p_3^{(n)}}} e^{p_3^{(n)} t} \psi_2^{(2)}(p_3^{(n)}) \right\} J_0\left(\frac{j_{0n}}{R} r\right) \tag{33b}$$

$\star$ Precursor density equation : from (12) (33c)

The general transient solution comes out NATURALLY as being expressed by means of a Hankel series expansion of both fluxes and precursors.

This representation is the **MOST APPROPRIATE ONE**: ITS TIME ASYMPTOTIC behaviour is represented through a single spatial eigenfunction, for both fluxes and precursors.

The time asymptotic energy spectrum, together with the precursors content per neutron present depends on the associated fundamental time eigenvalue.

PART III

A Second Application of the Theory

The Multigroup Approach to the Neutron Diffusion in a Fluid Core: Spherical Geometry

A contribution of
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A) The classical theory with standing fuel.

$$\begin{cases} \frac{1}{v_1}\frac{\partial \Phi_1(\mathbf{r},t)}{\partial t} = \\ = \nabla\cdot\left[D_1\nabla\Phi_1(\mathbf{r},t)\right] - \Sigma_{r1}\Phi_1(\mathbf{r},t) + (1-\beta)\chi_1\left[(\nu\Sigma_f)_1\Phi_1(\mathbf{r},t) + (\nu\Sigma_f)_2\Phi_2(\mathbf{r},t)\right] + \\ +\chi_{d,1}\lambda C(\mathbf{r},t) + S_{ext,1}(\mathbf{r},t)\ ; \\ \frac{1}{v_2}\frac{\partial \Phi_2(\mathbf{r},t)}{\partial t} = \\ = \nabla\cdot\left[D_2\nabla\Phi_2(\mathbf{r},t)\right] - \Sigma_{a2}\Phi_2(\mathbf{r},t) + (1-\beta)\chi_2\left[(\nu\Sigma_f)_1\Phi_1(\mathbf{r},t) + (\nu\Sigma_f)_2\Phi_2(\mathbf{r},t)\right] + \\ +\chi_{d,2}\lambda C(\mathbf{r},t) + p\Sigma_{r1}\Phi_1(\mathbf{r},t) + S_{ext,2}(\mathbf{r},t)\ ; \\ \frac{\partial C(\mathbf{r},t)}{\partial t} = -\lambda C(\mathbf{r},t) + \beta\left[(\nu\Sigma_f)_1\Phi_1(\mathbf{r},t) + (\nu\Sigma_f)_2\Phi_2(\mathbf{r},t)\right]\ ; \quad \mathbf{r}\in V, \end{cases} \tag{1}$$

Standard nomenclature has been adopted. V is the non reentrant tridimensional reactor region, whose contour is ∂V. Both fluxes vanish on ∂V.

B) New formulation of the theory accounting for fluid fuel and precursor diffusion.

$$\begin{aligned} \frac{\partial C(\mathbf{r},t)}{\partial t} &= \mathcal{D}_c\nabla^2 C(\mathbf{r},t) - \lambda C(\mathbf{r},t) + \\ &+\beta\left[(\nu\Sigma_f)_1\Phi_1(\mathbf{r},t) + (\nu\Sigma_f)_2\Phi_2(\mathbf{r},t)\right] \end{aligned} \tag{2}$$

where $\mathcal{D}_c$ $[cm^2/s]$ is the diffusion coefficient of the precursors inside the fluid core.

A boundary condition is now required for $C(r,t)$, at the (internal) wall of the core vessel. The most appropriate, when fuel extraction and reentrance do not occur, is ($\mathbf{n}$ is the outward normal to ∂V):

$$\mathcal{D}_c\nabla C(\mathbf{r}_s,t)\cdot\mathbf{n}(\mathbf{r}_s) = 0, \qquad \mathbf{r}_s\in\partial V. \tag{3}$$

Meaning: the heavy radioactive fission products cannot leak through the vessel wall.

Core Averaging of Eq. (2), after defining:

$$C(t) \doteq \frac{1}{V}\iiint_V C(\mathbf{r},t)\,d\mathbf{r},$$

$$\hat{\varphi}_j(t) \doteq \frac{1}{V} \iiint_V \Phi_j(\mathbf{r}')\, d\mathbf{r}', \qquad (j = 1,2)\ :$$

$$\frac{dC(t)}{dt} = \frac{1}{V} \iiint_V \mathcal{D}_c \nabla^2 C(\mathbf{r},t)\, d\mathbf{r} - \lambda C(t) +$$

$$+\beta \left[(\nu\Sigma_f)_1\, \hat{\varphi}_1(t) + (\nu\Sigma_f)_2\, \hat{\varphi}_2(t) \right]$$

Due to the boundary condition (3), the first term at the r.h.s. is zero, as it can be proved using Gauss's theorem.

C) Accounting for the fuel mixing.

A violent mixing of the fuel, taking place for cooling needs, has the same effect as $\mathcal{D}_c \to \infty$.

As a consequence, no gradient of precursor concentration will be allowed anymore in the core. So $C(\mathbf{r},t)$ must be **coincident** everywhere **with its average** value $C(t)$.

D) The consequences of the external circulation.

If an outside circulation of the violently mixed fuel is taking place, then the time-dependent balance of the homogenized precursor density despite the impervious vessel boundary becomes:

$$\frac{dC(t)}{dt} = -\lambda C(t) + \beta \left[(\nu\Sigma_f)_1\, \hat{\varphi}_1(t) + (\nu\Sigma_f)_2\, \hat{\varphi}_2(t) \right] +$$
$$-f\ C(t) + f\ C(t-\theta) \cdot e^{-\lambda\theta}\ ,$$

where:

i) f is the fraction of the fuel mass being circulated per second and

ii) θ is the average time spent by the fluid fuel in its solenoidal trip through the heat exchangers.

Due to the fuel homogenization the neutron balance equations take now a form even simpler than the previous one.

E) The stationary case with mixing and outside circulation.

The stationary counterpart of the above eq. is:

$$0=\beta\left[(\nu\Sigma_f)_1\hat{\varphi}_1+(\nu\Sigma_f)_2\hat{\varphi}_2\right]-C\left[\lambda+f\cdot(1-e^{-\lambda\theta})\right] ,$$

$$\lambda C=\frac{\beta\left[(\nu\Sigma_f)_1\hat{\varphi}_1+(\nu\Sigma_f)_2\hat{\varphi}_2\right]}{\left[1+\dfrac{f}{\lambda}\cdot(1-e^{-\lambda\theta})\right]}\doteq$$

$$\doteq\beta^*\cdot\left[(\nu\Sigma_f)_1\hat{\varphi}_1+(\nu\Sigma_f)_2\hat{\varphi}_2\right]$$

Implication: if circulation takes place, only the fraction

$$\frac{\beta^*}{\beta}=\frac{1}{[1+f\ /\ \lambda\cdot(1-e^{-\lambda\theta})]}$$

of the fission neutrons undergoing delayed emission are actually released inside the system.

F) The simplest stationary model for treating mixing and external circulation of the fuel.

Simplifying hypothesis: no fast fission occurs and the bare, homogeneous reactor is strictly thermal. So:

$(\nu\Sigma_f)_1=0;\quad \chi_1=\chi_{d,1}=1;\quad \chi_2=\chi_{d,2}=0.$

Defining:

$k_\infty\doteq\eta\cdot f\cdot p,$ where $\eta\doteq\dfrac{(\nu\Sigma_f)_2^{(U)}}{\Sigma_{a2}^{(U)}},\quad f\doteq\dfrac{\Sigma_{a2}^{(U)}}{\Sigma_{a2}},\quad p\doteq\dfrac{\Sigma_{1\to 2}}{\Sigma_{r1}}$, (U means "fuel").

Due to the very particular result now available for the precursor balance, i.e.

$$\lambda C(\mathbf{r})=\lambda C\equiv\frac{k_\infty}{p}\Sigma_{a2}\,\beta^*\cdot\hat{\varphi}_2\ ;\quad \forall\mathbf{r}\in V,$$

the stationary system, equivalent to eqs. (1), takes the much simpler form, integro-differential with respect to the space variables:

$$\left\{\begin{array}{l}\nabla^2\Phi_1(\mathbf{r})-\dfrac{1}{L_1^2}\Phi_1(\mathbf{r})+\dfrac{k_\infty}{D_1\cdot p}\Sigma_{a2}\left[(1-\beta)\,\Phi_2(\mathbf{r})+\beta^*\cdot\hat{\varphi}_2\right]\\ =-\dfrac{S_{ext,1}(\mathbf{r})}{D_1}\ ;\\ \nabla^2\Phi_2(\mathbf{r})-\dfrac{1}{L_2^2}\Phi_2(\mathbf{r})+p\dfrac{\Sigma_{r1}}{D_2}\Phi_1(\mathbf{r})=-\dfrac{S_{ext,2}(\mathbf{r})}{D_2}\qquad \mathbf{r}\in V\ .\end{array}\right.\tag{4}$$

A matrix formulation of this problem is:

$$\boxed{\left\{\nabla^2 \mathbf{I}_{2\times 2} + [\mathbf{A}(\beta)] + [\mathcal{I}(\beta^*)]\right\} \mid \Phi(\mathbf{r})\rangle = \mid S_{\text{ext}}(\mathbf{r})\rangle} \tag{5}$$

where:

$$\mid \Phi(\mathbf{r})\rangle = \left| \begin{matrix} \Phi_1(\mathbf{r}) \\ \Phi_2(\mathbf{r}) \end{matrix} \right\rangle \tag{6}$$

$$\mid S_{\text{ext}}(\mathbf{r})\rangle = \left| \begin{matrix} -S_{\text{ext},1}(\mathbf{r})/D_1 \\ -S_{\text{ext},2}(\mathbf{r})/D_2 \end{matrix} \right\rangle \quad . \tag{7}$$

$$\mathbf{A}(\beta) \doteq \begin{bmatrix} -\dfrac{1}{L_1^2} & \dfrac{k_\infty(1-\beta)}{p}\dfrac{\Sigma_{a2}}{D_1} \\ p\dfrac{\Sigma_{r1}}{D_2} & -\dfrac{1}{L_2^2} \end{bmatrix} \doteq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; \tag{8}$$

$$\nabla^2 \mathbf{I}_{2\times 2} \doteq \begin{bmatrix} \nabla^2 \ldots & 0 \\ 0 & \nabla^2 \ldots \end{bmatrix} ;$$

$$\mathcal{I}(\beta^*) \doteq \begin{bmatrix} 0 & \dfrac{\beta^* k_\infty}{p}\dfrac{\Sigma_{a2}}{D_1}\dfrac{1}{V}\iiint_V \cdots d\mathbf{r}' \\ 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} 0 & c_{12}\cdot \dfrac{1}{V}\iiint_V \cdots d\mathbf{r}' \\ 0 & 0 \end{bmatrix} . \tag{9}$$

G) The search for criticality.

It involves the investigation about the conditions under which, in absence of external source, the integro-differential operator $[\mathbf{O}]$

$$[\mathbf{O}] \doteq \nabla^2 \mathbf{I}_{2\times 2} + [\mathbf{A}(\beta)] + [\mathcal{I}(\beta^*)] \tag{10}$$

possesses a non trivial solution, belonging to the null eigenvalue:

$$[\mathbf{O}] \mid \Phi(\mathbf{r})\rangle = 0 \cdot |\Phi(\mathbf{r})\rangle \ . \tag{11}$$

To solve eq. (11) we let the unknown vector $[\mathcal{I}(\beta^*)] \mid \Phi(\mathbf{r})\rangle$ play formally the role of a "virtually" external source, now called $\mid S_{\text{int}}^d\rangle$:

$$-[\mathcal{I}(\beta^*)] \mid \Phi(\mathbf{r})\rangle \doteq \mid S_{\text{int}}^d\rangle = \left| \begin{matrix} -\dfrac{\beta^* k_\infty}{p}\dfrac{\Sigma_{a2}}{D_1}\dfrac{1}{V}\iiint_V \Phi_2(\mathbf{r})\, d\mathbf{r} \\ 0 \end{matrix} \right\rangle \equiv \hat{\varphi}_2 \cdot \left| \begin{matrix} -c_{12} \\ 0 \end{matrix} \right\rangle , \tag{12}$$

so that we will have to solve, first of all,

$$\left\{\nabla^2 \mathbf{I}_{2\times 2} + [\mathbf{A}(\beta)]\right\} \cdot \mid \Phi(\mathbf{r})\rangle = \mid S^d_{\text{int}}\rangle \ . \tag{13}$$

Let us ignore for the moment that $\mid S^d_{\text{int}}\rangle$ is unknown and adopt the stationary version of the kinetic pseudopotential algorithm .

Auxiliary problem and its adjoint:

$$\begin{array}{l} [\mathbf{A}(\beta)] \mid \psi_k\rangle = \gamma_k \cdot \mid \psi_k\rangle \\ \langle \psi_h \mid [\mathbf{A}(\beta)] = \eta_h \cdot \langle \psi_h \mid \end{array} \quad . \tag{14}$$

both leading now to the same secular equation.

The eigenvalues are:

$$\begin{array}{l} \gamma_1 = \dfrac{(a_{11}+a_{22}) + \left[(a_{11}-a_{22})^2 + 4a_{12}a_{21}\right]^{\frac{1}{2}}}{2} \equiv \eta_1 \doteq \mu^2(\beta) \ ; \\ \gamma_2 = \dfrac{(a_{11}+a_{22}) - \left[(a_{11}-a_{22})^2 + 4a_{12}a_{21}\right]^{\frac{1}{2}}}{2} \equiv \eta_2 \doteq -\nu^2(\beta) \ ; \end{array} \tag{15}$$

$$\mid \psi_k\rangle = \left| \begin{array}{c} 1 \\ \dfrac{\gamma_k - a_{11}}{a_{12}} \end{array} \right\rangle \doteq \left| \begin{array}{c} 1 \\ \psi_k^{(2)} \end{array} \right\rangle , \qquad (k=1,2) \ , \tag{16}$$

$$\langle \psi_k | = \left\langle \begin{array}{cc} 1 & \dfrac{\gamma_k - a_{11}}{a_{21}} \end{array} \right| \frac{1}{N_k} \doteq \left\langle \begin{array}{cc} \psi_k^{\dagger\,(1)} & \psi_k^{\dagger\,(2)} \end{array} \right| \ , \qquad (k=1,2) \ . \tag{17}$$

$$N_h = 1 + \frac{(\gamma_h - a_{11})^2}{a_{12}a_{21}}, \qquad (h=1,2) \ . \tag{18}$$

Properties of the base: biorthogonality and completeness:

$$\langle \psi_h \mid \psi_k\rangle = \delta_{hk} \ , \tag{19}$$

$$\sum_{k=1}^{2} \mid \psi_k\rangle \cdot \langle \psi_k \mid = \mathbf{I}_{2\times 2} \ . \tag{20}$$

Let us change the unknown functions $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ into the new ones $f_1(\mathbf{r})$ and $f_2(\mathbf{r})$:

$$|\Phi(\mathbf{r})\rangle = f_1(\mathbf{r}) \cdot \mid \psi_1\rangle + f_2(\mathbf{r}) \cdot \mid \psi_2\rangle \tag{21}$$

$$| S^d_{\rm int}\rangle = \langle\psi_1 \mid S^d_{\rm int}\rangle \cdot |\psi_1\rangle + \langle\psi_2 \mid S^d_{\rm int}\rangle \cdot |\psi_2\rangle \doteq s_1 \mid \psi_1\rangle + s_2 \mid \psi_2\rangle \ . \qquad (22)$$

Ignore for the moment that s_1 and s_2 are actually unknown. After substituting into eq. (13) we get:

$$\begin{aligned} &\nabla^2 f_1(\mathbf{r}) \cdot |\psi_1\rangle + \nabla^2 f_2(\mathbf{r}) \mid \psi_2\rangle + \mu^2 \cdot f_1(\mathbf{r}) \ |\psi_1\rangle - \nu^2 \cdot f_2(\mathbf{r}) \ |\psi_2\rangle = \\ &\qquad = s_1|\psi_1\rangle + s_2|\psi_2\rangle. \end{aligned} \qquad (23)$$

The formally uncoupled equations for the potentials are:

$$\begin{aligned} \nabla^2 f_1(\mathbf{r}) + \mu^2 f_1(\mathbf{r}) &= s_1 \ ; \\ \nabla^2 f_2(\mathbf{r}) - \nu^2 f_2(\mathbf{r}) &= s_2 \ . \end{aligned} \qquad (24)$$

To let the space-independent source components explicit:

$$s_k = \langle\psi_k \mid S^d_{\rm int}\rangle = \left\langle \ \psi_k^{\dagger\,(1)} \qquad \psi_k^{\dagger\,(2)} \ \right\| \left. \begin{matrix} -c_{12}\ \hat\varphi_2 \\ \\ 0 \end{matrix} \right\rangle \doteq g_k \cdot \hat\varphi_2 \qquad (25)$$

where $g_k \doteq -c_{12}/N_k$. Assuming, as usually verified:

$$\mu^2(\beta) > 0, \qquad -\nu^2(\beta) < 0 \ ,$$

and spherical geometry:

$$\begin{aligned} f_1\,(r) &= A_1 \frac{\sin{(\mu r)}}{r} + B_1 \frac{\cos{(\mu r)}}{r} + g_1 \frac{\hat\varphi_2}{\mu^2} \ ; \\ f_2\,(r) &= A_2 \frac{\sinh{(\nu r)}}{r} + B_2 \frac{\cosh{(\nu r)}}{r} - g_2 \frac{\hat\varphi_2}{\nu^2} \ . \end{aligned} \qquad (26)$$

To make both fluxes regular at the center of the sphere it is required that:

$$\begin{cases} \lim\limits_{r\to 0} \left[B_1 \cos(\mu r) + B_2 \cosh(\nu r)\right] = 0 \\ \\ \lim\limits_{r\to 0} \left[B_1 \psi_1^{(2)} \cos(\mu r) + B_2 \psi_2^{(2)} \cosh(\nu r)\right] = 0 \ , \end{cases} \qquad (27)$$

hence:

$$\begin{cases} B_1 + B_2 = 0 \ ; \\ \\ B_1 \psi_1^{(2)} + B_2 \psi_2^{(2)} = 0 \ . \end{cases}$$

Thus, from the linear independence of the eigenvectors $\mid \psi_k\rangle$ of the base:

$$\Longrightarrow \qquad B_1 = B_2 = 0 \quad , \qquad (28)$$

a sufficient, but also necessary, condition fro flux regularity. Furthermore, from the boundary conditions on both fluxes and the linear independence of the $| \psi_j \rangle' s$:

$$\Longrightarrow \qquad f_1(R) = f_2(R) = 0 \ ,$$

implying

$$A_1 = -\frac{g_1}{\mu^2}\hat{\varphi}_2 \frac{R}{\sin(\mu R)} \ ; \ A_2 = \frac{g_2}{\nu^2}\hat{\varphi}_2 \frac{R}{\sinh(\nu R)} \ . \tag{29}$$

Finally, the <u>formal</u> output of the theory produces the following fluxes:

$$\Phi_1(r) = \hat{\varphi}_2 \cdot \left\{ \frac{g_1}{\mu^2} \cdot \left[\left[1 - \frac{R}{\sin(\mu R)}\frac{\sin(\mu r)}{r}\right]\right] - \frac{g_2}{\nu^2} \cdot \left\{\left\{1 - \frac{R}{\sinh(\nu R)}\frac{\sinh(\nu r)}{r}\right\}\right\}\right\} ;$$

$$\Phi_2(r) = \hat{\varphi}_2 \cdot \left\{ \psi_1^{(2)}\frac{g_1}{\mu^2} \cdot \left[\left[1 - \frac{R}{\sin(\mu R)}\frac{\sin(\mu r)}{r}\right]\right] - \psi_2^{(2)}\frac{g_2}{\nu^2} \cdot \left\{\left\{1 - \frac{R}{\sinh(\nu R)}\frac{\sinh(\nu r)}{r}\right\}\right\}\right\} \tag{30}$$

or in a shorter, easily identifiable, writing:

$$\Phi_1(r) = \hat{\varphi}_2 \cdot \left\{ \frac{g_1}{\mu^2} \cdot [[\cdots]] - \frac{g_2}{\nu^2} \cdot \{\{\cdots\}\} \right\} ;$$
$$\Phi_2(r) = \hat{\varphi}_2 \cdot \left\{ \psi_1^{(2)}\frac{g_1}{\mu^2} \cdot [[\cdots]] - \psi_2^{(2)}\frac{g_2}{\nu^2} \cdot \{\{\cdots\}\} \right\} . \tag{31}$$

Properties of this formal solution:

a) $\Phi_1(r)$ and $\Phi_2(r)$ are both proportional to $\hat{\varphi}_2$, the <u>still unknown functional</u> of the solution;

b) $\Phi_1(R) = \Phi_2(R) = 0$, for all finite values of $\hat{\varphi}_2$, as required;

c) both fluxes are limited $\forall \ r \in V$ and, in particular, for $r \to 0$, for every finite value of $\hat{\varphi}_2$.

H) Selfconsistency of the theory.

The selfconsistency of the theory is not yet achieved at the present stage of the development. It has to be explicitly imposed. It will be established, after a space averaging of the representation of $\Phi_2(r)$ just derived, by imposing the identity between the l.h.s. average of the second of the eqs. (31) and the $\hat{\varphi}_2$ entering its averaged r.h.s. Selfconsistency can thus be guarantied if and only if the following equation is satisfied:

$$\boxed{1 = \psi_1^{(2)}\frac{g_1}{\mu^2}\,[[\overline{\cdots}]] - \psi_2^{(2)}\frac{g_2}{\nu^2}\,\{\{\overline{\cdots}\}\}} \tag{32}$$

where:

$$[[\because]] \doteq \frac{1}{V} \cdot \int_0^R \left[1 - \frac{R}{\sin(\mu R)} \frac{\sin(\mu r)}{r}\right] \cdot 4\pi r^2 dr \ ;$$

$$\{\{\because\}\} \doteq \frac{1}{V} \cdot \int_0^R \left\{1 - \frac{R}{\sinh(\nu R)} \frac{\sinh(\nu r)}{r}\right\} \cdot 4\pi r^2 dr \ .$$

The above formula, relating with each other material and geometrical parameters, is playing the role of a critical equation for this reactor. It is actually establishing also the mandatory coupling between the pseudo-potentials entering the theory: so they will be no longer uncorrelated.
Once (32) has been satisfied, the fluxes will take the new selfconsistent form:

$$\Phi_1(r) = A \left\{ \frac{g_1}{\mu^2} \cdot [[\cdots]] - \frac{g_2}{\nu^2} \{\{\cdots\}\} \right\} \ ;$$
$$\Phi_2(r) = A \left\{ \psi_1^{(2)} \frac{g_1}{\mu^2} [[\cdots]] - \psi_2^{(2)} \frac{g_2}{\nu^2} \{\{\cdots\}\} \right\} ,$$

wherein just a "critical set of values" of material and geometrical parameters do enter while all values of the power factor A are acceptable in principle.

L) A significant integral invariant of the fundamental eigenstate.

Once criticality has been established, it can be proved that the following formula, including volume and surface integrals of both fluxes, holds:

$$\boxed{k_\infty \cdot P_{NL}^{(1)}(\beta, \beta^*) \cdot P_{NL}^{(2)}(\beta) = 1} \qquad (33)$$

where:

$$P_{NL}^{(1)}(\beta, \beta^*) \doteq \frac{\iiint_V \Sigma_{r1} \Phi_1(\mathbf{r})\, d\mathbf{r}}{\iiint_V \Sigma_{r1} \Phi_1(\mathbf{r})\, d\mathbf{r} + \oint_{\partial V} J_1(\mathbf{r}_s) \cdot \mathbf{n}\; dA + \alpha \cdot V \cdot \hat{\varphi}_2} ,$$

$$P_{NL}^{(2)}(\beta) \doteq \frac{\iiint_V \Sigma_{a2} \Phi_2(\mathbf{r})\, d\mathbf{r}}{\iiint_V \Sigma_{a2} \Phi_2(\mathbf{r})\, d\mathbf{r} + \oint_{\partial V} J_2(\mathbf{r}_s) \cdot \mathbf{n}\; dA} ,$$

and $\alpha \doteq \frac{k_\infty}{p} \Sigma_{a2} (\beta - \beta^*)$. In spherical geometry with radial symmetry the above formulae take an even simpler form.

The proof of eq. (33) involves, of course, taking into account the critical equation that allows, for instance, the elimination of $\{\{\cdot\therefore\cdot\}\}$.

The results for the $P_{NL}^{(j)}$ $(j = 1, 2)$ are:

$$P_{NL}^{(1)}(\beta, \beta^*) = \frac{\kappa_1^2 \left(\frac{g_1}{\mu^2} [[\cdot\therefore\cdot]] + \frac{1}{\psi_2^{(2)}} - \frac{\psi_1^{(2)}}{\psi_2^{(2)}} \frac{g_1}{\mu^2} [[\cdot\therefore\cdot]] \right)}{\kappa_1^2 \left(\frac{g_1}{\mu^2} [[\cdot\therefore\cdot]] + \frac{1}{\psi_2^{(2)}} - \frac{\psi_1^{(2)}}{\psi_2^{(2)}} \frac{g_1}{\mu^2} [[\cdot\therefore\cdot]] \right) - g_1 - g_2 + g_1 [[\cdot\therefore\cdot]] - \frac{\nu^2}{\psi_2^{(2)}} + \frac{\psi_1^{(2)}}{\psi_2^{(2)}} \frac{\nu^2 \cdot g_1}{\mu^2} [[\cdot\therefore\cdot]] + \frac{\alpha}{D_1}}$$

and

$$P_{NL}^{(2)}(\beta) = \frac{\kappa_2^2}{\kappa_2^2 - g_1 \psi_1^{(2)} + g_1 \psi_1^{(2)} [[\cdot\therefore\cdot]] - g_2 \psi_2^{(2)} - \nu^2 + \frac{\psi_1^{(2)}}{\mu^2} \nu^2 g_1 [[\cdot\therefore\cdot]]}$$

where $\kappa_1^2 \doteq \frac{\Sigma_{r1}}{D_1}$, $\kappa_2^2 \doteq \frac{\Sigma_{a2}}{D_2}$.

M) The subcritical response to a stationary neutron source: neutron amplification.

Under subcritical condition a stationary source can sustain fluxes of limited values;

These fluxes can be derived from a selfconsistent application of the pseudo-potential theory, too.

Starting from the pseudo-potentials:

$$f_1(r) = A_1 \frac{\sin(\mu r)}{r} + B_1 \frac{\cos(\mu r)}{r} + g_1 \frac{\hat{\varphi}_2}{\mu^2} + \frac{s_{1,\text{ext}}}{\mu^2} ;$$
$$f_2(r) = A_2 \frac{\sinh(\nu r)}{r} + B_2 \frac{\cosh(\nu r)}{r} - g_2 \frac{\hat{\varphi}_2}{\nu^2} - \frac{s_{2,\text{ext}}}{\nu^2} ,$$

where the constants $s_{k,\text{ext}}$'s are now the actually known, constant projections of the external source on the $|\,\psi_k\rangle$'s:

$$s_{k,\text{ext}} = \langle \psi_k \mid S_{\text{ext}} \rangle .$$

According to the previous proof, the conditions $B_1 = B_2 = 0$ are still to be taken as *sufficient, but also necessary ones,* for the regularity of both fluxes inside the whole device.

From the boundary requirement $f_1(R) = f_2(R) = 0$ we deduce:

$$A_1 = -\frac{(g_1\hat{\varphi}_2 + s_{1,\text{ext}})}{\mu^2}\frac{R}{\sin(\mu R)} ;$$

$$A_2 = \frac{(g_2\hat{\varphi}_2 + s_{2,\text{ext}})}{\nu^2}\frac{R}{\sinh(\nu R)} ,$$

and the following, still not physically consistent fluxes:

$$\Phi_1(r) = \frac{(g_1\hat{\varphi}_2 + s_{1,\text{ext}})}{\mu^2}[[\cdots]] - \frac{(g_2\hat{\varphi}_2 + s_{2,\text{ext}})}{\nu^2}\{\{\cdots\}\}$$

$$\Phi_2(r) = \psi_1^{(2)}\frac{(g_1\hat{\varphi}_2 + s_{1,\text{ext}})}{\mu^2}[[\cdots]] - \psi_2^{(2)}\frac{(g_2\hat{\varphi}_2 + s_{2,\text{ext}})}{\nu^2}\{\{\cdots\}\} .$$

By rearranging the expressions of $\Phi_1(r)$ and $\Phi_2(r)$ we obtain:

$$\Phi_1(r) = \hat{\varphi}_2 \cdot \left\{\frac{g_1}{\mu^2}\cdot[[\cdots]] - \frac{g_2}{\nu^2}\cdot\{\{\cdots\}\}\right\} + \frac{s_{1,\text{ext}}}{\mu^2}[[\cdots]] - \frac{s_{2,\text{ext}}}{\nu^2}\{\{\cdots\}\}$$

$$\Phi_2(r) = \hat{\varphi}_2 \cdot \left\{\psi_1^{(2)}\frac{g_1}{\mu^2}\cdot[[\cdots]] - \psi_2^{(2)}\frac{g_2}{\nu^2}\cdot\{\{\cdots\}\}\right\} + \psi_1^{(2)}\frac{s_{1,\text{ext}}}{\mu^2}[[\cdots]] - \psi_2^{(2)}\frac{s_{2,\text{ext}}}{\nu^2}\{\{\cdots\}\} . \tag{34}$$

By imposing the <u>self-consistency condition</u> it is easy to find out that $\hat{\varphi}_2$ has to be proportional to the intensity of the external source, as expected, according to:

$$\hat{\varphi}_2 = \frac{\psi_1^{(2)}\ \frac{s_{1,\text{ext}}}{\mu^2}[[\because]] - \psi_2^{(2)}\ \frac{s_{2,\text{ext}}}{\nu^2}\{\{\because\}\}}{1 - \left(\psi_1^{(2)}\frac{g_1}{\mu^2}[[\because]] - \psi_2^{(2)}\frac{g_2}{\nu^2}\{\{\because\}\}\right)} . \tag{35}$$

After inserting into formulae (34) the above expression of $\hat{\varphi}_2$, both radial profiles of the subcritical fluxes are finally obtained in their <u>rigorous, selfconsistent form.</u>

We note that:

i) both fluxes are proportional to the intensity of the external source;

ii) the energy spectrum is source- and space-dependent and differs substantially from the critical one;

iii) the "amplification" of the neutrons injected by the source can be easily evaluated.

N) The experiment of approach to criticality and neutron energy spectrum.

By considering eqs. (34) and (35) it is easy to recognize that, when the initially subcritical, injected reactor approaches its critical state, i.e. when:

$$\left(\psi_1^{(2)}\frac{g_1}{\mu^2}\,[[\cdots]] - \psi_2^{(2)}\frac{g_2}{\nu^2}\,\{\{\cdots\}\}\right) \longrightarrow 1^-,$$

the last terms at the of r.h.s.'s of both of eqs. (34) become more and more negligeable with respect to the ones in front of which $\hat{\varphi}_2$ appears as a factor.

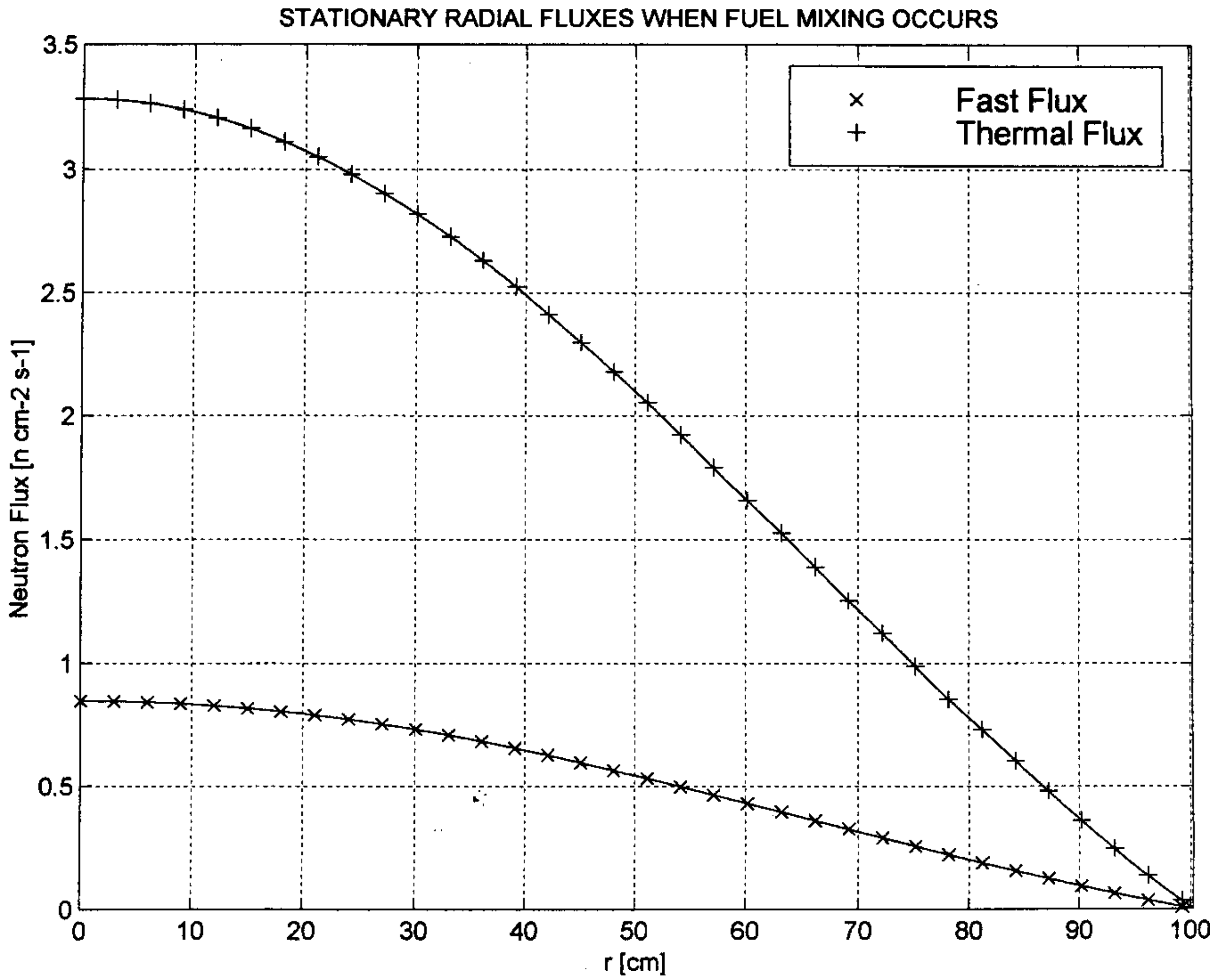
As a consequence, when we get closer to critical conditions:

i) both fluxes tend to infinity, their shapes approaching the critical ones;

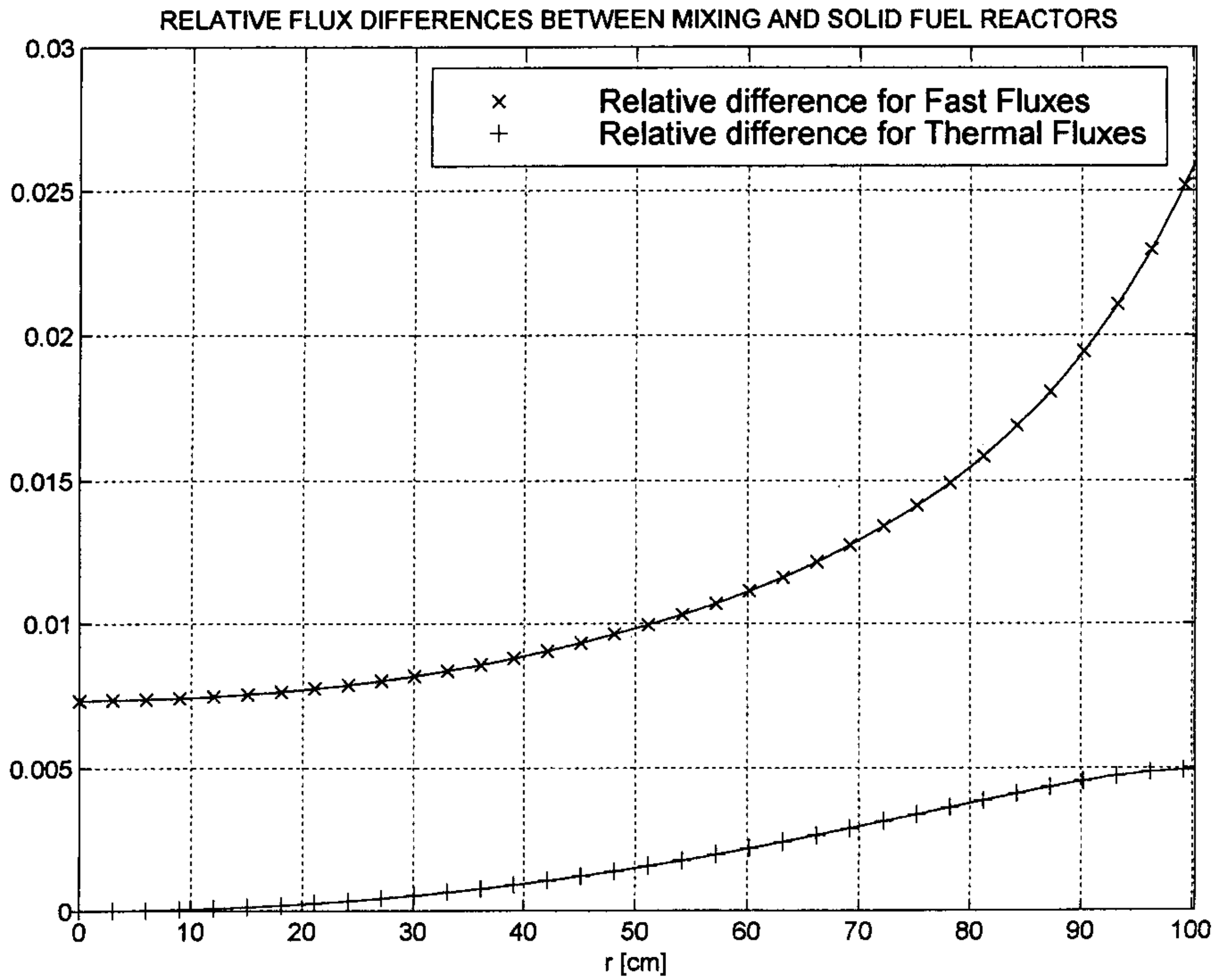
ii) the energy spectrum $\left(\frac{\Phi_1(r)}{\Phi_2(r)}\right)$ tends to be more and more similar to the one of the critical reactor, independently of the intensity, spectrum and space location of the forcing source, i.e.:

$$\left(\frac{\Phi_1(r)}{\Phi_2(r)}\right) \longrightarrow \left(\frac{\Phi_1(r)}{\Phi_2(r)}\right)_{\text{Critical}} \equiv \left.\frac{\frac{g_1}{\mu^2}\,[[\cdots]] - \frac{g_2}{\nu^2}\,\{\{\cdots\}\}}{\psi_1^{(2)}\frac{g_1}{\mu^2}\,[[\cdots]] - \psi_2^{(2)}\frac{g_2}{\nu^2}\,\{\{\cdots\}\}}\right|_{\text{Critical}} \quad ;$$

iii) the number of neutrons leaking out of the system per injected neutron tends to infinity.

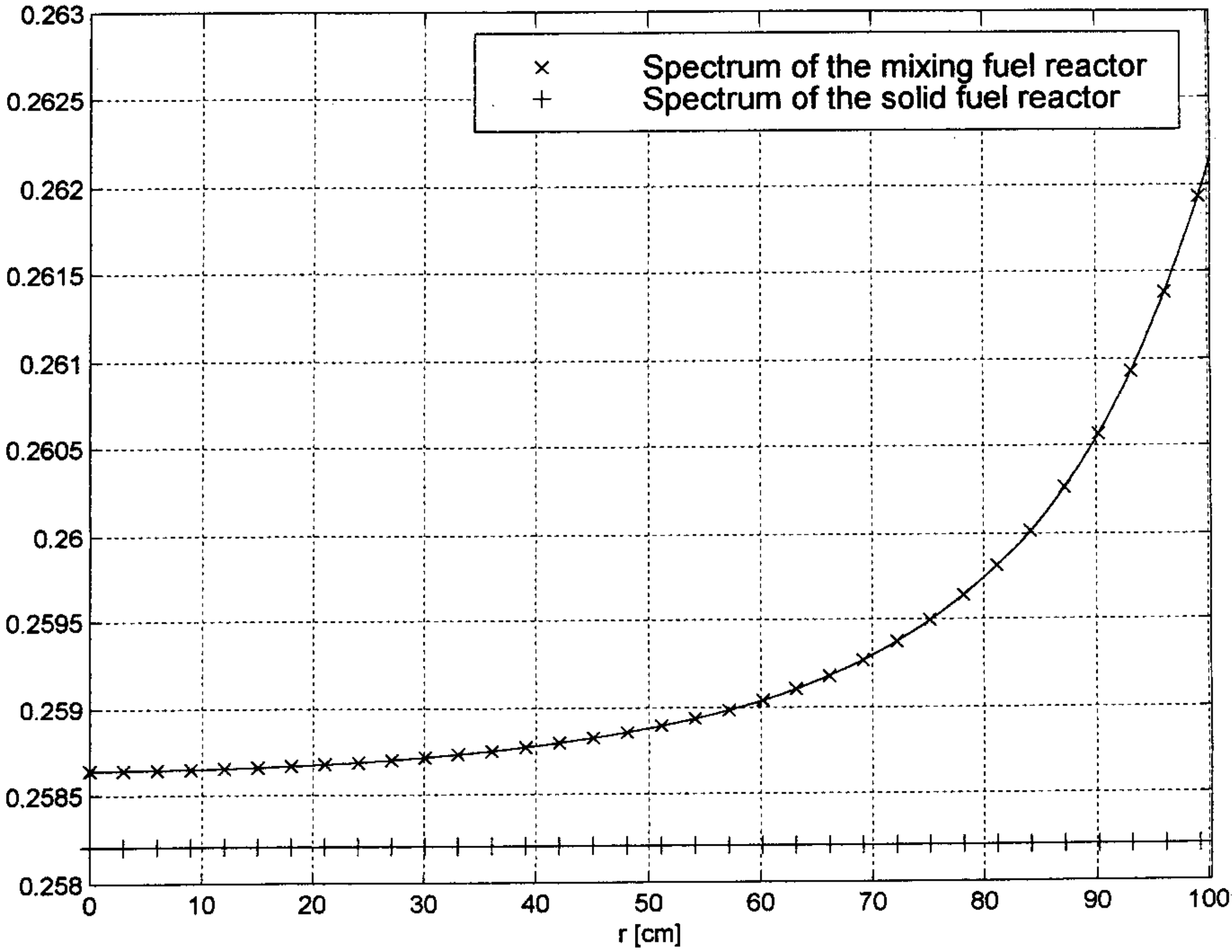

STATIONARY RADIAL FLUXES WHEN FUEL MIXING OCCURS
Fast Flux
Thermal Flux
Neutron Flux [n cm-2 s-1]
r [cm]
3.5
3
2.5
2
1.5
1
0.5
0
10
20
30
40
50
60
70
80
90
100

$D_2O + UO_2SO_4$

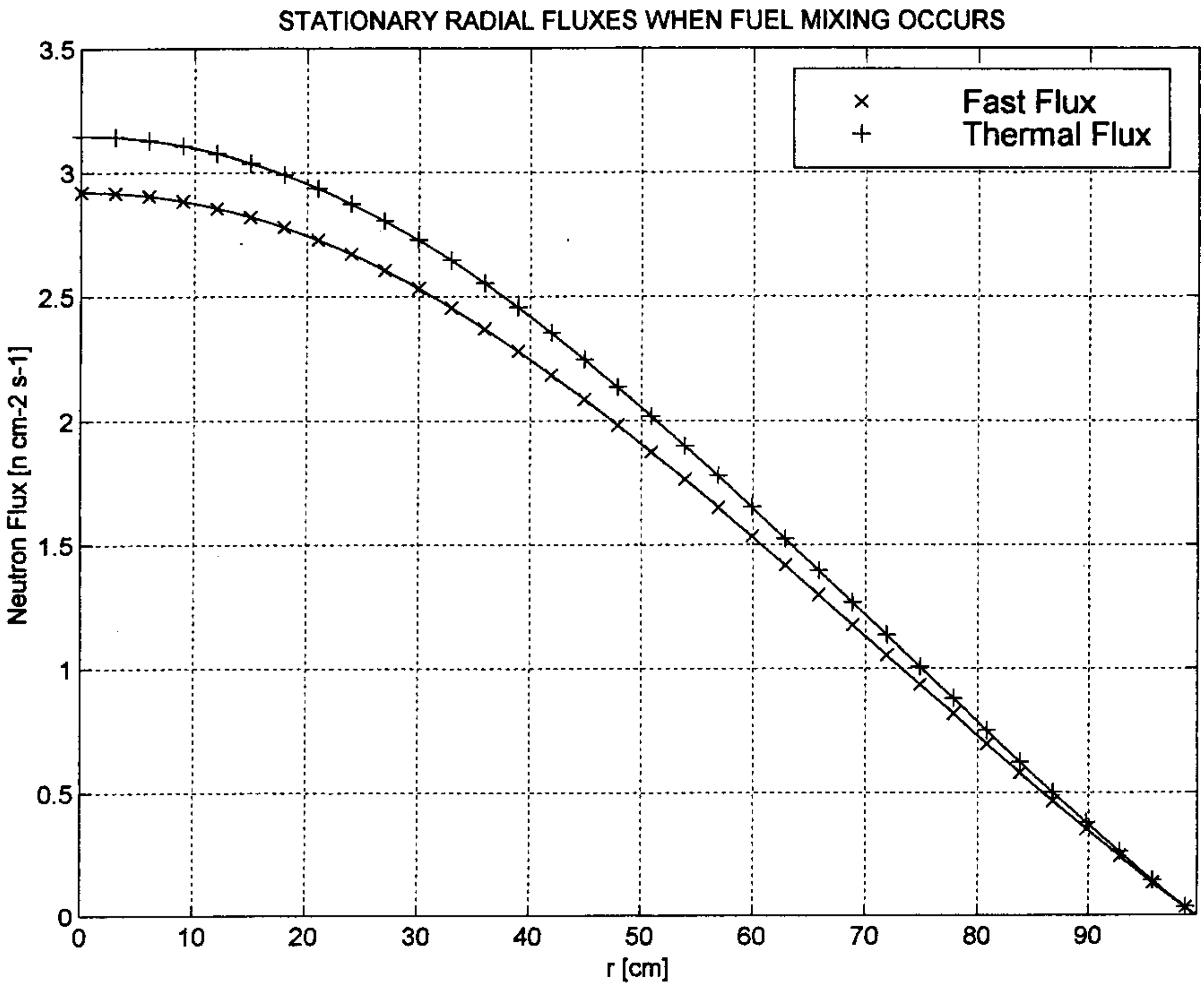

RELATIVE FLUX DIFFERENCES BETWEEN MIXING AND SOLID FUEL REACTORS
Relative difference for Fast Fluxes
Relative difference for Thermal Fluxes
0.03
0.025
0.02
0.015
0.01
0.005
0
0
10
20
30
40
50
60
70
80
90
100
r [cm]

$D_2O + UO_2SO_4$

SPECTRUM OF THE MIXING AND SOLID FUEL REACTOR

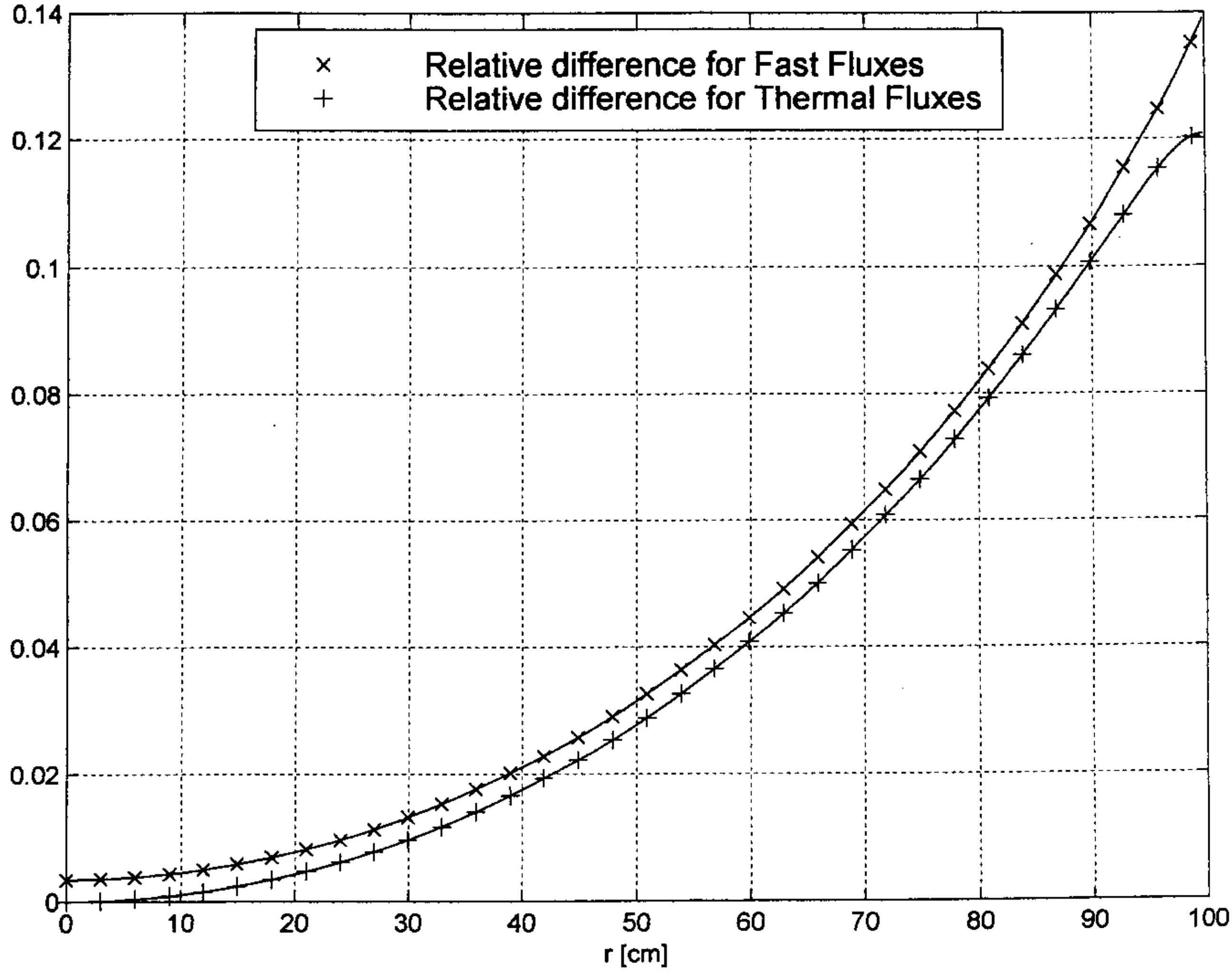


$D_2O + UO_2SO_4$

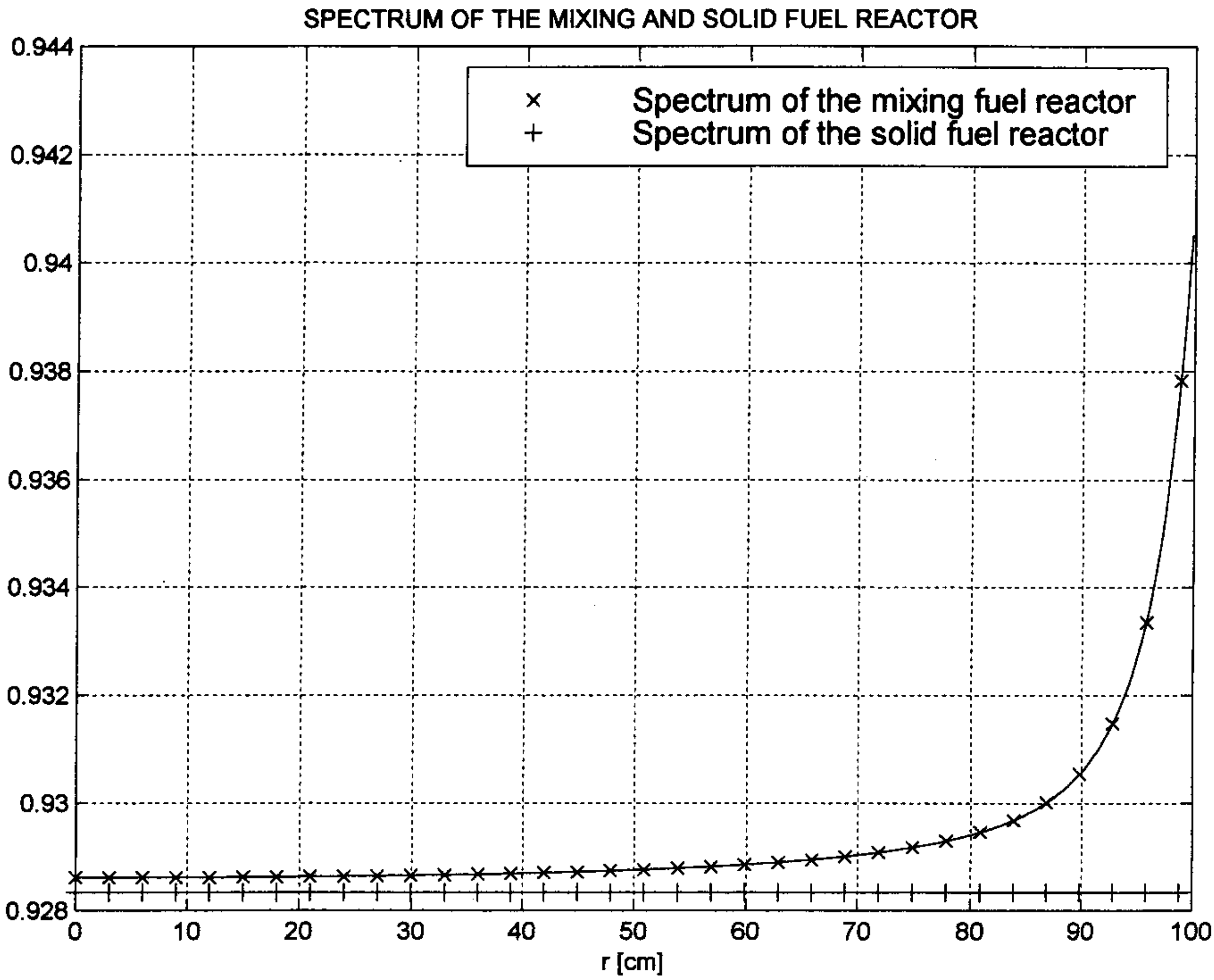

STATIONARY RADIAL FLUXES WHEN FUEL MIXING OCCURS
Fast Flux
Thermal Flux
Neutron Flux [n cm-2 s-1]
r [cm]
3.5
3
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90

$H_2O + UO_2SO_4$

RELATIVE FLUX DIFFERENCES BETWEEN MIXING AND SOLID FUEL REACTORS



$H_2O + UO_2SO_4$



SPECTRUM OF THE MIXING AND SOLID FUEL REACTOR
Spectrum of the mixing fuel reactor
Spectrum of the solid fuel reactor
0.944
0.942
0.94
0.938
0.936
0.934
0.932
0.93
0.928
0
10
20
30
40
50
60
70
80
90
100
r [cm]

$H_2O + UO_2SO_4$

PART IV

A Third Application of the Theory

The Neutron Amplifier Theory

A contribution of:

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Diffusion equation for fast neutrons in the internal core

$$\frac{1}{v_F} \cdot \frac{\partial \Phi_F(r,t)}{\partial t} = D_F \cdot \nabla^2 \Phi_F(r,t) - \Sigma_{r,F} \cdot \Phi_F(r,t)$$

$$+\bar{\nu} \cdot \Sigma_{f,T} \cdot (1 - \bar{\beta}) \cdot \Phi_T(r,t) + \lambda_c \cdot C(t) + \lambda_o \cdot C(r,t) + S_{F;ext}(r,t); \qquad \text{(a)}$$

Diffusion equation for fast neutrons in the reflector

$$\frac{1}{v_F} \cdot \frac{\partial \tilde{\Phi}_F(r,t)}{\partial t} = \tilde{D}_F \cdot \nabla^2 \tilde{\Phi}_F(r,t) - \tilde{\Sigma}_{r,F} \cdot \tilde{\Phi}_F(r,t) + \tilde{S}_{F;ext}(r,t); \qquad \text{(ã)}$$

Diffusion equation for thermal neutrons in the internal core

$$\frac{1}{v_T} \cdot \frac{\partial \Phi_T(r,t)}{\partial t} = D_T \cdot \nabla^2 \Phi_T(r,t) +$$

$$-\Sigma_{a,T} \cdot \Phi_T(r,t) + \Sigma_{F \to T} \cdot \Phi_F(r,t) + S_{T;ext}(r,t); \qquad \text{(b)}$$

Diffusion equation for thermal neutrons in the reflector

$$\frac{1}{v_T} \cdot \frac{\partial \tilde{\Phi}_T(r,t)}{\partial t} = \tilde{D}_T \cdot \nabla^2 \tilde{\Phi}_T(r,t) +$$

$$-\tilde{\Sigma}_{a,T} \cdot \tilde{\Phi}_T(r,t) + \tilde{\Sigma}_{F \to T} \cdot \tilde{\Phi}_F(r,t) + \tilde{S}_{T;ext}(r,t) \qquad (\tilde{\text{b}})$$

Time-evolution of the precursors embedded in the solid phase fuel

$$\frac{\partial C(r,t)}{\partial t} = (1-\xi_c)\cdot\beta_o\cdot\nu_o\cdot\Sigma_{f,T}\cdot\Phi_T(r,t)-\lambda_o\cdot C(r,t); \qquad \text{(c)}$$

Time-evolution of the precursors dilueted in the fluid phase fuel

$$\frac{d\mathbf{C}(t)}{dt} = -\lambda_c\cdot\mathbf{C}(t)-f\cdot[\mathbf{C}(t)-\mathbf{C}(t-\theta)\cdot e^{-\lambda_c\cdot\theta}]+\beta_c\cdot\xi_c\cdot\nu_c\cdot\Sigma_{f,T}\cdot\hat{\varphi}_T(t); \qquad \text{(d)}$$

where:

$$\hat{\varphi}_T(t) \doteq \frac{1}{V}\cdot\int_0^R \Phi_T(r',t)\cdot 4\pi r'^2\cdot dr'$$

Time-evolution of the precursors located in the spectrum converter

$$\frac{dC_s(t)}{dt} = -\lambda_s\cdot C_s(t)+\beta_s\cdot\eta\cdot 4\pi R^2\cdot\left[\zeta_i\cdot\gamma_i\cdot\Phi_T(R^-,t)+\zeta_e\cdot\gamma_e\cdot\tilde{\Phi}_T(R^+,t)\right]. \qquad \text{(e)}$$

Initial conditions

$$\Phi_F(r,t=0)=\Phi_F^0(r),\forall r\in[0,R[\,;\qquad \Phi_T(r,t=0)=\Phi_T^0(r),\forall r\in[0,R[\,;$$

$$\tilde{\Phi}_F(r,t=0)=\tilde{\Phi}_F^0(r),\forall r\in\,]R,\tilde{R}]\,;\qquad \tilde{\Phi}_T(r,t=0)=\tilde{\Phi}_T^0(r),\forall r\in\,]R,\tilde{R}]\,;$$

$$C(r,t=0)=C^0(r),\forall r\in[0,R[\,;\qquad \mathbf{C}(t=0)=\mathbf{C}^0;\;\; C_s(t=0)=C_s^0.$$

Interface and boundary conditions for fluxes

Vanishing of fluxes for $r=\tilde{R}$

$$\lim_{r\to\tilde{R}}\tilde{\Phi}_F(r,t)=\tilde{\Phi}_F(\tilde{R},t)=0,\qquad \forall t\geq 0;$$

$$\lim_{r\to\tilde{R}}\tilde{\Phi}_T(r,t)=\tilde{\Phi}_T(\tilde{R},t)=0,\qquad \forall t\geq 0.$$

Boundary conditions for thermal flux at the two faces of the spectrum converter:

$$\lim_{r \to R^-} \left[-D_T \cdot \frac{\partial \Phi_T(r,t)}{\partial r} - \gamma_i \cdot \Phi_T(r,t) \right] = 0, \qquad \forall t > 0;$$

$$\lim_{r \to R^+} \left[\tilde{D}_T \cdot \frac{\partial \tilde{\Phi}_T(r,t)}{\partial r} - \gamma_e \cdot \tilde{\Phi}_T(r,t) \right] = 0, \qquad \forall t > 0,$$

they are the equivalent of:

$$-D_T \cdot \frac{\partial \Phi_T(r,t)}{\partial r} \Big|_{r=R^-} = \gamma_i \cdot \Phi_T(R^-, t);$$

$$-\tilde{D}_T \cdot \frac{\partial \tilde{\Phi}_T(r,t)}{\partial r} \Big|_{r=R^+} = -\gamma_e \cdot \tilde{\Phi}_T(R^+, t).$$

Matrix formulation of the Amplifier's evolution problem

$$\boxed{\frac{\partial}{\partial t} \left| \Xi(r,t) \right\rangle = O \left| \Xi(r,t) \right\rangle + \left| S(r,t) \right\rangle}$$

Fast flux continuity at the interface $r = R$

$$\lim_{r \to R^-} \Phi_F(r,t) = \lim_{r \to R^+} \tilde{\Phi}_F(r,t) \qquad \forall t > 0.$$

Jump of the fast neutron current at the spectrum converter location $r = R$: it is the equivalent of introducing a singular spherical source on the shell $r = R$.

$$\lim_{r \to R^+} \tilde{J}_F(r,t) - \lim_{r \to R^-} J_F(r,t) =$$

$$= \eta \cdot (1 - \beta_s) \cdot \left[\zeta_i \cdot \gamma_i \cdot \Phi_T(R^-, t) + \zeta_e \cdot \gamma_e \cdot \tilde{\Phi}_T(R^+, t)\right] + \frac{\lambda_s \cdot C_s(t)}{4\pi R^2}, \qquad \forall t >$$

Solving procedure

The method of Laplace's transformation with respect to the time

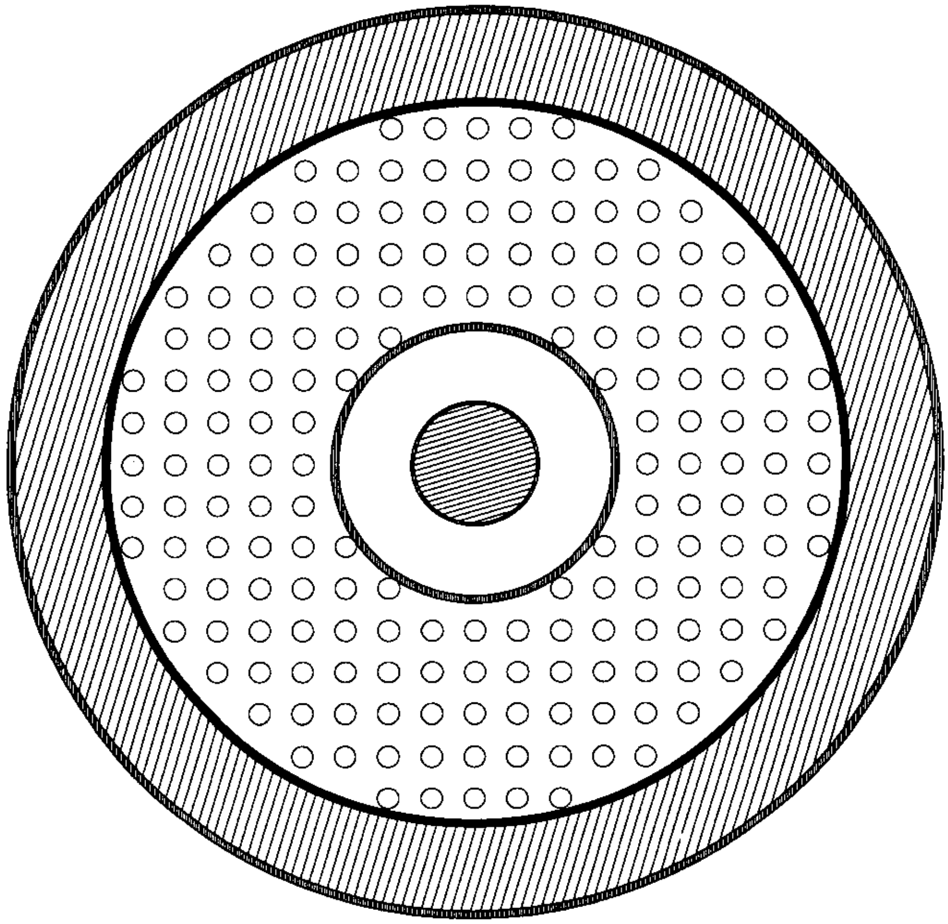
$$
\begin{cases}
\nabla^2\Phi_{F;\mathcal{L}}(r,p) - \dfrac{1}{\Lambda_F^2(p)}\cdot\Phi_{F;\mathcal{L}}(r,p) = \\
= -\dfrac{1}{D_F}\left\{\Sigma_{f,T}\left[\bar{\nu}\cdot\left(1-\bar{\beta}\right)+\dfrac{\lambda_o}{p+\lambda_o}(1-\xi_c)\beta_o\nu_o\right]\cdot\Phi_{T;\mathcal{L}}(r,p)+\right. \\
\left.+G\,(p)\cdot\hat{\varphi}_{T;\mathcal{L}}(p)+S_{F;\mathcal{L}}(r,p)+\dfrac{1}{v_F}\cdot\Phi_F^0(r)+\dfrac{\lambda_o}{p+\lambda_o}C^0(r)+g\,(p)\cdot\mathbf{C}^0\right\} \quad \text{(a)} \\
\\
\nabla^2\Phi_{T;\mathcal{L}}(r,p) - \dfrac{1}{\Lambda_T^2(p)}\cdot\Phi_{T;\mathcal{L}}(r,p) + \dfrac{1}{D_T}\cdot\Sigma_{F\to T}\cdot\Phi_{F;\mathcal{L}}(r,p) = \\
= -\dfrac{1}{D_T}\left\{+S_{T;\mathcal{L}}(r,p)+\dfrac{1}{v_T}\cdot\Phi_T^0(r)\right\}. \quad \text{(b)}
\end{cases}
$$

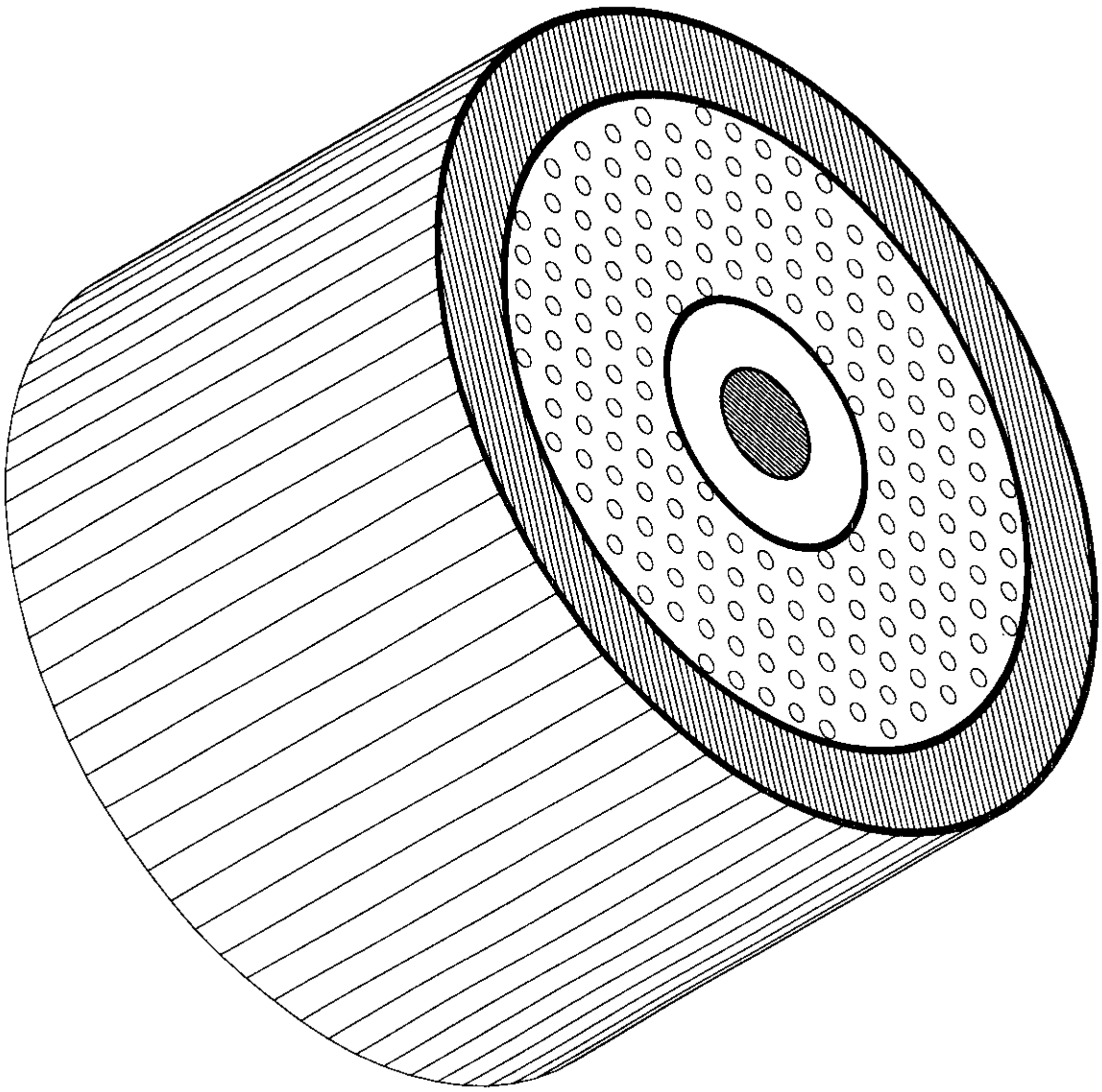
The formal solution by means of kinetic pseudopotentials

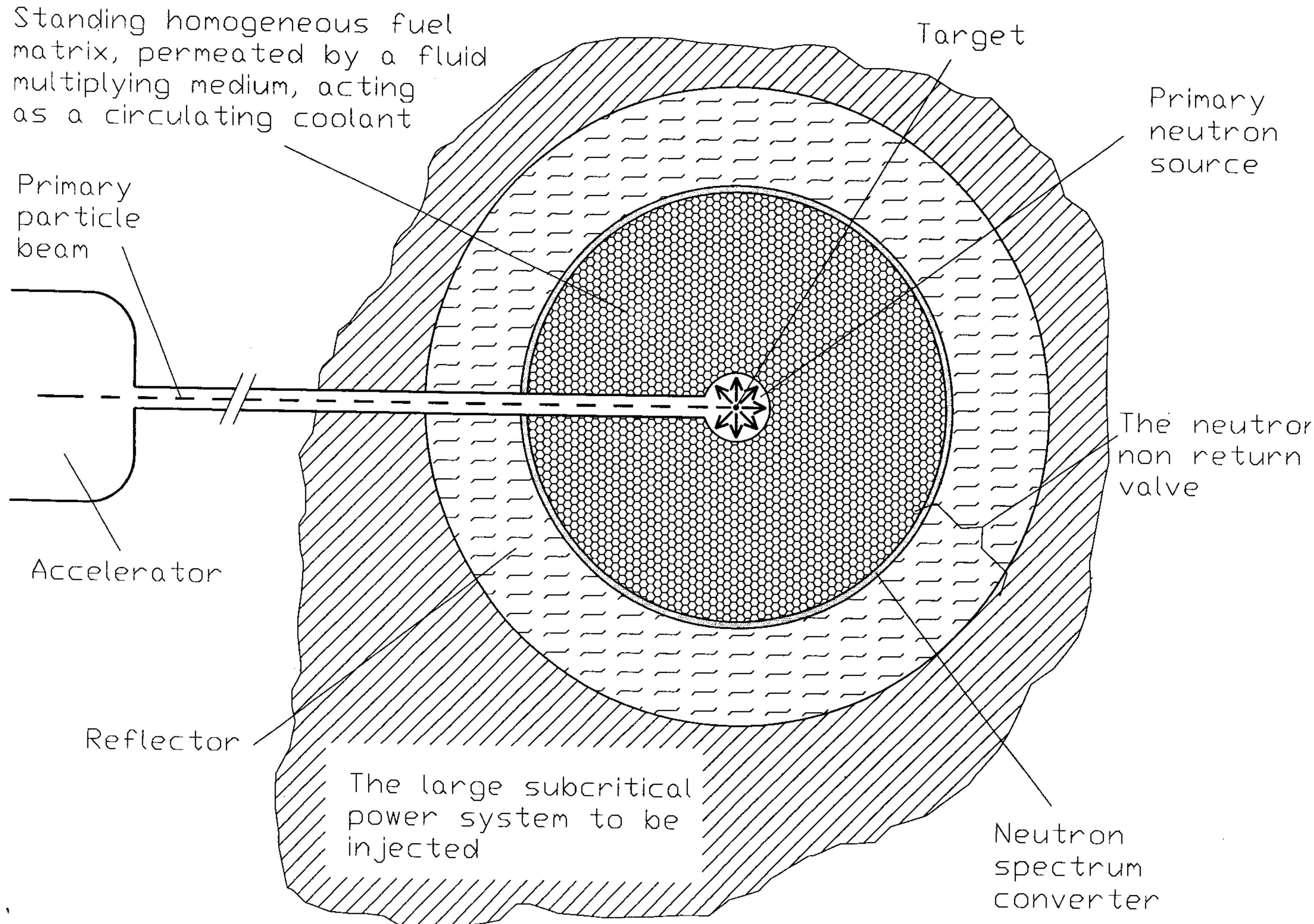
The requirement for self-consistency

The Super-Inhour Equation of the unidirectional neutron amplifier

The physical interpretation of the results






Standing homogeneous fuel matrix, permeated by a fluid multiplying medium, acting as a circulating coolant
Target
Primary neutron source
Primary particle beam
The neutron non return valve
Accelerator
Reflector
The large subcritical power system to be injected
Neutron spectrum converter