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SPRING WORKSHOP ON SUPERSTRINGS AND RELATED MATTERS

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SOLITONS AND INSTANTONS IN NON-COMMUTATIVE GAUGE THEORIES

Lecture III

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Lecture 3

Instantons in non-commutative gauge theory

- Four dimensions

- $$S = \int \mathrm{Tr}\, F_{\mu\nu} * F_{\nu\mu}\, d^4x$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu$$

- Could be matrix-valued

$$F_{\mu\nu}{}^i{}_j = \partial_\mu A_\nu{}^i{}_j - \partial_\nu A_\mu{}^i{}_j + A_\mu{}^i{}_k * A^k_{\nu j} - A^i_{\nu k} * A^k_{\mu j}$$

- Recall: $a * b(x) = e^{i\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\eta^j}} a(\xi) b(\eta)\Big|_{\xi=\eta=x}$

- Equations of motion

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + A_\mu * F_{\mu\nu} - F_{\mu\nu} * A_\mu = 0$$

- Hard to solve!

- Instantons:

(1) $F_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \Rightarrow D_\mu F_{\mu\nu} = 0$

Easier to solve!

Euclidean signature

- How to look for solutions of (1)?

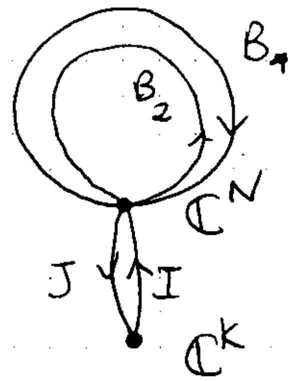
Instanton charge $N = -\frac{1}{8\pi^2} \int \mathrm{Tr}\, F \wedge F$

- Commutative case

ADHM (Atiyah - Drinfeld - Hitchin - Manin) construction

We study $U(k)$ instantons of charge N

QUIVER DIAGRAM



$V = \mathbb{C}^N$

$W = \mathbb{C}^k$

- B_1, B_2 – $N \times N$ complex matrices
- I : $k \times N$, J : $N \times K$ – complex matrices
- <u>ADHM</u> eq's $\qquad \mu^c = \bar{\mu}^c = \mu^r = 0$

$$\mu^c = [B_1, B_2] + IJ$$

$$\mu^r = [B_1, B_1^+] + [B_2, B_2^+] + II^+ - J^+J$$

<u>ADHM gauge group</u> $U(N)$ $\qquad B_1, B_2$ – in adjoint

I – fundamental

- J – antifundamental

Given a solution (B_1, B_2, I, J) to ADHM eq's

- <u>Complex of vector spaces</u>

$$z = (z_1, z_2) \in \mathbb{C}^2$$

$$V \xrightarrow{\sigma_z} V \otimes \mathbb{C}^2 \oplus W \xrightarrow{\tau_z} V$$

- $$\sigma_z = \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ J \end{pmatrix}$$

- $$\tau_z = \begin{pmatrix} -(B_2 - z_2) & -(z_1 - B_1) & I \end{pmatrix}$$

- ADHM eq's imply
$$\tau_z \sigma_z = 0 \qquad \forall z$$
$$\tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z$$

Constructing the gauge field

$$\mathcal{D}_z^{+} = \begin{pmatrix} \tau_z \\ \sigma_z^{+} \end{pmatrix} \; : \; V \otimes \mathbb{C}^2 \oplus W \longrightarrow V \otimes \mathbb{C}^2$$

"Dirac operator"

- $\mathcal{D}_z^{+} \Psi_z = 0 \qquad \Psi_z : W \to V \otimes \mathbb{C}^2 \oplus W$

- Normalize $\quad \Psi_z^{+} \Psi_z = \mathrm{Id}_W \qquad$ (possibility of doing this is a subtle matter)

- Gauge field

$$A_\mu = \Psi_z^{+} \partial_\mu \Psi_z$$

Solves $\qquad F_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} = 0 \qquad !!$

- Moduli space of instantons

(Recall our discussion of M5 branes)

$$\mathcal{M}_{k,N} = \{ (B_1, B_2, I, J) \mid \mu^{r} = 0, \mu^{c} = 0 \} / U(N)$$

$$B_{1,2} \to g^{-1} B_{1,2}\, g$$

$$I \to g^{-1} I$$

$$J \to J g$$

hyperkähler quotient

Example $k = 2$, $N = 1$

B_1, B_2 – complex numbers

a copy of $\mathbb{R}^4$

$$I = (i_1 \quad i_2)$$

$$J = (j_1 \quad j_2)^t$$

ADHM eq's IN THIS CASE GIVE

- $$\begin{cases} |i_1|^2 + |i_2|^2 = |j_1|^2 + |j_2|^2 \\ i_1 j_1 + i_2 j_2 = 0 \end{cases}$$

$$i \to e^{i\alpha} i \qquad j \to e^{-i\alpha} j \qquad \text{(gauge)}$$

- Let $X = i_1 j_1 = -i_2 j_2$

  $Y = i_1 j_2 \qquad Z = i_2 j_1$

- Then $X^2 = -YZ$

  and, given (X, Y, Z) obeying ~~this~~

  one gets unique solution to ADHM up to (gauge)

SO, THE MODULI SPACE

$$\mathcal{M}_{2,1} \approx \mathbb{R}^4 \times \text{A CONE} \quad X^2 + YZ = 0$$
$$\text{CONE} \cong \mathbb{R}^4/\mathbb{Z}_2 \qquad (X, Y, Z) \in \mathbb{C}^3$$

THE POINT $X = Y = Z$ — SINGULAR

- WHAT DOES THIS SINGULARITY MEAN PHYSICALLY ?

- LET US LOOK AT THE GAUGE FIELD :

$$\mathcal{D}_z^+ = \begin{pmatrix} z_2 - b_2 & b_1 - z_1 & i_1 & i_2 \\ \bar{b}_1 - \bar{z}_1 & \bar{b}_2 - \bar{z}_2 & \bar{j}_1 & \bar{j}_2 \end{pmatrix}$$

$$\mathcal{D}_z^+ \Psi_z = 0 \qquad \Psi_z = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}$$

ψ_i, χ_i are 1×2 matrices

- Shift $b \to 0$ by redefining z

$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ is a 2×2 matrix

$$i = (i_1 \;\; i_2) \qquad j = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}$$
$$(1 \times 2) \qquad\qquad (2 \times 1)$$

$$\psi_1 = \frac{z_1 j^+ - \bar{z}_2 i}{r^2} \chi$$

$$\psi_2 = \frac{z_2 j^+ + \bar{z}_1 i}{r^2} \chi$$

- Normalization gives

$$\chi^+ \left(1 + \frac{jj^+ + i^+ i}{r^2}\right) \chi = 1_{2\times 2}$$

$$\Rightarrow \chi = \frac{r}{\left(r^2 + jj^+ + i^+ i\right)^{1/2}}$$

(choice of gauge $\chi^+ = \chi$)

- We can rotate the vectors i and j by the $SU(2)$ transformation into the form

$$i = (\rho \quad 0) \qquad j = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \qquad \rho \geq 0$$

Then: $i^+ i + jj^+ = \rho^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- and The gauge field comes out to be

- $$A = \frac{\rho^2}{r^2(r^2+\rho^2)} \begin{pmatrix} \frac{1}{2}\left(z_2 d\bar{z}_2 + z_1 d\bar{z}_1 - \bar{z}_2 dz_2 - \bar{z}_1 dz_1\right) & z_1 dz_2 - z_2 d \\ \bar{z}_2 d\bar{z}_1 - \bar{z}_1 d z_2 & -\frac{1}{2}(\;\cdots\;) \end{pmatrix}$$

So it is traceless, anti hermitian, and :

- $F = dA + A \wedge A$ is anti-self-dual.
- BPST INSTANTON

Moreover

- $$\mathrm{Tr}\, F \wedge F \sim \frac{\rho^2}{(\rho^2 + x^2)^3}\, d^4x$$

- Now let us return to our moduli space $\mathcal{M}_{1,2}$.
- $$\rho^2 = \tfrac{1}{2}\left(i^+ i + j j^+\right)$$
- and $\mathcal{M}_{2,1} \sim \{X^2 + YZ = 0\} \times \mathbb{C}^2$
- $X = Y = Z = 0$ was the orbifold ($\mathbb{R}^4/\mathbb{Z}_2$ type) singularity point
- yet it corresponds to $\rho^2 = 0$
- POINT-LIKE INSTANTON
- So, the singularities of $\mathcal{M}_{K,N}$ come from point-like instantons

- In our discussion of the M5 brane quantization we came upon a susy QM whose target space was $\mathcal{M}_{k,N}$

- SQM on a space with singularities (even orbifold ones) is ill-defined (or, rather, behaves irregularly when the singularities are smoothed out)

- Instead of SQM on $\mathcal{M}_{k,N}$ one could look at 2d sigma model on $\mathcal{M}_{k,N}$ ($\mathcal{M}_{k,N}$ being hyperkähler $\Rightarrow$ σ-model has $(4,4)$ susy actually SCFT)

Such sigma model arizes in the study of D1–D5 system, little string theory, ...

- Superconformal sigma model on a space with orbifold singularity is secretly a sigma model on a blowup orbifold singularity

- SO, $\mathcal{M}_{K,N} \to \widetilde{\mathcal{M}}_{K,N}$

  THERE INDEED EXISTS A HYPERKÄHLER SMOOTH MANIFOLD $\widetilde{\mathcal{M}}_{K,N}$ WHICH IS a RESOLUTION OF SINGULARITIES OF $\mathcal{M}_{K,N}$.

- IN PARTICULAR, THE ORBIFOLD POINTS ARE REPLACED BY TWO-SPHERES, WHOSE SIZE ζ IS A MODULUS OF $\widetilde{\mathcal{M}}_{K,N}$

  $\zeta \to 0$ $\widetilde{\mathcal{M}}_{K,N}$ BECOMES $\mathcal{M}_{K,N}$

- IN OUR EXAMPLE OF BPST

$\mathcal{M}_{1,1}$

$$\tilde{\mathcal{M}}_{1,1} = \mathbb{R}^4 \times \begin{cases} i_1 j_1 + i_2 j_2 = 0 \\ |i_1|^2 + |i_2|^2 - \left(|j_1|^2 + |j_2|^2\right) = 5 \end{cases}$$

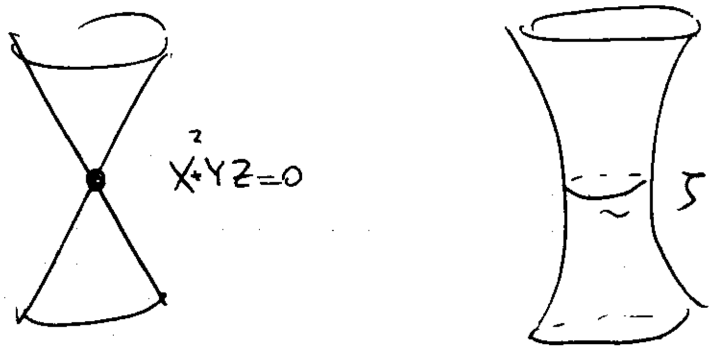
- NOW THE "BAD" POINT $i = j = 0$ IS ABSENT.

IF WE WANT TO MINIMIZE $|i|^2 + |j|^2$ WE CAN SET

$j = 0$, THEN

$$\begin{cases} |i_1|^2 + |i_2|^2 = 5 \\ (i_1, i_2) \to (i_1 e^{i\alpha}, i_2 e^{i\alpha}) \end{cases} = \text{COPY OF } \mathbb{CP}^1$$

- SO, THE CONICAL SINGULARITY IS REPLACED BY a SPHERE



$\mathcal{M}_{1,2}$ $\quad\quad$ $\tilde{\mathcal{M}}_{1,2}$

- FOR GENERAL k, N

REPLACE ADHM EQ'S

$\mu^c = 0 \quad \mu^r = 0$ BY

$\mu^c = 0 \quad \mu^r = \zeta \cdot \mathbb{1}_{N\times N}$

FI TERM – CONSISTENT WITH BOTH $U(N)$ gauge and HYPERKÄHLER STRUCTURE $\sim (4,4)$ SUSY

- $\widetilde{\mathcal{M}}_{K,N}$ is nice smooth space, but what does it parameterize?

Recall that out of (B, I, J) we constructed (τ_z, σ_z) and ψ_z, and

$$A_\mu = \psi_z^\dagger \partial_\mu \psi_z$$

- AND, A BEING (ASD) DEPENDED CRUCIALLY ON ADHM EQ'S

$$\begin{cases} \tau_z \sigma_z = 0 \qquad \sigma_z^\dagger \tau_z^\dagger = 0 \\ \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z \end{cases}$$

- BY $\mathcal{M}_{K,N} \to \widetilde{\mathcal{M}}_{K,N}$ WE ALTERED

$$\tau_z \tau_z^\dagger - \sigma_z^\dagger \sigma_z \sim \zeta \neq 0$$

- HOWEVER, WE ASSUMED THAT

$$[z_1, z_1^+] = [z_0, z_0^+] = 0$$ IN THE LAST EQ'N.

- SUPPOSE THAT WE WISH TO WORK NOT OVER $\mathbb{R}^4$ BUT OVER

NON-COMMUTATIVE $\mathbb{R}^4$

$$[z_1, z_1^+] + [z_2, z_2^+] = -\zeta \neq 0$$

$$[z_1, z_2] = [z_1^+, z_2^+] = 0$$

- THEN ADHM EQ'S HOLD AGAIN AND WE CAN PROCEED WITH CONSTRUCTING INSTANTONS AGAIN!

- EXCERSIZE: REWORK $k=2$, $N=1$ EXAMPLE FOR

$$[z_1, z_1^\dagger] = \zeta_- - \tfrac{1}{2}\zeta$$
$$[z_2, z_2^\dagger] = -\zeta_- + \tfrac{1}{2}\zeta$$

- NOTE, THAT $\widetilde{\mathcal{M}}_{1,N}$ IS A NON-EMPTY MANIFOLD (RESOLUTION OF DIAGONALS IN THE SYMM. PRODUCT $(\mathbb{C}^2)^N/S_N$)

ABELIAN INSTANTONS ?

- EXPLICIT CALCULATION

- RELATION TO CALOGERO - MOSER MANY - BODY SYSTEM