united nations educational, scientific and cultural organization

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SMR.1221 - 4

SPRING WORKSHOP ON SUPERSTRINGS AND RELATED MATTERS

27 March - 4 April 2000

SOLITONS AND INSTANTONS IN NON-COMMUTATIVE GAUGE THEORIES

Lectures I & II

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Plan of the mini-course

1) Introduction and Motivation

1a) Open Strings in background B-field

D-branes with $B \neq 0$

OPE of vertex operators $\longrightarrow$ star product in the Seiberg-Witten scaling limit

Non-commutative Yang-Mills theory from strings with $B \neq 0$

1b) M(atrix) theory with $C \neq 0$

M-theory on T^2 in DLCQ

Non-commutative torus

1c) M5-branes in M-theory with $C \neq 0$

DLCQ of M5-branes

Instanton moduli spaces and their resolution

1d) Kontsevich's deformation quantization

1e) Witten's open string field theory

2) A little bit of mathematical motivation (just mention)

2a) Analysis on non-Hausdorff spaces

Realization in 5d Chern-Simons theories

2b) Quantum groups, CFT

Lecture 1 and 2 will be devoted to explanations of the points above with the emphasis on 1

Lecture 3 and 4 : 3) From 1c) we shall ~~[illegible]~~ evolve into

3a) the study of instantons, their moduli spaces,

3a') (in 2d sigma models and 4d gauge theories)

3b) ADHM construction

3c) Finally — resolution of singularities of instanton moduli space.

3d) Interpretation of the resolution of singularities via Non-commutative gauge theory.

3e) Careful analysis $\Rightarrow$ Space-time Foam,

3f) its NC realization

D0 – D4 interpretation

4) Then, to justify the title "Solitons ..." we shall

4a) consider the monopoles : BPS equations

4b) Nahm's construction $\sim$ D1 – D3

4c) NC deformation, Explicit Solution in $U(1)$ case,

4d) Space – time interpretation

Lectures 1 and 2

Introduction and Motivation I.

- It has been widely appreciated by mathematicians (Gelfand, Grothendieck, von Neumann ...)

  that Geometric properties of the space X can be encoded in the properties of the algebra $C(X)$ of continious functions on X with values in $\mathbb{C}$.

  Actually $C(X)$ knows about topology, one can go to $C^\infty(X)$ or $\Omega^\bullet(X)$ (DeRham complex)

  to encode geometry.

- The algebra $A = C(X)$ is associative, commutative, unital
  $(f(gh) = (fg)h)$ $(fg = gf)$ $(f = 1 \in A)$

- The points of X $\longrightarrow$ maximal ideals in A $(f \in I_x \Leftrightarrow f(x) = 0)$
  $\longrightarrow$ irreducible representations of A
  $(R_x(f) = f(x)\cdot \quad)$
  $R_x \approx \mathbb{C}$

- Vector bundles over X = projective modules over A

(if $f \in A$ and
if σ is a section of E $\Rightarrow$ $f \cdot \sigma$ is also
a section)

(For any vector bundle E there exists
a bundle F, such that $E \oplus F$ = trivial bundle $\simeq A^{\oplus l}$

Hence $\Gamma(E)$ is a projective module over A

- Non-commutative geometry relaxes the
condition on A to be commutative
and tries to develop "geometrical"
intuition about associative algebras

WHY DO WE CARE?

Introduction and motivation Ia

Strings in background B-fields

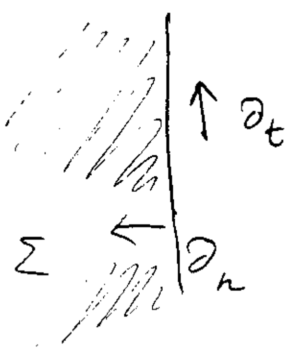
Open strings !

- Flat space, metric g_{ij}
- Constant NS B-field B_{ij}
- Dp-branes are present (so that B_{ij} ; i,j along Dp-brane cannot be gauged away)

World sheet action (bosonic string first)

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \left(g_{ij}\, \partial_a X^i \partial^a X^j - 2\pi i\alpha' B_{ij}\, \epsilon^{ab}\, \partial_a X^i \partial_b X^j \right) =$$

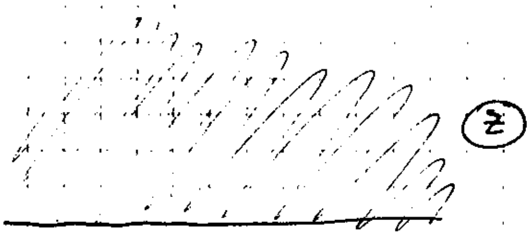
$$= \frac{1}{4\pi\alpha'}\int_\Sigma g_{ij}\, \partial_a X^i \partial^a X^j - \frac{i}{2}\int_{\partial\Sigma} B_{ij}\, X^i \partial_t X^j$$



Eq's of motion $\Rightarrow$ Boundary conditions:

$$g_{ij}\, \partial_n X^j + 2\pi i\alpha' B_{ij}\, \partial_t X^j \Big|_{\partial\Sigma} = 0$$

Take Σ – disc, $\approx$ upper half-plane



$$g_{ij}\,(\partial-\bar{\partial})X^j + 2\pi\alpha' B_{ij}\,(\partial+\bar{\partial})\,X^j\,\Big|_{z=\bar{z}} = 0$$

$$\langle X^i(z)\;X^j(z')\rangle = -\alpha'\Big[\; g^{ij}\log\left|\frac{z-z'}{z-\bar{z}'}\right| +$$

$$+\, G^{ij}\log|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\,\theta^{ij}\log\frac{z-\bar{z}'}{\bar{z}-z'} + D^{ij}\Big]$$

D^{ij} – $\mathrm{const}_{z,z'}$

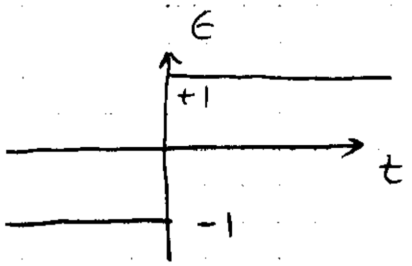
$$G^{ij} = \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_S = \left(\frac{1}{g+2\pi\alpha' B}\,g\,\frac{1}{g-2\pi\alpha' B}\right)^{ij}$$

$$\theta^{ij} = 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_A = -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}\,B\,\frac{1}{g-2\pi\alpha' B}\right)^{ij}$$

- Open string vertex operators:

$z = \bar{z} = t$

$$\langle x^i(t)\, x^j(t') \rangle = -\alpha' G^{ij} \log(t-t')^2 + \frac{i}{2}\theta^{ij}\epsilon(t-t')$$



$$[x^i(t), x^j(t)] = T\left(x^i(t)\, x^j(t-0) - x^i(t)\, x^j(t+0)\right) = i\theta$$

X^i BEHAVE AS COORDINATES ON NONCOMMUTATIVE SPACE!

- Limit of SW: $\alpha' \to 0$ G, θ - fixed

OPE of tachyon operators

$V_p(t) = \, :e^{ipx}(t):$

$$V_p(t)\, V_q(t') \sim (t-t')^{2\alpha' G^{ij} p_i q_j} e^{-\frac{i}{2}\theta^{ij} p_i q_j} V_{p+q}(t')$$

$$\longrightarrow e^{-\frac{i}{2}\theta^{ij} p_i q_j} V_{p+q}(t') \quad \text{in SW limit}$$

- The algebra of tachyonic operators —

  Moyal algebra:

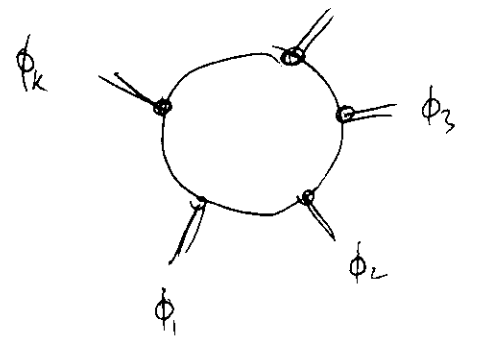
$$f * g\,(x) = \exp\left(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\eta^j}\right) f(\xi)g(\eta)\Big|_{\xi=\eta}.$$

- Effective action

$$\int d^{p+1}x\,\sqrt{\det G}\ \mathrm{Tr}\ \partial^{n_1}\phi_1\,\partial^{n_2}\phi_2\ldots\partial^{n_k}\phi_k \qquad (B=0)$$

$$\Downarrow$$

$$\int d^{p+1}x\,\sqrt{\det G}\ \mathrm{Tr}\ \partial^{n_1}\phi_1 * \partial^{n_2}\phi_2 * \cdots * \partial^{n_k}\phi_k \qquad \left(\begin{array}{l} B\neq 0 \\ \text{no SW limit}\end{array}\right)$$



Gauge theory from string theory

- Background gauge fields = interaction

$$S \to S_0 - i \int dt \; A_i(X)\, \partial_t X^i$$

($F \to 0$ at $X \to \infty$ therefore B cannot be gauged away)

- Gauge transformations

NAIVELY : $\delta A_i = \partial_i \lambda$

$\lambda(x)$ is WHERE OUR TACHYON ANALYSIS WILL HELP

$$\delta \int DX \; e^{-S_0 + i\int A_i dx^i} \approx \qquad i\int \partial_t \lambda \, dt$$

$$\approx \int DX \; e^{-S_0 + i\int A_i dx^i} \; i\int \partial_i \lambda \, dx^i =$$

$$= \int DX \, e^{-S_0} \left(i\int \partial_t \lambda \, dt + (-1) \underbrace{\int A_i dx^i \int \partial_t \lambda \, dt}_{\text{need's regularization}} + \dots \right)$$

$$\int dt_1 dt_2 \; : A_i(x(t_1))\, \partial_t X^i(t_1) : \; : \partial_t \lambda\,(x(t_2)) :$$

POINT SPLITTING

$$= \int dt \; : A_i(x(t))\, \partial_t X^i(t) : \; : \lambda(X(t+0)) - \lambda(X(t-0)) :$$

- AS $\quad A(x(t)) \cdot \lambda(x(t+0)) \sim A * \lambda$

$$A(x(t)) \; \lambda(x(t-0)) \sim \lambda * A$$

- and $\quad A * \lambda - \lambda * A \neq 0$

- WE SEE THAT

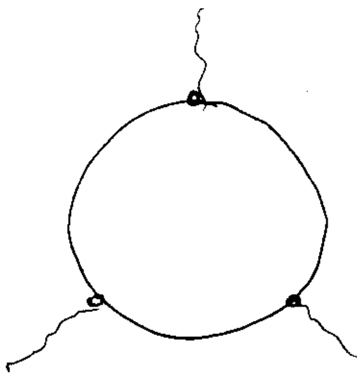
$$\delta A_i = \partial_i \lambda$$

- IS NOT A SYMMETRY OF WILSON LOOP IN THE POINT SPLITTING REGULARIZATION!

- INSTEAD : $\delta A_i = \partial_i \lambda + i A_i * \lambda - i A_i * \lambda$ (works in all orders in A)

THIS IS THE GAUGE INVARIANCE OF NON-COMMUTATIVE GAUGE THEORY

(1



$\Longleftrightarrow$ 3 pt function in NC Yang-Mills theory

$\alpha' \to 0$ limit

$$\xi_i^{(a)} \partial X^i e^{ipX(t)} = V^{(a)}(t)$$ — vertex operator corresp. to gauge field

$\xi_i p_j G^{ij} = 0$ $\quad p_i p_j G^{ij} = 0$ — physical vertex op's

NC YM:

$$\frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2} G_s} \int \sqrt{\det G}\; G^{ii'} G^{jj'} \,\mathrm{Tr}\, F_{ij} * F_{i'j'}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i - i A_i * A_j + i A_j * A_i$$

- Gauge invariance of $\mathrm{Tr}\, F*F$: is due to $\int A*B = \int B*A$

Introduction and motivation I b)

M − THEORY WITH $C \neq 0$

- The B-field of STRING THEORY COMES FROM THE THREEFORM GAUGE FIELD OF 11 D SUGRA

- $B_{\mu\nu} = C_{11\,\mu\nu}$

- M−THEORY, COMPACTIFIED ON A NULL CIRCLE

  $X^- = \frac{1}{2}(x^0 - x^1)$

  $X^- \sim X^- + 2\pi R$

  IS DESCRIBED BY A MATRIX THEORY =

  SUPERSYMMETRIC QUANTUM MECHANICS WITH 16 SUPERCHARGES (DEFINED UNIQUELY UP TO A CHOICE OF ASSOCIATIVE ALGEBRA)

$M = 1, \dots, 9$

- $$\mathcal{H} = \sum_{M} \mathrm{Tr}\, P_M^2 + \sum_{M<N} \mathrm{Tr}\, [X^M, X^N]^2 + \mathrm{Tr}\, \bar{\Psi} [X^M \Gamma_M, \Psi]$$

HAMILTONIAN + GAUSS LAW : $[P_M, X^M] + [\bar{\Psi}, \Psi] = 0$

$\bar{\Psi}^{\dot\alpha}$, Ψ_α – 16 COMPONENT SPINOR (MAYORANA-WEYL)

Γ_M – DIRAC MATRICES of $SO(9)$

- X^M, Ψ^M ARE USUALLY TAKEN TO BE

  $N \times N$ matrices, i.e. THE

  ASSOCIATIVE ALGEBRA

- $A = Mat_N(\mathbb{C})$ IS INVOLVED

- HOWEVER, MORE GENERAL ALGEBRAS CAN APPEAR —

  THERE ARE AS MANY ALGEBRAS AS MANY MAX. SUSY BACKGROUNDS OF M-THEORY

TORUS COMPACTIFICATION. T^d

- $x^M \sim x^M + 2\pi R_M$ $M = 2, \dots, d+2$

$\uparrow$

SPACE COORDINATES

- MATRIX RESPONSE TO THAT

$$\begin{cases} X^M + 2\pi R_M = U_M^{-1} X^M U_M \\ X^N = U_M^{-1} X^N U_M \qquad M \neq N \end{cases}$$

WHERE U_M's ARE UNITARY

- OF COURSE, WITH $Mat_N(\mathbb{C})$ WE CANNOT SOLVE THESE EQUATIONS

- BUT, WITH ALGEBRA OF FUNCTIONS ON A (DUAL) TORUS WE CAN:

$$X^M \sim \partial_M + A_M(\sigma)$$

$$U_M \sim \exp i \sigma_M R_M$$

(WE OMIT THE MATRIX INDICES)

- THE GAUGE TRANSFORMING MATRICES

U_M HAVE THE PROPERTY THAT

$Z_{MN} = U_M U_N U_M^{-1} U_N^{-1}$ COMMUTE WITH X^M

THESE ARE CENTER ELEMENTS.

ABOVE — $Z_{MN} = 1$, BUT IN GENERAL ONE COULD HAVE

$$Z_{MN} = \exp(2\pi i \theta_{MN}) \neq 1$$

- NON-COMMUTATIVE TORUS

$$\theta_{MN} \sim C_{-MN}$$

Introduction and Motivation Ic)

M5 - BRANES WITH $C \neq 0$

M5 BRANE IS THE "SOLITON" OF M-THEORY WHICH HAS (2,0) SIX DIMENSIONAL SUPERCONFORMAL THEORY ON ITS WORLDVOLUME.

SINGLE M5 CARRIES A TENSOR MULTIPLET WITH 5 SCALARS AND SELF-DUAL TENSOR FIELD

$$H^{+} = \qquad H = dB \qquad B_{\mu\nu}$$

$$B \sim B + d\lambda$$

WHEN SEVERAL M5'S COINCIDE THE "TENSOR" SYMMETRY BECOMES "NON-ABELIAN" AND MYSTERIOUS.

- IF M5 IS COMPACTIFIED ON A CIRCLE OF RADIUS R
- THEN THE AMBIENT M-THEORY BECOMES IIA STRING

WITH STRING COUPLING

$$g_s \approx \left(\frac{R}{L_{PL}}\right)^{3/2} \qquad \ell_s = \frac{L_{Pl}^{3/2}}{R^{1/2}}$$

- WHILE **M5** -BRANE BECOMES

D4 BRANE WITH ITS WORLDVOLUME GAUGE THEORY (WITH 16 SUSY'S)

$$g_{YM}^2 \sim g_s \ell_s = R$$

- IF WE HAD N UNITS OF MOMENTUM ALONG THE CIRCLE, THEN
- D4 BRANE IS SUPPLIED WITH
- N D0 BRANES $\longrightarrow$ INSTANTONS IN GAUGE THEORY

- NOW LET US MAKE THE CIRCLE LIGHTLIKE (EXCERCIZE : $\Leftrightarrow$ $R \to 0$)

COUPLING $\longrightarrow 0$ BUT INSTANTON CHARGE $\neq 0$ $\Rightarrow$

- WE GET ESSENTIALLY A QUANTUM MECHANICS ON THE MODULI SPACE OF INSTANTONS ON $\mathbb{R}^4$

- FOR k M5 BRANES
  WITH N units of momentum

- SQM with target $\mathcal{M}_{k,N}$ –
  – moduli space of $U(k)$ charge N instantons

- $S = \int d^4x \; \mathrm{Tr}\, F_{\mu\nu} F_{\mu'\nu'} \, g^{\mu\mu'} g^{\nu\nu'} \sqrt{g}$

  min. of $S \Rightarrow F_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$

  (anti) instantons

- k = rank of the matrices

- $N = -\frac{1}{8\pi^2} \int \mathrm{Tr}\, F \wedge F =$

  $= -\frac{1}{8\pi^2} \int \varepsilon_{\mu\nu\lambda\rho} \mathrm{Tr} F^{\mu\nu} F^{\lambda\rho}$

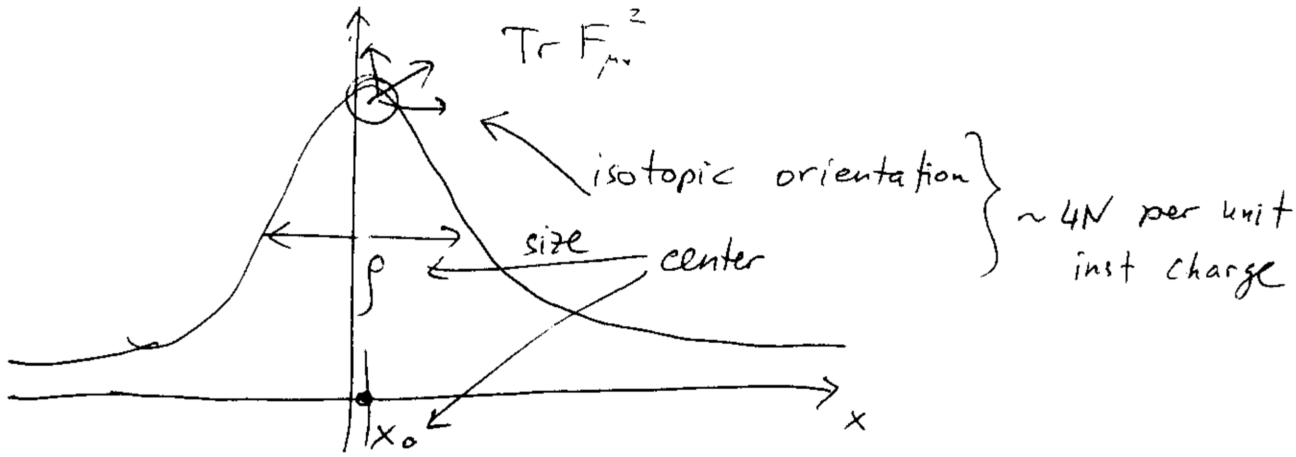
- $\dim \mathcal{M}_{k,N} = 4kN$

  $\uparrow$

  hyperkähler manifold

- SINGULAR

- TYPICAL INSTANTON



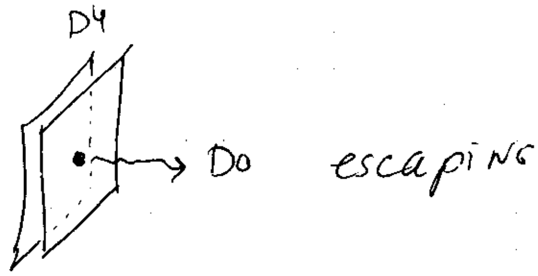
- SHRINKING INSTANTON

$\rho \to 0$

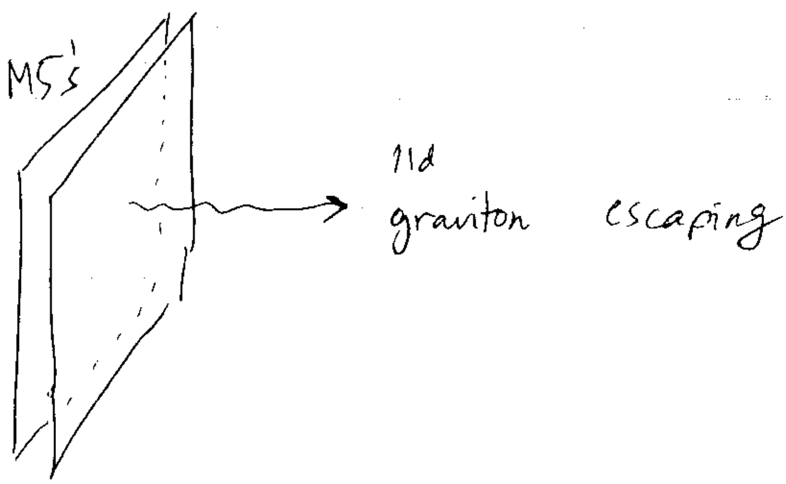
$$\mathrm{Tr}\, F_{\mu\nu}^2 \to \delta^{(4)}(x - x_0)$$

- SO, QM IS NOT GOING TO BE WELL-DEFINE

- SPACE - TIME PICTURE (TEN DIM'S)



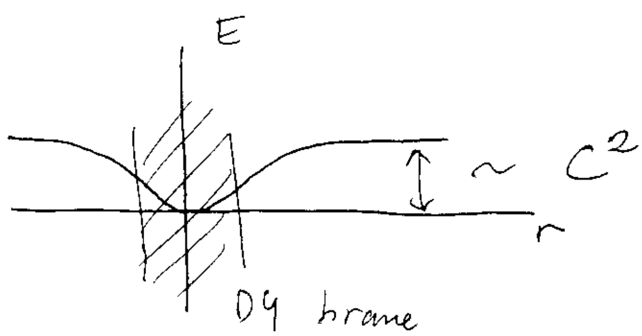
(ELEVEN DIM'S)



- IT TURNS OUT, THAT

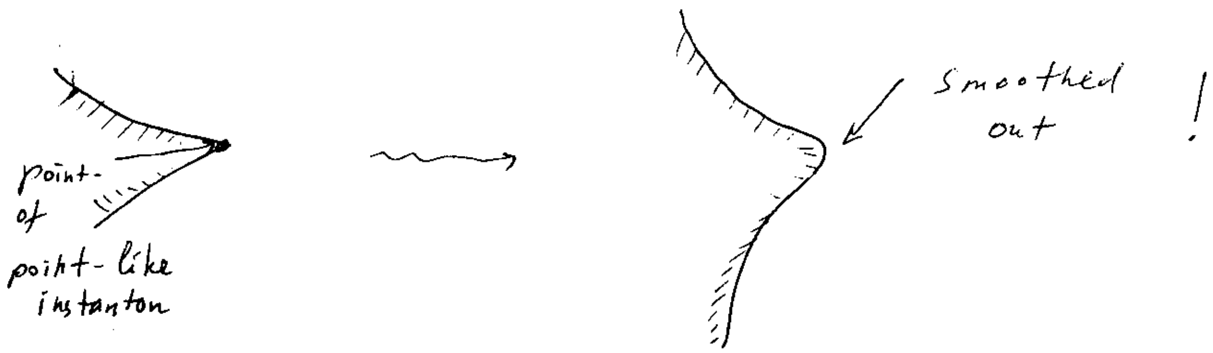
BY TURNING ON $C_{-\mu\nu}$ ($\mu\nu$ along D4)

THE D0-BRANES ARE GLUED TO D4



- WHAT HAPPENS ON THE QM SIDE?

$$\mathcal{M}_{k,N} \rightarrow \widetilde{\mathcal{M}}_{k,N}$$



- WHAT HAPPENED TO INSTANTONS?

THEY BECAME INSTANTONS ON NON-COMMUTATIVE $\mathbb{R}^4$

$$[X_\mu, X_\nu] \sim C_{-\mu\nu}$$

- AS $\Delta X_\mu \Delta X_\nu \sim C_{-\mu\nu} > 0$

NOTHING CAN SHRINK!

Introduction and motivation Id)

Kontsevich' deformation quantization

- Started out as mathematical question
- X - Poisson manifold
- $\{f, g\} = \pi^{ij} \partial_i f \, \partial_j g$
- $\{f, g\} = -\{g, f\}$
- $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$
- $A = C^\infty(X)$ - commutative algebra
- deform it into non-commutative associative

$$f g \longrightarrow f * g = fg + B_2^{ij} \partial_i f \partial_j g + B_3^{ijk} \partial_{ij}^2 f \partial_k g + \ldots + B_4^{ijkl} \partial_{ij}^2 f \, \partial_{kl}^2 g + \ldots$$

$$(f * g) * h = f * (g * h)$$

- Up to "gauge transformation"

$$f \mapsto f + D_1^i \partial_i f + D_2^{ij} \partial^2_{ij} f + \cdots$$

- turns out : B_2 up to D's $\Leftrightarrow$

Poisson structures.

- Question : fix π. Is there a set of B's ?

- Kontsevich found a positive answer to that Q, using open topological string

- Namely, he found a string action $\mathcal{L}$, such that functions on X are the physical open string vertex operators

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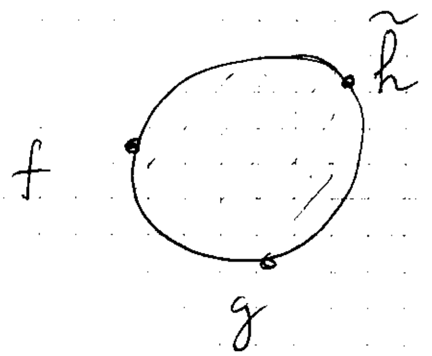

$$= \int_X f \star g \; \tilde{h}$$

- $\int f(X(0))\, g(X(1))\, \tilde{h}(X(\infty)) \quad \exp(-\mathcal{L})$

- $\mathcal{L} = \mathcal{L}_{free} + \int \pi^{ij} \, p_i \wedge p_j \; + \partial\pi \ldots \; + \partial^2\pi \ldots$

$$\mathcal{L}_{free} = \int p_i \wedge dX^i \; + \; \text{gauge fixing}$$

- Perturbation expansion in π of $\langle f(X(0))\, g(X(1))\, \tilde{h}(X(\infty)) \rangle$

- IS THE FORMULA FOR DEFORMATION QUANTIZATION

- ASSOCIATIVITY $* \longrightarrow$ CONSEQUENCE OF THE Q_{BRST}

- IT IS INTERESTING THAT THIS TOPOLOGICAL STRING IS A CLOSE COUSIN OF THE IIB (TWISTED) STRING ON Calabi-Yau manifold ($X^{\mathbb{C}}$ - complexification of X)

$$\{\pi\} \longleftrightarrow h^{-2,0}$$