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SPRING WORKSHOP ON SUPERSTRINGS AND RELATED MATTERS

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SOLITONS AND INSTANTONS IN NON-COMMUTATIVE GAUGE THEORIES

Lecture IV

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Lecture IV

"Monopoles in non-commutative gauge theory"

(based on D. Gross, N. Nekrasov, to appear)

Monopoles — static minima of the energy, non-singular field configurations, non-trivial magnetic charge

gauge group G, charged matter ϕ,

$\phi \neq 0$ breaks $G \to H$.

if at infinity

the map $[\phi] : S^2 \to G/H$

is non-trivial in $\pi_2(G/H)$

(related to $\pi_1(H)$ — magnetic charges for H)

then there exists a minimum of

$$\mathcal{E} = \int d^3x \, \mathrm{Tr}\, F_{\mu\nu}^2 + \| D_\mu \phi \|^2 + V(\phi) + \ldots$$

More SPECIFICALLY,

LET ϕ BE A HIGGS FIELD

IN THE ADJOINT REPRESENTATION,

THEN

$$\mathcal{E} = \frac{1}{2}\int \mathrm{Tr}\, F_{\mu\nu}^2 + \mathrm{Tr}\,(D_\mu \phi)^2 =$$

$$= \int \frac{1}{2}\,\mathrm{Tr}\left(F_{\mu\nu} \pm \frac{1}{2}\varepsilon_{\mu\nu\lambda} D_\lambda \phi\right)^2 \mp \int d\, \mathrm{Tr}\, \phi F$$

For fixed $\int_{S^2_\infty} \mathrm{Tr}\, \phi F$ given

The absolute minimum of $\mathcal{E}$ (if achieved) is at

$$F = \pm * D\phi \quad , \text{or} \quad F_{\mu\nu} = \pm \varepsilon_{\mu\nu\lambda} D_\lambda \phi$$

$\boxed{\text{Bogomolny Equations}}$

For example, for $G = SU(2)$ $H = U(1)$

$$\|\phi\|^2 \to \frac{a^2}{2} \quad \text{at} \quad r \to \infty$$

$$\phi = \frac{1}{2}\left(\frac{a}{\tanh(ra)} - \frac{1}{r}\right)\sigma_3$$

't-Hooft – Polyakov monopole

(note that at $r \to \infty$

$$\phi \sim \frac{1}{2}\left(a - \frac{1}{r}\right)\sigma_3 \quad - \left(\begin{array}{c}\text{abelian}\\ \text{monopole}\end{array}\right)$$

at $r \to 0$ $\quad \phi \to 0$ $\quad$ SU(2) restored)

If, on the other hand we formally put $G = U(1)$

then $D_\mu \phi = \partial_\mu \phi$ and

$F_{\mu\nu} = \pm \frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\lambda \phi$ has a

SINGULAR solution

$$\phi = \pm \frac{1}{r} \qquad A = \frac{1}{2}(1 \pm \cos\theta)\, d\varphi$$

(ill-defined at $r=0$)

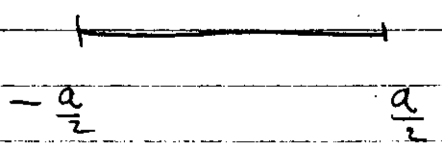
How does one solve the monopole equations for higher magnetic charges?

<u>Analogue of ADHM</u> :

<u>NAHM's Equations</u>

$G = SU(2)$

Take interval $\left(-\frac{a}{2}, \frac{a}{2}\right)$



$T_i(z) \qquad -\frac{a}{2} < z \leq \frac{a}{2}$

$\downarrow$ $k \times k$ matrices $\qquad T_i(z) = T_i^{\dagger}(z)$.

Equations:

$$\partial_z T_i = i \; \varepsilon_{ijk} T_j T_k$$

Bndry cndt's : $z \to z_0$ $\quad z_0 = \pm \frac{a}{2}$

$$T_i(z) \longrightarrow \frac{t_i}{z - z_0} + \text{reg.}$$

$$[t_i, t_j] = i \varepsilon_{ijk} t_k$$

t_i must form an IRREDUCIBLE
k- DIM'L REPRESENTATION
OF SU(2)

Then LOOK FOR A FUNDAMENTAL
(NON-SINGULAR) SOLUTION TO

$$\partial_z \psi - \tau_i \sigma_i \psi = 0$$

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \qquad \tau_i = T_i(z) + x_i$$

$\Psi_\pm$ are $k \times 2$ matrices

NORMALIZE Ψ AS

$$\int_{-a/2}^{+a/2} dz \quad \Psi^+ \Psi = I_{2\times 2}$$

THEN:

$$A_\mu = \int_{-a/2}^{+a/2} dz \, \Psi^+ \partial_\mu \Psi$$

$$\Phi = \int_{-a/2}^{+a/2} dz \quad z \, \Psi^+ \Psi$$

BOGOMOLNY EQ'S ON (A, Φ)

FOLLOW FROM $\partial_z T_i = i \varepsilon_{ijk} T_j T_k$

REMARK REPLACE $\left(-\frac{a}{2}, \frac{a}{2}\right)$ BY

$$(-\infty, a)$$

$\Psi_\pm$ BY 1×1 MATRICES

THEN, FOR $T_i = x_{i0}$ $\quad i = 1, 2, 3$

ONE GETS :
$$\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} \sqrt{x_3 - x_{30} + |\vec{r} - \vec{r}_0|} \\ \dfrac{x_1 - x_{10} + i(x_2 - x_{20})}{\sqrt{x_3 - x_{30} + |\vec{r} - \vec{r}_0|}} \end{pmatrix} e^{|\vec{r} - \vec{r}_0|(z-a}$$

$$\phi = a - \frac{1}{2|\vec{r} - \vec{r}_0|}$$

$\vec{A}$ — magnetic monopole centered at $\vec{r}_0 = (x_{10}, x_{20}, x_{30})$

SUPPOSE WE WANTED
TO LOOK FOR THE MONOPOLE
SOLUTIONS IN THE
NON-COMMUTATIVE GAUGE THEORY.

JUST AS IN THE CASE
WITH INSTANTONS THE MAIN
THING IS TO KEEP THE
EQUATION ON

$$\mathcal{T}_i = T_i + x_i$$

KEPT INTACT

THEREFORE IF

SPACE COORDINATES OBEY

$$[X_1, X_2] = -i\theta \qquad \theta > 0$$

$$[X_1, X_3] = [X_2, X_3] = 0$$

THEN NAHM'S EQUATIONS

ARE MODIFIED TO:

$$\partial_z T_i = i\,\varepsilon_{ijk}\, T_j T_k + \theta\, \delta_{i3} \qquad (*)$$

UNLIKE INSTANTON CASE, (*)

ARE TRIVIAL TO SOLVE

$$T_i^{\text{non-comm}}(z) = T_i^{\text{comm}}(z) + \theta z\, \delta_{i3}$$

solution to ordinary Nahm

EXPLICITLY CHARGE ONE:

$T_{1,2} = 0$, $T_3 = \theta z$ (shift of T_i's by a constant)

AFTER SOME ALGEBRA

WITH $a = \frac{1}{\sqrt{2\theta}}(x_1 - i x_2)$

$$a^{+} = \frac{1}{\sqrt{2\theta}}(x_1 + i x_2)$$

$$b = \frac{1}{\sqrt{2\theta}}(\partial_z + x_3 + \theta z)$$

$$b^{+} = \frac{1}{\sqrt{2\theta}}(-\partial_z + x_3 + \theta z)$$

$$[b, b^{+}] = [a, a^{+}] = 1, \quad [a, b] = [a^{+}, b^{+}] = \ldots = 0$$

ONE GETS :

$$\Phi = \Phi(x_3, \overset{N}{a^{+}a}) = \frac{\zeta_{N+1}}{\zeta_N} - \frac{\zeta_{N+2}}{\zeta_{N+1}} =$$

$$= \partial_3 \log \xi_N$$

$$\zeta_N = \int_0^{\infty} t^N e^{-\frac{t^2}{2} + 2 t x_3} dt$$

$$\xi_N = \sqrt{\frac{\zeta_N}{\zeta_{N+1}}}$$

$$A_3 = 0$$

$$A_1 + i A_2 = 2 a^{+} \left(1 - \frac{\xi_N}{\xi_{N+1}}\right)$$

$$A_1 - i A_2 = 2 \left(1 - \frac{\xi_N}{\xi_{N+1}}\right) a$$

CHECK BOGOMOLNY EQUATIONS

DISCOVER "BNDRY" TERM

$$\frac{1}{2}\varepsilon_{\mu\nu\lambda} D_\mu \phi + F_{\nu\lambda} = \frac{1}{2}\varepsilon_{\mu\nu 3}\, \partial_3 \frac{1}{\zeta_0} \quad \delta_{N,0}$$

ORIGIN: IN F_{12} ONE

HAS A TERM

$$F_{12} = \quad \dots \quad + 2\left(a^{+} \frac{\zeta_N}{\zeta_{N+1}}\, a\, \frac{\zeta_{N+1}}{\zeta_N} \right)$$

WHILE IN

$D_3\phi$ THERE IS A TERM $-2N \dfrac{\zeta_{N-1}\zeta_{N+1}}{\zeta_N^2}$

THESE TWO TERMS ARE IDENTICAL (13)

FOR $N>0$

BUT AS $N \to 0$ $\quad a|0\rangle = 0$

WHILE

$$\lim_{N\to 0} N S_{N-1} \neq 0$$

HENCE THE <u>DIFFERENCE</u> AT $N=0$

<u>CRUCIAL FOR CONSISTENCY.</u>

As $r^2 = X_3^2 + N \to 0$ $\quad \phi \sim -\frac{1}{2r}$

$N>0$

$F \sim$ usual $U(1)$ monopole gauge field

THEREFORE, WITHOUT $\delta_{N,0}$ TERM WE WOULD GET

$$0 < \int F_{\mu\nu}^2 + D_\mu\phi * D_\mu\phi \sim \int d(\phi * F + F * \phi) \to 0$$

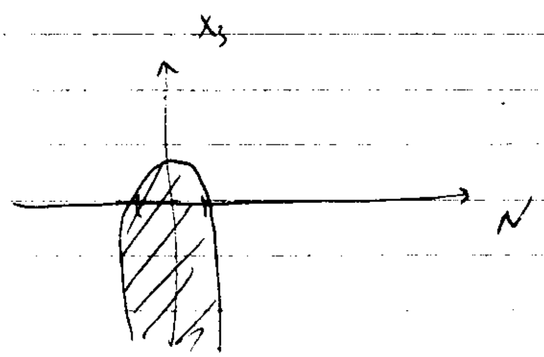
$\delta_{N,0}$

WITH THIS TERM

$$\mathcal{E} \sim \sum_N \delta_{N,0} \int \frac{dx_3 \; t(x_3)}{g_{YM}^2 \sqrt{\theta}}$$

$$t(x_3) = \left(\partial_3 \frac{1}{\zeta_0} \right)^2 \sim \begin{cases} \text{const.} & x_3 \ll 0 \\ \sim x_3^2 e^{-4x_3^2} & x_3 \gg 0 \end{cases}$$

Energy distribution



IF TIME PERMITS

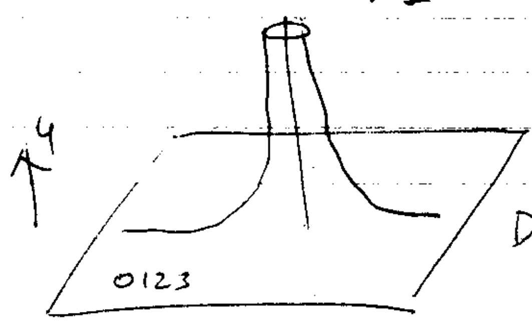
I WILL EXPLAIN

D-BRANE REALIZATION

AND MATCHING OF THIS

PICTURE WITH D1-D3 CONFIGURATION

F1 — fundamental string



$X^4 \sim \frac{1}{r}$ ← D3 BRANE PROFILE, LOOKS LIKE STRING

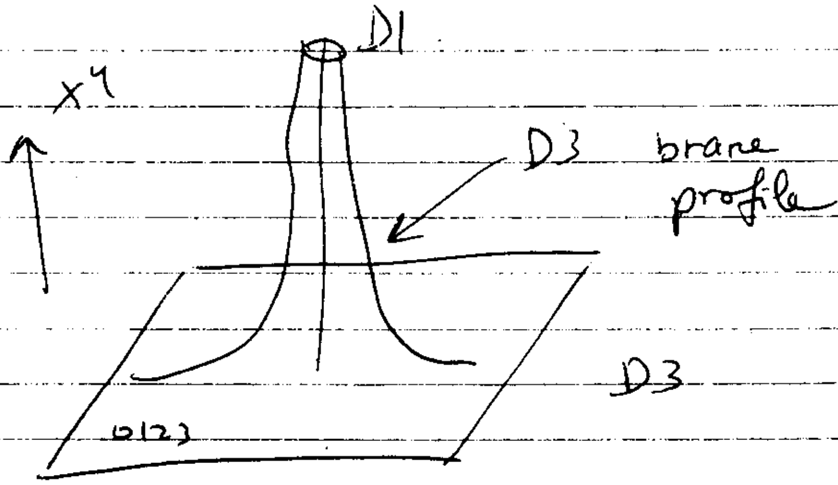
ELECTRIC CHARGE

$\vec{E} \sim \vec{\nabla}\left(-\frac{1}{r}\right)$

$\vec{E} + \vec{\nabla} X^4 = 0 \Rightarrow$ SUSY

$\vec{A} = 0$

$A_0 = -\frac{1}{r}$



MAGNETIC CHARGE

$$\vec{B} + \vec{\nabla} X^4 = 0 \qquad \Rightarrow \text{SUSY}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad A_0 = 0$$

TURN ON B-field $B_{12} = \theta$

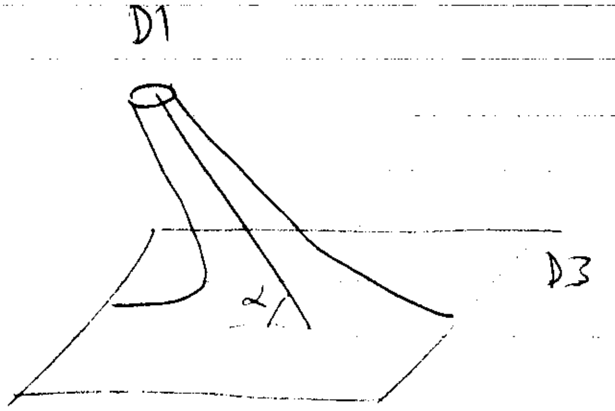
SUSY

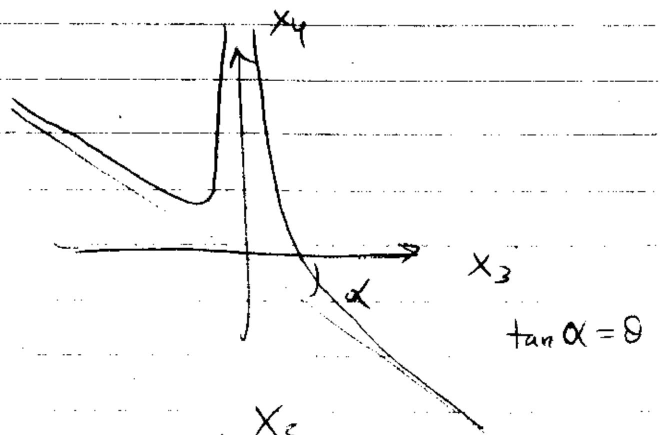
$$*(F+B) + D_A\phi = 0$$

$$F_{12} + \theta + \partial_3\phi = 0$$

$$F_{13} + \partial_2\phi = 0 \qquad F_{23} + \partial_1\phi = 0$$

$$X_4 = \phi = -\theta X_3 + \frac{1}{r}$$





ENERGY DENSITY :

EXACT MATCH !