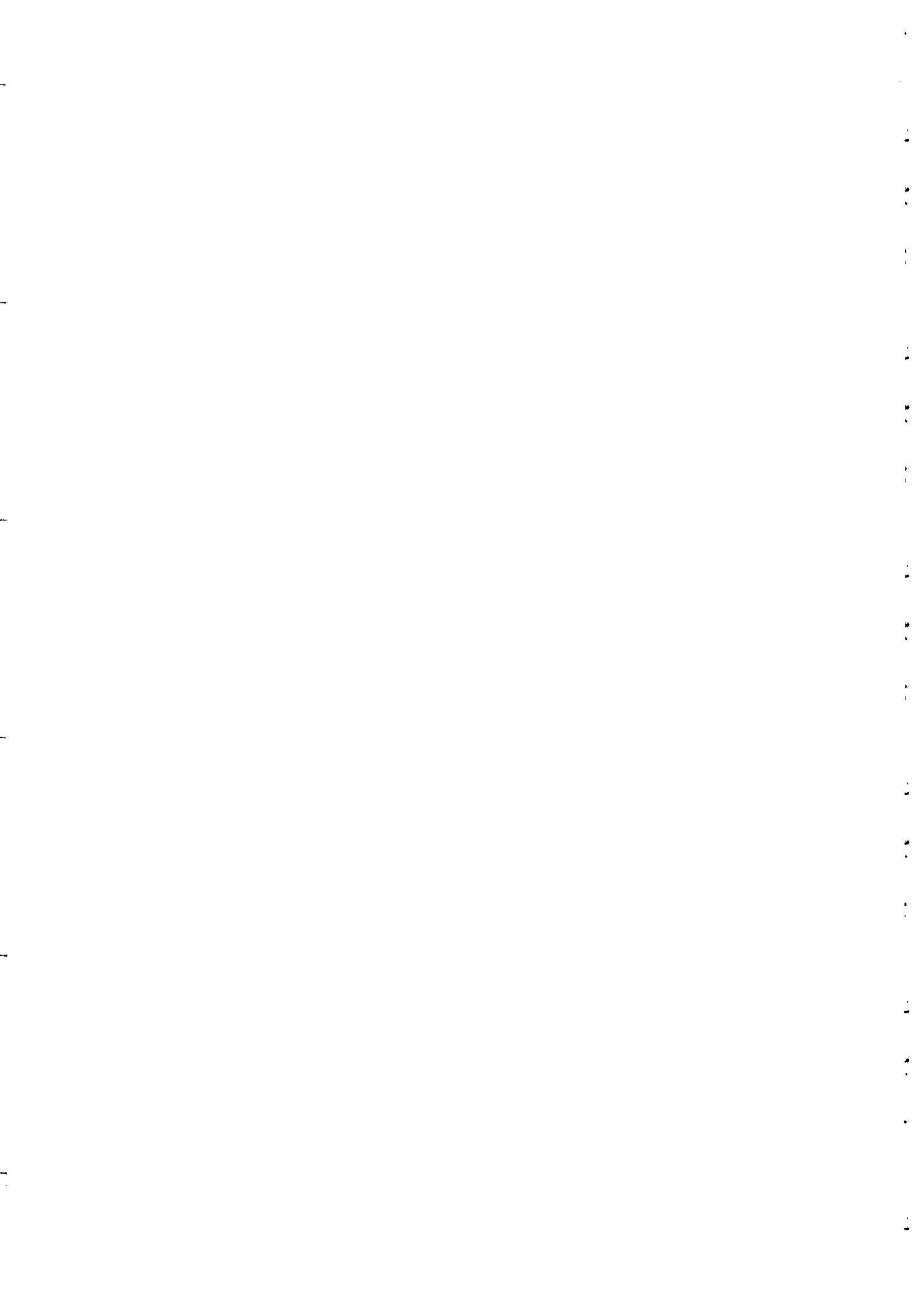# SPRING WORKSHOP ON SUPERSTRINGS AND RELATED MATTERS

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LECTURES ON NON-BPS DIRICHLET BRANES

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Lectures on Non-BPS Dirichlet branes[*]

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Abstract

A comprehensive introduction to the boundary state approach to Dirichlet branes is given. Various examples of BPS and non-BPS Dirichlet branes are discussed. In particular, many details of the non-BPS states in the duality of Type IIA on K3 and the heterotic string on T^4 are worked out.

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1   Introduction

The past few years have seen a tremendous increase in our understanding of the dynamics of superstring theory. In particular it has become apparent that the five ten-dimensional theories, together with an eleven-dimensional theory (M-theory), are different limits in moduli space of some unifying description. A crucial ingredient in understanding the relation between the different perturbative descriptions has been the realisation that the *solitonic* objects that define the relevant degrees of freedom at strong coupling are *Dirichletbranes* that have an alternative description in terms of open string theory [1, 2, 3].

The D-branes that were first analysed were BPS states that break half the (spacetime) supersymmetry. It has now been realised, however, that because of their description in terms of open strings, D-branes can be constructed and analysed in much more general situations. In fact, D-branes are essentially described by a boundary conformal field theory [4, 5, 6, 7, 8, 9] (see also [10, 11, 12, 13, 14, 15] for earlier work in this direction), the consistency conditions of which are not related to spacetime supersymmetry [16, 17, 18]. In an independent development, D-branes that break supersymmetry have been constructed in terms of bound states of branes and anti-branes by Sen [19, 20, 21, 22, 23, 24] (see also [25] for a good review). This beautiful construction has been interpreted in terms of K-theory by Witten [26], and this has opened the way for a more mathematical treatment of D-branes [27, 28, 29]. It has also led the way to new insights into the nature of the instability that is described by the open string tachyon [30].

The motivation for studying D-branes that do not preserve spacetime supersymmetry (and that are therefore sometimes called non-BPS D-branes) is at least four-fold. First, in order to understand the strong/weak coupling dualities of supersymmetric string theories in more detail, it is important to analyse how these dualities act on states that are not BPS saturated. After all, the behaviour of the BPS states at arbitrary coupling is essentially determined by spacetime supersymmetry (provided that it remains unbroken for all values of the coupling constant), and thus one is not really probing the underlying string theory unless one also understands how non-BPS states behave at strong coupling. The dualities typically map perturbative states to non-perturbative (D-brane type) states, and thus one will naturally encounter non-BPS D-branes in these considerations.

The second motivation is related to the question of whether string duality should intrinsically only apply to supersymmetric string theories, or whether also non-supersymmetric theories should be related by duality. This is certainly, *a priori*, an open question[1]: it is conceivable that spacetime supersymmetry is a crucial ingredient without which there is no reason to believe that these dualities should exist, but it is also conceivable that spacetime supersymmetry is just a convenient tool that allows one to use sophisticated arguments and techniques to verify conjectures that are otherwise difficult to check. Dirichlet branes play a central rôle in the understanding of string dualities, and if one wants to make progress on this question, it is important to develop techniques to analyse and describe Dirichlet branes without reference to spacetime supersymmetry.

Thirdly, one of the interesting implications of the Maldacena conjecture [37] is that one can obtain non-trivial predictions about field theory from string theory. In the original

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[1]Recently, some suggestive proposals have however been made [16, 31, 32, 33, 34, 35, 36].

formulation this was applied to supersymmetric string and field theories, but it is very tempting to believe that similar insights may be gained for non-supersymmetric theories. This line of thought has been developed recently, starting with a series of papers by Klebanov & Tseytlin [18].

Finally, non-BPS D-branes offer the intriguing possibility of string compactifications in which supersymmetry is preserved in the bulk but broken on the brane. Various models involving branes and anti-branes have been constructed recently [38, 39, 40, 41], but it is presumably also possible to construct interesting models involving non-BPS D-branes. The fact that at specific points in the moduli space their spectrum is bose-fermi degenerate may be of significance in this context [42].

The main aim of these lectures is to explain the boundary state approach to D-branes, and to give some applications of it, in particular to the construction of non-BPS D-branes. The structure of the lectures is as follows. In section 2 we explain carefully the underpinnings of the boundary state approach and apply it to the simplest case, Type IIA/IIB and Type 0A/0B. In section 3 we use the techniques that we have developed to construct one of the simplest non-BPS D-branes — the non-BPS D-particle of the orbifold of Type IIB by $(-1)^{F_L}\mathcal{I}_4$ — in detail. If we compactify this theory on a 4-torus, it is T-dual to Type IIA at the orbifold point of K3, which in turn is S-dual to the heterotic string on T^4. This connection (and in particular the various non-BPS states in this duality) are analysed in detail in section 4.

2 The boundary state approach

Suppose we are given a closed string theory. We can ask the question whether it is consistent to add to this theory additional open string sectors in such a way that the resulting open- and closed theory is consistent. The different open string sectors that we can add are characterised by the *boundary conditions* that we impose on the end-points of the open strings. Conventional open strings have *Neumann* boundary conditions at either end; if we denote by $X^\mu(t,s)$ the coordinate field, where $t \in \mathbb{R}$ and $s \in [0,\pi]$ are the time and space coordinates on the world-sheet of the open string, then this is the condition that

$$\partial_s X^\mu(t,0) = \partial_s X^\mu(t,\pi) = 0 \,. \tag{2.1}$$

We can also consider open strings whose boundary condition at one or both ends is of *Dirichlet* type, *i.e.*

$$X^\nu(t,0) = a^\nu \,, \tag{2.2}$$

where a^ν is a constant. Finally, we can consider open strings that satisfy Neumann boundary conditions for some of the X^μ, and Dirichlet boundary conditions for the others

$$\begin{aligned} \partial_s X^\mu(t,0) &= 0 & \mu &= 0,\ldots,p \\ X^\nu(t,0) &= a^\nu & \nu &= p+1,\ldots,9 \,. \end{aligned} \tag{2.3}$$

The endpoint of such an open string is then constrained to lie on a submanifold (a hyperplane of dimension $p + 1$), whose position in the ambient space is described by a^ν;

3

this submanifold is then called the Dirichlet p-brane or Dp-brane for short. The different boundary conditions of the open string are in one-to-one correspondence with the different D-branes. We can therefore rephrase the above question as the question of which D-branes can be consistently defined in a given closed string theory.

The idea of the boundary state approach to D-branes is to represent a D-brane as a coherent (boundary) state of the underlying closed string theory. The key ingredient in this approach is *world-sheet duality* that allows one to rewrite the above conditions (that are defined in terms of the coordinate function of the open string) in terms of the coordinate function of the closed string. At first, the coordinate functions of the open and the closed string theory are not related at all: the world-sheet of the open string is an infinite strip, whilst the world-sheet of the closed string has the topology of a cylinder. In particular, the closed string world-sheet is parametrised by τ and σ, where $\tau \in \mathbb{R}$ is the time variable, and σ is a *periodic* space-variable $\sigma \in [0, 2\pi]$ (where $\sigma = 0$ is identified with $\sigma = 2\pi$).

Suppose now that we consider an open string that has definite boundary conditions at either end (and can therefore be thought of as stretching between two not necessarily different D-branes). If we determine the 1-loop partition function of this open string, we have to identify the time coordinate periodically (and integrate over all periodicities). The open-string world-sheet has then the topology of a cylinder, where the periodic variable is t, and s takes values in a finite interval (from $s = 0$ to $s = \pi$). Because of world-sheet duality, we can re-interpret this world-sheet as being a closed string world-sheet if we identify t with σ (up to normalisation) and s with τ; from the point of view of the closed string, the diagram then corresponds to a tree-diagram between two external states.
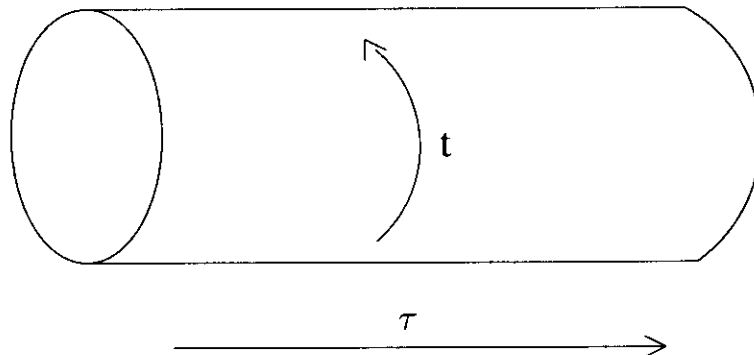

Figure 1: World-sheet duality

Furthermore, the boundary condition on the ends of the open string become conditions that must be satisfied by the external states; since we exchange (t, s) with (σ, τ) these conditions then become

$$
\begin{aligned}
\partial_\tau X^\mu(\sigma, 0)|Dp\rangle &= 0 & \mu &= 0, \ldots, p \\
X^\nu(\sigma, 0)|Dp\rangle &= a^\nu |Dp\rangle & \nu &= p + 1, \ldots, 9,
\end{aligned}
\tag{2.4}
$$

and a similar relation for $s = \tau = \pi$. Here we have assumed that the boundary condition at $s = \tau = 0$ corresponds to those of a Dp-brane.

It is useful to rewrite these conditions in terms of the modes of the closed string theory. To this end, let us recall that the coordinate field in the closed string theory can be written as

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma), \qquad (2.5)$$

where in terms of modes,

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau + \sigma) + \frac{i}{2}\sum_{n \neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau+\sigma)} \qquad (2.6)$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau - \sigma) + \frac{i}{2}\sum_{n \neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}. \qquad (2.7)$$

These modes satisfy the commutation relations

$$\begin{aligned}
\left[\alpha_m^\mu, \alpha_n^\nu\right] &= m\,\delta^{\mu\nu}\delta_{m,-n} \\
\left[\alpha_m^\mu, \tilde{\alpha}_n^\nu\right] &= 0 \\
\left[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\right] &= m\,\delta^{\mu\nu}\delta_{m,-n}\,.
\end{aligned} \qquad (2.8)$$

In terms of modes the conditions (2.4) then become

$$\begin{aligned}
p^\mu|Dp\rangle &= 0 & \mu &= 0,\ldots,p \\
\left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu\right)|Dp\rangle &= 0 & \mu &= 0,\ldots,p \\
\left(\alpha_n^\nu - \tilde{\alpha}_{-n}^\nu\right)|Dp\rangle &= 0 & \nu &= p+1,\ldots,9 \\
x^\nu|Dp\rangle &= a^\nu|Dp\rangle & \nu &= p+1,\ldots,9\,.
\end{aligned} \qquad (2.9)$$

The boundary conditions for the fermions are more difficult to establish. Ultimately they are determined by the condition that the closed string tree diagram reproduces, upon world-sheet duality, the open string loop diagram (see (2.31) – (2.34) below). In order to formulate the appropriate condition, it is necessary to introduce an additional parameter η (that corresponds to the different spin structures), and the relevant equations are then

$$\begin{aligned}
\left(\psi_r^\mu + i\eta\tilde{\psi}_{-r}^\mu\right)|Bp,\eta\rangle &= 0 & \mu &= 0,\ldots,p \\
\left(\psi_r^\nu - i\eta\tilde{\psi}_{-r}^\nu\right)|Bp,\eta\rangle &= 0 & \nu &= p+1,\ldots,9\,.
\end{aligned} \qquad (2.10)$$

The actual D-brane state $|Dp\rangle$ is then a linear combination of the boundary states $|Bp,\eta\rangle$ with $\eta = \pm$. It is also worth pointing out that the equations can be solved separately for the different closed string sectors of the theory (i.e. the NS-NS and the R-R sector, as well as the corresponding twisted sectors if we are dealing with an orbifold theory). We shall therefore, in the following, usually denote by $|Bp,\eta\rangle$ the solution in a specific sector of the theory; the D-brane state is then a certain linear combination of the boundary states in the different sectors and with $\eta = \pm$.

In the following we shall always work in the NS-R formalism; we shall also, for simplicity, work in light-cone gauge, and we shall always choose the two light-cone directions to be $\mu = 0, 1$.[2] The boundary conditions in both light-cone directions will be taken

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[2]For a good introduction to the covariant approach see the lecture notes by Di Vecchia and Liccardo [43].

to be Dirichlet; the boundary states we describe are therefore really D-instantons ($i.e.$ they satisfy a Dirichlet boundary condition in the time direction). However, by means of a double Wick rotation, these states can be transformed into states whose boundary conditions are specified as above [7]. In these conventions we necessarily restrict ourselves to D-branes with at least two Dirichlet directions; thus we can only describe Dp-branes with $-1 \le p \le 7$. Also, since the two light-cone directions are always Dirichlet, only $p-2$ of the transverse directions satisfy a Dirichlet boundary condition in order for the state to describe the Wick rotate of a Dp-brane; thus in these conventions the boundary states that combine to define a Dp-brane are characterised by the following conditions

$$
\begin{array}{llll}
p^\mu |Bp, \eta\rangle &=& 0 & \mu = 2, \ldots, p+2 \\
\left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu\right) |Bp, \eta\rangle &=& 0 & \mu = 2, \ldots, p+2 \\
\left(\alpha_n^\nu - \tilde{\alpha}_{-n}^\nu\right) |Bp, \eta\rangle &=& 0 & \nu = p+3, \ldots, 9 \\
x^\nu |Bp, \eta\rangle &=& a^\nu |Bp, \eta\rangle & \nu = 0, 1, p+3, \ldots, 9 \\
\left(\psi_r^\mu + i\eta \tilde{\psi}_{-r}^\mu\right) |Bp, \eta\rangle &=& 0 & \mu = 2, \ldots, p+2 \\
\left(\psi_r^\nu - i\eta \tilde{\psi}_{-r}^\nu\right) |Bp, \eta\rangle &=& 0 & \nu = p+3, \ldots, 9.
\end{array}
\tag{2.11}
$$

It is actually not difficult to describe the solution to these equations. In each (left-right-symmetric) sector of the theory, and for each choice of η, the unique solution is of the form

$$
|Bp, \mathbf{a}, \eta\rangle = \mathcal{N} \int \prod_{\nu = 0, 1, p+3, \ldots, 9} dk^\nu e^{ik^\nu a^\nu} |\widehat{Bp, \mathbf{k}}, \eta\rangle,
\tag{2.12}
$$

where $\mathcal{N}$ is a normalisation constant that will be determined further below, and $|\widehat{Bp, \mathbf{k}}, \eta\rangle$ is the coherent state

$$
\begin{aligned}
|\widehat{Bp, \mathbf{k}}, \eta\rangle &= \exp\left\{ \sum_{n>0} \left[-\frac{1}{n} \sum_{\mu=2}^{p+2} \alpha_{-n}^\mu \tilde{\alpha}_{-n}^\mu + \frac{1}{n} \sum_{\mu=p+3}^{9} \alpha_{-n}^\mu \tilde{\alpha}_{-n}^\mu \right] \right. \\
&\quad \left. + i\eta \sum_{r>0} \left[-\sum_{\mu=2}^{p+2} \psi_{-r}^\mu \tilde{\psi}_{-r}^\mu + \sum_{\mu=p+3}^{9} \psi_{-r}^\mu \tilde{\psi}_{-r}^\mu \right] \right\} |Bp, \mathbf{k}, \eta\rangle^{(0)}.
\end{aligned}
\tag{2.13}
$$

The ground state is a momentum eigenstate with eigenvalue $\mathbf{k}$, where $k^\mu = 0$ for $\mu = 2, \ldots, p+2$; in the NS-NS sector, it is the unique tachyonic ground state, whereas in the R-R sector, it is determined by the condition (2.10) with $r = 0$, $i.e.$

$$
\begin{array}{llll}
\left(\psi_0^\mu + i\eta \tilde{\psi}_0^\mu\right) |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} &=& 0 & \mu = 2, \ldots, p+2 \\
\left(\psi_0^\nu - i\eta \tilde{\psi}_0^\nu\right) |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} &=& 0 & \nu = p+3, \ldots, 9.
\end{array}
\tag{2.14}
$$

If the theory under consideration is an orbifold theory (such as the theory we shall discuss later), there are also similar boundary states in the corresponding twisted sectors. The actual D-brane state is then a certain linear combination of these states in the different sectors of the theory and for both values of η; it is characterised by three properties [16, 32]:

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(i) The boundary state is a physical state of the closed string theory, *i.e.* it is GSO-invariant, and invariant under orbifold and orientifold projections where appropriate.[3]

(ii) The open string amplitude obtained by world-sheet duality from the closed string exchange between any two D-branes constitutes an *open string partition function*, *i.e.* it corresponds to a trace over a set of open string states of the open string time-evolution operator.

(iii) The open strings that are introduced in this way have *consistent string field interactions* with the original closed strings.

One is usually also interested in D-branes that are *stable*; a necessary condition for this is that the spectrum of open strings that begin and end on the *same* D-brane is free of tachyons. If the underlying theory is supersymmetric, one may sometimes also want to impose the condition that the D-branes preserve some part of the supersymmetry, and that they are therefore *BPS saturated*; this requires that the spectrum of open strings beginning and ending on the D-brane is supersymmetric. However, there exist interesting D-branes in supersymmetric theories that are stable but not BPS [20, 21, 17, 26, 44, 45, 25, 46, 47, 48]; some examples of these will be described later.

The first condition is usually relatively easy to check, although it requires care in all sectors that have fermionic zero modes. (We shall describe the relevant subtleties in some detail for the case of Type IIA and Type IIB in subsection 2.2.) The second condition is in essence equivalent to the statement that world-sheet duality holds. It is a very powerful constraint that determines the normalisations of the different boundary states (as we shall show in the next subsection). This condition can be formulated in terms of the conformal field theory and is sometimes referred to as Cardy's condition (see also subsection 2.4). The third condition is very difficult to check in detail; as far as I am aware, there is only one theory (namely the two Type 0A and 0B theories) for which it seems to imply constraints that go beyond (i) and (ii).

It should be stressed that the above conditions are *intrinsic consistency conditions* of an interacting string (field) theory; in particular, they are more fundamental than spacetime supersymmetry, and also apply in cases where spacetime supersymmetry is broken or absent.

2.1 World-sheet duality

Before describing some examples in detail, it is useful to illustrate the condition of world-sheet duality more quantitatively (since the same calculation will be needed for essentially all models). The closed string tree diagram that is represented in Figure 1 is described by

$$\int_0^\infty dl \, \langle Dq | e^{-lH_c} | Dp \rangle \,, \tag{2.15}$$

---

[3]Since we are working in light-cone gauge, we do not have to impose any other constraints.

where H_c is the closed string Hamiltonian in light cone gauge,

$$H_c = \pi \mathbf{k}^2 + 2\pi \sum_{\mu=2,\ldots,9} \left[\sum_{n=1}^{\infty} (\alpha_{-n}^{\mu} \alpha_n^{\mu} + \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_n^{\mu}) + \sum_{r>0} r(\psi_{-r}^{\mu} \psi_r^{\mu} + \tilde{\psi}_{-r}^{\mu} \tilde{\psi}_r^{\mu}) \right] + 2\pi C_c . \quad (2.16)$$

The constant C_c takes the value -1 in the NS-NS, and 0 in the R-R sector. Under the substitution $t = 1/2l$, this integral should become the open string one-loop amplitude that is given by

$$\int_0^{\infty} \frac{dt}{2t} \text{Tr}(e^{-2tH_o} \mathcal{P}) , \quad (2.17)$$

where $\mathcal{P}$ is an appropriate projection operator, and H_o is the open string Hamiltonian given as

$$H_o = \pi \vec{p}^2 + \frac{1}{4\pi} \vec{w}^2 + \pi \sum_{\mu=2,\ldots,9} [\sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_n^{\mu} + \sum_{r>0} r\psi_{-r}^{\mu} \psi_r^{\mu}] + \pi C_o . \quad (2.18)$$

Here $\vec{p}$ denotes the open string momentum along the directions for which the string has Neumann (N) boundary conditions at both ends, $\vec{w}$ is the difference between the two end-points of the open string, and α_n^{μ} and ψ_r^{μ} are the bosonic and fermionic open string oscillators, respectively; they satisfy the commutation relations

$$[\alpha_m^{\mu}, \alpha_n^{\nu}] = m \, \delta^{\mu\nu} \delta_{m,-n}, \qquad \{\psi_r^{\mu}, \psi_s^{\nu}\} = \delta^{\mu\nu} \delta_{r,-s} . \quad (2.19)$$

For coordinates satisfying the same boundary condition at both ends of the open string (i.e. both Neumann (N) or both Dirichlet (D)) n always takes integer values, whereas r takes integer (integer $+ \frac{1}{2}$) values in the R (NS) sector. On the other hand, for coordinates satisfying different boundary conditions at the two ends of the open string (one D and one N) n takes integer$+\frac{1}{2}$ values and r takes integer $+\frac{1}{2}$ (integer) values in the R (NS) sector. The normal ordering constant C_o vanishes in the R-sector and is equal to $-\frac{1}{2} + \frac{s}{8}$ in the NS sector (in $\alpha' = 1$ units) where s denotes the number of coordinates satisfying D-N boundary conditions. The trace, denoted by Tr, is taken over the full Fock space of the open string, and also includes an integral over the various momenta.

The calculation (2.15) can be performed separately for the different boundary states in the different components since the overlap between states from different sectors vanishes. For definiteness let us consider one specific example in some detail, the tree exchange between two Dp-brane boundary states in the NS-NS sector. (The result for the other sectors will be given below.) Thus we want to consider the amplitude

$$\int_0^{\infty} dl \, \langle Bp, \mathbf{a_1}, \eta | e^{-lH_c} | Bp, \mathbf{a_2}, \eta \rangle_{\text{NS-NS}} , \quad (2.20)$$

where $|Bp, \mathbf{a}, \eta\rangle_{\text{NS-NS}}$ is the coherent state in the NS-NS sector given in (2.12). The momentum integral gives a Gaussian integral that can be performed, and the amplitude becomes[4]

$$\mathcal{N}_{\text{NS-NS}}^2 \int_0^{\infty} dl \, l^{-\frac{9-p}{2}} e^{-\frac{(\mathbf{a_1}-\mathbf{a_2})^2}{4\pi l}} \langle \widehat{Bp, 0}, \eta | e^{-lH_c} | \widehat{Bp, 0}, \eta \rangle_{\text{NS-NS}} . \quad (2.21)$$

---

[4]It is convenient to choose the convention that the amplitude is bilinear so that the prefactor is $\mathcal{N}^2$ rather than $\mathcal{N}\bar{\mathcal{N}}$. On momentum eigenstates the amplitude then satisfies $\langle k_1 | k_2 \rangle = \delta(k_1 - k_2)$.

In order to determine the overlap between the two coherent states, we observe that the states of the form

$$\prod_i \frac{1}{l_i!} \left(\frac{1}{n_i} \alpha^{\mu_i}_{-n_i} \tilde{\alpha}^{\mu_i}_{-n_i} \right)^{l_i} |0\rangle , \tag{2.22}$$

where $n_i \geq n_{i+1}$ and if $n_i = n_{i+1}$ then $\mu_i < \mu_{i+1}$, form an *orthonormal basis* for the space generated by the modes $\alpha^\mu_n \tilde{\alpha}^\mu_n$ and similarly for

$$\prod_i \frac{1}{l_i!} \left(i\eta \psi^{\mu_i}_{-r_i} \tilde{\psi}^{\mu_i}_{-r_i} \right)^{l_i} |0\rangle . \tag{2.23}$$

Here the bilinear inner product is defined by the relation

$$(\alpha^\mu_n \phi, \chi) = -(\phi, \alpha^\mu_{-n}\chi), \qquad (\psi^\mu_n \phi, \chi) = i(\phi, \psi^\mu_{-n}\chi), \tag{2.24}$$

and similarly for $\tilde{\alpha}^\mu_n$ and $\tilde{\psi}^\mu_r$ together with the normalisation

$$\left(|0\rangle, |0\rangle \right) = 1 . \tag{2.25}$$

It is then easy to see that the above amplitude becomes

$$\mathcal{N}^2_{\text{NS-NS}} \int_0^\infty dl \; l^{-\frac{9-p}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2}{4\pi l}} \frac{f_3^8(q)}{f_1^8(q)} , \tag{2.26}$$

where $q = e^{-2\pi l}$, and the functions f_i are defined as in [4]

$$f_1(q) = q^{\frac{1}{12}} \prod_{n=1}^\infty (1 - q^{2n}) ,$$

$$f_2(q) = \sqrt{2} q^{\frac{1}{12}} \prod_{n=1}^\infty (1 + q^{2n}) ,$$

$$f_3(q) = q^{-\frac{1}{24}} \prod_{n=1}^\infty (1 + q^{2n-1}) ,$$

$$f_4(q) = q^{-\frac{1}{24}} \prod_{n=1}^\infty (1 - q^{2n-1}) . \tag{2.27}$$

Next we substitute $t = 1/2l$, and using the transformation properties of the f_i functions,

$$\begin{array}{llll}
f_1(e^{-\pi/t}) &=& \sqrt{t} f_1(e^{-\pi t}), & f_2(e^{-\pi/t}) &=& f_4(e^{-\pi t}), \\
f_3(e^{-\pi/t}) &=& f_3(e^{-\pi t}), & f_4(e^{-\pi/t}) &=& f_2(e^{-\pi t}),
\end{array} \tag{2.28}$$

the above integral becomes

$$\mathcal{N}^2_{\text{NS-NS}} 2^{\frac{9-p}{2}} \int_0^\infty \frac{dt}{2t} t^{-\frac{(p+1)}{2}} e^{-\frac{(\mathbf{a}_1 - \mathbf{a}_2)^2}{2\pi} t} \frac{f_3^8(\tilde{q})}{f_1^8(\tilde{q})} , \tag{2.29}$$

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where $\tilde{q} = e^{-\pi t}$. This is to be compared with the open string one-loop amplitude

$$\int_0^\infty \frac{dt}{2t} \mathrm{Tr}_{\mathrm{NS}}(e^{-2tH_o}) = \frac{V_{p+1}}{(2\pi)^{p+1}} \int_0^\infty \frac{dt}{2t}(2t)^{-\frac{(p+1)}{2}} e^{-\frac{(a_1-a_2)^2}{2\pi}t} \frac{f_3^8(\tilde{q})}{f_1^8(\tilde{q})}, \qquad (2.30)$$

where V_{p+1} is the world-volume of the brane, which together with the factor of $(2t)^{-\frac{(p+1)}{2}}$ comes from the momentum integration. Thus we find that

$$\int dl \, \langle Bp, \eta | \, e^{-lH_{closed}} \, | Bp, \eta \rangle_{\mathrm{NS\text{-}NS}} = \mathcal{N}_{\mathrm{NS\text{-}NS}}^2 \frac{32(2\pi)^{p+1}}{V_{p+1}} \int \frac{dt}{2t} \mathrm{Tr}_{\mathrm{NS}}\left[e^{-tH_o}\right]. \qquad (2.31)$$

Similarly we find

$$\int dl \, \langle Bp, \eta | \, e^{-lH_{closed}} \, | Bp, -\eta \rangle_{\mathrm{NS\text{-}NS}} = \mathcal{N}_{\mathrm{NS\text{-}NS}}^2 \frac{32(2\pi)^{p+1}}{V_{p+1}} \int \frac{dt}{2t} \mathrm{Tr}_{\mathrm{R}}\left[e^{-tH_o}\right], \qquad (2.32)$$

$$\int dl \, \langle Bp, \eta | \, e^{-lH_{closed}} \, | Bp, \eta \rangle_{\mathrm{R\text{-}R}} = -\frac{\mathcal{N}_{\mathrm{R\text{-}R}}^2}{16} \frac{32(2\pi)^{p+1}}{V_{p+1}} \int \frac{dt}{2t} \mathrm{Tr}_{\mathrm{NS}}\left[(-1)^F e^{-tH_o}\right], \qquad (2.33)$$

and

$$\int dl \, \langle Bp, \eta | \, e^{-lH_{closed}} \, | Bp, -\eta \rangle_{\mathrm{R\text{-}R}} = 0 = -\frac{\mathcal{N}_{\mathrm{R\text{-}R}}^2}{16} \frac{32(2\pi)^{p+1}}{V_{p+1}} \int \frac{dt}{2t} \mathrm{Tr}_{\mathrm{R}}\left[(-1)^F e^{-tH_o}\right]. \qquad (2.34)$$

We learn from this that we can satisfy world-sheet duality provided we include appropriate combinations of boundary states and choose their normalisations correctly. We have now assembled the necessary ingredients to work out some examples in detail.

2.2 A first example: Type IIA and IIB

Let us first consider the familiar case of the Type IIA and Type IIB theories. The spectra of these theories is given by

$$\begin{aligned}
\mathrm{IIA}: \quad & (\mathrm{NS+}, \mathrm{NS+}) \oplus (\mathrm{R+}, \mathrm{R-}) \oplus (\mathrm{NS+}, \mathrm{R-}) \oplus (\mathrm{R+}, \mathrm{NS+}) \\
\mathrm{IIB}: \quad & (\mathrm{NS+}, \mathrm{NS+}) \oplus (\mathrm{R+}, \mathrm{R+}) \oplus (\mathrm{NS+}, \mathrm{R+}) \oplus (\mathrm{R+}, \mathrm{NS+}),
\end{aligned} \qquad (2.35)$$

where the signs refer to the eigenvalues of $(-1)^F$ and $(-1)^{\tilde{F}}$, respectively. In particular, the NS-NS sector is the same for the two theories, and consists of those states for which both $(-1)^F$ and $(-1)^{\tilde{F}}$ have eigenvalue $+1$. Given that the tachyonic ground state has eigenvalue -1 under both $(-1)^F$ and $(-1)^{\tilde{F}}$, the boundary state given by (2.12) and (2.13) transforms as

$$\begin{aligned}
(-1)^F |Bp, \mathbf{a}, \eta \rangle &= -|Bp, \mathbf{a}, -\eta \rangle \\
(-1)^{\tilde{F}} |Bp, \mathbf{a}, \eta \rangle &= -|Bp, \mathbf{a}, -\eta \rangle.
\end{aligned}$$

Thus

$$|Bp, \mathbf{a} \rangle_{\mathrm{NS\text{-}NS}} = (|Bp, \mathbf{a}, + \rangle_{\mathrm{NS\text{-}NS}} - |Bp, \mathbf{a}, - \rangle_{\mathrm{NS\text{-}NS}}) \qquad (2.36)$$

is a GSO-invariant state for all p. It follows from (2.31) and (2.32) that this state does not describe a *stable* D-brane by itself since the open string that begins and ends on $|Bp, \mathrm{a}\rangle_{\text{NS-NS}}$ consists of an unprojected NS and R sector, and therefore contains a tachyon in its spectrum. In fact (2.36) with

$$\mathcal{N}^2_{\text{NS-NS}} = \frac{1}{64} \frac{V_{p+1}}{(2\pi)^{p+1}} \tag{2.37}$$

describes the unstable Dp-brane for p odd (even) in Type IIA (IIB) that was considered by Sen in his construction of non-BPS D-branes [24, 25]; the unstable D9-brane of Type IIA was also used by Hořava in his discussion of the K-theory of Type IIA [27].

In order to obtain a stable D-brane, we have to add to (2.36) a boundary state in the R-R sector; since the R-R sector involves fermionic zero modes, the discussion of GSO-invariance is somewhat delicate, and we need to introduce a little bit of notation. Let us define the modes

$$\psi^\mu_\pm = \frac{1}{\sqrt{2}} \left(\psi^\mu_0 \pm i\tilde{\psi}^\mu_0 \right) , \tag{2.38}$$

which satisfy the anti-commutation relations

$$\{\psi^\mu_\pm, \psi^\nu_\pm\} = 0 , \qquad \{\psi^\mu_+, \psi^\nu_-\} = \delta^{\mu\nu} , \tag{2.39}$$

as follows from the fact that both the left- and right-moving fermion modes satisfy the Clifford algebras,

$$\begin{aligned}
\{\psi^\mu_r, \psi^\nu_s\} &= \delta^{\mu\nu}\delta_{r,-s} \\
\{\psi^\mu_r, \tilde{\psi}^\nu_s\} &= 0 \\
\{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} &= \delta^{\mu\nu}\delta_{r,-s} .
\end{aligned} \tag{2.40}$$

In terms of $\psi^\mu_\pm$ (2.14) can be rewritten as

$$\begin{aligned}
\psi^\mu_\eta |Bp, \mathbf{k}, \eta\rangle^{(0)}_{\text{R-R}} &= 0 & \mu &= 2, \ldots, p+2 \\
\psi^\nu_{-\eta} |Bp, \mathbf{k}, \eta\rangle^{(0)}_{\text{R-R}} &= 0 & \nu &= p+3, \ldots, 9 .
\end{aligned} \tag{2.41}$$

Because of the anti-commutation relations (2.39) we can define

$$|Bp, \mathbf{k}, +\rangle^{(0)}_{\text{R-R}} = \prod_{\mu=2}^{p+2} \psi^\mu_+ \prod_{\nu=p+3}^{9} \psi^\nu_- |Bp, \mathbf{k}, -\rangle^{(0)}_{\text{R-R}} , \tag{2.42}$$

and then it follows that

$$|Bp, \mathbf{k}, -\rangle^{(0)}_{\text{R-R}} = \prod_{\mu=2}^{p+2} \psi^\mu_- \prod_{\nu=p+3}^{9} \psi^\nu_+ |Bp, \mathbf{k}, +\rangle^{(0)}_{\text{R-R}} . \tag{2.43}$$

On the ground states the GSO-operators take the form

$$(-1)^F = \prod_{\mu=2}^{9} (\sqrt{2}\psi^\mu_0) = \prod_{\mu=2}^{9} (\psi^\mu_+ + \psi^\mu_-) , \tag{2.44}$$

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and

$$(-1)^{\widetilde{F}} = \prod_{\mu=2}^{9} (\sqrt{2}\widetilde{\psi}_0^\mu) = \prod_{\mu=2}^{9} (\psi_+^\mu - \psi_-^\mu) . \tag{2.45}$$

Taking these equations together we then find that

$$(-1)^{F} |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} = |Bp, \mathbf{k}, -\eta\rangle_{\text{R-R}}^{(0)} \tag{2.46}$$

$$(-1)^{\widetilde{F}} |Bp, \mathbf{k}, \eta\rangle_{\text{R-R}}^{(0)} = (-1)^{p+1} |Bp, \mathbf{k}, -\eta\rangle_{\text{R-R}}^{(0)} . \tag{2.47}$$

The action on the non-zero modes is as before, and therefore the action of the GSO-operators on the whole boundary states is given by

$$(-1)^{F} |Bp, \mathbf{a}, \eta\rangle_{\text{R-R}} = |Bp, \mathbf{a}, -\eta\rangle_{\text{R-R}} \tag{2.48}$$

$$(-1)^{\widetilde{F}} |Bp, \mathbf{a}, \eta\rangle_{\text{R-R}} = (-1)^{p+1} |Bp, \mathbf{a}, -\eta\rangle_{\text{R-R}} . \tag{2.49}$$

It follows from the first equation that the only potentially GSO-invariant boundary state is of the form

$$|Bp, \mathbf{a}\rangle_{\text{R-R}} = (|Bp, \mathbf{a}, +\rangle_{\text{R-R}} + |Bp, \mathbf{a}, -\rangle_{\text{R-R}}) , \tag{2.50}$$

and the second equation implies that it is actually GSO-invariant if p is even (odd) in the case of Type IIA (IIB). In this case we can find a GSO-invariant boundary state

$$|Dp, \mathbf{a}\rangle = |Bp, \mathbf{a}\rangle_{\text{NS-NS}} + |Bp, \mathbf{a}\rangle_{\text{R-R}} . \tag{2.51}$$

This state satisfies world-sheet duality provided we choose

$$\mathcal{N}_{\text{NS-NS}}^2 = \frac{1}{128} \frac{V_{p+1}}{(2\pi)^{p+1}} \qquad \mathcal{N}_{\text{R-R}}^2 = -\frac{1}{8} \frac{V_{p+1}}{(2\pi)^{p+1}} . \tag{2.52}$$

This gives rise to an open string consisting of a GSO-projected NS and R sector; in particular, the GSO-projection removes the open string tachyon from the NS sector, and the D-brane is stable. The D-brane is also BPS since the open string spectrum is supersymmetric.

Actually the condition of world-sheet duality does not specify the relative sign between the NS-NS and the R-R component in (2.51) since only the square of the normalisation constant enters the calculation.[5] The opposite choice of the sign defines the anti-brane that is also BPS by itself; however, the combination of a brane and an anti-brane breaks supersymmetry since the two states preserve disjoint sets of supercharges. This can also be seen from the present point of view: the open string that stretches between a brane and an anti-brane consists of a NS and a R-sector whose GSO-projection is opposite to that of the brane-brane or anti-brane-anti-brane open string. In particular, the open string tachyon from the NS sector survives the projection; the system is therefore unstable, and

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[5]It also, obviously, does not specify the overall sign, but this is just the familiar ambiguity in the definition of quantum mechanical states.

certainly does not preserve supersymmetry. It is also possible to analyse the action of the supercharges on the boundary states directly [6].

Finally, we should like to stress that the above analysis shows not only that Type IIA and Type IIB has BPS Dp-branes for the appropriate values of p, but also that these are the only *stable* D-branes of Type IIA and Type IIB. This is not necessarily the case — as we shall see below, some theories possess stable D-branes that are not BPS.

2.3 A second example: Type 0A and 0B

As a second example let us examine the D-brane spectrum of Type 0A and Type 0B [16, 18, 49, 32]. These theories can be obtained from Type IIA and IIB as an orbifold by $(-1)^{F^s}$, where F^s is the space-time fermion number. The effect of $(-1)^{F^s}$ in the untwisted sector is to retain the bosons (*i.e.* the states in the NS-NS and R-R sectors) and to remove the fermions (*i.e.* the states in the NS-R and R-NS sectors). In the two remaining sectors, the GSO projection acts in the usual way

$$
\begin{aligned}
\text{NS-NS:} \quad & P_{GSO,U} = \tfrac{1}{4}\left(1 + (-1)^F\right)\left(1 + (-1)^{\widetilde{F}}\right) \\
\text{R-R:} \quad & P_{GSO,U} = \tfrac{1}{4}\left(1 + (-1)^F\right)\left(1 \pm (-1)^{\widetilde{F}}\right),
\end{aligned}
\tag{2.53}
$$

where the $+$ sign corresponds to Type IIB, and the $-$ sign to Type IIA. In the twisted sector, the effect of $(-1)^{F^s}$ is to reverse the GSO projection for both left and right-moving sectors. In addition only the states invariant under $(-1)^{F^s}$ (*i.e.* the bosons) are retained. Thus the states in the twisted sector are again in the NS-NS and the R-R sector, but their GSO projection is now

$$
\begin{aligned}
\text{NS-NS:} \quad & P_{GSO,T} = \tfrac{1}{4}\left(1 - (-1)^F\right)\left(1 - (-1)^{\widetilde{F}}\right) \\
\text{R-R:} \quad & P_{GSO,T} = \tfrac{1}{4}\left(1 - (-1)^F\right)\left(1 \mp (-1)^{\widetilde{F}}\right),
\end{aligned}
\tag{2.54}
$$

where now the $-$ sign corresponds to Type IIB, and the $+$ sign to Type IIA. Taking (2.53) and (2.54) together, we can describe the spectrum of Type 0A and Type 0B more compactly as the subspaces of the NS-NS and R-R sectors that are invariant under the GSO-projection

$$
\begin{aligned}
\text{NS-NS:} \quad & P_{GSO} = \tfrac{1}{2}\left(1 + (-1)^{F+\widetilde{F}}\right) \\
\text{R-R:} \quad & P_{GSO} = \tfrac{1}{2}\left(1 \pm (-1)^{F+\widetilde{F}}\right).
\end{aligned}
\tag{2.55}
$$

The resulting spectrum is thus given by

$$
\begin{aligned}
\textbf{0A}: \quad & (\text{NS}+,\text{NS}+) \oplus (\text{NS}-,\text{NS}-) \oplus (\text{R}+,\text{R}-) \oplus (\text{R}-,\text{R}+) \\
\textbf{0B}: \quad & (\text{NS}+,\text{NS}+) \oplus (\text{NS}-,\text{NS}-) \oplus (\text{R}+,\text{R}+) \oplus (\text{R}-,\text{R}-).
\end{aligned}
\tag{2.56}
$$

The NS-NS sector is the same for the two theories: in particular, the low lying states consist of the ground state tachyon (that is invariant under (2.55) since it is invariant under (2.54)), and the bosonic part of the supergravity multiplet, *i.e.* the graviton, Kalb-Ramond 2-form, and dilaton. On the other hand, the R-R sector is different for the two

theories (as is familiar from Type IIA and Type IIB). There are no tachyonic states, and the massless states transform as

$$\textbf{0A}: \qquad (\mathbf{8_s} \otimes \mathbf{8_c}) \oplus (\mathbf{8_c} \otimes \mathbf{8_s}) = 2 \cdot \mathbf{8_v} + 2 \cdot \mathbf{56},$$
$$\textbf{0B}: \qquad (\mathbf{8_s} \otimes \mathbf{8_s}) \oplus (\mathbf{8_c} \otimes \mathbf{8_c}) = 2 \cdot \mathbf{1} + 2 \cdot \mathbf{28} + \mathbf{70}. \qquad (2.57)$$

In the case of Type 0A, the theory has two 1-forms and two 3-forms in the R-R sector, whereas Type 0B has two scalars, two 2-forms, and a 4-form (with an unrestricted 5-form field strength). The states in the R-R sector of Type 0A and Type 0B are therefore doubled compared to those in Type IIA and Type IIB. One may therefore expect that the D-brane spectrum of these theories is also doubled.

In the NS-NS sector, each boundary state $|Bp, \mathbf{a}, \eta\rangle$ is by itself GSO-invariant; the most general GSO-invariant boundary state in the NS-NS sector is therefore

$$|Bp, \mathbf{a}\rangle_{\text{NS-NS}} = \alpha_+ |Bp, \mathbf{a}, +\rangle_{\text{NS-NS}} + \alpha_- |Bp, \mathbf{a}, -\rangle_{\text{NS-NS}}. \qquad (2.58)$$

If $\alpha_+ \alpha_- \neq 0$, it follows from (2.32) that the open string that begins and ends on the *same* boundary state contains space-time fermions. Since the closed string sector only consists of bosons, this presumably means that the open-closed vertex of the string field theory cannot be consistently defined; thus condition (iii) suggests that we have to have $\alpha_+ \alpha_- = 0$.[6] Thus there are two consistent NS-NS boundary states, and they are given by $|Bp, \mathbf{a}, +\rangle_{\text{NS-NS}}$ and $|Bp, \mathbf{a}, -\rangle_{\text{NS-NS}}$. As before, neither of them is stable since the open string that begins and ends on this state has a tachyon from the unprojected open string NS sector. In oder to stabilise the brane, we have to add a boundary state in the R-R sector, but as before, these are only GSO-invariant provided that p is even (odd) for Type 0A (0B). For each such p we are then left with four different D-brane states (together with their anti-branes)

$$|Dp, \mathbf{a}, \eta, \eta'\rangle = |Bp, \mathbf{a}, \eta\rangle_{\text{NS-NS}} + |Bp, \mathbf{a}, \eta'\rangle_{\text{R-R}}, \qquad (2.59)$$

where

$$\mathcal{N}^2_{\text{NS-NS}} = \frac{1}{64} \frac{V_{p+1}}{(2\pi)^{p+1}}, \qquad \mathcal{N}^2_{\text{R-R}} = -\frac{1}{4} \frac{V_{p+1}}{(2\pi)^{p+1}}. \qquad (2.60)$$

However not all of these branes are *mutually* consistent: the open string between the boundary state $|Dp, \mathbf{a}, +, +\rangle$ and $|Dp, \mathbf{b}, -, +\rangle$ consists of a R-sector together with a NS-sector with a $(-1)^F$ insertion, and likewise for $|Dp, \mathbf{a}, -, -\rangle$ and $|Dp, \mathbf{b}, +, -\rangle$. Thus there are only two mutually consistent D-brane states for each allowed value of p which we can take to be

$$|Dp, \mathbf{a}, +, +\rangle \qquad \text{and} \qquad |Dp, \mathbf{a}, -, -\rangle. \qquad (2.61)$$

These D-branes have played an important role in recent attempts to extend the Maldacena conjecture [37] to certain backgrounds of Type 0B string theory [18].

---

[6] If the theory actually possesses one brane with $\alpha_+ \alpha_- \neq 0$, the mutual consistency condition between different branes to be discussed below implies that the theory has only one stable brane (and anti-brane) for each allowed value of p; this would also seem to be in conflict with the doubled R-R spectrum of the theory.

2.4 More abstract point of view: Conformal field theory with boundaries

The construction of D-branes in terms of boundary states that we have described above can be understood, from a more abstract point of view, as the construction of a conformal field theory on a world-sheet with a boundary [8]. Given a conformal field theory that is defined on closed world-sheets (*i.e.* on closed Riemann surfaces), we can ask the question whether we can extend the definition of this conformal field theory to world-sheets that have boundaries. The prototype geometry of such a world-sheet is an infinite strip that we take to be parametrised by (t, s), where $0 \leq s \leq \pi$ and $t \in \mathbb{R}$.

As before, we can then consider the situation where the strip is made periodic in the t-direction with period $2\pi T$. The manifold is then topologically an annulus, and the relevant partition function becomes

$$Z_{\alpha\beta}(\tilde{q}) = \text{Tr } \tilde{q}^{H_{\alpha\beta}} , \tag{2.62}$$

where $\tilde{q} = e^{-2\pi T}$, α and β label the boundary conditions at either end of the strip, and $H_{\alpha\beta}$ is the corresponding Hamiltonian. This partition function can be expressed in terms of the representations of the chiral algebra of the conformal field theory (see for example [50] for an introduction to these matters),

$$Z_{\alpha\beta}(\tilde{q}) = \sum_i n^i_{\alpha\beta} \chi_i(\tilde{q}) , \tag{2.63}$$

where $\chi_i(\tilde{q})$ is the *character* of the representation labelled by i,

$$\chi_i(\tilde{q}) = \tilde{q}^{-\frac{c}{24}} \text{Tr}_i \, \tilde{q}^{L_0} , \tag{2.64}$$

and c is the central charge of the corresponding Virasoro algebra.

Under world-sheet duality, *i.e.* the modular transformation $T \mapsto 1/T$, each character transforms as

$$\chi_i(\tilde{q}) = \sum_j S^j_i \chi_j(q) , \tag{2.65}$$

where $q = e^{-2\pi/T}$, and thus (2.63) becomes

$$Z_{\alpha\beta}(\tilde{q}) = \sum_{ij} n^i_{\alpha\beta} S^j_i \chi_j(q) . \tag{2.66}$$

This should then again be interpreted as the cylinder diagram between external (boundary) states of the original closed conformal field theory. The closed string trace will give rise to a character of the chiral algebra provided that each boundary state satisfies the condition

$$\left(S_n - (-1)^{h_S} \bar{S}_{-n} \right) |B\alpha\rangle = 0 , \tag{2.67}$$

where S is an arbitrary (quasi-primary) field of the chiral algebra, and h_S is its conformal weight. In particular, choosing $L = S$ we have the condition

$$\left(L_n - \bar{L}_{-n} \right) |B\alpha\rangle = 0 \tag{2.68}$$

which is just the condition that the boundary preserves the conformal invariance. A solution to these conditions has been constructed by Ishibashi [51] and Onogi and Ishibashi [52], and the corresponding coherent states are sometimes called Ishibashi states. The actual boundary states are linear combinations of these Ishibashi states, where the (relative) normalisations are determined by the condition that the numbers $n^i_{\alpha\beta}$ in (2.63) are non-negative integers for all choices of α and β. For left-right symmetric rational conformal field theories (for which the chiral algebra has only finitely many irreducible representations), explicit solutions to these constraints have been found by Cardy [8]. Finally, the string field theory condition (iii) is related to the condition that the sewing relations of the conformal field theory are satisfied [9].

For the examples of free theories (such as the bosonic Veneziano model),[7] the condition (2.67) for $S = \partial X^\mu$ (where $h_S = 1$) is just the condition that the boundary state represents a space-time spanning D-brane; the different boundary states (that are labelled by α in the above) are then the different position eigenstates (labelled by a).

In order to describe boundary states that correspond to D-branes other than space-time spanning D-branes, the above analysis has to be generalised slightly. In fact, it is actually not necessary to demand that (2.67) holds, but it is sufficient to impose

$$\left(S_n - (-1)^{h_S}\rho(\bar{S}_{-n})\right)|B\alpha\rangle = 0, \qquad (2.69)$$

where ρ is an automorphism of the chiral algebra that leaves the conformal field L invariant (so that (2.68) is not modified). With this modification, the abstract approach accounts for all D-branes in the above model. However, it can also be generalised to theories on curved spaces that do not possess free bosons (and for which the definition of a Neumann or Dirichlet boundary condition is somewhat ambiguous). In particular, this analysis has been performed for the Gepner models [54, 55, 56] and the WZW theories [57, 58, 59, 60, 61, 62].

It should be stressed though, that the conformal field theory analysis that we have just sketched usually applies to the whole conformal field theory spectrum. For theories with world-sheet supersymmetry on the other hand, the spectrum that is relevant for string theory consists only of a certain subspace of the conformal field theory spectrum, namely of those states that are invariant under the GSO-projection. Thus the conformal field theory approach has to be slightly modified to take this into account. More significantly, the sewing conditions of the conformal field theory only guarantee a consistent definition of the various amplitudes for the full conformal field theory, but it is *a priori* not clear whether they are sufficient to guarantee the consistency on the GSO-invariant subspace of string theory, *i.e.* the string field theoretic consistency conditions (see (iii) above).[8] At any rate, at least for the free theories that we are considering in these lectures, most of the subtleties concern the nature of the GSO-projection, and therefore go beyond at least the naive conformal field theory analysis.

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[7]This theory is obviously not rational, and thus Cardy's solution does not directly apply; see however [53].

[8]This was, by the way, already pointed out in [9].

3   The non-BPS D-particle in IIB/$(-1)^{F_L}\mathcal{I}_4$

Up to now we have described a general construction of D-branes that does not rely on space-time supersymmetry. We want to apply this technique now to the construction of stable non-BPS D-branes. From the point of view that is presented in these lectures, the simplest stable non-BPS D-brane is presumably the D-particle of a certain orbifold of Type IIB [17, 20]; this shall be the topic of this section.

As was pointed out by Sen some time ago [19], duality symmetries in string theory sometimes predict the existence of solitonic states which are not BPS, but are stable due to the fact that they are the lightest states carrying a given set of charge quantum numbers. One example Sen considered concerns the orientifold [63, 15] of Type IIB by $\Omega\mathcal{I}_4$ where $\mathcal{I}_4$ is the inversion of four coordinates. This theory is dual to the orbifold of Type IIB by $(-1)^{F_L}\mathcal{I}_4$, where F_L is the left-moving spacetime fermion number. As we shall explain below, the spectrum of the orbifold contains in the twisted sector a massless vector multiplet of $\mathcal{N} = (1,1)$ supersymmetry in $D = 6$, and this implies that the orbifold fixed-plane corresponds to a (mirror) pair of D5-branes on top of an orientifold 5-plane [19]. Because of the orientifold projection, the massless states of the string stretching between the two D5-branes is removed, and the gauge group is reduced from $U(2)$ to $SO(2)$. The lightest state that is charged under the $SO(2)$ is then the first excited open string state of the string stretching between the two D5-branes: in the open string NS-sector the first excited states are

$$
\begin{array}{ll}
\psi^\mu_{-3/2}|0\rangle & 8 \text{ states} \\
\alpha^\mu_{-1}\psi^\nu_{-1/2}|0\rangle & 8\cdot 8 = 64 \text{ states} \\
\psi^\mu_{-1/2}\psi^\nu_{-1/2}\psi^\rho_{-1/2}|0\rangle & \binom{8}{3} = 56 \text{ states}
\end{array}
\tag{3.1}
$$

and in the R-sector the relevant states are

$$
\begin{array}{ll}
\alpha^\mu_{-1}|8_s\rangle & 8\cdot 8 = 64 \text{ states} \\
\psi^\mu_{-1}|8_c\rangle & 8\cdot 8 = 64 \text{ states}.
\end{array}
\tag{3.2}
$$

Thus there are altogether 128 bosons and 128 fermions which form a long (non-BPS) multiplet of the $\mathcal{N} = (1,1)$ $D = 6$ supersymmetry algebra.

Since these states are stable, one should expect that the dual (orbifold) theory also contains a stable multiplet of states that is charged under this $SO(2)$. However, since they are not BPS, the corresponding state cannot be a BPS D-brane; in fact, as we shall show below, the orbifold theory possesses a stable non-BPS D-particle that is stuck to the orbifold fixed plane and satisfies all the required properties.

3.1   The spectrum of the orbifold

Let us first describe the orbifold of the Type IIB theory in some detail. For simplicity we shall consider the uncompactified theory, i.e. the orbifold of $R^{9,1}/(-1)^{F_L}\mathcal{I}_4$. Let us choose the convention that $\mathcal{I}_4$ inverts the four spatial coordinates, $x^6,\ldots,x^9$. The fixed points under $\mathcal{I}_4$ form a 5-plane at $x^6 = x^7 = x^8 = x^9 = 0$, which extends along the coordinates $x^2,\ldots,x^5$, as well as the light-cone coordinates x^0,x^1. In light-cone gauge, type IIB

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string theory has 16 dynamical supersymmetries and 16 kinematical supersymmetries. The former transform under the transverse $SO(8)$ as

$$Q \sim \mathbf{8_s} , \quad \tilde{Q} \sim \mathbf{8_s} . \tag{3.3}$$

The orbifold breaks the transverse $SO(8)$ into $SO(4)_S \times SO(4)_R$, where the $SO(4)_S$ factor corresponds to rotations of $(x^2, \ldots, x^5)$, and the $SO(4)_R$ factor to rotations of $(x^6, \ldots, x^9)$. The above supercharges therefore decompose as

$$\mathbf{8_s} \longrightarrow ((\mathbf{2, 1}), (\mathbf{2, 1})) + ((\mathbf{1, 2}), (\mathbf{1, 2})) , \tag{3.4}$$

where we have written the representations of $SO(4)$ in terms of $SO(4) \simeq SU(2) \times SU(2)$. The operator $\mathcal{I}_4$ reverses the sign of the vector representation of $SO(4)_R$ (the $(\mathbf{2, 2})$), and we therefore choose its action on the $SO(4)_R$ spinors as

$$\mathcal{I}_4 : \quad \begin{cases} (\mathbf{2, 1}) & \to & -(\mathbf{2, 1}) \\ (\mathbf{1, 2}) & \to & (\mathbf{1, 2}) \end{cases} . \tag{3.5}$$

The action of $(-1)^{F_L}$ is simply

$$(-1)^{F_L} : \quad Q \to -Q , \quad \tilde{Q} \to \tilde{Q} , \tag{3.6}$$

and the surviving supersymmetries thus transform as

$$Q \sim ((\mathbf{2, 1}), (\mathbf{2, 1})) , \quad \tilde{Q} \sim ((\mathbf{1, 2}), (\mathbf{1, 2})) . \tag{3.7}$$

From the point of view of the 5-plane world-volume this is (dynamical, light-cone) $\mathcal{N} = (1, 1)$ supersymmetry[9].

The unbroken supersymmetry of these models can also be determined by analysing which states in the (untwisted) sector are invariant under the orbifold projection. The NS-NS sector is the same for both IIA and IIB, and it consists in ten dimensions of a graviton g_{MN} (35 physical degrees of freedom), a Kalb-Ramond 2-form B_{MN} (28) and a dilaton ϕ (1). In six dimensions, the graviton gives rise to a 6d graviton $g_{\mu\nu}$ (9), four vectors $g_{\mu i}$ (16) and ten scalars g_{ij} (10). The Kalb-Ramond 2-form gives rise to a 6d 2-form $B_{\mu\nu}$ (6), four vectors $B_{\mu i}$ (16), and six scalars B_{ij} (6). Under $\mathcal{I}_4(-1)^{F_L}$ (or $\mathcal{I}_4$), the vectors are all removed, and we retain a 6d graviton, a 6d 2-form and 17 scalars.

The R-R sector of Type IIB in ten dimensions consists of a 4-form with a self-dual 5-form field strength (35), a 2-form (28) and a scalar (1). In six dimensions, the 4-form becomes one scalar (1), four vectors (16) and three 2-forms (18); the 2-form becomes a 2-form (6), four vectors (16) and six scalars (6), whilst the scalar remains a scalar. If we orbifold by $\mathcal{I}_4(-1)^{F_L}$, we retain the eight vectors, and remove the scalars and the 2-forms; thus we have a graviton, a 2-form, four vectors and one scalar (which combine into a supergravity multiplet of $\mathcal{N} = (1, 1)$) together with 4 vectors and 16 scalars (which combine into four vector multiplets of $\mathcal{N} = (1, 1)$).[10]

---

[9]The same orbifold of type IIA would yield $\mathcal{N} = (2, 0)$ supersymmetry.

[10]A convenient summary of the various supermultiplets can be found in [64].

On the other hand, if we orbifold by $\mathcal{I}_4$, we retain the four 2-forms and eight additional scalars. Thus we have a graviton and 5 2-forms with self-dual 3-form field strengths (that combine into a supergravity multiplet of $\mathcal{N} = (2,0)$) together with 5 2-forms with anti-self-dual 3-form field strengths and 25 scalars (which combine into five tensor multiplets of $\mathcal{N} = (2,0)$).

The analysis for Type IIA is analogous. The R-R sector in ten dimensions consists of a 3-form (56) and a 1-form (8). In six dimensions, the 3-form becomes seven vectors (28), four 2-forms (24) and four scalars (4), whilst the 1-form becomes a vector (4) and four scalars (4). If we orbifold by $\mathcal{I}_4(-1)^{F_L}$, we retain the four 2-forms and the eight scalars, and therefore have the same massless states as in the IIB orbifold by $\mathcal{I}_4$ giving $\mathcal{N} = (2,0)$ supersymmetry; if we orbifold by $\mathcal{I}_4$, we retain the eight vectors, and thus obtain the same massless states as in the IIB orbifold by $\mathcal{I}_4(-1)^{F_L}$ giving $\mathcal{N} = (1,1)$ supersymmetry.

In addition to the untwisted sectors, the theory also contains a twisted sector that is localised at the 5-plane. In the twisted sector the various oscillators are moded as

$$
\begin{aligned}
\text{twisted NS}: \quad & n \in \left\{ \begin{array}{ll} \mathbb{Z} & \mu = 2,\ldots,5 \\ \mathbb{Z} + 1/2 & \mu = 6,\ldots,9 \end{array} \right. \qquad r \in \left\{ \begin{array}{ll} \mathbb{Z} + 1/2 & \mu = 2,\ldots,5 \\ \mathbb{Z} & \mu = 6,\ldots,9 \end{array} \right. \\
\text{twisted R}: \quad & n \in \left\{ \begin{array}{ll} \mathbb{Z} & \mu = 2,\ldots,5 \\ \mathbb{Z} + 1/2 & \mu = 6,\ldots,9 \end{array} \right. \qquad r \in \left\{ \begin{array}{ll} \mathbb{Z} & \mu = 2,\ldots,5 \\ \mathbb{Z} + 1/2 & \mu = 6,\ldots,9 \, . \end{array} \right.
\end{aligned} \tag{3.8}
$$

The ground state energy vanishes in both the R and NS sectors, and they both contain four fermionic zero modes that transform in the vector representation of $SO(4)_S$ and $SO(4)_R$, respectively. Consequently the twisted NS-NS and R-R ground states transform as

$$
((\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2})) \otimes ((\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2})) \, , \tag{3.9}
$$

where the charges correspond to $SO(4)_S$ ($SO(4)_R$) in the R-R (NS-NS) sector. The unique massless representation of $D = 6$ $\mathcal{N} = (1,1)$ supersymmetry (other than the gravity multiplet) is the vector multiplet

$$
((\mathbf{2},\mathbf{2}),(\mathbf{1},\mathbf{1})) + ((\mathbf{1},\mathbf{1}),(\mathbf{2},\mathbf{2})) + \text{fermions} \, . \tag{3.10}
$$

In order to preserve supersymmetry, we therefore have to choose the GSO-projections in all twisted sectors to be of the form

$$
P_{GSO,T} = \frac{1}{4}\left(1 - (-1)^F\right)\left(1 + (-1)^{\tilde{F}}\right) . \tag{3.11}
$$

This agrees with what we would have expected from standard orbifold techniques, namely that the effect of $(-1)^{F_L}$ is to change the left-GSO projection in the twisted sector. In addition, the spectrum of the twisted sector must be projected onto a subspace with either $(-1)^{F_L}\mathcal{I}_4 = +1$ or $(-1)^{F_L}\mathcal{I}_4 = -1$ (in the untwisted sector only $+1$ is allowed). Since twisted NS-NS (R-R) states are even (odd) under $(-1)^{F_L}$, and $\mathcal{I}_4$ reverses the sign of the vector of $SO(4)_R$ (and leaves the vector of $SO(4)_S$ invariant), we conclude that in the present case the twisted sector states are odd under $(-1)^{F_L}\mathcal{I}_4$.

Having described the spectrum and the GSO projections of the various sectors in some detail, we can now analyse whether a D-brane boundary state with the appropriate

properties exists. Since the non-BPS state in the orientifold theory is localised at the orientifold plane, one would expect that the corresponding non-BPS D-brane should be a D0-brane that is stuck to the orbifold fixed plane; we shall therefore analyse in the following whether such a D-brane state exists. For definiteness we shall assume that the D0-brane is oriented in such a way that it satisfies a Neumann boundary condition along the x^2 direction.

In the (untwisted) NS-NS sector the action of $(-1)^{F_L}$ is trivial, and $\mathcal{I}_4$ acts on the boundary state given in (2.36) as

$$\mathcal{I}_4|Bp, \mathbf{a}\rangle_{\text{NS-NS}} = |Bp, \mathcal{I}_4\mathbf{a}\rangle_{\text{NS-NS}}, \tag{3.12}$$

since $\mathcal{I}_4$ acts in the same way on left- and right-movers. If $\mathbf{a} = \mathbf{a_0}$ lies on the fixed plane, $\mathcal{I}_4\mathbf{a_0} = \mathbf{a_0}$, and the boundary state is invariant. Thus we have a physical $p = 0$ NS-NS boundary state

$$|U0, \mathbf{a_0}\rangle = |B0, \mathbf{a_0}\rangle_{\text{NS-NS}}. \tag{3.13}$$

On the other hand the $p = 0$ R-R boundary state is not physical since, as we saw in section 2.2, it is not invariant under the GSO-projection.[11]

In the twisted sector, the boundary state is of the same form as described before, except that now the moding of the different fields is as described in (3.8). Since there are only bosonic zero modes for $\mu = 0, 1, 2, 3, 4, 5$, and since x^2 is a Neumann direction, the position of the D0-brane boundary state is described by a 5-dimensional vector $\mathbf{b}$ that can be identified with $\mathbf{a_0}$. Both the twisted NS-NS and the twisted R-R sector contain fermionic zero modes, and the ground state of the D0-brane boundary state therefore has to satisfy

$$\psi_\eta^\nu|B0, \mathbf{a_0}, -\eta\rangle_{\text{NS-NS,T}}^{(0)} = 0 \qquad \text{for } \nu = 6, 7, 8, 9, \tag{3.14}$$

in the twisted NS-NS sector, and

$$
\begin{aligned}
\psi_\eta^2|B0, \eta\rangle_{\text{R-R,T}}^{(0)} &= 0 \\
\psi_\eta^\mu|B0, -\eta\rangle_{\text{R-R,T}}^{(0)} &= 0 \quad \text{for } \mu = 3, 4, 5,
\end{aligned}
\tag{3.15}
$$

in the twisted R-R sector. On the ground states, the GSO operators act as

$$
\begin{aligned}
\text{twisted NS-NS}: &\quad (-1)^F = \Pi_{\mu=6}^9(\sqrt{2}\psi_0^\mu), \quad (-1)^{\tilde{F}} = \Pi_{\mu=6}^9(\sqrt{2}\tilde{\psi}_0^\mu) \\
\text{twisted R-R}: &\quad (-1)^F = \Pi_{\mu=2}^5(\sqrt{2}\psi_0^\mu), \quad (-1)^{\tilde{F}} = \Pi_{\mu=2}^5(\sqrt{2}\tilde{\psi}_0^\mu).
\end{aligned}
\tag{3.16}
$$

Using the same arguments as before in section 2.2 we then find

$$
\begin{aligned}
(-1)^F|B0, \mathbf{a_0}, \eta\rangle_{\text{NS-NS,T}} &= |B0, \mathbf{a_0}, -\eta\rangle_{\text{NS-NS,T}}, \\
(-1)^{\tilde{F}}|B0, \mathbf{a_0}, \eta\rangle_{\text{NS-NS,T}} &= +|B0, \mathbf{a_0}, -\eta\rangle_{\text{NS-NS,T}},
\end{aligned}
\tag{3.17}
$$

---

[11]This boundary state is actually also not invariant under $(-1)^{F_L}\mathcal{I}_4$, as follows from the analysis of [20].

and

$$(-1)^F |B0, \mathbf{a_0}, \eta\rangle_{\text{R-R,T}} = |B0, \mathbf{a_0}, -\eta\rangle_{\text{R-R,T}},$$
$$(-1)^{\widetilde{F}} |B0, \mathbf{a_0}, \eta\rangle_{\text{R-R,T}} = -|B0, \mathbf{a_0}, -\eta\rangle_{\text{R-R,T}}. \tag{3.18}$$

Because of (3.11) it then follows that only the combination

$$|T0, \mathbf{a_0}\rangle = (|B0, \mathbf{a_0}, +\rangle_{\text{R-R,T}} + |B0, \mathbf{a_0}, -\rangle_{\text{R-R,T}}), \tag{3.19}$$

in the twisted R-R sector survives the GSO-projection, and that no combination of twisted NS-NS sector boundary states is GSO invariant. In addition, the ground states of the twisted R-R sector boundary state are odd under $(-1)^{F_L} \mathcal{I}_4$, as they are precisely the vector states of $SO(4)_S$ that arise in the twisted sector. We therefore have one further physical boundary state, and the total D-particle state is of the form

$$|\widehat{D0}, \mathbf{a_0}\rangle = |U0, \mathbf{a_0}\rangle + |T0, \mathbf{a_0}\rangle. \tag{3.20}$$

We can then determine the cylinder diagram for a closed string that begins and ends on the D-particle, and we find that

$$\int_0^\infty dl \langle \widehat{D0}, \mathbf{a_0} | e^{-lH_c} | \widehat{D0}, \mathbf{a_0}\rangle$$
$$= \int_0^\infty \frac{dt}{t^{3/2}} \left\{ 2^{9/2} \mathcal{N}_{\text{NS-NS}}^2 \frac{f_3^8(\tilde{q}) - f_2^8(\tilde{q})}{f_1^8(\tilde{q})} + 2^{1/2} \mathcal{N}_{\text{R-R,T}}^2 \frac{f_3^4(\tilde{q}) f_4^4(\tilde{q})}{f_1^4(\tilde{q}) f_2^4(\tilde{q})} \right\}, \tag{3.21}$$

where f_i is defined as in (2.27). Thus if we choose

$$\mathcal{N}_{\text{NS-NS}}^2 = \frac{1}{128} \frac{V_1}{(2\pi)}, \qquad \mathcal{N}_{\text{R-R,T}}^2 = -\frac{1}{2} \frac{V_1}{(2\pi)}, \tag{3.22}$$

we obtain (compare [20])

$$\int dl \langle D0, \mathbf{a_0} | e^{-lH_c} | D0, \mathbf{a_0}\rangle = \frac{V_1}{2\pi} \int \frac{dt}{2t} \text{Tr}_{\text{NS-R}} \left[\frac{1}{2} (1 + (-1)^F \mathcal{I}_4) e^{-2tH_o} \right]. \tag{3.23}$$

The open string spectrum thus consists of a NS and a R sector, and both are projected by $1/2(1 + (-1)^F \mathcal{I}_4)$. The tachyon of the NS sector is even under $\mathcal{I}_4$ but odd under $(-1)^F$, and is therefore removed from the spectrum. This indicates that the D-particle is stable.

In addition, 4 massless states are removed from the NS sector, leaving 4 massless bosons, and the R sector contains 8 massless fermions. Including the zero modes in the light-cone directions,[12] this gives the D-particle 5 bosonic zero modes and 16 fermionic zero modes. The former reflect the fact that the D-particle is restricted to move within the 5-plane, and the latter give rise to a long ($2^8 = 256$-dimensional) representation of the six-dimensional $\mathcal{N} = (1, 1)$ supersymmetry algebra. Finally, the D-particle is charged

---

[12]When counting the zero modes of a D-brane one must include the light-cone directions as well as the physical (transverse) massless states of the open string. See for example [65] for a discussion of the type IIB D-string.

under the vector field in the twisted R-R sector. We have therefore managed to construct a boundary state that possesses all the properties that we expected to find from the S-dual description.

Sen has proposed a different realisation for this state as the ground state of a D-string anti-D-string system [20]. In order to describe this construction, it is useful to consider the theory where at least one of the four circles that are inverted by the action of $\mathcal{I}_4$ is compact. (This will serve as a preparation for the following section where we consider the T-dual of the configuration where all four circles are compactified.) Let us then consider a D1-brane anti-D1-brane pair that wraps around this compact circle, x^6, say. In the reduced space, the branes stretch from the fixed point at $x^6 = 0$ to the fixed point at $x^6 = \pi R^6$.

As we have seen before, the ground state of the open string between the brane and the anti-brane is a tachyon; this indicates that the system is unstable to decay into the vacuum. However, we can consider the configuration where we switch on a $\mathbb{Z}_2$ Wilson line on either the brane or the anti-brane. This implies that the tachyon changes sign as we go around the circle, and thus the ground state energy of the open string is given by

$$m^2 = -\frac{1}{2} + \frac{1}{R_6^2} \, . \tag{3.24}$$

In particular, the ground state of the open string is only tachyonic if $R_6 > \sqrt{2}$; on the other hand, for $R_6 < \sqrt{2}$ the ground state of the open string is massive, and the brane anti-brane system is stable.

As we shall see in the next section, the non-trivial $\mathbb{Z}_2$ Wilson line implies that the twisted R-R charge at the endpoints of the D1-brane have opposite sign. Thus the combined system of the brane anti-brane pair only carries twisted R-R charge at one end (but not the other); it also does not carry any untwisted R-R charge, and therefore has the same charges as the non-BPS D-particle that we have just discussed (see Figure 2). This
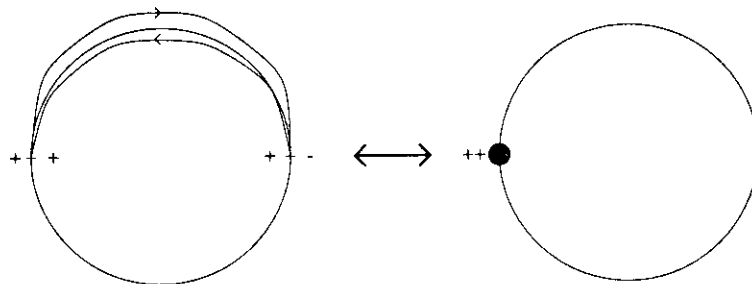

Figure 2: D1-brane anti-brane pair with a relative $\mathbb{Z}_2$ Wilson line and the D0-brane.

suggests that the brane anti-brane pair decays into the D-particle if $R_6 > \sqrt{2}$. This interpretation is further supported by the fact that for $R_6 < \sqrt{2}$, the open string that begins and ends on the D-particle contains a tachyon, and thus indicates that the D-particle is not the stable state. Indeed, the projection $1/2(1 + (-1)^F \mathcal{I}_4)$ removes the tachyon with winding number 0, but the anti-symmetric combination of winding number $w = \pm 1$ is

contained in the spectrum; this state has mass

$$m^2 = -\frac{1}{2} + \left(\frac{R_6}{2}\right)^2 \tag{3.25}$$

and thus becomes tachyonic if $R_6 < \sqrt{2}$. One can also compare the mass and the R-R charge of the two states, and they agree indeed.

4 Non-BPS states in Heterotic – Type II duality

If we consider the compactification of the above IIB orbifold on a 4-torus (on which $\mathcal{I}_4$ acts) then the theory is T-dual to

$$\text{IIB on } T^4/(-1)^{F_L}\mathcal{I}_4 \; \xleftrightarrow{\text{T}} \; \text{IIA on } T^4/\mathcal{I}_4 \,. \tag{4.1}$$

The orbifold of $T^4/\mathcal{I}_4$ describes a special point in the moduli space of K3 surfaces, the so-called orbifold point. On the other hand, Type IIA on K3 is known to be S-dual to the heterotic string on T^4 [70].

Under T-duality, the stable non-BPS D0-brane of the Type IIB orbifold becomes a stable non-BPS D1-brane of Type IIA on K3; it is then natural to ask whether one can identify the corresponding non-BPS state in the heterotic theory. This is actually an interesting problem in its own right since both theories of the dual pair are quantitatively under control, and one can make detailed comparisons. The following discussion, except for section 4.3.2 that has not been discussed before, follows closely [44] (see also [66]).

4.1 The setup

Let us first explain the conventions of the orbifold of the Type IIA theory. In the untwisted sector, the GSO-projections are given as in (2.35). If we denote the compact coordinates along which $\mathcal{I}_4$ acts by $x^6, \ldots, x^9$, the moding of the fields in the twisted sectors is as in (3.8). Furthermore, the GSO-projections in the relevant twisted sectors are given by

$$\text{twisted NS-NS} \quad \tfrac{1}{4}\Big(1 - (-1)^F\Big)\Big(1 - (-1)^{\tilde{F}}\Big)\,, \tag{4.2}$$

$$\text{twisted R-R} \quad \tfrac{1}{4}\Big(1 - (-1)^F\Big)\Big(1 + (-1)^{\tilde{F}}\Big) \tag{4.3}$$

Since the theory has $D = 6$ $\mathcal{N} = (1,1)$ supersymmetry, the states in the massless R-R sector must form a vector, and thus the GSO-projection must be the same as for the T-dual IIB/$(-1)^{F_L}\mathcal{I}_4$ orbifold. Consistency with the operator product expansion, in particular the OPE

$$\text{R-R} \times \text{R-R;T} = \text{NS-NS;T} \tag{4.4}$$

then determines the GSO-projection in the twisted NS-NS sector; in fact, since the GSO-projection of Type IIA and Type IIB are opposite in the untwisted R-R sector, the same must hold in the twisted NS-NS sector.[13]

---

[13]For a recent discussion of the subtleties associated with the choice of the GSO-projections in the twisted sectors see [67].

Next let us recall the precise relation between type IIA at the orbifold point of K3 and the heterotic string on T^4; the following discussion follows closely [68]. Let us denote the radii of the compactified coordinates by R_{Ai} and R_{hi} for Type IIA and the heterotic string, respectively. The sequence of dualities relating the two theories is given by

$$\text{het } T^4 \xrightarrow{S} \text{I } T^4 \xrightarrow{T^4} \text{IIB } T^4/\mathbb{Z}_2' \xrightarrow{S} \text{IIB } T^4/\mathbb{Z}_2'' \xrightarrow{T} \text{IIA } T^4/\mathbb{Z}_2, \qquad (4.5)$$

where the various $\mathbb{Z}_2$ groups are

$$\mathbb{Z}_2' = (1, \Omega \mathcal{I}_4) \quad \mathbb{Z}_2'' = (1, (-1)^{F_L} \mathcal{I}_4) \quad \mathbb{Z}_2 = (1, \mathcal{I}_4). \qquad (4.6)$$

Here $\mathcal{I}_4$ reflects all four compact directions, Ω reverses world-sheet parity, and F_L is the left-moving part of the spacetime fermion number. The first step is ten-dimensional S-duality between the $(SO(32))$ heterotic string and the type I string [69], which relates the (ten-dimensional) couplings and radii as[14]

$$g_I \propto g_h^{-1} \qquad R_{Ij} \propto g_h^{-1/2} R_{hj}. \qquad (4.7)$$

The second step consists of four T-duality transformations on the four circles, resulting in the new parameters

$$\begin{aligned} g' &= V_I^{-1} g_I &\propto V_h^{-1} g_h \\ R_j' &= R_{Ij}^{-1} &\propto g_h^{1/2} R_{hj}^{-1}, \end{aligned} \qquad (4.8)$$

where $V_I = \prod_j R_{Ij}$ and $V_h = \prod_j R_{hj}$ denote the volumes (divided by $(2\pi)^4$) of the T^4 in the type I and heterotic strings, respectively. This theory has 16 orientifold fixed points. In order for the dilaton to be a constant, the R-R charges have to be canceled locally, *i.e.* one pair of D5-branes has to be situated at each orientifold 5-plane. In terms of the original heterotic theory, this means that suitable Wilson lines must be switched on to break $SO(32)$ (or $E_8 \times E_8$) to $U(1)^{16}$; this will be further discussed below. The third step is S-duality of type IIB. The new parameters are given by

$$\begin{aligned} g'' &= g'^{-1} &\propto V_h g_h^{-1} \\ R_j'' &= g'^{-1/2} R_j' &\propto V_h^{1/2} R_{hj}^{-1}. \end{aligned} \qquad (4.9)$$

Finally, the fourth step is T-duality along one of the compact directions, say x^6. The resulting theory is type IIA on a K3 in the orbifold limit. The coupling constants and radii are given by

$$\begin{aligned} g_A &= g'' (R_6'')^{-1} &= g_h^{-1} R_{h6} V_h^{1/2} \\ R_{Aj} &= R_j'' &= 2 V_h^{1/2} R_{hj}^{-1} \quad \text{for } j \neq 6 \\ R_{A6} &= (R_6'')^{-1} &= 2^{-1} V_h^{-1/2} R_{h6}, \end{aligned} \qquad (4.10)$$

where we have now included the numerical factors (that will be shown below to reproduce the correct masses for the BPS-states).[15] In addition, the metrics in the low energy effective theories are related as [70]

$$G_{\mu\nu}^A = V_h g_h^{-2} G_{\mu\nu}^h. \qquad (4.11)$$

---

[14]Numerical factors are omitted until the last step.

[15]In our conventions $\alpha_h' = 1/2$, $\alpha_A' = 1$.

The corresponding point in the moduli space of the heterotic theory has $B = 0$ and Wilson lines that can be determined in analogy with the duality between the heterotic string on S^1 and type IIA on $S^1/\Omega\mathcal{I}_1$ (type IA). A constant dilaton background for the latter requires the Wilson line $A = ((\frac{1}{2})^8, 0^8)$ in the former [71, 72, 73], resulting in the gauge group $SO(16) \times SO(16)$. The sixteen entries in the Wilson line describe the positions of the D8-branes along the interval in type IA. This suggests that the four Wilson lines in our case should be

$$
\begin{aligned}
A^9 &= \left(\left(\frac{1}{2}\right)^8, 0^8 \right) \\
A^8 &= \left(\left(\frac{1}{2}\right)^4, 0^4, \left(\frac{1}{2}\right)^4, 0^4 \right) \\
A^7 &= \left(\left(\frac{1}{2}\right)^2, 0^2, \left(\frac{1}{2}\right)^2, 0^2, \left(\frac{1}{2}\right)^2, 0^2, \left(\frac{1}{2}\right)^2, 0^2 \right) \\
A^6 &= \left(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0 \right),
\end{aligned}
\tag{4.12}
$$

so that there is precisely one pair of D-branes at each of the sixteen orientifold planes. Indeed, this configuration of Wilson lines breaks the gauge group $SO(32)$ to $SO(2)^{16} \sim U(1)^{16}$, and there are no other massless gauge particles that are charged under the Cartan subalgebra of $SO(32)$. To see this, recall that the momenta of the compactified heterotic string are given as [74]

$$
\begin{aligned}
\mathbf{P}_L &= (P_L, p_L) = \left(V_K + A^i_K w_i \,, \; \frac{p^i}{2R_i} + w^i R_i \right) \\
\mathbf{P}_R &= \quad p_R = \left(\frac{p^i}{2R_i} - w^i R_i \right),
\end{aligned}
\tag{4.13}
$$

where p^i is the physical momentum in the compact directions

$$
p^i = n^i + B^{ij} w_j - V^K A^i_K - \frac{1}{2} A^i_K A^j_K w_j \,,
\tag{4.14}
$$

$w_i, n_i \in \mathbb{Z}$ are elements of the compactification lattice $\Gamma^{4,4}$, and V^K is an element of the internal lattice Γ^{16}. For a given momentum $(\mathbf{P}_L, \mathbf{P}_R)$, a physical state can exist provided the level matching condition

$$
\frac{1}{2} \mathbf{P}_L^2 + N_L - 1 = \frac{1}{2} \mathbf{P}_R^2 + N_R - c_R
\tag{4.15}
$$

is satisfied, where N_L and N_R are the left- and right-moving excitation numbers, and $c_R = 1/2$ ($c_R = 0$) for the right-moving NS (R) sector. The state is BPS if $N_R = c_R$ [78], and its mass is given by

$$
\frac{1}{4} m_h^2 = \left(\frac{1}{2} \mathbf{P}_L^2 + N_L - 1 \right) + \left(\frac{1}{2} \mathbf{P}_R^2 + N_R - c_R \right) = \mathbf{P}_R^2 + 2(N_R - c_R) \,.
\tag{4.16}
$$

The massless states of the gravity multiplet and the Cartan subalgebra have $N_L = 1$ and $\mathbf{P}_L = \mathbf{P}_R = 0$. Additional massless gauge bosons would have to have $N_L = 0$, and therefore $\mathbf{P}_L^2 = 2$. If $w_i = 0$ for all i, this requires $V^2 = 2$ and $p_i = 0$. The possible choices for V are then simply the roots of $SO(32)$, and it is easy to see that for each root at least one of the inner products $V^K A_K^i$ is half-integer; thus $p^i \in \mathbb{Z} + 1/2$ cannot vanish, and the state is massive. On the other hand, if $w_i \neq 0$ for at least one i, the above requires $(V + Aw)^2 < 2$, and it follows that $V + Aw = 0$, $i.e.$ that the massless gauge particle is not charged under the Cartan subalgebra of $SO(32)$.

4.2 BPS states

In order to test the above identification further, it is useful to relate some of the perturbative BPS states of the heterotic string to D-brane states in IIA on $T^4/\mathbb{Z}_2$, and to compare their masses. Let us start with the simplest case – a bulk D-particle. This state is charged only under the bulk $U(1)$ corresponding to the ten-dimensional R-R one-form $C_{R-R}^{(1)}$. It can be described by the boundary state

$$|D0; \mathbf{a}, \mathbf{b}, \epsilon_1\rangle = \Big(|B0; \mathbf{a}, \mathbf{b}\rangle_{\text{NS-NS}} + \epsilon_1 |B0; \mathbf{a}, \mathbf{b}\rangle_{\text{R-R}}\Big)$$
$$+ \Big(|B0; \mathbf{a}, -\mathbf{b}\rangle_{\text{NS-NS}} + \epsilon_1 |B0; \mathbf{a}, -\mathbf{b}\rangle_{\text{R-R}}\Big), \qquad (4.17)$$

where $\mathbf{a}$ denotes the position along the uncompactified directions for which the D-brane has Dirichlet boundary conditions, $i.e.$ x^0, x^1, x^3, x^4, x^5, and $\mathbf{b}$ denotes the position along the compacitified directions, $x^6, \ldots, x^9$. Since the directions $x^6, \ldots, x^9$ are compact, the corresponding momenta are quantised, $k^i = m_i/R_{Ai}$, and the momentum integrals are replaced by sums; thus the boundary state becomes

$$|B0, \mathbf{a}, \mathbf{b}, \eta\rangle = \mathcal{N} \int \prod_{\nu=0,1,3,\ldots,5} dk^\nu e^{ik^\nu a^\nu} \left(\prod_{i=6}^{9} \sum_{m_i \in \mathbb{Z}} e^{im_i b^i/R_{Ai}} \right) |B0, \widehat{\mathbf{k}, \mathbf{m}}, \eta\rangle, \qquad (4.18)$$

where $|B0, \mathbf{k}, \mathbf{m}, \eta\rangle$ is given by the same formula as in (2.13). The GSO-invariant boundary state $|B0; \mathbf{a}, \mathbf{b}\rangle$ is then again given as in (2.36) and (2.50). (Since we are dealing with the untwisted sector of a Type IIA orbifold, the R-R sector boundary state with p even is GSO invariant.) The state in (4.17) is manifestly also invariant under the orbifold operator $\mathcal{I}_4$ since it is the symmetric combination of a D0-brane state together with its image under $\mathcal{I}_4$.

In order to determine the correct normalisation of the different boundary states we have to perform a similar calculation as before in the case of the uncompactified Type IIA and Type IIB theory. There are, however, two minor modifications. Firstly, since the momenta along the four compact directions are quantised, one cannot simply do the Gaussian integral; instead, one is left with a momentum sum that can be transformed, using the Poisson resummation,

$$\sum_{m \in \mathbb{Z}} e^{-\pi l(m/R)^2} = \frac{R}{\sqrt{l}} \sum_{n \in \mathbb{Z}} e^{-2l\pi(nR)^2}, \qquad (4.19)$$

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into a winding sum which in turn appears in the open string trace.[16] Secondly, the open string that one obtains from (4.17) will have four sectors (depending on whether each end of the string is at $(\mathbf{a}, \mathbf{b})$ or at $(\mathbf{a}, -\mathbf{b})$), each of which consists of

$$[\text{NS - R}] \frac{1}{2}\left(1 + (-1)^F\right). \tag{4.20}$$

However, under the action of $\mathcal{I}_4$, the four sectors are pairwise identified, and therefore only half as many open string states survive. Taking this into account, the normalisation of the boundary states in (4.17) turn out to be

$$R_{A6}R_{A7}R_{A8}R_{A9}\mathcal{N}^2_{\text{NS-NS}} = \frac{1}{2}\frac{1}{128}\frac{V_1}{(2\pi)}, \qquad R_{A6}R_{A7}R_{A8}R_{A9}\mathcal{N}^2_{\text{R-R}} = -\frac{1}{2}\frac{1}{8}\frac{V_1}{(2\pi)}. \tag{4.21}$$

As before, $\epsilon_1 = \pm 1$ differentiates a D-particle from an anti-D-particle.

The corresponding state in the heterotic string has trivial winding ($w_i = 0$) and momentum ($V = 0$, $p^i = 0$), except for $p_6 = \epsilon_1$. Level matching then requires that $N_L = 1$, and therefore the state is really a Kaluza-Klein excitation of either the gravity multiplet or one of the vector multiplets in the Cartan subalgebra. Its mass is given by (4.16)

$$m_h(D0) = \frac{1}{R_{h6}}. \tag{4.22}$$

The corresponding mass in type IIA can be found using (4.10) and (4.11), and turns out to be

$$m_A(D0) = V_h^{-1/2}g_h m_h(D0) = \frac{1}{g_A}. \tag{4.23}$$

This is in complete agreement with the mass of a D-particle.

Next consider the D-particle that is stuck at one of the fixed planes (which we may assume to be the fixed plane with $\mathbf{b} = 0$). The mass and the bulk R-R charge of this D-particle is precisely half of that of the bulk D-particle that we discussed above; it is therefore sometimes called a 'fractional' D-particle [75]. It also carries unit charge with respect to the twisted R-R $U(1)$ at the fixed plane. The corresponding boundary state is then

$$|D0, \mathbf{a}; \epsilon_1, \epsilon_2\rangle = |B0; \mathbf{a}\rangle_{\text{NS-NS}} + \epsilon_1|B0; \mathbf{a}\rangle_{\text{R-R}} + \epsilon_2|B0; \mathbf{a}\rangle_{\text{NS-NS;T}} + \epsilon_1\epsilon_2|B0; \mathbf{a}\rangle_{\text{R-R,T}}. \tag{4.24}$$

As we have seen above, the boundary states in the untwisted sector are GSO- and orbifold-invariant. As regards the boundary states in the twisted sectors, the analysis is completely analogous to the analysis of the previous section, the only difference being that the GSO-projection in the twisted NS-NS sector is now opposite to what it was there; as a consequence the D0-brane boundary state is also GSO-invariant in that sector.

The normalisation of the boundary states in the untwisted sector is as for the case of the bulk D0-brane above,

$$R_{A6}R_{A7}R_{A8}R_{A9}\mathcal{N}^2_{\text{NS-NS}} = \frac{1}{2}\frac{1}{128}\frac{V_1}{(2\pi)}, \qquad R_{A6}R_{A7}R_{A8}R_{A9}\mathcal{N}^2_{\text{R-R}} = -\frac{1}{2}\frac{1}{8}\frac{V_1}{(2\pi)}, \tag{4.25}$$

---

[16]More details on this can be found in [20] and [42].

and in the twisted sectors it is

$$\mathcal{N}^2_{\text{NS-NS;T}} = \frac{1}{16} \frac{V_1}{(2\pi)}, \qquad \mathcal{N}^2_{\text{R-R;T}} = -\frac{1}{4} \frac{V_1}{(2\pi)}. \tag{4.26}$$

With these normalisations, the open string between two such D-particles is given by

$$[\text{NS - R}] \frac{1}{4}\left(1 + \epsilon_1\epsilon_1'(-1)^F\right)\left(1 + \epsilon_2\epsilon_2'\mathcal{I}_4\right). \tag{4.27}$$

If we consider the limit of the bulk D0-brane state as $\mathbf{b} \to 0$, *i.e.* as the bulk D-particle approaches the fixed plane, the normalisation of the boundary states of the bulk brane is indeed *twice* that of the corresponding components of the fractional brane. This demonstrates explicitly that the mass and the untwisted R-R charge of the bulk brane is indeed twice that of the fractional brane.

As before, $\epsilon_1 = \pm 1$ and $\epsilon_1\epsilon_2 = \pm 1$ determine the sign of the bulk and the twisted charges of the fractional brane, respectively. In the blow up of the orbifold to a smooth K3, the fractional D-particle corresponds to a D2-brane which wraps a supersymmetric cycle [76]. In the orbifold limit the area of this cycle vanishes, but the corresponding state is not massless, since the two-form field $B^{(2)}$ has a non-vanishing integral around the cycle [77]. In fact $B = 1/2$, and the resulting state carries one unit of twisted charge coming from the membrane itself, and one half unit of bulk charge coming from the D2-brane world-volume action term $\int d^3\sigma\, C^{(1)}_{R-R} \wedge (F^{(2)} + B^{(2)})$. At each fixed point there are four such states, corresponding to the two possible orientations of the membrane, and the possibility of having $F = 0$ or $F = \pm 1$ (as F must be integral, the state always has a non-vanishing bulk charge). These are the four possible fractional D-particles of (4.24). Since there are sixteen orbifold fixed planes, there are a total of 64 such states.

In the heterotic string these correspond to states with internal weight vectors of the form

$$V = \pm(0^{2n}, 1, \pm 1, 0^{14-2n}) \qquad (n = 0, \ldots, 7), \tag{4.28}$$

and vanishing winding and internal momentum, except for $p_6 = \pm 1/2$. The sixteen twisted $U(1)$ charges in the IIA picture correspond to symmetric and anti-symmetric combinations of the $(2n + 1)$'st and $(2n + 2)$'nd Cartan $U(1)$ charges in the heterotic picture. It follows from the heterotic mass formula (4.16) that the mass of these states is

$$m_h(D0_f) = \frac{1}{2R_{h6}}. \tag{4.29}$$

As before, this corresponds to the mass

$$m_A(D0_f) = V_h^{-1/2}\, g_h\, m_h(D0_f) = \frac{1}{2g_A}, \tag{4.30}$$

in the orbifold of type IIA, and is thus in complete agreement with the mass of a fractional D-particle.

Additional BPS states are obtained by wrapping D2-branes around non-vanishing supersymmetric 2-cycles, and by wrapping D4-branes around the entire compact space. One can compute the mass of each of these states, and thus find the corresponding state in the heterotic string. Let us briefly summarise the results:

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(i) A D2-brane that wraps the cycle (x^i, x^j) where $i \neq j$ and $i, j \in \{7, 8, 9\}$ has mass $m_A = R_{Ai} R_{Aj}/(2g_A)$; in heterotic units this corresponds to $m_h = 2R_{hk}$, where $k \in \{7, 8, 9\}$ is not equal to either i or j. The corresponding heterotic state has $w_k = \pm 1$, $p^l = 0$, $(V \pm A_k)^2 = 2$, and $N_L = 0$.

(ii) A D2-brane that wraps the cycle (x^i, x^6), where i is either $7, 8$ or 9, has mass $m_A = R_{Ai} R_{A6}/(2g_A)$; in heterotic units this corresponds to $m_h = 1/(2R_{hi})$. The corresponding heterotic state therefore has $p^i = \pm 1/2$, $w^j = 0$, $V^2 = 2$, and $N_L = 0$.

(iii) A D4-brane wrapping the entire compact space has mass $m_A = \prod_i R_{Ai}/(2g_A)$; in heterotic units this corresponds to $m_h = 2R_{h6}$. The corresponding heterotic state therefore has $w_6 = \pm 1$, $p^l = 0$, $(V \pm A_4)^2 = 2$, and $N_L = 0$.

4.3 Non-BPS states

The heterotic string also contains non-BPS states that are stable in certain domains of the moduli space. One should therefore expect that these states can also be seen in the dual type IIA theory, and that they correspond to non-BPS branes. Of course, since non-BPS states are not protected by supersymmetry against quantum corrections to their mass, the analysis below will only hold for $g_h \ll 1$ and $g_A \ll 1$ in the heterotic and type IIA theory, respectively.

4.3.1 Non-BPS D-string

The simplest examples of this kind are the heterotic states with vanishing winding and momenta $(w_i = p_i = 0)$, and weight vectors given by

$$
\begin{aligned}
V &= (0^m, \pm 2, 0^{15-m}) \\
V' &= (0^{2m}, \pm 1, \pm 1, 0^{2n}, \pm 1, \pm 1, 0^{12-2n-2m}) \, .
\end{aligned}
\tag{4.31}
$$

The results of the previous section indicate that these states are charged under precisely two $U(1)$'s associated with two fixed points in IIA, and are uncharged with respect to any of the other $U(1)$'s. There are four states for each pair of $U(1)$'s, carrying ± 1 charges with respect to the two $U(1)$'s. In all cases $V^2 = 4$, and we must choose $N_R = c_R + 1$ to satisfy level-matching. These states are therefore *not* BPS, and transform in long multiplets of the $D = 6$ $\mathcal{N} = (1, 1)$ supersymmetry algebra. Their mass is given by

$$
m_h = 2\sqrt{2} \, ,
\tag{4.32}
$$

as follows from (4.16); in particular, the mass is independent of the radii.

On the other hand, these states carry the same charges as two BPS states of the form discussed in the previous section (where the charge with respect to the spacetime $U(1)$'s is chosen to be opposite for the two states), and they might therefore decay into them. Whether or not the decay occurs depends on the values of the radii, since the masses of

the BPS states depend on them. In particular, the first state in (4.31) carries the same charges as the two BPS states with $p_6 = \pm 1/2$, and weight vectors of the form

$$
\begin{aligned}
V_1 &= (0^{2n}, 1, 1, 0^{14-2n}) \\
V_2 &= (0^{2n}, 1, -1, 0^{14-2n}),
\end{aligned}
\tag{4.33}
$$

where we have assumed that m is even and written $m = 2n$; if m is odd, the two weight vectors are

$$
\begin{aligned}
V_1 &= (0^{2n}, 1, 1, 0^{14-2n}) \\
V_2 &= -(0^{2n}, 1, -1, 0^{14-2n}),
\end{aligned}
\tag{4.34}
$$

where $m = 2n+1$. The mass of each of these states is $1/(2R_{h6})$, and the decay is therefore energetically forbidden when

$$
R_{h6} < \frac{1}{2\sqrt{2}}.
\tag{4.35}
$$

More generally, the above non-BPS state has the same charges as two BPS states with $w_i = 0$, and internal weight vectors

$$
\begin{aligned}
V_1 &= \pm \left(0^m, 1, 0^k, 1, 0^{14-m-k}\right) \\
V_2 &= \pm \left(0^m, 1, 0^k, -1, 0^{14-m-k}\right),
\end{aligned}
\tag{4.36}
$$

where again the non-vanishing internal momenta are chosen to be opposite for the two states. The lightest states of this form have a single non-vanishing momentum, $p_i = \pm 1/2$ for one of $i = 6, 7, 8, 9$, and their mass is $1/(2R_{hi})$. Provided that

$$
R_{hi} < \frac{1}{2\sqrt{2}} \qquad i = 6, 7, 8, 9,
\tag{4.37}
$$

the non-BPS state cannot decay into any of these pairs of BPS states, and it should therefore be stable. Similar statements also hold for the non-BPS states of the second kind in (4.31).

We should therefore expect that the IIA theory possesses a non-BPS D-brane that has the appropriate charges and multiplicities. This state is easily constructed: it is a non-BPS D-string that stretches between the two fixed planes into whose fractional D-particles it can potentially decay. Let us for simplicity consider the state that stretches along x^6 from the origin to the fixed plane with coordinates $(\pi R_{A6}, 0, 0, 0)$, and let us denote the transverse position by $\mathbf{c}$ (where $\mathbf{c}$ has non-trivial coordinates along x^0, x^1, x^3, x^4, x^5). Then the boundary state is given as

$$
|\widehat{D1}, \mathbf{c}; \theta, \epsilon\rangle = |B1, \mathbf{c}; \theta\rangle_{\text{NS-NS}} + \epsilon \left(|B1, \mathbf{c}; 0\rangle_{\text{R-R;T}} + e^{i\theta}|B1, \mathbf{c}; (\pi R_{A6}, 0, 0, 0)\rangle_{\text{R-R;T}}\right),
\tag{4.38}
$$

where θ denotes a Wilson line which originates from the fact that the x^6 direction is compact so that the NS-NS vacuum can be characterised by a winding number w_6.[17] In

---

[17]The relevant closed string Hamiltonian contains then also an additional term $v^2/(4\pi)$, where v is the winding length.

fact, the boundary state $|B1, \mathbf{c}; \theta\rangle_{\text{NS-NS}}$ is defined by

$$|B1, \mathbf{c}; \theta\rangle_{\text{NS-NS}} = \sum_{w_6} e^{i\theta w_6} |B1, \mathbf{c}; w_6\rangle_{\text{NS-NS}}, \tag{4.39}$$

where $|B1, \mathbf{c}; w_6\rangle_{\text{NS-NS}}$ is given by (2.36), (4.18) and (2.13) except that $|B1, \mathbf{k}, \eta\rangle^{(0)}$ in (2.13) is now replaced by

$$|B1, \mathbf{k}, w_6\rangle^{(0)}. \tag{4.40}$$

This tachyonic ground state has winding number w_6 along the x^6 direction, and momentum equal to k^i for $i \neq 6$. Because it describes a D1-brane with a Neumann direction along x^2, we also have that $k^2 = 0$; furthermore the momenta k^i for $i = 7, 8, 9$ are again quantised. This boundary state is (as before) obviously invariant under the GSO-projection; invariance under the orbifold projection requires that $\theta = 0$ or $\theta = \pi$ (since $\mathcal{I}_4$ maps $w_6 \mapsto -w_6$.) The correct normalisation will turn out to be

$$R_{A7} R_{A8} R_{A9} \mathcal{N}_{\text{NS-NS}}^2 = \frac{1}{64} \frac{V_2}{(2\pi)^2}, \tag{4.41}$$

where $V_2 = \pi R_{A6} V_1$, with V_1 being the volume along the x^2-direction along which the D1-brane has a Neumann boundary condition.

The two boundary states in the twisted R-R sector are localised at different fixed planes, and are otherwise standard boundary states. Since the twisted R-R sector does not have any fermionic zero modes in the x^6 direction, the ground state satisfies the same zero mode condition as the D0-brane boundary state discussed above; this also implies that it is GSO-invariant. The parameter ϵ takes the values ± 1, and determines the sign of the twisted R-R charge at one end of the D1-brane. The correct normalisation will turn out to be

$$\mathcal{N}_{\text{R-R}}^2 = -\frac{1}{4} \frac{V_1}{(2\pi)}, \tag{4.42}$$

where V_1 is the world-volume along the x^2 direction.

In order to describe the corresponding open string it is convenient to use a different description for the orbifold [20]. Let us denote, as before, by $\mathcal{I}_4$ the reflection of the four coordinates $x^6, \ldots, x^9$, and let $\mathcal{I}_4'$ be defined by

$$\mathcal{I}_4' : \begin{cases} x^i \mapsto -x^i & \text{if } i \neq 6, \\ x^6 \mapsto 2\pi R_{A6} - x^6. \end{cases} \tag{4.43}$$

Let us consider the compactification where initially the radius of the sixth circle is $2R_{A6}$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of this theory that is generated by $\mathcal{I}_4$ and $\mathcal{I}_4'$ is then equivalent to the above orbifold. In order to see this we observe that $\mathcal{I}_4$ and $\mathcal{I}_4'$ commute with each other, and that both are of order two. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold can therefore equivalently be described as the $\mathcal{I}_4$-orbifold of the $\mathcal{I}_4' \mathcal{I}_4$-orbifold; however, $\mathcal{I}_4 \mathcal{I}_4'$ is the translation $x^6 \mapsto x^6 + 2\pi R_{A6}$, and its effect is simply to reduce the radius from $2R_{A6}$ to R_{A6}.

For the above choice of normalisation constants, the spectrum of open strings that begin and end on the above D-string is then given as

$$[\text{NS - R}] \ \frac{1}{4} \left(1 + (-1)^F \mathcal{I}_4\right) \left(1 + (-1)^F \mathcal{I}_4'\right), \tag{4.44}$$

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where the terms that involve $\mathcal{I}_4$ come from the twisted R-R sector localised at $\mathbf{0}$, the terms involving $\mathcal{I}_4'$ come from the twisted R-R sector localised at $(\pi R_{A6}, 0, 0, 0)$, and the remaining terms arise from the untwisted NS-NS sector. (More specifically, the term with the unit operator corresponds to the contribution where w_6 is even, whereas the term $1/4(-1)^F\mathcal{I}_4(-1)^F\mathcal{I}_4' = 1/4(x^6 \mapsto x^6 + 2\pi R_{A6})$ comes from the terms with w_6 odd.

Since θ and ϵ can only take two different values each, there are four different D-strings for each pair of orbifold points. These four D-strings are only charged under the two twisted sector $U(1)$s associated to the two fixed planes, and the four different configurations correspond to the four different sign combinations for the two charges. These charges are of the same magnitude as those of the fractional D-particles, since the ground state of twisted R-R sector contribution satisfies the same zero-mode condition, and has the same normalisation (compare (4.26) and (4.42)). Furthermore, it follows from (4.44) that the D-strings have sixteen (rather than eight) fermionic zero modes, and therefore transform in long multiplets of the $D = 6$, $\mathcal{N} = (1,1)$ supersymmetry algebra. These states therefore have exactly the correct properties to correspond to the above non-BPS states of the heterotic theory.
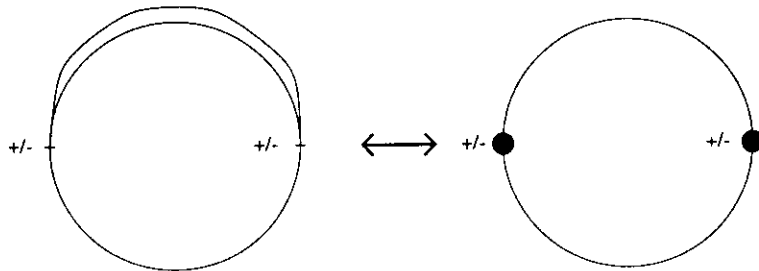
Figure 3: The non-BPS D1-brane and the two fractional D0-branes into which it can decay.

The open string NS sector in (4.44) contains a tachyon. However, since the tachyon is $(-1)^F$-odd, and since $\mathcal{I}_4$ reverses the sign of the momentum along the D-string, the zero-momentum component of the tachyon field on the D-string is projected out. Furthermore, since $\mathcal{I}_4\mathcal{I}_4'$ acts as $x^6 \to x^6 + 2\pi R_{A6}$, the half-odd-integer momentum components are also removed, leaving a lowest mode of unit momentum. As a consequence, the mass of the tachyon is shifted to

$$m_T^2 = -\frac{1}{2} + \frac{1}{R_{A6}^2}. \qquad (4.45)$$

For $R_{A6} < \sqrt{2}$ the tachyon is actually massive, and thus the non-BPS D1-brane is stable. On the other hand, for $R_{A6} > \sqrt{2}$ the configuration is unstable and decays into the configuration of two D-particles that sit at either end of the interval. These D-particles carry opposite untwisted R-R charge, and their twisted R-R charge is determined in terms of the twisted R-R charge of the D-string at either end.

One can also understand this instability from the point of view of the two fractional BPS D-particles. Since they carry opposite untwisted R-R charge, the open string between

them consists of

$$[\text{NS - R}] \frac{1}{4}\left(1 - (-1)^F\right)\left(1 \pm \mathcal{I}_4\right).$$

(4.46)

The ground state of the open string NS sector therefore has a mass

$$m^2 = -\frac{1}{2} + (\pi R_{A6} T_0)^2 = -\frac{1}{2} + \left(\frac{R_{A6}}{2}\right)^2,$$

(4.47)

and so becomes tachyonic for $R_{A6} < \sqrt{2}$, indicating an instability to decay into the non-BPS D-string. The D-string can therefore be thought of as a bound state of two fractional BPS D-particles located at different fixed planes. This is also confirmed by the fact that the classical mass of the D-string (4.52) is smaller than that of two fractional D-particles (4.30) when

$$R_{A6} < \sqrt{2},$$

(4.48)

and thus the D-string is stable against decay into two fractional D-particles in this regime (see Figure 3). In terms of the heterotic string, this decay channel corresponds to (4.33). Given the duality relation (4.10), the domain of stability of the non-BPS D-string (4.48) becomes in terms of the heterotic moduli

$$V_h^{-1/2} R_{h6} < 2\sqrt{2}.$$

(4.49)

Thus the D-string is stable provided that R_{h6} is sufficiently small; this agrees qualitatively with the regime of stability in the heterotic theory (4.35). (Since we are dealing with non-BPS states, one should not expect that these regimes of stability match precisely!)

Other decay channels become available to the D-string when the other distances R_{Ai} ($i = 7, 8, 9$) become small. In particular, the D-string along x^6 can decay into a pair of D2-branes carrying opposite bulk charges, i.e. a D2-brane and an anti-D2-brane, and wrapping the (x^i, x^6) cycle.
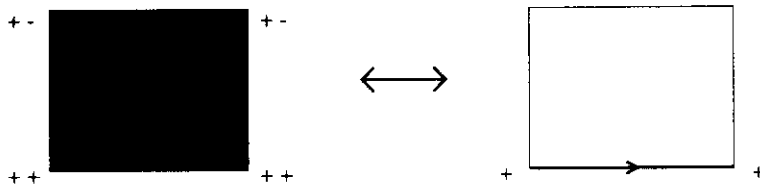

Figure 4: A D2-brane anti-brane pair and the non-BPS D1-brane into which it can decay. The twisted R-R charge of each D2-brane at each of the four corners is one half of that of the non-BPS D1-brane.

Since the mass of each D2-brane in the orbifold metric is $R_{Ai} R_{A6}/(2g_A)$, the D-string is stable in this channel when

$$R_{Ai} > \frac{1}{\sqrt{2}} \quad (i = 7, 8, 9).$$

(4.50)

The D-string can therefore also be thought of as a bound state of two BPS D2-branes. This decay channel can also be understood from the appearance of a tachyon on the D-string

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carrying one unit of winding in the x^i direction, when $R_{Ai} < 1/\sqrt{2}$ [23], or alternatively from the appearance of a tachyon between the two D2-branes when $R_{Ai} > 1/\sqrt{2}$. In terms of the heterotic string, these decay channels are described by (4.36). Using the duality relation (4.10) as before, (4.50) then becomes

$$V_h^{-1/2} R_{hj} < 2\sqrt{2} \qquad \text{for } j \neq 6 \,. \tag{4.51}$$

Thus the D-string is stable against this decay provided that R_{hj} is sufficiently small, and this agrees again qualitatively with the heterotic domain of stability (4.37). A similar analysis can also be performed for D-strings that stretch between any two fixed points.

One can also compare the mass of the non-BPS D1-brane with that of the dual heterotic state. As we mentioned before, one should not expect that these are related exactly by the duality map since for non-BPS states the masses are not protected from quantum corrections. Indeed, the classical mass of the above non-BPS D1-brane is given by

$$m_A(\widehat{D1}) = \frac{R_{A6}}{\sqrt{2}g_A} \,, \tag{4.52}$$

where the factor of $\sqrt{2}$ comes from the fact that the normalisation of the untwisted NS-NS component (4.41) is by a factor of $\sqrt{2}$ larger than that of the standard BPS D-brane of Type II (2.52). In heterotic units, this mass is $\propto 1/V_h$, and therefore does not agree with (4.32).

In the blow up of the orbifold to a smooth K3, the non-BPS D-strings correspond to membranes wrapping pairs of shrinking 2-cycles. Since such curves do not have holomorphic representatives, the states are non-BPS. For each pair of 2-cycles there are four states, associated with the different orientations of the membrane; the membrane can wrap both cycles with the same orientation, or with opposite orientation. In either case the net bulk charge due to $B = 1/2$ can be made to vanish by turning on an appropriate world-volume gauge field strength ($F = \pm 1$ in the first case, and $F = 0$ in the second). The decay of the non-BPS D-string into a pair of fractional BPS D-particles is described in this picture as the decay of this membrane into two separate membranes, that wrap individually around the two 2-cycles. It would be interesting to understand in more detail how non-BPS branes behave away from the orbifold point; first steps in this direction have recently been taken by [79].

Finally, the entire discussion also has a parallel in the T-dual theory that we analysed in the previous section. The non-BPS D-string that stretches along x^6 is mapped under T-duality to the non-BPS D-particle of the IIB orbifold. (The two different values $\theta = 0, \pi$ correspond to the two possible positions, and ϵ to the sign of the charge of the D-particle.) The non-BPS D-string can be formed as a bound state of a fractional D-particle and a fractional anti-D-particle (see Figure 3). Under T-duality, the D-particle anti-D-particle pair becomes a pair of a BPS D-string and an anti-D-string of the IIB theory that stretch along the x^6 direction; since the D-particles sit on different fixed planes, the BPS D-strings have a relative Wilson line. Thus the non-BPS D-particle can be understood as the bound state of a D1-brane anti-D1-brane pair with a relative Wilson line; this

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reproduces precisely the construction of Sen [20].[18] By T-duality it follows that the D-particle is stable provided that

$$R_i \geq \frac{1}{\sqrt{2}} \qquad i = 6, 7, 8, 9. \tag{4.53}$$

Similarly the other decay channels can also be related to decay channels considered by Sen.

4.3.2 Non-BPS D3-brane

In addition to the non-BPS D1-brane, the IIA theory also has a non-BPS D3-brane for which an analogous analysis applies. The corresponding boundary state has a component in the untwisted NS-NS sector, and a component in each of the eight twisted R-R sectors that are localised at the vertices of the cube along which the D3-brane stretches. The D3-brane is characterised by three $\mathbb{Z}_2$ Wilson lines (that determine the relative signs of the twisted R-R charge at the different end-points), and one additional sign (that determines the overall sign of the twisted R-R charges). The states in the twisted R-R sector are again GSO-invariant, since their ground state satisfies the same fermionic zero mode conditions as the D0-brane state. Furthermore, a careful analysis of the boundary state reveals [47] that the non-BPS D3-brane carries at each corner precisely one half of the twisted R-R sector charge of a fractional D0-brane. This normalisation is consistent with the decay process of the non-BPS D3-brane into a D2-brane anti-brane pair (see Figure 5) that is the analogue of the decay process of Figure 3.[19] The non-BPS D3-brane is stable against
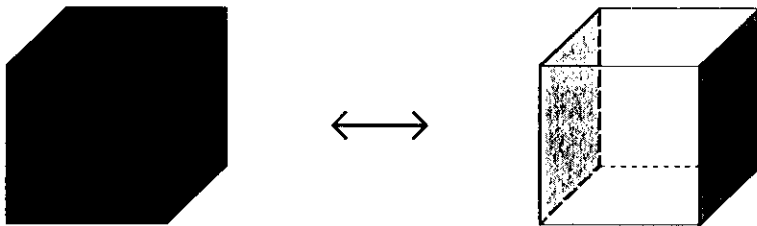

Figure 5: A non-BPS D3-brane and the D2-brane anti-brane pair into which it can decay.

this decay provided that the three radii along which it stretches are each smaller than $\sqrt{2}$. There is also a decay channel along which the D3-brane can decay into a D4-brane anti-D4-brane pair (this is the analogue of the decay process of Figure 4), and the non-BPS D3-brane is stable against this decay process provided that the transverse radius is larger than $1/\sqrt{2}$.

In order to identify the corresponding states in the heterotic string it is convenient to consider the different non-BPS D3-brane states (that are characterised by four signs

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[18]In the previous section we considered the uncompactified theory where all radii are infinite; in this regime the D-particle is stable.

[19]Indeed, the decay process of Figure 3 implies that the twisted R-R charge of each end of a non-BPS D1-brane is the same as that of a BPS D0-brane, and the decay process of Figure 4 implies that this charge is twice as large as the twisted R-R charge of a BPS D2-brane at each of its four corners.

and their position in the T^4) in conjunction with those non-BPS D3-brane states that correspond to the configuration where a non-BPS D1-brane is embedded within the D3-brane.[20] Since the magnitude of the twisted R-R charge at the end-point of the non-BPS D1-brane is twice that of the non-BPS D3-brane, the sign of the twisted R-R charge of the bound state differs at two vertices from that of the original D3-brane. Proceeding in this way, we can thus obtain D3-brane states with $2^7 = 128$ different sign combinations at the eight end-points. (For conventional D3-branes, the number of combinations was only $2^4 = 16$!) In addition we can localise the D3-brane in $2 \cdot 15 = 30$ different ways: there are fifteen different direction vectors between the vertices of the unit cell, and we can choose the D3-brane to be orthogonal to any one of them; for each such orientation, we can then localise the D3-brane at two different positions. Taking all of this together we are therefore looking for $30 \cdot 2^7$ states in the heterotic theory.

The states that correspond to these non-BPS D3-branes in the dual heterotic theory must be charged under eight of the sixteen $U(1)$s that are described following (4.28), but not under any of the other $U(1)$s. The charge with respect to each of these eight $U(1)$s must be precisely half of that of the states in (4.28). Furthermore, for each allowed set of eight such $U(1)$s (there are 30 such sets that correspond to the different localisations of the D3-brane) there are 128 such states that differ by the signs of the charges at the end points. Heterotic states with these properties can be found as follows: there are 128 states that are only charged under the first eight $U(1)$s, and the corresponding internal weight vectors are of the form

$$\left(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, 0^8\right), \tag{4.54}$$

where $\mathbf{a_i}$ is a two-dimensional vector which is equal to one of the following four vectors

$$\mathbf{e_1} = (1,0), \qquad \mathbf{e_2} = (-1,0), \qquad \mathbf{f_1} = (0,1), \qquad \mathbf{f_2} = (0,-1). \tag{4.55}$$

Of the $4^4 = 256$ combinations only those are allowed (*i.e.* have integer inner product with A^6) where an even number of the $\mathbf{a_i}$ are equal to $\mathbf{e_1}$ or $\mathbf{e_2}$ (and an even number of the $\mathbf{a_i}$ are equal to $\mathbf{f_1}$ or $\mathbf{f_2}$); this reduces the number of possibilities by half to the desired 128. It is not difficult to check that all of these states are only charged under the first eight $U(1)$s (provided we choose the momentum and winding numbers appropriately), and that the magnitude of the corresponding charge is precisely half that of the states in (4.28). Furthermore, these are the only states with this property.

We can similarly construct states that are charged under eight $U(1)$s by choosing different positions for the four $\mathbf{a_i}$ vectors in the sixteen dimensional space. Since the resulting states should not be charged under any other $U(1)$s, we have to demand that the internal weight vectors have integer inner product with all four Wilson lines; the

---

[20]One can presumably describe this configuration also as a non-BPS D3-brane with a non-trivial electric flux. It would be interesting to understand this in more detail.

possible configurations are then

$$
\begin{array}{ll}
\left(a, a, a, a, 0, 0, 0, 0\right), & \left(0, 0, 0, 0, a, a, a, a\right), \\
\left(a, a, 0, 0, a, a, 0, 0\right), & \left(0, 0, a, a, 0, 0, a, a\right), \\
\left(a, a, 0, 0, 0, 0, a, a\right), & \left(0, 0, a, a, a, a, 0, 0\right), \\
\left(a, 0, a, 0, a, 0, a, 0\right), & \left(0, a, 0, a, 0, a, 0, a\right), \\
\left(a, 0, a, 0, 0, a, 0, a\right), & \left(0, a, 0, a, a, 0, a, 0\right), \\
\left(a, 0, 0, a, a, 0, 0, a\right), & \left(0, a, a, 0, 0, a, a, 0\right), \\
\left(a, 0, 0, a, 0, a, a, 0\right), & \left(0, a, a, 0, a, 0, 0, a\right).
\end{array}
\tag{4.56}
$$

There are fourteen different such classes of states, and this construction accounts therefore for $14 \cdot 128$ states.

The remaining $16 \cdot 128$ states correspond to states in the *spinor representation* of $SO(32)$. These are the states whose internal weight vectors are of the form

$$
\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right),
\tag{4.57}
$$

where the number of + signs is even. Each of these 2^{15} states is charged under eight of the sixteen internal $U(1)$s. In order for the state to be uncharged under any other $U(1)$, we have to demand again that the inner product of the internal weight vector with each of the four Wilson lines is integral. For each Wilson line, this condition selects one half of the states, and since the four conditions are independent of each other, the number of states that have this property for all four Wilson lines is $2^{11} = 16 \cdot 128$. Together with the above $14 \cdot 128$ states we have therefore found all $30 \cdot 128$ states that correspond to D3-branes (including those that contain D1-branes within). It is also easy to see that these are all the states in the heterotic theory that have the above properties!

As we have seen above, there are $30 \cdot 16$ conventional non-BPS D3-brane configurations; these are mapped under T-duality (of all four circles) to the various non-BPS D1-brane configurations that we have discussed before; their number is

$$
4 \cdot \binom{16}{2} = 30 \cdot 16
\tag{4.58}
$$

and is therefore in agreement with the above. The remaining bound states of non-BPS D3-branes with non-BPS D1-branes are mapped into themselves under T-duality.

One can also analyse the stability of these non-BPS states in both theories. For example, the spinor state with internal weight vector

$$
\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)
\tag{4.59}
$$

has the same charges as the two BPS states with momenta

$$
\begin{aligned}
(P_L, p_L; p_R)_1 &= \left(-\tfrac{1}{2}, (\tfrac{1}{2})^6, -\tfrac{1}{2}, 0^8\right), R_{h9}; -R_{h9}\right) \\
(P_L, p_L; p_R)_2 &= \left((0^8, -\tfrac{1}{2}, (\tfrac{1}{2})^6, -\tfrac{1}{2}), -R_{h9}; R_{h9}\right).
\end{aligned}
\tag{4.60}
$$

The mass of all of the above non-BPS states in the heterotic theory is $m_h = 2\sqrt{2}$, whereas the mass of each of the two BPS states in (4.60) is $m_h = 2R_{h9}$; thus the non-BPS state (4.59) is stable against the decay into (4.60) provided that

$$R_{h9} > \frac{1}{\sqrt{2}} \,. \tag{4.61}$$

The two BPS states in (4.60) correspond, in the IIA theory, to two D2-branes that extend along the x^7, x^8 plane (this follows from the analysis at the end of section 4.2.), and the non-BPS D3-brane extends along the x^7, x^8, x^9 directions. The decay process that we are considering is therefore that depicted in Figure 5. The mass of the D3-brane is

$$m_A(D3) = \frac{1}{\sqrt{2}g_A} R_{A7} R_{A8} R_{A9} \,, \tag{4.62}$$

whereas the mass of each of the two D2-branes is

$$m_A(D2) = \frac{1}{2g_A} R_{A7} R_{A8} \,. \tag{4.63}$$

The non-BPS D3-brane is therefore stable against this decay process provided that

$$R_{A9} < \sqrt{2} \,. \tag{4.64}$$

(This is by the way what was already mentioned following Figure 5.) In terms of the heterotic theory the last equation becomes

$$V_h^{-\frac{1}{2}} R_{h9} > \sqrt{2} \,. \tag{4.65}$$

Again, this agrees qualitatively with (4.61). The other cases are similar.

4.3.3 Bose-Fermi degeneracy

BPS D-branes carrying identical charges do not exert any force on each other, and can be at equilibrium at all distances. This is a consequence of supersymmetry, and reflects the fact that the spectrum of open strings living on the world volume of the system has exact degeneracy between bosonic and fermionic states at all mass levels. As a result the partition function of the open strings, which corresponds to the negative of the interaction energy of the pair of D-branes, vanishes identically.

A non-BPS D-brane (such as the D-branes we have analysed above) breaks supersymmetry and the spectrum of open strings that begin and end on it does in general not have exact Bose-Fermi degeneracy. The open string partition function, and hence the interaction energy of a pair of such D-branes, is then not zero. The D-branes then exert a force on each other, and the system is not in equilibrium.

It was observed in [42] that the partition function depends non-trivially on the moduli (in particular the four radii), and that there exist special points in the moduli space where the spectrum develops exact Bose-Fermi degeneracy. For definiteness let us consider the

case of the non-BPS D-particle of the IIB orbifold. We are interested in the situation where all four directions along which the orbifold acts are compact; the boundary state for the D-particle is then given as in the previous section, except that the momentum integrals along $x^6 \ldots, x^9$ are replaced by sums, and that the normalisation constant in the untwisted NS-NS sector is changed to

$$R_6 R_7 R_8 R_9 \mathcal{N}^2_{\text{NS-NS}} = \frac{1}{128} \frac{V_1}{(2\pi)}. \tag{4.66}$$

(Details of this can again be found in [42].) The open string partition function is then given by

$$Z = \frac{1}{2} \int \frac{dt}{2t} \frac{V_1}{(2\pi)} (2t)^{-\frac{1}{2}} \left[\frac{f_4(\tilde{q})^8}{f_1(\tilde{q})^8} \left(\prod_{i=6}^{9} \sum_{n_i \in \mathbb{Z}} \tilde{q}^{2R_i^2 n_i^2} \right) - 4 \cdot \frac{f_3(\tilde{q})^4 f_4(\tilde{q})^4}{f_1(\tilde{q})^4 f_2(\tilde{q})^4} \right]. \tag{4.67}$$

Let us now consider the critical case where $R_i = \frac{1}{\sqrt{2}}$ for each $i = 6, 7, 8, 9$. In this case we get

$$\sum_{n_i \in \mathbb{Z}} \tilde{q}^{2R_i^2 n_i^2} = \sum_{n \in \mathbb{Z}} \tilde{q}^{n^2}. \tag{4.68}$$

Using the sum and the product representation of the Jacobi ϑ-function $\vartheta_3(0|\tau)$ [80],

$$\vartheta_3(0|\tau) = \sum_{n \in \mathbb{Z}} \tilde{q}^{n^2} = \prod_{n=1}^{\infty} (1 - \tilde{q}^{2n})(1 + \tilde{q}^{2n-1})^2 = f_1(\tilde{q}) f_3^2(\tilde{q}), \tag{4.69}$$

where $\tilde{q} = e^{2\pi i \tau}$, and the identity

$$f_4(\tilde{q}) \frac{1}{\sqrt{2}} f_2(\tilde{q}) f_3(\tilde{q}) = 1, \tag{4.70}$$

we get

$$\sum_{n \in \mathbb{Z}} \tilde{q}^{n^2} = \sqrt{2} \frac{f_1(\tilde{q}) f_3(\tilde{q})}{f_2(\tilde{q}) f_4(\tilde{q})}. \tag{4.71}$$

Using Eqs. (4.68) and (4.71), (4.67) then becomes

$$Z = 0. \tag{4.72}$$

Since the integrand of Z vanishes for all t, this shows that there is exact degeneracy between bosonic and fermionic open string states at all mass level, although the brane is non-BPS.

The critical radii where the spectrum of open strings develops exact bose-fermi degeneracy correspond precisely to the values below which the non-BPS D-brane becomes unstable against decay into a pair of BPS branes [20]. This is not a coincidence: For $R_i > \frac{1}{\sqrt{2}}$ the massless spectrum in light-cone gauge contains four bosonic states, but eight fermionic states. In order to have Bose-Fermi degeneracy at the massless level, we need four extra massless bosonic states: these are the would-be tachyons that precisely become massless at the critical point.

We can use this result to conclude that when $R_6 = R_7 = R_8 = R_9 = \frac{1}{\sqrt{2}}$, the force between a pair of non-BPS D-particles vanishes at all distances. To see this we note that if we consider a pair of such branes separated by a distance r in any of the non-compact directions transverse to the brane, then the partition function of open strings stretched from one of the branes to another is given by the same expression as (4.67) except for an overall extra factor of $\tilde{q}^{r^2/2\pi^2}$ in the integrand, reflecting the energy associated with the tension of the open string stretched over a distance r. Thus at the critical radius the partition function vanishes, reflecting that the potential energy $V(r)$ between the pair of branes (which is equal to negative of the partition function) vanishes identically for all r.

Since $\sum_{n_i \in \mathbb{Z}} \tilde{q}^{2R_i^2 n_i^2}$ is a monotonically decreasing function of R_i (as $0 < \tilde{q} < 1$), we see that for $R_i > \frac{1}{\sqrt{2}}$ the integrand of Eq. (4.67) is a negative definite function. Thus $V(r)$ is positive definite. Furthermore since $V(r)$ only depends on r via $\tilde{q}^{r^2/2\pi^2}$, it follows by the same argument that $V'(r)$ is negative, and hence that $V(r)$ is a monotonically decreasing function of r. Thus for $R_i > \frac{1}{\sqrt{2}}$, where the non-BPS brane is stable, the interaction between a pair of such branes is repulsive at all distances.

Acknowledgements

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