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Geometry of minimal rational curves on Fano manifolds

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Abstract

This lecture is an introduction to my joint project with N. Mok where we develop a geometric theory of Fano manifolds of Picard number 1 by studying the collection of tangent directions of minimal rational curves through a generic point. After a sketch of some historical background, the fundamental object of this project, the variety of minimal rational tangents, is defined and various examples are examined. Then some results on the variety of minimal rational tangents are discussed including an extension theorem for holomorphic maps preserving the geometric structure. Some applications of this theory to the stability of the tangent bundles and the rigidity of generically finite morphisms are given.

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Contents

A Fano manifold is a smooth projective variety whose anti-canonical class K^{-1} is ample. In dimension 1, the Riemann sphere $\mathbf{P}_1$ is the only example of a Fano manifold. In higher dimensions, two possible sources of complication exist, as usual:

(i) The product of two Fano manifolds is again a Fano manifold, or more generally, many fiber bundles over Fano manifolds with Fano fibers are themselves Fano.

(ii) The blow-up of a Fano manifold with a suitable center is again Fano.

To handle complications of these sorts in higher dimensions, the minimal model program has been developed since 1980's. Many aspects of this program were surveyed by several authors, e.g., (e.g.[KM]), and I believe that some of these will be touched upon in some other lecture series of this school. For uniruled varieties, there is another machinery in handling these matters, developed in 1990's based on the concept of rationally connected varieties, which was nicely surveyed in [Kl] and [Mi]. For these reasons, I will exclude problems of this sort completely from our discussion, by assuming that our Fano manifolds have Picard number 1.

There have been a large number of works on Fano manifolds of Picard number 1, including a complete classification for dimension 3 and those with high indices (see [IP] and the references therein). Here, the index of a Fano manifold means the positive integer representing the anti-canonical bundle, when we identify the Picard group of the Fano manifold with $\mathbf{Z}$. The methods employed in these works include the classical method of double projections, adjunction theory, vector bundle techniques, as well as methods coming from the minimal model program.

In the last few years, Ngaiming Mok and I have been trying to develop a geometric theory of Fano manifolds of Picard number 1 from a different perspective, by using rational curves of minimal degree on Fano manifolds. Undoubtedly, the importance of rational curves in the study of Fano manifolds is well-known and most works mentioned above also use rational curves extensively as one of the main geometric tools. What is new in our study is an emphasis on the subvariety in the projectivized tangent spaces of the Fano manifold defined by the tangent directions of rational curves. Before starting a systematic discussion, let me roughly describe some motivations and history surrounding this idea.

The story begins with two related conjectures which were outstanding in 1970's. Both were proposed as a generalization of the uniformization of Riemann surfaces for the genus zero case. The first one, Frankel conjecture, was proposed as a differential geometric generalization and the next one, Hartshorne conjecture, was proposed as an algebraic geometric generalization:

Frankel conjecture *A compact Kähler manifold with positive holomorphic bisectional curvature is the projective space.*

Hartshorne conjecture *A projective manifold with ample tangent bundle is the projective space.*

Hartshorne conjecture implies Frankel conjecture. Frankel conjecture was solved by Siu and Yau ([SY]) and Hartshorne conjecture was solved by Mori ([Mo1]). These two proofs are of completely different nature. The method of Siu-Yau depends heavily on the positive curvature condition and seems very difficult to be generalized in the study of other Fano manifolds. On the other hand, Mori's work really provided a new ground for the study of higher dimensional Fano manifolds. Mori established the fundamental fact that a Fano manifold is uniruled, namely, there exists a rational curve through each point. When the tangent bundle is ample, he recovered the projective space by studying the space of all rational curves of minimal degree through a given point on the manifold. This last step of his proof, recovering the Fano manifold from the information on rational curves of minimal degree is what concerns us most here. Under the assumption of ample tangent bundle, this was an easy step in Mori's work. But the story gets more exciting when we look at more general Fano manifolds.

Around the time of the resolution of Frankel conjecture, the following generalization to the case of semi-positive curvature was proposed, in essence, by Siu and Yau:

Generalized Frankel conjecture *A Fano manifold with a Kähler metric of non-negative holomorphic bisectional curvature is a Hermitian symmetric space.*

Before explaining what a Hermitian symmetric space is, let me state the corresponding generalization of Hartshorne conjecture, due to Campana and Peternell ([CP]):

Campana-Peternell Conjecture *A Fano manifold with nef tangent bundle is a rational homogeneous space.*

Here 'nef tangent bundle' means roughly that the manifold has non-negative curvature in algebro-geometric sense. See [Pe] for a precise definition and a good survey of a circle of related results, including above mentioned conjectures. A rational homogeneous space means a homogeneous Fano manifold. We can write it as G/P for a complex semi-simple Lie group G and a parabolic subgroup P. When P is a maximal parabolic subgroup and the isotropy action of P on the tangent space at a base point is irreducible, we say that G/P is an irreducible Hermitian symmetric space. A Hermitian symmetric space means the product of finitely many irreducible Hermitian symmetric spaces.

Generalized Frankel conjecture was solved by Mok ([Mk1]). As Mori did in the proof of Hartshorne conjecture, Mok considered the space $\mathcal{K}_x$ of all rational curves of minimal degree through a given point x on the Fano manifold X with non-negative holomorphic bisectional curvature, and tried to recover the Hermitian symmetric spaces using it. For this, he examined the subvariety $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ consisting of tangent directions to members of $\mathcal{K}_x$. In Mori's case, $\mathcal{C}_x$ is just the total projectivized tangent space $\mathbf{P}T_x(X)$ and its role was negligible. But in Mok's case, it was essential to consider $\mathcal{C}_x$, not just $\mathcal{K}_x$. What made him to study this subvariety of the projectivized tangent space? Well, he wanted to recover Hermitian symmetric spaces and until then, the only characterization of Hermitian symmetric spaces without assuming homogeneity of the manifold is the one based on Berger's classification of holonomy groups ([Be], [Si]). Berger's work is for general Riemannian manifolds. Here I will just state the Fano case.

Berger's Theorem *If the holonomy group of a Kähler metric on a Fano manifold X at a point x does not act transitively on $\mathbf{P}T_x(X)$, then X is a Hermitian symmetric space different from the projective space.*

With some oversimplification, Mok's idea is to show that the subvariety $\mathcal{C}_x$ is invariant under the action of the holonomy group of a suitable deformation of the given Kähler metric of non-negative holomorphic bisectional curvature. Then by Berger's theorem, one can recover Hermitian symmetric spaces.

A natural approach to Campana-Peternell conjecture is to try to translate Mok's proof into algebraic geometry. The variety $\mathcal{C}_x$ is already defined algebraically and the concept 'non-negative curvature' has a natural algebraic analogue. The major problem is an analogue of Berger's theorem. 'Holonomy group' is a purely differential geometric concept. When a Riemannian metic is given, there exists a natural way of translating a tangent vector along a given arc, called 'Levi-Civita connection'. The parallel translation along a closed arc starting from and ending at x induces a linear transformation of $T_x(X)$. The holonomy group is the subgroup of $GL(T_x(X))$ generated by the elements coming from parallel translations along all possible closed arcs. One definitely needs a Riemannian metric or at least a connection to define this concept. Thus for Campana-Peternell conjecture, an approach in this direction looks difficult. After all, rational homogeneous spaces other than Hermitian symmetric spaces cannot be characterized by their holonomy. Perhaps, one needs a completely different approach from Mok's?

Fortunately, I had a chance to discuss the problem with my adviser, Yum-Tong Siu, when I was a graduate student in the early 1990's, and he gave me an interesting advice. He encouraged me to develop "algebro-geometric holonomy theory", drawing my attention to the following analogies.

Riemannian geometry	*Algebraic geometry*
geodesics	rational curves of minimal degree
parallel translations along arcs	splitting of $T(X)$ along rational curves
holonomy group	?

The main role of a metric in the holonomy theory is to give parallel translations along arcs. Siu's suggestion is that the splitting of tangent bundle of X along a rational curve should be an analogue of parallel translation. Of course, a major problem is to understand in what sense this is an analogue. Although I am still not sure what algebro-geometric holonomy is, his suggestion makes me realize the significance of studying the splittings of $T(X)$ along various rational curves. This has been a continuous source of motivation and inspiration for the project that I want to discuss in this lecture series.

The analogy of geodesics in Riemannian geometry and rational curves of minimal degree in algebraic geometry is actually not so direct. In Riemannian geometry, a geodesic exists in any given direction. But on a Fano manifold different from the projective space, rational curves of minimal degree exist only in special directions. In fact, the hero of our story will be the subvariety $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ consisting of tangent directions to rational curves of minimal degree through a generic point $x \in X$. Roughly speaking, these special directions can be viewed as the direction where the manifold is most positively curved. Thus $\mathcal{C}_x$ is the total $\mathbf{P}T_x(X)$ for the projective space, but is a strictly smaller subvariety for other Fano manifolds of Picard number 1. In a problem like Campana-Peternell conjecture, where we want to generalize a known result for the projective space, we have to use this smaller set of positive directions to imitate what was done with the full supply of positive directions. To achieve this we will exploit the projective geometry of the projective subvariety $\mathcal{C}_x \subset \mathbf{P}T_x(X)$. In other words, the subvariety $\mathcal{C}_x$ of tangent directions to rational curves of minimal degree is equipped with a natural projective embedding and the projective geometric properties of this embedding will be a key ingredient of our study. This is reminiscent of what micro-local analysts do over the cotangent space to use special 'elliptic' directions of a non-elliptic operator to get 'sub-elliptic' result resembling that of fully elliptic operator. In this sense, what we are going to do can be said to belong to "micro-local projective geometry".

I have to warn you that these grandiose names "algebro-geometric holonomy theory" and "micro-local projective geometry" should not be taken too literally. I introduce these terms just because they may help you grasp the main philosophy of the project. In reality, our project at the present stage cannot even be called a theory at all. I will just give some examples and discuss some results which follow from the projective geometric study of $\mathcal{C}_x$ combined with the splitting of the tangent bundles. In concluding this introduction, I want to say that although Campana-Peternell conjecture was the original motivation of this study, we won't talk about it again in this lecture series. At present, the conjecture seems to be far away from our perspective and I will not be surprised if it is solved by a completely different approach from ours.

Some of the material below overlaps with the survey article [HM4], which has a slightly more complex analytic flavor. However, there were some significant new developments (especially Section 3 below) after that article was written, so the perspective of this lecture series is somewhat different from that survey. After all, this lecture series is aimed at explaining the basic geometric ideas, rather than surveying recent works, although I will mention most of our recent works at least briefly.

This note consists of five sections. In the first section, we give the definition of the variety of minimal rational tangents, which is the central object of the study. We will give several examples. In the second section, the problem of linear nondegeneracy of the variety of minimal rational tangents is treated with some applications to the simplicity and the stability of the tangent bundles. The third section treats one of the most fundamental results of this study, Cartan-Fubini type extension theorem. The fourth section explains some applications of the extension theorem, with the help of the concept of the variety of distinguished tangents. In the last section, we discuss some open questions.

1 Variety of minimal rational tangents

1.1 Basics on deformation theory of curves

Our main tool is basic deformation theory of curves on a projective manifold. Let us recall it briefly. [Kl] is a standard and comprehensive reference for this.

Let C be a smooth curve on a projective manifold X of dimension n. Given a deformation C_t of $C = C_0$, its initial velocity $\frac{d}{dt}|_{t=0}C_t$ defines a section of the normal bundle N_C. In this way, the vector space $H^0(C, N_C)$ can be viewed as the (virtual) tangent space to the Hilbert scheme of curves on X at the point corresponding to the curve C. Conversely, given a section $\sigma \in H^0(C, N_C)$, σ can be realized as the initial velocity of some deformation C_t of C if certain cohomology classes in $H^1(C, N_C)$, called the obstructions, vanish. In particular, if $H^1(C, N_C) = 0$, then any section of $H^0(C, N_C)$ can be realized as the initial velocity of a deformation of C. This means that any virtual tangent vector to the Hilbert scheme at the point corresponding to C is actually the tangent space to a holomorphic arc in the Hilbert scheme, and so the Hilbert scheme is smooth at that point.

Now fix a point $x \in C$. Given a deformation C_t of C with the point x fixed, i.e. $x \in C_t$ for all t, its initial velocity as a section of the normal bundle of C vanishes at x. In other words, it defines an element of $H^0(C, N_C \otimes \mathbf{m}_x)$ where $\mathbf{m}_x$ denotes the maximal ideal of the point x. So the vector space $H^0(C, N_C \otimes \mathbf{m}_x)$ is the tangent space to the Hilbert scheme of curves on X passing through the point x. Just as above, if $H^1(C, N_C \otimes \mathbf{m}_x) = 0$, then any section can be realized as the initial velocity of some deformation of C fixing x and the Hilbert scheme of curves passing through x is smooth at the point corresponding to C.

When C is not smooth, its deformation theory becomes more complicated. In this lecture series, the singular curves we will deal with are only rational curves and for rational curves, it is easier to consider parametrized rational curves first, by which we mean a morphism $f : \mathbf{P}_1 \to X$ which is birational over its image. Often, we will not distinguish the parametrized rational curve from its image $f(\mathbf{P}_1)$, and just call f a rational curve. The initial velocity for a deformation f_t of f defines a section of the tangent bundle $f^*T(X)$ over $\mathbf{P}_1$. So $H^0(\mathbf{P}_1, f^*T(X))$ is the virtual tangent space to the space $Hom(\mathbf{P}_1, X)$ of parametrized rational curves on X at the point corresponding to f. If $H^1(\mathbf{P}_1, f^*T(X)) = 0$, this space is smooth at that point and any section of $f^*T(X)$ can be realized as the initial velocity of an actual deformation. Now fix a base point $o \in \mathbf{P}_1$ and let $x = f(o)$. By the same argument as before, $H^0(\mathbf{P}_1, f^*T(X) \otimes \mathbf{m}_o)$ is the tangent space to the space $Hom((\mathbf{P}_1; o), (X; x))$ of parametrized rational curves sending o to x. If $H^1(\mathbf{P}_1, f^*T(X) \otimes \mathbf{m}_o) = 0$, then $Hom((\mathbf{P}_1; o), (X; x))$ is smooth at the point corresponding to f and any section of $f^*T(X)$ vanishing at o can be realized as the initial velocity of a deformation of f sending o to x.

Recall that any vector bundle on $\mathbf{P}_1$ is isomorphic to a direct sum of line bundles. In particular, for any parametrized rational curve $f : \mathbf{P}_1 \to X$, the pull-back of the tangent bundle $T(X)$ of X splits as

$$f^*T(X) \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$$

for some integers $a_1 \geq \cdots \geq a_n$. The parametrized rational curve f or its image $f(\mathbf{P}_1)$ is said to be **free**, if all the integers $a_1, \ldots, a_n$ are nonnegative. A free rational curve can be deformed to cover a Zariski open subset in X because any section of $f^*T(X)$ can be extended to a family of deformations of the parametrized rational curve f from the vanishing of

$$H^1(\mathbf{P}_1, f^*T(X)) = H^1(\mathbf{P}_1, \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)).$$

1.2 Minimal rational curves

From now on, we assume that X is a Fano manifold of Picard number 1. We will use the following fundamental result of Mori ([Mo1], [Kl], [Mi]) without a proof.

Theorem 1.1 *X is uniruled, namely, there exists a rational curve through each point of X.*

From the countability of the number of components of the Hilbert scheme, there exists a countable set of proper subvarieties of X so that any rational curve through a point outside this countable set of proper subvarieties is free (See [Kl] for details.). Then the deformation of such free rational curves will cover a Zariski open subset of X. In particular, for a generic point $x \in X$, there exists a free rational curve through x.

The degree of $f^*K_X^{-1}$ will be called the anti-canonical degree of the parametrized rational curve $f : \mathbf{P}_1 \to X$. A free rational curve of minimal anti-canonical degree will be called a **minimal free rational curve**. A **minimal rational curve** is an effective 1-dimensional cycle in X which can be obtained as a deformation of minimal free rational curves. A **minimal rational component** is a component of the Chow space of X whose members are minimal rational curves.

A free rational curve is said to be **standard** if

$$T(X)|_{f(\mathbf{P}_1)} := f^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^{n-1-p}$$

where $p+2$ is the anti-canonical degree $K_X^{-1} \cdot f(\mathbf{P}_1)$. In this case, the unique $\mathcal{O}(2)$-factor of $f^*T(X)$ corresponds to the tangent bundle $T(\mathbf{P}_1)$ and f must be an immersion. The image $f(\mathbf{P}_1)$ may not be smooth but has only nodal singularities. The quotient bundle $f^*T(X)/T(\mathbf{P}_1)$ of a standard rational curve will be called its **normal bundle**. The following result of Mori ([Mo1]), which we will use without proof again, shows that there are lots of standard rational curves on X. Mori's method of proof is nowadays called 'bend-and-break' ([Kl]).

Theorem 1.2 *A generic member of a minimal rational component is standard.*

Choose a minimal rational component $\mathcal{K}$. For a generic point $x \in X$, let $\mathcal{K}_x$ be the normalization of the Chow space of members of $\mathcal{K}$ through x. The following is again essentially proved in [Mo1].

Theorem 1.3 *$\mathcal{K}_x$ is the union of finitely many smooth algebraic varieties of dimension p.*

Let us briefly recall the proof. First of all, note that any member of $\mathcal{K}_x$ is free from the genericity of x. A component of $Hom((\mathbf{P}_1; o), (X, x))$ belonging to $\mathcal{K}_x$ is smooth from $H^1(\mathbf{P}_1, f^*T(X) \otimes \mathbf{m}_o) = 0$ for a free rational curve f and of dimension

$$\begin{aligned} h^0(\mathbf{P}_1, f^*T(x) \otimes \mathbf{m}_o) &= h^0(\mathbf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}^p \oplus [\mathcal{O}(-1)]^{n-1-p}) \\ &= p+2. \end{aligned}$$

The normalized Chow space $\mathcal{K}_x$ can be obtained by taking a quotient of corresponding components of $Hom((\mathbf{P}_1; o), (X; x))$ by the 2-dimensional automorphism group of $\mathbf{P}_1$ fixing o and this quotient is nice enough to preserve the smoothness. See [Kl] or Section 2 of [HM2] for details.

1.3 Tangent map and variety of minimal rational tangents

A generic member of each component of $\mathcal{K}_x$ is a standard rational curve which is smooth at x. We define the **tangent map** at x as the rational map $\tau_x : \mathcal{K}_x \to \mathbf{P}T_x(X)$ which sends a member of $\mathcal{K}_x$ smooth at x to its tangent direction at x. Although the tangent map τ_x is fundamental in our study, we know very little about it for a general Fano manifold of Picard number 1. In fact, the only result we know of, which holds without any additional assumption on X, is the following.

Proposition 1.4 *For a Fano manifold X of Picard number 1 and any choice of minimal rational component $\mathcal{K}$, the tangent map τ_x at a generic point x is an immersion at the point of $\mathcal{K}_x$ corresponding to a standard rational curve. In particular, τ_x is generically finite over its image.*

Proof. Let $f : \mathbf{P}_1 \to X$ be a standard rational curve with $f(o) = x$, which is a member of $\mathcal{K}_x$. Then

$$\tau([f]) = \frac{df}{dz}|_{z=o} \in \mathbf{P}T_x(X)$$

where z is a local coordinate on $\mathbf{P}_1$ centered at o. Given a tangent vector

$$v \in T_{[f]}(\mathcal{K}_x) = H^0(\mathbf{P}_1, N_f \otimes \mathbf{m}_o)$$

where N_f denotes the normal bundle to $f(\mathbf{P}_1)$, we can find a deformation f_t of $f = f_0$ so that its initial velocity $\frac{df_t}{dt}|_{t=0}$ is v. Then the differential

$$d\tau : T_{[f]}(\mathcal{K}_x) \to T_{\tau([f])}(\mathbf{P}T_x(X))$$

sends v to

$$\begin{aligned} d\tau(v) &= \frac{d}{dt}|_{t=0}\frac{df_t}{dz}|_{z=o} \\ &= \frac{d}{dz}|_{z=o}\frac{df_t}{dt}|_{t=0} \\ &= \frac{dv}{dz}|_{z=o}. \end{aligned}$$

But from the splitting type of $N_f \otimes \mathbf{m}_o = \mathcal{O}^p \oplus \mathcal{O}(-1)^{n-1-p}$, a non-zero section cannot have vanishing differential. Thus $d\tau(v) \neq 0$. □

Let $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ be the strict image of the tangent map. $\mathcal{C}_x$ is called the **variety of minimal rational tangents** at x. This will be the central object of our study. Note that there are only finitely many possible choices of $\mathcal{K}$. Thus there are only finitely many possible choices of $\mathcal{C}_x$ for a generic point x. In this sense, $\mathcal{C}_x$ is a naturally defined subvariety which sits inside the projectivized tangent space. For a large class of examples, we can show that τ_x is an embedding by the following proposition.

Proposition 1.5 *Suppose X can be embedded in a projective space $\mathbf{P}_N$ so that through each point of X there exists a line in $\mathbf{P}_N$ which lies on X. Then the tangent map τ_x at a generic point $x \in X$ is an embedding and $\mathcal{C}_x$ is smooth.*

Proof. Clearly, minimal rational curves are lines in $\mathbf{P}_N$ lying on X. Since two distinct lines through x have distinct tangent vectors, the tangent map τ_x is a 1-to-1 morphism. It suffices to show that τ_x is an immersion. By Proposition 1.4, this is equivalent to showing that any line C through x is standard. Let

$$T(X)|_C \cong \mathcal{O}(a_1) \oplus \cdots \mathcal{O}(a_n)$$

with $a_1 \geq \cdots \geq a_n \geq 0$. We know that $a_1 \geq 2$ because $T(\mathbf{P}_1)$ is a subbundle of $T(X)|_C$ from the smoothness of C. On the other hand, $T(X)|_C$ is a subbundle of $T(\mathbf{P}_N)|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^n$. Thus $a_1 = 2, 1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and C is standard. □

The condition of Proposition 1.5 is satisfied by prime Fano manifolds of index $> \frac{n+1}{2}$. Recall that a **prime Fano manifold** is a Fano manifold of Picard number 1 so that the ample generator L of the Picard group is very ample. The index $i(X)$ of a Fano manifold is the positive integer satisfying $K_X^{-1} = i(X)L$ in the Picard group. For a minimal rational curve C,

$$\begin{aligned} n+1 &\geq p+2 \\ &= C \cdot K_X^{-1} \\ &= i(X)\, C \cdot L. \end{aligned}$$

Thus if $i(X) > \frac{n+1}{2}$, C must be a line under the embedding given by L.

1.4 Examples

Now let us look at some examples to get a feeling of what the variety of minimal rational tangents looks like.

1.4.1 Projective spaces

For the projective space $\mathbf{P}_n$, a minimal rational curve is just a line. For any point $x \in \mathbf{P}_n$, the Chow space $\mathcal{K}_x$ of all lines through x is isomorphic to $\mathbf{P}_{n-1}$ and the tangent map $\tau_x : \mathbf{P}_{n-1} \to \mathbf{P}T_x(\mathbf{P}_n)$ is an isomorphism. Thus the variety of minimal rational tangents at every x is the full projectivized tangent space $\mathbf{P}T_x(\mathbf{P}_n)$.

1.4.2 Fano hypersurfaces

A smooth hypersurface $X \subset \mathbf{P}_{n+1}$ of dimension $n \geq 3$ and degree $1 \leq d \leq n+1$ is a Fano manifold of Picard number 1. Let $F(t_0, \ldots, t_{n+1})$ be a homogeneous polynomial of degree d defining X and $x = [x_0, \ldots, x_{n+1}]$ be a generic point of X. A line through x is given by $[x_0 + \lambda y_0, \ldots, x_{n+1} + \lambda y_{n+1}]$ where $[y_0, \ldots, y_{n+1}]$ is a point of $\mathbf{P}_{n+1}$ and $\lambda \in \mathbf{C}$ is a parameter. This line lies on X if and only if

$$F(x_0 + \lambda y_0, \ldots, x_{n+1} + \lambda y_{n+1}) = 0$$

holds for all $\lambda \in \mathbf{C}$. Expanding in λ, we get

$$F(x) + \lambda \Delta_x(y)F(x) + \lambda^2 \frac{1}{2}(\Delta_x(y))^2 F(x) + \cdots + \lambda^d \frac{1}{d!}(\Delta_x(y))^d F(x) = 0$$

where

$$\Delta_x(y) := y_0 \frac{\partial}{\partial t_0} + \cdots + y_{n+1} \frac{\partial}{\partial t_{n+1}}.$$

Thus if $d \leq n$, the variety of minimal rational tangents at x is the smooth complete intersection defined by the system of equations in y

$$\begin{aligned} \Delta_x(y)F(x) &= 0 \\ (\Delta_x(y))^2 F(x) &= 0 \\ &\vdots \\ (\Delta_x(y))^d F(x) &= 0. \end{aligned}$$

Among these the first one is just the defining equation for $\mathbf{P}T_x(X)$. Thus $\mathcal{C}_x$ is a smooth complete intersection of multi-degree $(2, 3, \ldots, d)$ for $d \leq n$.

When $d = 2$, X is the hyperquadric $\mathbf{Q}_n$ which is homogeneous. $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ is a smooth hyperquadric $\mathbf{Q}_{n-2}$ for any $x \in X$.

When d is high but $d < n$, we get examples where the variety of minimal rational tangents is Calabi-Yau or of general type.

When $d = n$, $\mathcal{C}_x$ is a finite set of cardinality $n!$.

When $d = n+1$, there exists no line on X passing through a generic point. However there exist conics through a generic point ([Kl] V. 4.4.4) and $\mathcal{C}_x$ is a finite set.

1.4.3 Fano threefolds

From the classification of Fano threefolds of Picard number 1 ([Is]), excepting $\mathbf{P}_3$ and $\mathbf{Q}_3$, all the other Fano threefolds of Picard number 1 have finitely many rational curves of minimal degree through a generic point. When $\mathcal{C}_x$ is finite, it is obvious that $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.

1.4.4 Grassmannians

Let $X = Gr(s, V)$ be the Grassmannian of s-dimensional subspaces in an $(s+r)$-dimensional vector space V. Under the Plücker embedding, there exist lines on X. Thus τ_x is an embedding by Proposition 1.5. It is well-known that a line on $Gr(s, V)$ through a point corresponding to an s-dimensional subspace W of V is determined by a choice of a subspace W' of dimension $s-1$ contained in W and a subspace W'' of dimension $s+1$ containing W. The line consists of subspaces of dimension s which are containing W' and contained in W''. So $\mathcal{K}_x$ for x corresponding to W is isomorphic to $\mathbf{P}W^* \times \mathbf{P}(V/W)$ and the tangent map is just the Segre embedding

$$\tau_x : \mathcal{K}_x = \mathbf{P}W^* \times \mathbf{P}(V/W) \to \mathbf{P}T_x(X) = \mathbf{P}(W^* \otimes V/W).$$

There is another interesting way to see this. We have the universal bundle $\mathcal{S}$ and the universal quotient bundle $\mathcal{Q}$ on X so that $T(X) \cong \mathcal{S}^* \otimes \mathcal{Q}$. Suppose C is a line on X. Let

$$\begin{aligned} \mathcal{S}^*|_C &\cong \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_s) \\ \mathcal{Q}|_C &\cong \mathcal{O}(c_1) \oplus \cdots \oplus \mathcal{O}(c_r), \end{aligned}$$

with $b_1 \geq \cdots \geq b_s, c_1 \geq \cdots \geq c_r$. Since K_X^{-1} is $(s+r)$-times the Plücker bundle, the splitting of $T(X)$ over a line C is

$$\mathcal{S}^* \otimes \mathcal{Q}|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{s+r-2} \oplus \mathcal{O}^{sr-s-r+1}.$$

It follows that

$$\begin{aligned} \mathcal{S}^* &\cong \mathcal{O}(b) \oplus [\mathcal{O}(b-1)]^{s-1} \\ \mathcal{Q} &\cong \mathcal{O}(c) \oplus [\mathcal{O}(c-1)]^{r-1} \end{aligned}$$

with $b + c = 2$. In particular, the tangent bundle of C which is the unique $\mathcal{O}(2)$-factor in $T(X)|_C$ must be the pure tensor $\mathcal{O}(b) \otimes \mathcal{O}(c)$. So the variety of minimal rational tangents $\mathcal{C}_x$ is contained in the set of pure tensors in $T_x(X) = \mathcal{S}_x^* \otimes \mathcal{Q}_x$. Since the dimension of $\mathcal{C}_x$ is $s + r - 2$ which is equal to the dimension of the Segre variety in $\mathbf{P}(\mathcal{S}_x^* \otimes \mathcal{Q}_x)$, we see that $\mathcal{K}_x \cong \mathcal{C}_x$ is isomorphic to $\mathbf{P}_{s-1} \times \mathbf{P}_{r-1}$ and τ_x is just the Segre embedding.

1.4.5 Hermitian symmetric spaces

Generalizing the examples of hyperquadrics and Grassmannians, we have irreducible Hermitian symmetric spaces. An irreducible Hermitian symmetric space is a Fano manifold X of Picard number 1 which is homogeneous and the isotropy group at a base point $x \in X$ acts irreducibly on $T_x(X)$. The ample generator of the Picard group is very ample and lines exist under this embedding, so Proposition 1.5 applies. The isotropy group action at x has a unique closed orbit in $\mathbf{P}T_x(X)$, the highest weight orbit, and this is exactly the variety of minimal rational tangents at x. From this, it is easy to identify the variety of minimal rational tangents. Here is the list.

X	$\mathcal{K}_x \cong \mathcal{C}_x$	τ_x
$Gr(s,r)$	$\mathbf{P}_{s-1} \times \mathbf{P}_{r-1}$	Segre embedding
$SO(2r)/U(r)$	$Gr(2, r-2)$	Plücker embedding
$Sp(r)/U(r)$	$\mathbf{P}_{r-1}$	2nd Veronese embedding
$\mathbf{Q}_n$	$\mathbf{Q}_{n-2}$	standard embedding
Cayley plane	$SO(10)/U(5)$	minimal embedding
$E_7/E_6 \times U(1)$	Cayley plane	exceptional Severi embedding

Note the remarkable fact that $\mathcal{C}_x$ is a (not necessarily irreducible) Hermitian symmetric space of rank 2. Here the rank of the Hermitian symmetric space means the minimal degree of a rational curve in a generic tangent direction through a given point, measured with respect to the direct sum of generators of the Picard group of each irreducible factor of the Hermitian symmetric space. Various aspects of the geometry of Hermitian symmetric spaces related to the variety of minimal rational tangents are discussed in the book [Mk2], where the variety of minimal rational tangents is called the characteristic variety.

1.4.6 Homogeneous contact manifolds

A subbundle D of the tangent bundle of a complex manifold M is called a distribution on M. For each $x \in M$ and any two vectors $u, v \in D_x$, let $\tilde{u}, \tilde{v}$ be local sections of D extending u, v. Regarded as sections of $T(M)$, we can consider the bracket $[\tilde{u}, \tilde{v}]$ of two vector fields. The value of the vector field $[\tilde{u}, \tilde{v}]$ at x depends on the choice of the local extension $\tilde{u}, \tilde{v}$, but two different choices of local extensions result in the difference by a vector in D_x. Thus

$$[u, v] \;:=\; [\tilde{u}, \tilde{v}] \bmod D$$

gives a well-defined section $[,]$ of $Hom(\Lambda^2 D, T(M)/D)$. We will call it the **Frobenius bracket tensor** of D.

A **contact structure** on a complex manifold M of odd dimension $n = 2m + 1$ is a distribution $D \subset T(M)$ of rank $2m$ whose Frobenius bracket defines a symplectic form on D_x for each $x \in M$. The line bundle $L := T(M)/D$ is called the **contact line bundle.** The Frobenius bracket is just an L-valued symplectic form on D. This gives an isomorphism $D \cong D^* \otimes L$. It follows that $2c_1(D) = nL$ and $K_M^{-1} = mL$ in $Pic(M)$.

A **homogeneous contact manifold** is a homogeneous Fano manifold with a contact structure. The classification of homogeneous contact manifolds was done by Boothby ([Bo]) and for each complex simple Lie algebra $\mathbf{g}$, there exists a unique homogeneous contact manifold where the Lie group G of $\mathbf{g}$ acts transitively as holomorphic automorphisms preserving the contact structure. It is precisely the highest weight orbit, namely, the unique closed orbit, of the adjoint action of G on $\mathbf{Pg}$. For the simple Lie algebra of type A, the corresponding homogeneous contact manifold has Picard number 2, and it is the projectivization of the cotangent bundle of a projective space. Excepting this, all the other homogeneous contact manifolds have Picard number 1.

Let X be a homogeneous contact manifold of Picard number 1. If the contact line bundle L is not a generator of $Pic(X)$, then K_X^{-1} is $(n+1)$-multiple of some line bundle and so X is $\mathbf{P}_{2m+1}$ by Kobayashi-Ochiai characterization of the projective space. $\mathbf{P}_{2m+1}$ is the homogeneous contact manifold associated to the simple Lie algebra of type C.

Now let us assume that X is a homogeneous contact manifold of Picard number 1 and the contact line bundle L is a generator of the Picard group. It is known that for any homogeneous Fano manifold of Picard number 1, the generator of the Picard group is very ample and there are lines on the Fano manifold under this projective embedding. For X, the degree of the line with respect to $K_X^{-1} = mL$ is m. Thus $p = m - 2$. Restricting the exact sequence

$$0 \longrightarrow D \longrightarrow T(X) \longrightarrow L \longrightarrow 0$$

to a line C, we have

$$0 \longrightarrow D \longrightarrow \mathcal{O}(2) \oplus \mathcal{O}(1)^{m-2} \oplus \mathcal{O}^{m+2} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Thus the tangent bundle $T(C) = \mathcal{O}(2)$ must be contained in D. It follows that $\mathcal{C}_x \subset D_x$. In particular, the variety of minimal rational tangents is degenerate in $\mathbf{P}T_x(X)$.

Applying Proposition 1.5, we know that the variety of minimal rational tangents is a smooth subvariety. We can characterize X as a homogeneous Fano manifold of Picard number 1 where the isotropy representation on the tangent space has an irreducible submodule of codimension 1. Then the variety of minimal rational tangents is the unique closed orbit of the isotropy action on $\mathbf{P}D_x$ (see [Hw1] for details.). From this, one can easily identify the variety of minimal rational tangents. In fact, the homogeneous cone $\check{\mathcal{C}}_x \subset D_x$ of the variety of minimal rational tangents is a Lagrangian cone with respect to the symplectic structure on D_x induced by the Frobenius bracket. In other words, $\check{\mathcal{C}}_x$ has dimension m and the restriction of the symplectic form to the tangent spaces of $\check{\mathcal{C}}_x$ is zero. Here is the list.

$\mathfrak{g}$	$\mathcal{K}_x \cong \mathcal{C}_x$	$\tau_x : \mathcal{K}_x \to \mathbf{P}D_x$
$\mathbf{so}_{m+4}$	$\mathbf{P}_1 \times \mathbf{Q}_{m-2}$	Segre embedding
G_2	$\mathbf{P}_1$	twisted cubic
F_4	$Sp(3)/U(3)$	minimal embedding
E_6	$Gr(3,3)$	Plücker embedding
E_7	$SO(12)/U(6)$	minimal embedding
E_8	$E_7/E_6 \times U(1)$	minimal embedding

It is illuminating to see the $\mathbf{so}_{m+4}$-case more explicitly. In this case, the homogenous contact manifold X is just the variety of lines lying on a hyperquadric of dimension $m+2$. Let $[l] \in X$ be a point corresponding to a line $l \subset \mathbf{Q}_{m+2}$. A line in X passing through $[l]$ corresponds to a 1-dimensional deformation of l fixing a point, spanning a plane $\mathbf{P}_2 \subset \mathbf{Q}_{m+2}$. A plane in $\mathbf{Q}_{m+2}$ containing l is determined by a choice of a line through $\mathbf{P}T_y(l)$ in the variety of minimal rational tangents of $\mathbf{Q}_{m+2}$ at a point $y \in l$. Thus the set of planes in $\mathbf{Q}_{m+2}$ containing l is isomorphic to $\mathbf{Q}_{m-2}$. It follows that the variety of minimal rational tangents is isomorphic to $\mathbf{P}_1 \times \mathbf{Q}_{m-2}$ where the $\mathbf{P}_1$-factor comes from the choice of one fixed point on l.

The contact bundle D in this case can also be described explicitly. Note that the normal bundle N_l of l in $\mathbf{Q}_{m+2}$ is $\mathcal{O}(1)^m \oplus \mathcal{O}$ and the tangent space to X at $[l]$ is naturally isomorphic to

$$H^0(l, N_l) \cong H^0(\mathbf{P}_1, \mathcal{O}(1)^m \oplus \mathcal{O}),$$

which has dimension $2m+1$. Let $D_{[l]}$ be the subspace corresponding to $H^0(\mathbf{P}_1, \mathcal{O}(1)^m)$ which has dimension $2m$. This is precisely the contact distribution on X. A deformation of l fixing a point has initial velocity in $D_{[l]}$, which explains why $\mathcal{C}_{[l]}$ is contained in $D_{[l]}$.

1.4.7 Symplectic Grassmannians

The examples of irreducible Hermitian symmetric spaces and homogeneous contact manifolds may have given the impression that the variety of minimal rational tangents of a homogeneous Fano manifold is homogeneous itself. This is not the case for symplectic Grassmannian X of i-dimensional isotropic subspaces in a $2l$-dimensional symplectic vector space for $1 < i < l$.

Let V be a $2l$-dimensional vector space with a symplectic form ω. X is the set of all i-dimensional isotropic subspaces of V, $1 < i < l$. It is known that the dimension of X is $\frac{1}{2}(4l - 3i + 1)i$. There is a canonical inclusion $X \subset Gr(i, V)$ and lines on X are just lines of $Gr(i,V)$ lying on X. Fix an i-dimensional isotropic subspace $W \subset V$. A line on X through $[W]$ corresponds to a choice of an $(i-1)$-dimensional subspace $U \subset W$ and an $(i+1)$-dimensional subspace $U' \subset U^\perp$, where $U^\perp = \{v \in V, \omega(v,u) = 0, \forall u \in U\}$. Hence the variety of minimal rational tangents at $[W]$ can be identified as a subset of $\mathbf{P}W^* \times \mathbf{P}(V/W)$, defined by

$$\mathcal{C}_{[W]} = \{(\lambda, \mu) \in \mathbf{P}W^* \times \mathbf{P}(V/W),\ \omega(w,\mu) = 0 \text{ for all } w \in W \text{ satisfying } \lambda(w) = 0\}.$$

Projections to $\mathbf{P}W^*$ realize $\mathcal{C}_{[W]}$ as a $\mathbf{P}_{2l-2i}$-bundle on $\mathbf{P}_{i-1}$. It contains the trivial subbundle

$$\mathcal{H} = \{(\lambda,\mu) \in \mathcal{C}_{[W]},\ \lambda \in W^*, \mu \in W^\perp/W\} \cong \mathbf{P}_{i-1} \times \mathbf{P}_{2l-2i-1}.$$

Let F be a vector bundle on $\mathbf{P}W^*$ so that $\mathbf{P}F = \mathcal{C}_{[W]}$. $F \subset \mathbf{P}W^* \times (V/W)$ and F contains the trivial subbundle $\mathbf{P}W^* \times (W^\perp/W)$. The fiber of F at $\lambda \in W^*$ is

$$\{\mu \in V/W,\ \omega(w,\mu) = 0 \text{ if } \lambda(w) = 0\}.$$

Modulo the trivial subbundle $W^\perp/W$, the fiber can be identified with $\mathbf{C}\lambda$ by associating the linear functional $\omega(\cdot, v) \in W^*$ to v in the fiber of F. Thus the quotient line bundle of F modulo the trivial subbundle is the tautological bundle for $\mathbf{P}W^*$. It follows that $F \cong \mathcal{O}^{2l-2i} \oplus \mathcal{O}(-1)$.

The embedding $\mathcal{C}_{[W]} = \mathbf{P}F \subset \mathbf{P}W^* \times \mathbf{P}(V/W)$ restricts to the Segre embedding on $\mathcal{H}$. For a trivial bundle $\mathcal{O}^q$ on $\mathbf{P}_{i-1}$, the Segre embedding of the projectivized bundle is induced by the dual tautological line bundle of the projectivization $\mathbf{P}\mathcal{O}(-1)^q$. Thus the embedding of $\mathbf{P}F$ should be induced by the dual tautological bundle when we view it as $\mathbf{P}(\mathcal{O}(-1)^{2l-2i} \oplus \mathcal{O}(-2))$. Hence $\mathcal{C}_{[W]} \subset \mathbf{P}H^0(\mathbf{P}W^*, \mathcal{O}(2) \oplus \mathcal{O}(1)^{2l-2i})^*$ and the linear span of $\mathcal{H}$ corresponds to the kernel of $H^0(\mathbf{P}W^*, \mathcal{O}(2))$. Note that $h^0(\mathbf{P}_{i-1}, \mathcal{O}(2) \oplus \mathcal{O}(1)^{2l-2i}) = \frac{1}{2}i(i+1) + (2l-2i)i = \dim X$. Thus $\mathcal{K}_{[W]} \cong \mathbf{P}(\mathcal{O}(-1)^{2l-2i} \oplus \mathcal{O}(-2))$ as a projectivized vector bundle on $\mathbf{P}_{i-1} = \mathbf{P}W$ and the tangent map $\tau_{[W]}$ is the embedding defined by the complete linear system of the dual tautological line bundle of the projectivization.

The variety of minimal rational tangents $\mathbf{P}(\mathcal{O}(-1)^{2l-2i} \oplus \mathcal{O}(-2))$ is not homogeneous, but it is almost homogeneous, consisting of two orbits, the hypersurface $\mathcal{H}$ and its complement.

1.4.8 Moduli space of stable bundles of rank 2 on a projective algebraic curve

Let R be a projective algebraic curve of genus $g \geq 2$. A vector bundle W of rank 2 and degree d is stable if any line subbundle of W has degree strictly smaller than $\frac{d}{2}$. The moduli scheme $\mathcal{SU}_R(2,d)$ of stable bundles on R of rank 2 with a fixed determinant of odd degree d has a natural structure of $(3g-3)$-dimensional Fano manifold of Picard number 1 (See [NR] and the references therein.). Since $\mathcal{SU}_R(2,d)$ is isomorphic to $\mathcal{SU}_R(2,1)$ as long as d is odd, we will assume that $d = 1$. Then stability condition for W means that any line subbundle has non-positive degree.

When $g = 2$, $X = \mathcal{SU}_R(2,1)$ is a Fano threefold, in fact, the intersection of two quadrics in $\mathbf{P}_5$. So the variety of minimal rational tangents is just 4 points in the plane $\mathbf{P}T_x(X)$ given by the intersection of two conics.

Let us assume $g \geq 3$. Then a generic point $[W] \in X$ corresponds to a strongly stable bundle on R in the sense that any line subbundle of W has negative degree. There are special rational curves on X through $[W]$, called the Hecke curves, constructed by Narasimhan and Ramanan ([NR]). Let us recall their construction.

For a rank 2 bundle W, its projectivization $\mathbf{P}W$ is canonically isomorphic to the projectivization of its dual $\mathbf{P}W^*$. Given a point $\eta \in \mathbf{P}W$, we use the same letter η to denote the 1-dimensional spaces in W and W^* corresponding to η.

Let $[W] \in X$ be a strongly stable bundle. Let $\pi : \mathbf{P}W \to R$ be the natural projection. Given a point $\eta \in \mathbf{P}W$ with $y = \pi(\eta) \in R$, the canonical projection $W_y \to W_y/\eta$ defines a new rank 2 bundle W^η by

$$0 \longrightarrow W^\eta \longrightarrow W \longrightarrow \mathcal{O}_y \otimes (W_y/\eta) \longrightarrow 0.$$

Let V be $W^{\eta*}$, the dual of W^{η}. Then $det(V) = -det(W) + L_y$, where L_y is the line bundle on R corresponding to the divisor y. For each $\nu \in \mathbf{P}V_y$, we have

$$0 \longrightarrow V^{\nu} \longrightarrow V \longrightarrow \mathcal{O}_y \otimes (V_y/\nu) \longrightarrow 0.$$

$V^{\nu*}$ has the same determinant as W and one can check that V^{ν} is stable because W is strongly stable. Thus the family $\{V^{\nu*}, \nu \in \mathbf{P}V_y\}$ defines a rational curve on X, called the **Hecke curve** associated to $\eta \in \mathbf{P}W$. It is a smooth rational curve.

At the point y, the map $W^{\eta}_y \to W_y$ has 1-dimensional kernel. Let $\eta' \in \mathbf{P}V_y$ be its annihilator. Then the dual of $V^{\eta'}$ is isomorphic to W. So the Hecke curve associated to $\eta \in \mathbf{P}W$ is a rational curve through $[W] \in X$.

A more geometric way of describing this construction is as follows. Let $Bl_{\eta}(\mathbf{P}W)$ be the blow-up of the ruled surface $\mathbf{P}W$ at the point η. The exceptional divisor E_{η} is canonically isomorphic to $\mathbf{P}T_{\eta}(\mathbf{P}W)$. The strict transform of the fiber $\mathbf{P}W_y$ is a (-1)-curve on $Bl_{\eta}(\mathbf{P}W)$ and can be blown-down to get a new ruled surface $\mathbf{P}W^{\eta}$. For each choice of a tangent direction $\nu \in \mathbf{P}T_{\eta}(\mathbf{P}W) = E_{\eta} = \mathbf{P}W^{\eta}_y$, we blow-up $\mathbf{P}W^{\eta}$ at ν and then blow-down the strict transform of the fiber $\mathbf{P}W^{\eta}_y$ to get a new ruled surface $\mathbf{P}W^{\eta,\nu}$. The family of bundles $\{W^{\eta,\nu}, \nu \in \mathbf{P}T_{\eta}(\mathbf{P}W)\}$ is the Hecke curve and when ν is tangent to the fiber of $\mathbf{P}W \to R$, we recover W.

It turns out ([Hw3]) that Hecke curves are minimal rational curves of X and they are generic in a minimal rational component of X. I do not know whether there exists any other minimal rational component. It is known that X is a prime Fano manifold ([BV]), but Hecke curves have degree 2 with respect to the generator of $Pic(X)$. Different choices of $\eta \in \mathbf{P}W$ give different rational curves through $[W]$. Thus $\mathcal{K}_{[W]}$ is naturally isomorphic to the ruled surface $\mathbf{P}W$. The tangent map $\tau_{[W]} : \mathcal{K}_{[W]} \to \mathbf{P}T_{[W]}(X)$ can be described as follows. Let T^{rel} be the relative tangent bundle of the $\mathbf{P}_1$-bundle $\pi : \mathbf{P}W \to R$. It is easy to see that $\pi_* T^{rel} = ad(W)$, the traceless endomorphism bundle of W, and $R^1\pi_* T^{rel} = 0$. From the standard deformation theory of vector bundles, the tangent space to the moduli scheme $\mathcal{SU}_R(2,1)$ at $[W]$ is $H^1(R, ad(W))$. Thus the tangent map is a morphism from $\mathbf{P}W$ to $\mathbf{P}H^1(R, ad(W))$. Consider the line bundle $\pi^* K_R \otimes T^{rel}$ on $\mathbf{P}W$. From

$$\begin{aligned} H^0(\mathbf{P}W, \pi^* K_R \otimes T^{rel}) &= H^0(R, K_R \otimes ad(W)) \\ &= H^1(R, ad(W))^*, \end{aligned}$$

it is not difficult to see that $\tau_{[W]}$ is the morphism defined by the complete linear system associated to the line bundle $\pi^* K_R \otimes T^{rel}$. In fact, it is not difficult to show that this is an embedding for $g \geq 5$. I do not know whether it is an embedding for $g = 3, 4$.

In general, the moduli scheme $\mathcal{SU}_R(r,d)$ of stable rank r bundles of a fixed determinant of degree d is a Fano manifold of Picard number 1 if r and d are coprime. One can define Hecke curves similarly and the tangent map is an embedding given by a complete linear system when the genus is high enough. I expect that these are minimal rational curves, but do not have a proof yet.

2 Distribution defined by the linear span of the variety of minimal rational tangents

2.1 Introductory remarks

Let X be a Fano manifold of Picard number 1 and we fix a minimal rational component $\mathcal{K}$ so that the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ is defined at a generic point x of X. Our basic philosophy is that many problems on X can be solved by using the projective geometry of $\mathcal{C}_x$. This approach is effective only when we have some information about the projective geometry of $\mathcal{C}_x$. When a specific Fano manifold is given, sometimes it is not too difficult to describe the variety of minimal rational tangents explicitly, as we saw in the examples in Section 1. For this reason, our approach is so far most successful when dealing with problems concerning a specifically given Fano manifold.

However, quite often we have to deal with Fano manifolds which are not explicitly given and even for well-known Fano manifolds, there are many cases where explicit description of the variety of minimal rational tangents is hard. For this reason, it is natural to study the projective geometry of the variety of minimal rational tangents in a general setting. Unfortunately, essentially nothing is known about the variety of minimal rational tangents for a general Fano manifold X. But under a reasonable assumption on X, we can get non-trivial information on $\mathcal{C}_x$, as we have already seen in Proposition 1.5. In this section, we will give anther example of this, which is more involved.

One of the most basic question we can ask about a subvariety of the projective space is its nondegeneracy, namely, whether it is contained in a hyperplane or not. In many examples we have seen, the variety of minimal rational tangents is a nondegenerate subvariety of $\mathbf{P}T_x(X)$. However in the case of homogeneous contact manifolds different from the odd-dimensional projective space, it is contained in the contact hyperplane $\mathbf{P}D_x \subset \mathbf{P}T_x(X)$ and its affine cone is Lagrangian with respect to the symplectic form on D_x arising from the Frobenius bracket of the contact distribution $D \subset T(X)$. From these, one may expect the following:

(i) For many Fano manifolds of Picard number 1, the variety of minimal rational tangents at a generic point $x \in X$ is a nondegenerate subvariety of $\mathbf{P}T_x(X)$.

(ii) When $\mathcal{C}_x$ is degenerate in $\mathbf{P}T_x(X)$, let W_x be its linear span in $T_x(X)$. The collection of W_x's, as x varies over generic points of X, define a distribution on a Zariski open subset of X. Then the variety of minimal rational tangents has a special property with respect to the Frobenius bracket of this distribution.

The main goal of this section is to explain some results which confirm these two expectations partially. In addition to its intrinsic interest, I think the discussion of this topic is valuable because it shows very transparently

(a) the use of the condition on the Picard number of X;

(b) the use of the deformation theory of rational curves;

(c) the use of the projective geometry of the variety of minimal rational tangents.

While reading this section, please keep these three points in mind and examine how they are used.

2.2 Frobenius bracket of the distribution

Our main object of study is the distribution W which is defined on a Zariski open subset of X by the linear span of the variety of minimal rational tangents. The variety of minimal rational tangents is not necessarily irreducible and its linear span means the linear span of all the components. Note that we can always 'saturate' W to a distribution defined outside a set of codimension ≥ 2. More precisely, regard W as a subsheaf of the tangent bundle and consider the annihilator subsheaf $W^\perp \subset \Omega^1(X)$. Then the saturation of W is the subsheaf of $T(X)$ consisting of vectors annihilated by $W^\perp$. The saturation gives a distribution outside a subvariety E of codimension ≥ 2 which agrees with W on the open set where W is originally defined. From now on, W will denote this saturated distribution.

The following lemma is very convenient when discussing the property of W with respect to a generic minimal rational curve.

Lemma 2.1 *Given any subset $E \subset X$ of codimension ≥ 2, we can find a standard minimal rational curve disjoint from E.*

Proof. Choose a standard minimal rational curve C through a generic point $x \in X, x \notin E$. Let $N_C \cong \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$ be the normal bundle of C. Choose sections $\sigma_1, \ldots \sigma_p$ of N_C corresponding to independent sections of $\mathcal{O}(1)^p$ vanishing at x, and sections $\sigma_{p+1}, \ldots, \sigma_n$ of N_C which generate the trivial factors $\mathcal{O}^{n-1-p}$ of N_C. Since there is no obstruction, we have an $(n-1)$-dimensional deformations of C whose initial velocities are contained in the linear span of $\sigma_1, \ldots, \sigma_{n-1}$ in $H^0(C, N_C)$. Suppose all members of this $(n-1)$-dimensional family of curves intersect E. Since E has codimension ≥ 2, this means that we have a 1-dimensional subfamily passing through a given point $y \in E$. In particular, in the linear span of $\sigma_1, \ldots, \sigma_{n-1}$ in $H^0(C, N_C)$, there exists a non-zero section vanishing at y. But this is impossible because $\sigma_1, \ldots, \sigma_{n-1}$ are pointwise linearly independent outside x. Thus generic deformation of C, which is itself a standard minimal rational curve, is disjoint from E. □

Recall Frobenius theorem that the Frobenius bracket of a holomorphic distribution vanishes identically if and only if the distribution arises from tangent spaces of the leaves of a holomorphic foliation. In this case, we say that the distribution is **integrable**. The most basic fact about the distribution W is the following which is a consequence of the condition on the Picard number of X.

Proposition 2.2 *If W is a proper distribution, then it is not integrable.*

Proof. Suppose the variety of minimal rational tangents is degenerate and the distribution W is integrable, so that it defines a non-trivial foliation on $X - E$. The leaf of E through a generic point x of $X - E$ is a complex analytic submanifold in $X - E$. We want to show that this submanifold can be compactified to a subvariety of X. For this, we build-up a sequence of subvarieties $x = \mathcal{V}^0(x) \subset \mathcal{V}^1(x) \subset \mathcal{V}^2(x) \subset \cdots$ inductively by the rule

$$\mathcal{V}^i(x) \;:=\; \text{the closure of the union of members of } \mathcal{K} \text{ through generic points of } \mathcal{V}^{i-1}(x).$$

Clearly, $\mathcal{V}^i(x)$ has strictly bigger dimension than $\mathcal{V}^{i-1}(x)$ unless $\mathcal{V}^i(x) = \mathcal{V}^{i-1}(x)$. Thus $\mathcal{V}^n(x) = \mathcal{V}^{n+1}(x)$, namely, any member of $\mathcal{K}$ through a generic point of $\mathcal{V}^n(x)$ is already contained in $\mathcal{V}^n(x)$. Since all members of $\mathcal{K}$ through generic points of X are tangent to W, we see that the generic part of $\mathcal{V}^i(x)$ is contained in the leaf through x for all i. Thus if y is a generic point of $\mathcal{V}^n(x)$, then $T_y(\mathcal{V}^n(x)) \subset W_y$. On the other hand, from the property of $\mathcal{V}^n(x)$, $\mathcal{C}_y \subset T_y(\mathcal{V}^n(x))$. It follows that $T_y(\mathcal{V}^n(x)) = W_y$ and the variety $\mathcal{V}^n(x)$ must be the complex analytic closure of the leaf through x.

Thus $X - E$ is foliated by the leaves of the integrable distribution W which are algebraic subvarieties. So we have an algebraic fibration $\rho : X - E \to S$ over an algebraic variety S. By Lemma 2.1, a generic member C of $\mathcal{K}$ is contained in $X - E$ and by the definition of W, C must be contained in a fiber of ρ. Take a generic hypersurface H in S disjoint from $\rho(C)$. Then C is disjoint from the divisor $\rho^{-1}(H)$ in X, a contradiction to the Picard number. □

From Proposition 2.2, we see that the Frobenius bracket tensor $[,] : \wedge^2 W \to T(X-E)/W$ is not identically zero. Now we want to examine what are the special properties of the variety of minimal rational tangents with respect to $[,]$.

Let $C \subset X$ be a standard minimal rational curve through a generic point x. Then $\mathcal{C}_x$ is immersed at $\alpha := \mathbf{P}T_x(C)$ and we can consider the projective tangent space of $\mathcal{C}_x$ at α. This tangent space has a simple description in terms of the splitting of $T(X)|_C$.

Proposition 2.3 *Let $\hat{T}_\alpha \subset T_x(X)$ be the $(p+1)$-dimensional subspace corresponding to the positive factors of the splitting $T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^{n-1-p}$. Then $\mathbf{P}\hat{T}_\alpha$ is the projective tangent space of $\mathcal{C}_x$ at α.*

Proof. As in the proof of Proposition 1.4, the differential of the tangent map at a standard minimal rational curve $f : \mathbf{P}_1 \to X$ is equivalent to sending a section v of $f^*T(X)$ vanishing at x to $\frac{dv}{dz}$ with respect to the coordinate z on $\mathbf{P}_1$. Since v is a section of the positive part of the splitting $T(X)|_C$, $\frac{dv}{dz}$ is also in the positive part of the splitting. Considering that $\dim(\mathcal{C}_x) = p$ is equal to the dimension of $\mathbf{P}(\mathcal{O}(2) \oplus \mathcal{O}(1)^p)$, the result follows. □

This gives the following information about the Frobenius bracket of W:

Proposition 2.4 *Let $\mathcal{T}_x \subset \mathbf{P}\wedge^2 W_x$ be the subvariety consisting of lines of $\mathbf{P}W_x$ which are tangent to the smooth locus of $\mathcal{C}_x$. Then $\mathcal{T}_x$ is contained in the projectivization of the kernel of the Frobenius bracket tensor $[,] : \wedge^2 W_x \to T_x(X)/W_x$.*

Proof. We need to show that $[\alpha, \beta] = 0$ for any $\alpha \in W_x$ corresponding to a generic point of $\mathcal{C}_x$ and any $\beta \in \hat{T}_\alpha$. We may assume that both α and β are non-zero.

To prove this, it suffices to find a local complex analytic surface through x which is tangent to W in a neighborhood of x and whose tangent space at x contains α and β. Let C be a standard rational curve through x in the direction of α and fix a point $y \in C$ different from x. By the definition of $\hat{T}_\alpha$, β is a vector in the positive part of the splitting of $T(X)|_C$. Thus there exists a non-zero section σ of the normal bundle N_C so that $\sigma(y) = 0$ and $\sigma(x) = \beta$. Since $H^1(C, N_C \otimes \mathbf{m}_y) = 0$, we can find a deformation C_t of C fixing y whose initial velocity is exactly σ. The union of these curves C_t define a local analytic surface $\mathcal{S}$ through x whose tangent space at x is spanned by $T_x(C) = \alpha$ and $\sigma(x) = \beta$. Moreover its tangent space at z near x is spanned by $T_z(C_t)$ and $\sigma_t(z)$ where $\sigma_t \in H^0(C_t, N_{C_t} \otimes \mathbf{m}_y$ is the velocity of C_t at t. By Proposition 2.3, this implies that $\sigma_t(z)$ is in the projective tangent space of $\mathcal{C}_z$ at $\mathbf{P}T_z(C_t)$, hence in W_z. It follows that the surface $\mathcal{S}$ is tangent to W, as desired. □

Proposition 2.4 explains why the cone over the variety of minimal rational tangents is isotropic with respect to the symplectic form on the contact distribution D for the homogeneous contact manifolds different from the odd-dimensional projective spaces. In this sense, Proposition 2.4 fulfills our expectation (ii) mentioned at the beginning of this section.

2.3 Nondegeneracy of the variety of minimal rational tangents

Now I will turn to the expectation (i). So far, the only examples we have seen where the variety of minimal rational tangents is degenerate is the homogeneous contact manifold. In that example, $n = 2m+1$ and $p = m-1$ so that $p = \frac{n-3}{2}$. Note that X is embedded in a projective space by the ample generator of the Picard group and is covered by lines under this embedding. If we take a hyperplane section, we get a Fano manifold of one dimension low and the minimal rational curves are just lines which lie on this hyperplane section. So the variety of minimal rational tangents at a generic point will be a hyperplane section of the variety of minimal rational tangents for the homogeneous contact manifold. This way, hyperplane sections of the homogeneous contact manifold provide us new examples of Fano manifolds of Picard number 1 whose variety of minimal rational tangents is degenerate. By taking successive hyperplane sections as long as there are lines through generic point, we get many examples of Fano manifolds of Picard number 1 whose variety of minimal rational tangents is degenerate. In all these examples, we have $p \leq \frac{n-3}{2}$. Are there other examples where $p > \frac{n-3}{2}$? It would not be easy to find such examples, because of the following result.

Theorem 2.5 *Let X be a Fano manifold of Picard number 1 with $p > \frac{n-3}{2}$. If the variety of minimal rational tangents is smooth, it is nondegenerate in $\mathbf{P}T_x(X)$.*

The proof uses some deep result from projective geometry applied to $\mathcal{C}_x$. This argument shows the philosophy of 'micro-local projective geometry' nicely.

Let ω be an anti-symmetric bilinear form on a vector space W and $J \subset W$ be a cone which is isotropic with respect to ω. If ω is nondegenerate, i.e., symplectic, it is well-known that $\dim(J) \leq \frac{\dim(W)}{2}$. This needs not be true if ω is degenerate. However, it remains true if we assume that $\mathbf{P}J \subset \mathbf{P}W$ is a non-linear smooth subvariety:

Proposition 2.6 *Let W be a vector space and $J \subset W$ be a non-linear cone with $\dim(J) > \frac{\dim(W)}{2}$, such that $\mathbf{P}J$ is a smooth subvariety of $\mathbf{P}W$. Let $\mathcal{T} \subset \mathbf{P}\wedge^2 W$ be the variety of tangential lines of $\mathbf{P}J$. Then $\mathcal{T}$ is nondegenerate in $\mathbf{P}\wedge^2 W$.*

It seems that this was proved for the first time in [HM5], although it is a rather simple consequence of Zak's theorem on tangencies ([Za]):

Theorem 2.7 *Let $Z \subset \mathbf{P}_N$ be a non-linear smooth variety of dimension k and $\mathbf{P}_l \subset \mathbf{P}_N$ be an l-dimensional linear subspace. Then the set of points on Z at which $\mathbf{P}_l$ is tangent to Z is at most of dimension $l-k$.*

Proof of Proposition 2.6. Suppose $\mathcal{T}$ is degenerate and choose a non-zero element ω of $\wedge^2 W^*$, so that J is isotropic with respect to ω. Let $Q \subset W$ be the kernel of ω, namely,

$$Q = \{w \in W | \omega(w,v) = 0 \text{ for all } v \in W\}.$$

Let $\pi : W \to W/Q$ be the quotient. ω induces a symplectic form on W/Q. Let T be the tangent space of $\pi(J)$ at a generic point. Then $\mathbf{P}(\pi^{-1}(T))$ is a subspace of $\mathbf{P}W$ of dimension $\dim(Q) + \dim(\pi(J)) - 1$ which is tangent to $\mathbf{P}J$ along a subset of dimension $\dim(J) - \dim(\pi(J))$. By Zak's theorem on tangencies,

$$\begin{aligned}\dim(J) - \dim(\pi(J)) &\leq \dim(Q) + \dim(\pi(J)) - 1 - (\dim(J) - 1)\\ \dim(J) - \dim(\pi(J)) &\leq \frac{\dim(Q)}{2}.\end{aligned}$$

But $\pi(J)$ is an isotropic cone in the symplectic vector space W/Q, thus

$$\dim(\pi(J)) \leq \frac{\dim(W/Q)}{2}.$$

This gives the contradiction

$$\begin{aligned}\dim(J) - \dim(\pi(J)) &\geq \dim(J) - \frac{\dim(W/Q)}{2}\\ &= \dim(J) - \frac{\dim(W)}{2} + \frac{\dim(Q)}{2}\\ &> \frac{\dim(Q)}{2}\end{aligned}$$

where the last inequality follows from the assumption $\dim(J) > \frac{\dim(W)}{2}$. □

We are ready for the proof of Theorem 2.5.

Proof of Theorem 2.5. Assume that $\mathcal{C}_x$ is degenerate, defining the non-trivial distribution W of rank $m < n$. From the assumption that $\mathcal{C}_x$ is smooth and of dimension $p > \frac{n-3}{2}$, Proposition 2.4 and Proposition 2.6 imply that the Frobenius bracket tensor of W vanishes identically. This is a contradiction to the non-integrability of W, Proposition 2.2. □

Combined with Proposition 1.5, we get

Corollary 2.8 *For a prime Fano manifold of dimension n with index $> \frac{n+1}{2}$, the variety of minimal rational tangents is nondegenerate.*

2.4 Stability of tangent bundles

Now let me turn to some applications of the results we discussed. Recall that a vector bundle V on X is **simple** if $\dim H^0(X, End(V)) = 1$, in other words, if the scalar multiplications are the only endomorphisms of V.

Theorem 2.9 *For a Fano manifold X of Picard number 1, if the variety of minimal rational tangents is irreducible and nondegenerate for some choice of the minimal rational component, then the tangent bundle $T(X)$ is simple.*

Proof. Let ξ be an endomorphism of $T(X)$. Choose a minimal rational component $\mathcal{K}$ for which the variety of minimal rational tangents is irreducible and nondegenerate. Let $x \in X$ be a generic point and $v \in T_x(X)$ be a tangent vector to a standard minimal rational curve C through x. Let $\tilde{v}$ be a vector field on C extending v having two distinct zeroes. Then $\xi(\tilde{v})$ is a section of $T(X)|_C$ vanishing at two distinct points where $\tilde{v}$ vanishes. From the splitting type of $T(X)|_C$, either $\xi(\tilde{v})$ is identically zero or it is a vector field on C again which are proportional to $\tilde{v}$. It follows that v is an eigenvector of ξ in $T_x(X)$. Since this is true for any choice of v in $\mathcal{C}_x$ which is nondegenerate in $\mathbf{P}T_x(X)$, we see that every vector of $T_x(X)$ is an eigenvector of ξ, thus ξ acts as a scalar multiplication on $T_x(X)$ for generic $x \in X$. The eigenvalues must be constant along standard minimal rational curves because $\tilde{v}$ and $\xi(\tilde{v})$ are just a constant multiple of each other. It follows that ξ is just a scalar multiplication and $T(X)$ is simple. □

Theorem 2.9 can be applied to most examples we have seen in section 1. Combined with Corollary 2.8, we see that a prime Fano manifold of index $> \frac{n+1}{2}$ has simple tangent bundle. As a matter of fact, for prime Fano manifolds of index $> \frac{n+1}{2}$, we can refine the method further to show that their tangent bundles are stable ([Hw2], [HM5]).

Recall that a vector bundle V of rank r on X is **stable** if for any subsheaf $F \subset V$ of rank $k, 1 \leq k \leq r-1$, the inequality

$$\frac{c_1(F)\cdot(K_X^{-1})^{n-1}}{k} < \frac{c_1(V)\cdot(K_X^{-1})^{n-1}}{r}$$

holds. Since X has Picard number 1, we can check this inequality by restricting F and V to a generic standard minimal rational curves. Namely, given a torsion free sheaf F of rank k, define its slope $\mu(F)$ as the rational number $\frac{c_1(F)\cdot C}{k}$ for a generic standard minimal rational curve C. By Lemma 2.1, F is locally free on C, so the meaning of $c_1(F)\cdot C$ is clear. Then a vector bundle V of rank r is stable if for any subsheaf F of rank $k, 1 \leq k \leq r-1$, the inequality $\mu(F) < \mu(V)$ holds. In particular, if the tangent bundle $T(X)$ is not stable, there exists a subsheaf $F \subset T(X)$ with slope $\mu(F) \geq \frac{p+2}{n}$. If we choose F with maximal possible value of $\mu(F)$, it is easy to check that F defines an integrable distribution on a Zariski open subset of X. Let k be the rank of F. Using Proposition 2.3, the inequality

$$\frac{c_1(F)\cdot C}{k} \geq \frac{p+2}{n}$$

can be translated to the statement

$$\frac{\dim(\mathbf{P}F_x \cap \mathbf{P}\hat{T}_\alpha)+1}{k} \geq \frac{p+2}{n}.$$

In other words, the instability of $T(X)$ results in some excessive intersection property of the tangent spaces of the variety of minimal rational tangents with a linear subspace in $\mathbf{P}T_x(X)$:

Proposition 2.10 *Suppose the intersection of the projective tangent space at a generic point of $\mathcal{C}_x$ with any linear subspace in $\mathbf{P}T_x(X)$ has dimension $< \frac{k}{n}(p+2)-1$ where $k-1$ is the dimension of the linear subspace. Then the tangent bundle $T(X)$ is stable.*

Using this, one can show that the stability of the tangent bundles of many Fano manifolds of Picard number 1 ([Hw2]). For example, the stability of the tangent bundle of the moduli space of rank 2 bundles over a projective curve was proved using this argument ([Hw3]). The proof of the following Theorem was sketched in [HM5].

Theorem 2.11 *A prime Fano manifold X of index $> \frac{n+1}{2}$ has stable tangent bundle.*

Proof. Suppose not. By Proposition 2.10, we have a linear subspace $\mathbf{P}F \subset \mathbf{P}T_x(X)$ with $\dim(F) = k$ so that its intersection with the projective tangent space at a generic point of $\mathcal{C}_x$ has dimension $\geq \frac{k}{n}(p+2)-1$. Let

$$\psi : \mathbf{P}T_x(X) - \mathbf{P}F \to \mathbf{P}^{n-k-1}$$

be the projection from $\mathbf{P}F$ to a complementary linear space. Let q be the generic fiber dimension of $\psi|_{\mathcal{C}_x}$. By the assumption, $q \geq \frac{k}{n}(p+2)$.

Let T be the projective tangent space to $\psi(\mathcal{C}_x)$ at a generic point $\alpha \in \psi(\mathcal{C}_x)$. Then $\psi^{-1}(T)$ is a linear space of dimension

$$\dim(T) + k = p - q + k.$$

This linear space $\psi^{-1}(T)$ is tangent to Y along the fiber $(\psi|_{\mathcal{C}_x})^{-1}(\alpha)$. From Proposition 1.5, $\mathcal{C}_x$ is smooth. By Zak's theorem on tangencies,

$$\begin{aligned} \dim[(\psi|_{\mathcal{C}_x})^{-1}(\alpha)] &\leq \dim(\psi^{-1}(T)) - \dim(\mathcal{C}_x) \\ q &\leq (p-q+k) - p. \end{aligned}$$

So we get $q \leq \frac{k}{2}$. Combined with $q \geq \frac{k}{n}(p+2)$, we have $i(X) = p+2 \leq \frac{n}{2}$, a contradiction. □

3 Cartan-Fubini type extension theorem

3.1 Statement of Cartan-Fubini type extension theorem

When we looked at the examples of varieties of minimal rational tangents in Section 1, you may have noticed the striking fact that when two Fano manifolds of Picard number 1 are of different type, their varieties of minimal rational tangents are of quite different nature. This leads to the following natural question.

Question 3.1 *Let X be a Fano manifold of Picard number 1. Choose a minimal rational component $\mathcal{K}$ and let $\mathcal{C}_x$ be the variety of minimal rational tangents at a generic point. Does $\mathcal{C}_x$ determine X in the following sense?*

Let X' be any other Fano manifold of Picard number 1 with a choice of minimal rational component $\mathcal{K}'$ and let $\mathcal{C}'_{x'}$ be the variety of minimal rational tangents at a generic point. Suppose there are analytic open subsets $U \subset X, U' \subset X'$ with a biholomorphism $\varphi : U \to U'$ such that there exists an isomorphism $\psi : \mathbf{P}T(U) \to \mathbf{P}T(U')$ of projective bundles compatible with φ satisfying $\psi(\mathcal{C}_x) = \mathcal{C}'_{\varphi(x)}$ for each $x \in U$. Then X is biholomorphic to X'.

The answer depends on X. One example where the answer is no is the moduli space of rank 2 bundles of a fixed determinant of odd degree over a projective curve R of genus 2, which is equivalent to the complete intersection of two quadrics in $\mathbf{P}_5$. As we have seen in Section 1, the variety of minimal rational tangents for this moduli space is the union of 4 points in $\mathbf{P}_2 \cong \mathbf{P}T_x(X)$ which are the intersection of two conics. Thus the isomorphism type of $\mathcal{C}_x \subset \mathbf{P}_2$ is independent of x and also independent of the projective curve R, while the biregular type of X depends on R.

On the other hand, when X is the projective space $\mathbf{P}_n$, Cho and Miyaoka showed that the answer to Question 3.1 is yes ([CM]). Recently, Mok and I have found out that the answer is yes for any irreducible Hermitian symmetric space ([HM9]).

Unfortunately, for most examples of X, I do not know an answer to Question 3.1 and it seems rather difficult to guess what the answer will be for a given X, even for examples like Fano hypersurfaces. However, if we weaken the question by assuming that the isomorphism

ψ of the projectivized tangent bundle comes from the differential of φ, we can give an affirmative answer for many examples ([HM7]):

Theorem 3.2 *Let X be a Fano manifold of Picard number 1 with a choice of minimal rational component $\mathcal{K}$ so that the variety of minimal rational tangents $\mathcal{C}_x$ at a generic point is of positive dimension $p > 0$ and the Gauss map of $\mathcal{C}_x$ as a projective subvariety of $\mathbf{P}T_x(X)$ is generically finite. Let X' be any Fano manifold of Picard number 1 with a choice of minimal rational component $\mathcal{K}'$ for which we denote the variety of minimal rational tangents at a generic point x' by $\mathcal{C}'_{x'}$. Suppose there exist connected analytic open subsets $U \subset X, U' \subset X'$ and a biholomorphic map $\varphi : U \to U'$ so that the differential $\varphi_* : \mathbf{P}T(U) \to \mathbf{P}T(U')$ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}'_{\varphi(x)}$ for general $x \in U$. Then φ can be extended to a biholomorphic map $X \to X'$.*

Note the difference in the statement of this theorem from that of Question 3.1. The theorem requires a stronger condition that the isomorphism between the varieties of minimal rational tangents are induced by the differential of the map on the base. The statement of the theorem is stronger in the sense that the isomorphism between X and X' is an extension of the given map φ, while in Question 3.1, the isomorphism between X and X' may not be related to φ. For example, when X is the projective space, the answer to Question 3.1 is yes by Cho and Miyaoka, but Theorem 3.2 does not hold. In fact, for $X = X' = \mathbf{P}_n$, the condition $\varphi_*(\mathcal{C}_x) = \mathcal{C}'_{\varphi(x)}$ is satisfied by any biholomorphic map $\varphi : U \to U'$, because $\mathcal{C}_x = \mathbf{P}T_x(X)$. Note that in this case, the Gauss map is not generically finite. Recall that the Gauss map of non-linear smooth projective subvariety is generically finite ([GH]), in fact, finite ([Za]). Thus Theorem 3.2 can be applied to all the examples we have seen as long as $p > 0$, excepting the projective space. It may be true that the statement of the theorem holds for the case of $p = 0$, too. But our proof strongly depends on the condition $p > 0$.

There are some earlier results in the direction of the theorem. About eighty years ago, E. Cartan and G. Fubini had initiated the study of a related problem for hypersurfaces in the projective spaces. For this reason, we call Theorem 3.2 as **Cartan-Fubini type extension theorem**. The study of Cartan and Fubini was completed with modern rigor by Jensen-Musso ([JM]). Meanwhile Ochiai proved Theorem 3.2 when both X and X' are Hermitian symmetric spaces ([Oc]) and this was generalized to other homogeneous Fano manifolds by Yamaguchi ([Ya]). The method employed by Jensen-Musso is that of moving frames and Ochiai-Yamaguchi's works are in terms of Lie algebra cohomologies. Our method is completely different. It uses deformation of rational curves and analytic continuation.

3.2 Proof of Cartan-Fubini type extension theorem

I will explain the main ideas of the proof of Theorem 3.2, which consists of the following 4 steps.

Step 1 To show that the map φ sends local pieces of members of $\mathcal{K}$ in U to local pieces of members of $\mathcal{K}'$ in U'.

Step 2 To extend the domain of definition of φ from the analytic open set U to an etale open set.

Step 3 To extend the domain of definition of the map from an etale open set to a Zariski open set.

Step 4 To extend the domain of definition of the map from a Zariski open set to the entire Fano manifold X.

Before going into details, let me point out that the condition $p > 0$ is used only in Step 2 and Step 3, while the Gauss map condition is used, in essence, only in Step 1. Step 1 and 4 hold also for $p = 0$ case.

For Step 1, we start with some definitions. Given a distribution $\mathcal{D}$ on a complex manifold M, regarded as a subsheaf of the tangent sheaf, its **Cauchy characteristic** is the subsheaf defined by

$$Ch(\mathcal{D}) \;:=\; \{\text{ local sections } f \text{ of } \mathcal{D} \text{ satisfying } [f,g] = 0 \text{ for all local sections } g \text{ of } \mathcal{D}\}$$

where $[f,g]$ denotes the Frobenius bracket for the distribution $\mathcal{D}$. $Ch(\mathcal{D})$ is an integrable distribution over an open set where it is locally free. The following lemma can be checked by a direct computation with the bracket, which we leave as an exercise.

Lemma 3.3 *Let $g : M \to N$ be a submersion of complex manifolds so that the fibers of g define a distribution $Ker(dg)$ on M. Let $\mathcal{D}$ be a distribution on N. For the pull-back distribution $g^*\mathcal{D}$ defined by*

$$(g^*\mathcal{D})_m \;=\; (dg)^{-1}(\mathcal{D}_{g(m)})$$

where $dg : T_m(M) \to T_{g(m)}(N)$ is the differential of g at the point $m \in M$. we have

$$Ker(dg) \;\subset\; Ch(g^*\mathcal{D}).$$

Let $\mathcal{C} \subset \mathbf{P}T(X)$ be the closure of the union of $\mathcal{C}_x$'s as x varies over generic points of X. Consider the universal family $\rho : \mathcal{U} \to \mathcal{K}$ over the component $\mathcal{K}$ of the Chow scheme ([Kl], I) and the associated morphism $\mu : \mathcal{U} \to X$. Then the normalization of the fiber $\mu^{-1}(x)$ corresponds to our $\mathcal{K}_x$. Thus we can take union of the tangent maps $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ to get a rational map $\tau : \mathcal{U} \to \mathcal{C}$, which is generically finite. The images under τ of the fibers of ρ on $\mathcal{U}$ induce a multi-valued foliation by curves on generic part of $\mathcal{C}$. Let us denote this multi-valued foliation by $\mathcal{F}$. A leaf of $\mathcal{F}$ is the lift of a minimal rational curve to its tangent vectors. If we restrict to a local analytic open subset of $\mathcal{C}$, we can choose a 'branch' of this multi-valued foliation to get a genuine foliation on analytic open subset.

Now we define a distribution $\mathcal{P}$ of rank $2p+1$ on generic part of $\mathcal{C}$ by defining its fiber at $\alpha \in \mathcal{C}$ as

$$\mathcal{P}_\alpha \;:=\; (d\pi)^{-1}(\hat{T}_\alpha)$$

where $d\pi : T_\alpha(\mathcal{C}) \to T_x(X)$ is the differential of the natural projection $\pi : \mathcal{C} \to X$ at $\alpha \in \mathcal{C}, x = \pi(\alpha)$ and $\hat{T}_\alpha \subset T_x(X)$ is the linear tangent space of $\mathcal{C}_x$ at α as in Proposition 2.3. Note that this distribution $\mathcal{P}$ on $\mathcal{C}$ over an analytic open set $U \subset X$ is completely determined by the information of the embedding $\mathcal{C} \subset \mathbf{P}T(X)$ over U. Similarly, the embedding $\mathcal{C}' \subset \mathbf{P}T(X')$ induces a distribution $\mathcal{P}'$ on $\mathcal{C}'$. Then $\varphi_*(\mathcal{C}|_U) \subset \mathcal{C}'$ implies that φ_* sends $\mathcal{P}$ to $\mathcal{P}'$.

Recall that at a generic point $[C]$ of $\mathcal{K}$ corresponding to a standard minimal rational curve C, the tangent space of $\mathcal{K}$ is canonically isomorphic to $H^0(C, N_C)$. From

$$N_C \;\cong\; \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$$

there exists a natural subspace $\mathcal{Q}_{[C]} \subset H^0(C, N_C)$ corresponding to the sections of $\mathcal{O}(1)^p$-part of N_C. This defines a natural distribution $\mathcal{Q}$ on generic part of $\mathcal{K}$.

Choose an analytic open subset O in $\mathcal{U}$ so that $\tau|_O$ is biholomorphic. We regard O as an open subset of $\mathcal{C}$. $\mathcal{F}$ can be regarded as a univalent foliation on O corresponding to the fibers of ρ. From Proposition 2.3, $\mathcal{P} = \rho^*\mathcal{Q}$. From Lemma 3.3, $\mathcal{F} \subset Ch(\mathcal{P})$. In fact, the following holds.

Proposition 3.4 *If $\mathcal{C}_x$ has generically finite Gauss map, then $\mathcal{F} = Ch(\mathcal{P})$.*

This is the part where the Gauss map condition is used. Instead of giving a detailed proof of this, let me explain the main idea. First let us examine what the condition on Gauss map means. It is perhaps easier to look at the affine case. So let $Z \subset \mathbf{C}^n$ be an affine variety of dimension m and let $z \in Z$ be a generic smooth point. Let $z_1, \ldots, z_m$ be a local coordinate system of Z at z and $w_1, \ldots, w_n$ be an affine coordinate system on $\mathbf{C}^n$. The Gauss map of Z is just associating to z its tangent space $T_z(Z)$. If the Gauss map is not generically finite, its differential has kernel in a neighborhood of z. Let $V \in T_z(Z)$ be in the kernel of the differential of the Gauss map. This means that in the direction of V, the tangent spaces $T_x(Z)$ remain constant to the first order as x varies in a neighborhood of z. In particular, for any local vector field ω on Z, regarded as a section of $T(\mathbf{C}^n)$ restricted to Z,

$$\omega = a_1(z_1, \ldots, z_m)\frac{\partial}{\partial w_1} + \cdots + a_n(z_1, \ldots, z_m)\frac{\partial}{\partial w_n},$$

its derivative in the direction of V

$$V\omega = V(a_1(z_1, \ldots, z_m))\frac{\partial}{\partial w_1} + \cdots + V(a_n(z_1, \ldots, z_m))\frac{\partial}{\partial w_n}$$

remains tangent to Z at z. Conversely, one can see that if V is a tangent vector to Z at z so that $V\omega(z) \in T_z(Z)$ for any local vector field ω of Z, then V is in the kernel of the differential of the Gauss map. This can be applied to a projective subvariety of $\mathbf{P}_{n-1}$ by taking its affine cone. Using this interpretation, the proof of Proposition 3.4 goes as follows.

First one shows that if there exists a vector in $Ch(\mathcal{P})_\alpha$ which is not in $\mathcal{F}_\alpha$, where α is a generic point of $\mathcal{C}$, then there must exist such a vector V tangent to the fibers of $\pi : \mathcal{C} \to X$, namely, $V \in T_\alpha(\mathcal{C}_x)$ where $x = \pi(\alpha)$. This follows from a simple manipulation of Jacobi identity for the brackets of vector fields. Now the condition $V \in Ch(\mathcal{P})_\alpha$ can be written, by some abuse of notation, as

$$[V, \mathcal{P}] \subset \mathcal{P},$$

and this implies

$$[V, \mathcal{P} \cap T(\mathbf{P}T_x(X))] \subset \mathcal{P} \cap T(\mathbf{P}T_x(X)).$$

Combined with the fact

$$\mathcal{P}_\alpha \cap T_\alpha(\mathbf{P}T_x(X)) = T_\alpha(\mathcal{C}_x),$$

we get

$$[V, T(\mathcal{C}_x)] \subset T(\mathcal{C}_x).$$

Thus $V\omega(\alpha) \in T_\alpha(\mathcal{C}_x)$ for any local vector field ω on $\mathcal{C}_x$ near α and V must be in the kernel of the Gauss map. So the assumption on the Gauss map gives $V = 0$.

Now for Step 1 of the proof of Theorem 3.2 can be achieved as follows. As mentioned before, $\varphi_* : \mathcal{C}|_U \to \mathcal{C}'|_{U'}$ sends $\mathcal{P}$ to $\mathcal{P}'$. By Proposition 3.4, this implies that φ_* sends $\mathcal{F}$ to $\mathcal{F}'$, which means that local pieces of members of $\mathcal{K}$ are sent to local pieces of members of $\mathcal{K}'$.

Once Step 1 is done, the rest is extending a map which sends pieces of members of $\mathcal{K}$ to pieces of members of $\mathcal{K}'$. This is done by an analytic continuation along minimal rational curves, in the following way. Suppose C is a standard minimal rational curve intersecting the open subset U. φ is defined on $C \cap U$ and we want to extend it to other points on C. To define the extension at a point $y \in C$, consider a deformation C_t of C fixing the point y. This is where we need to have $p > 0$. If $p = 0$ there exists no non-trivial deformation fixing a point. Now consider the local pieces $U \cap C_t$. By Step 1, $\varphi(U \cap C_t)$ is a local piece of some minimal rational curve C'_t belonging to $\mathcal{K}'$. These curves C'_t have a unique common point y' and we define y' as the image of y. The common point y' exists because it exists when y is chosen to be inside U. It is unique because C'_t's do not have deformations fixing two or more points. In fact, if such a deformation exists, then its initial velocity is a section of the normal bundle of a standard minimal rational curve vanishing at two or more points, a contradiction to the splitting type. This way we can extend φ along standard minimal rational curves intersecting U. This enlarges the domain of definition of φ to a bigger open set $\hat{U}$. Applying the same argument to $\hat{U}$, we can analytically continue along standard minimal rational curves intersecting $\hat{U}$. We can repeat this procedure until the domain of definition covers a Zariski open subset in X. But there is a gap in this extension argument. A point outside U may belong to different standard minimal rational curves intersecting U. So when we carry out the analytic continuation, we end up with a multi-valued extension of φ. So what we get at the end is a multi-valued extension of φ over an etale open subset $\tilde{U}$ of X, namely a quasi-projective variety $\tilde{U}$ with an etale morphism $\nu : \tilde{U} \to X$ covering a Zariski open subset of X and a morphism $\tilde{\varphi} : \tilde{U} \to X'$ extending φ. This completes Step 2. Here, a priori, one has to worry about the possibility of an essential analytic singularity for $\tilde{\varphi}$. But this can be easily taken care of, because the analytic continuation can be done along the whole minimal rational curve. We skip the details.

Step 3 is to extend $\tilde{\varphi}$ to a morphism Φ_o defined on a Zariski open subset X_o of X. To do this, we have to reduce the multi-valuedness of $\tilde{\varphi}$. First of all, we can reduce the multi-valuedness of $\tilde{\varphi}$ by identifying two points u_1 and $u_2 \in \tilde{U}$ if $\nu(u_1) = \nu(u_2)$ and $\tilde{\varphi}(u_1) = \tilde{\varphi}(u_2)$. So let us assume that there is no such two distinct points. Then we claim that ν must be 1-to-1, which implies that $\tilde{U}$ is Zariski open in X. Suppose not. Choose a standard minimal rational curve $C \subset X$ generically and pick a generic point $x \in C$. Then there exists an irreducible component $\tilde{C}$ of $\nu^{-1}(C)$ containing a pair of points $u_1, u_2 \in \tilde{U}$ with $\nu(u_1) = \nu(u_2)$ and $\tilde{\varphi}(u_1) \neq \tilde{\varphi}(u_2)$, from the following Lemma.

Lemma 3.5 *Let $\pi : Y \to X$ be a generically finite morphism from a normal variety Y onto a Fano manifold X with Picard number 1. Suppose for a generic standard rational curve $C \subset X$ belonging to a chosen minimal rational component $\mathcal{K}$, each component of the inverse image $\pi^{-1}(C)$ is birational to C by π. Then $\pi : Y \to X$ itself is birational.*

Proof. Suppose π is not birational. We can choose a ramification divisor $R \subset Y$ of π so that $\pi(R)$ is a divisor in X. By genericity of C, we may assume that $\pi^{-1}(C)$ lies on the smooth part of the normal variety Y. Let C_1 be any irreducible component of $\pi^{-1}(C)$. Then C_1 is also a rational curve and deformations of C_1 give deformations of C since $\pi|_{C_1}$ is birational. It follows that the space of deformations of C and the space of deformations of C_1 have equal dimensions. So we have $K_Y \cdot C_1 = K_X \cdot C$. This implies C_1 is disjoint from the ramification divisor $R \subset Y$. Since this holds for any component C_1 of $\pi^{-1}(C)$, C is disjoint from the divisor $\pi(R)$, a contradiction to the assumption that X is of Picard number 1. □

Now let C_t be a deformation of C with x fixed, which exists by $p > 0$. Then their inverse images under ν contains components $\tilde{C}_t$ which are deformation of $\tilde{C}$ fixing u_1 and u_2. Then their images under $\tilde{\varphi}$ define a family of standard rational curves in X' fixing two distinct point $\varphi(u_1)$ and $\varphi(u_2)$, a contradiction. This finishes Step 3.

By applying the same extension to $\varphi^{-1} : U' \to U$, we see that the rational map Φ_o obtained in Step 3 is birational. For Step 4, first we claim that Φ_o has no exceptional set of codimension 1. Suppose not and let $E \subset X$ be an exceptional set of codimension 1 which is contracted to a set Z of codimension ≥ 2. From the Picard number condition, all members of $\mathcal{K}$ intersect E. It follows that generic members of $\mathcal{K}'$ must intersect Z, a contradiction to Lemma 2.1.

Now that Φ_o defines a biholomorphism between X and X' outside sets of codimension ≥ 2, we can push sections of powers of K_X^{-1} to sections of powers of $K_{X'}^{-1}$ inducing an isomorphism $H^0(X, mK_X^{-1}) = H^0(X', mK_{X'}^{-1})$ for all m. Since X and X' are Fano, they are isomorphic by this map, finishing Step 4. This completes the proof of Theorem 3.2.

Let me remark that for homogeneous Fano manifolds, the analytic continuation in Step 2 can be carried out using $\mathbf{C}^*$-actions. See [HM3] or [Mk3] for this approach. Also in [Mk3], a more differential geometric treatment of Step 1 is given for Hermitian symmetric spaces, which might be helpful in understanding the key idea.

3.3 Curvature

We will discuss some applications of Theorem 3.2 in the next section where the conditions on the map φ is, in a sense, guaranteed a priori. Before this, I want to discuss how one can ever check the condition on φ in a setting where it is a priori not obvious. This has to do with understanding the difference between Question 3.1 and Theorem 3.2. We may state the problem as follows.

Let X and X' be Fano manifolds of Picard number 1 with minimal rational components $\mathcal{K}$ and $\mathcal{K}'$, respectively. Suppose there exist analytic open subsets $U \subset X, U' \subset X'$ with an isomorphism $\psi : \mathbf{P}T(U) \to \mathbf{P}T(U')$ which sends $\mathcal{C}|_U$ to $\mathcal{C}'|_{U'}$ isomorphically. Does there exist open subsets $U_o \subset U, U'_o \subset U'$ and a biholomorphic map $\varphi : U_o \to U'_o$ so that $\varphi_*(\mathcal{C}|_{U_o}) = \mathcal{C}'_{U'_o}$?

This is a question in local differential geometry. I want to illustrate this point for the case when X is the smooth hyperquadric in $\mathbf{P}_{n+1}$. As we saw in Section 1, the variety of minimal rational tangents is a $(n-2)$-dimensional hyperquadric in $\mathbf{P}T_x(X)$. Since X is homogeneous, this is the case for every point $x \in X$. Thus we are given a subbundle of $\mathbf{P}T(X)$ with fibers isomorphic to hyperquadrics. This is a 'conformal structure' on X.

In general, a **conformal structure** on a complex manifold M is a vector bundle morphism $\sigma : Sym^2 T(M) \to L$ for a line bundle L, which gives a nondegenerate symmetric bilinear form at each fiber $T_x(M)$. The hyperquadric in $T_x(M)$ defined by the zero locus of σ is called the **null-cone** at x. Equivalently, a conformal structure on M is just a fiber subbundle $\mathcal{C} \subset \mathbf{P}T(M)$ whose fiber at each point $m \in M$ is a smooth hyperquadric $\mathcal{C}_x \subset \mathbf{P}T_m(M)$.

After choosing a local trivialization of L, σ can be regarded locally as a holomorphic Riemannian metric (e.g. [Le]).

$$\sigma \simeq \sum_{i,j=1}^{n} g_{ij}(z) dz^i \otimes dz^j$$

in some local holomorphic coordinate system $z_1, \ldots, z_n$. Here (g_{ij}) is a nondegenerate symmetric matrix. Just following the usual formalism of Riemannian geometry, we can define the Levi-Civita connection

$$\Gamma^i_{jk} := \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{kl}}{\partial z_j} + \frac{\partial g_{jl}}{\partial z_k} - \frac{\partial g_{jk}}{\partial z_l}\right)$$

and the curvature tensor

$$R^i_{jkl} := \frac{\partial \Gamma^i_{jl}}{\partial z^k} - \frac{\partial \Gamma^i_{jk}}{\partial z^l} + \sum_{\mu}(\Gamma^{\mu}_{jl}\Gamma^i_{\mu k} - \Gamma^{\mu}_{jk}\Gamma^i_{\mu l})$$

which is a local holomorphic tensor antisymmetric in k and l. Similarly we can define geodesics as local holomorphic curves satisfying the geodesic equation

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j}\Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$$

All these notions depend on the choice of the local trivialization of the line bundle L. However certain part of the curvature tensor, named **Weyl tensor**, is independent of the local trivialization. Geodesics which are tangent to the null-cone, called **null-geodesics**, are independent of the local trivialization of L. If the Weyl tensor vanishes, we say that the conformal structure is flat. In this case, there exists a local trivialization of L with respect to which the local Riemannian metric is flat. In other words, there exists a local holomorphic coordinate system, called a flattening coordinate system, with respect to which

$$\sigma \simeq \sum_i dz^i \otimes dz^i.$$

Unfortunately, I do not know of a good reference for this standard fact in conformal differential geometry. A proof in a more general setting is given in [Gu] which is summarized in the first section of [HM1] in a language more friendly to algebraic geometers.

For the hyperquadric, the conformal structure given by the variety of minimal rational tangents is flat. This can be seen by an explicit choice of a flattening coordinate system. As a matter of fact, the open cell defined as the complement of a singular hyperplane section of the hyperquadric in $\mathbf{P}_{n+1}$ carries a natural affine coordinate system which flattens the conformal structure. This is an example of what is called 'Harish-Chandra coordinate system' on

Hermitian symmetric spaces ([HM3], [Mk2]). Minimal rational curves are precisely the null geodesics.

When we apply Theorem 3.2 to the hyperquadric X, the condition that $\varphi_*\mathcal{C}_x = \mathcal{C}'_{\varphi(x)}$ means that the conformal structure defined at generic points of X' by the variety of minimal rational tangents is flat. The difference between Question 3.1 and Theorem 3.2 in this case is exactly the Weyl tensor. In this sense, this difference, in general, can be viewed as a notion of curvature and what Question 3.1 is implicitly asking is whether the 'geometric structure' given by the variety of minimal rational tangents is flat, namely the curvature vanishes.

Even in the case of the conformal structure, it is not easy to show that the Weyl tensor vanishes for X'. To give you some idea how the curvature can be handled by minimal rational curves, I want to discuss a special case. Let X be a Fano manifold of Picard number 1 whose variety of minimal rational tangents is a hyperquadric. Thus a conformal structure is given on a Zariski open subset of X. Now make the strong assumption that this conformal structure extends to the whole X. Under this assumption, we will show that the Weyl tensor

$$W \in H^0(X, \Lambda^2 T^*(X) \otimes End(T(X)))$$

vanishes identically. For a standard minimal rational curve C,

$$T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-2} \oplus \mathcal{O}$$

because $p = \dim(\mathcal{C}_x) = n-2$. To show that W vanishes at a generic point $x \in X$, we need to show the vanishing of $W(u \wedge v) \in End(T_x(X))$ for all possible choices of $u, v \in T_x(X)$. But we know that $\Lambda^2 T_x(X)$ is spanned by elements of the form $\alpha \wedge \beta$ where α is a generic point of $\mathcal{C}_x$ and β is in the tangent space to $\mathcal{C}_x$ at α from Proposition 2.6. Let C be a standard minimal rational curve in the direction of α. Let $\tilde{\alpha}$ be a section of $T(C) \subset T(X)|_C$ which is non-zero at x. Let $\tilde{\beta}$ be a section of $T(X)|_C$ which has value β at x. By Proposition 2.3, we know that $\tilde{\beta}$ is a section of $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-2}$ and we can choose it to have a zero at some point $y \in C$. Let us consider $W(\tilde{\alpha}, \tilde{\beta})$ as a section of $End(T(X))|_C$. It has three zeros, two coming from that of $\tilde{\alpha}$ and one coming from that of $\tilde{\beta}$. But

$$End(T(X))|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{2n-4} \oplus \mathcal{O}^{n^2-4n+6} \oplus [\mathcal{O}(-1)]^{2n-4} \oplus \mathcal{O}(-2)$$

cannot have a non-zero section with three zeros. Thus $W(\alpha, \beta)$ vanishes at x and consequently W vanishes on X.

A similar argument as above works for any Hermitian symmetric space where 'the conformal structure' is replaced by the natural geometric structure on the Hermitian symmetric space ([HM1]). For example, one can show that if the tangent bundle of a Fano manifold of Picard number 1 is isomorphic to the tensor product of two vector bundles of rank ≥ 2 (this geometric structure is called 'Grassmann-spinor structure' in [Ma]), then the Fano manifold is the Grassmannian.

Of course, the assumption on the extendability of the conformal structure (or other geometric structure modeled on a Hermitian symmetric space) arising from the variety of minimal rational tangents to the whole X is a strong one. However, it is possible to show that such an extension indeed exists and give an affirmative answer to Question 3.1 for Hermitian symmetric spaces ([HM9]).

4 Applications to generically finite morphisms over Fano manifolds of Picard number 1

4.1 Varieties of distinguished tangents

In this section, we will discuss certain rigidity properties of a generically finite morphism $f : Y \to X$ from a projective manifold Y onto a Fano manifold X of Picard number 1. Our strategy is to study the inverse images of minimal rational curves on X under f. These curves are, in general, not rational. This leads us to seek a generalization of the variety of minimal rational tangents for non-rational curves. A natural generalization is the following. For a given projective manifold Y, fix a component $\mathcal{M}$ of the Chow scheme of curves. For a generic point $y \in Y$, let $\mathcal{M}_y$ be the subscheme corresponding to members of $\mathcal{M}$ passing through y, which we assume to be non-empty. We have the tangent map $\tau_y : \mathcal{M}_y \to \mathbf{P}T_y(Y)$ defined at those points corresponding to members of $\mathcal{M}_y$ which are smooth at y. Then the closure of the image of τ_y would play the role of the variety of minimal rational tangents. But to have an interesting theory, the image of τ_y should be a proper subvariety of $\mathbf{P}T_y(Y)$. Unfortunately, I do not know of a suitable setting for non-rational curves where this image is a proper subvariety. (If anyone knows a non-trivial example, please let me know.) To remedy this, we will refine our definition so that even when the image is not proper, some proper subvariety can be defined naturally. This is done by considering a natural stratification associated with the tangent map and the proper subvariety will be defined as a stratum.

Consider a morphism $h : M \to Z$ where M, Z are quasi-projective varieties. The **h-stratification** of M is a decomposition $M = M_1 \cup \cdots \cup M_k$ of M into a disjoint union of quasi-projective subvarieties which is induced by h and satisfies the following conditions.

h1 Each M_i is smooth and its image $h(M_i)$ is also smooth.

h2 For any tangent vector v to $h(M_i)$, we can find a local holomorphic arc in M_i whose image under h is tangent to v.

h3 When a connected Lie group acts on M and Z, and h is equivariant under these actions, each M_i is invariant under the group action.

It is easy to see how to construct such a stratification. Repeatedly using the usual stratification of a variety into smooth locus and singular loci, we can always find a stratification satisfying **h1**. We can stratify each stratum further by the rank of the restriction of h to the stratum to achieve the condition **h2**. But after this new stratification, **h1** may be violated. Then we apply the singular loci stratification to each stratum again. After a finitely many steps of applying these two stratifying procedures, we end up with the stratification satisfying both **h1** and **h2**. Since this procedure is canonical, **h3** is automatic.

Now consider the stratification arising from the tangent map. Given a smooth projective variety Y and a point $y \in Y$, choose an irreducible component $\mathcal{N}$ of the Chow scheme of curves on Y passing through y. Let $\mathcal{N}' \subset \mathcal{N}$ be the open subscheme corresponding to curves smooth at y. We consider only the case when $\mathcal{N}' \neq \emptyset$. Let $\underline{\mathcal{N}}'$ be the underlying quasi-projective variety of $\mathcal{N}'$. We can decompose $\underline{\mathcal{N}}'$ into disjoint union of finitely many quasi-projective subvarieties, $N^1 \cup \cdots \cup N^l$, according to the geometric genus of the curves corresponding to points of $\underline{\mathcal{N}}'$. So curves corresponding to $N^j, 1 \leq j \leq l$ have the same geometric genus. For a choice of N^j, define the tangent morphism $\Phi : N^j \to \mathbf{P}T_y(Y)$

which assigns to a curve smooth at y its tangent direction at y. Now we can consider the Φ-stratification of $N^j = M_1^j \cup \cdots \cup M_k^j$. The closure of the image $\Phi(M_i^j)$ is called a **variety of distinguished tangents**. In other words, a subvariety of $\mathbf{P}T_y(Y)$ is a variety of distinguished tangents if it is the closure of the image $\Phi(M_i^j)$ for some choices of $\mathcal{N}$, N^j and M_i^j. The following three properties are important.

d1 For given Y and y, there are only countably many varieties of distinguished tangents in $\mathbf{P}T_y(Y)$.

d2 Let $\mathcal{D} \subset \mathbf{P}T_y(Y)$ be a variety of distinguished tangents. Then for any tangent vector v to $\mathcal{D}$, we can find a family of curves l_t belonging to $\mathcal{N}$ smooth at y so that the derivative of the tangent directions $\mathbf{P}T(l_t) \in \mathbf{P}T_y(Y)$ at $t = 0$ is v.

d3 Suppose a connected Lie group P acts on Y fixing y. Then any variety of distinguished tangents in $\mathbf{P}T_y(Y)$ is P-invariant under the isotropy action of P on $\mathbf{P}T_y(Y)$.

d1 follows from the fact that there are only countably many irreducible components of the Chow scheme. **d2** follows from the property **h2** of h-stratification. **d3** follows from the property **h3** of h-stratification. The property **d1** is the key to the rigidity result we will discuss. **d2** is one of the key point of the definition of varieties of distinguished tangents. Unlike the standard minimal rational curves, it is very rare that we have a good information on the normal bundle of high genus curves. As a result, their deformation theory can be very tricky. But **d2** automatically takes care of obstructions to deformations. **d3** is useful in the study of homogeneous spaces.

4.2 Pull-back of the variety of minimal rational tangents under a generically finite morphism

Given a curve $l \subset Y$ and a smooth point $y \in Y$, there exists a unique variety of distinguished tangents in $\mathbf{P}T_y(Y)$ determined by the component of the Chow scheme and the stratum of the Φ-stratification containing l. It will be denoted by $\mathcal{D}_y(l)$. The next Proposition is a direct consequence of **d1**.

Proposition 4.1 *Let $l_z, z \in Z$ be a family of curves through $y \in Y$ parametrized by an irreducible variety Z so that l_z is smooth at y for a generic z. Let $\mathcal{Z} \subset \mathbf{P}T_y(Y)$ be the closure of the union of the tangent directions $\mathbf{P}T_y(l_z)$ for generic $z \in Z$. Then $\mathcal{Z} \subset \mathcal{D}_y(l_z)$ for a generic $z \in Z$.*

Proof. $\mathcal{Z} \subset \cup_{z\in Z}\mathcal{D}_y(l_z)$ by definition. But $\mathcal{Z}$ is irreducible and the union is a countable union by **d1**, from which the result follows. □

Let $y \in Y$ be a sufficiently general point. When $l \subset Y$ is a smooth curve through y and N_l^* is its conormal bundle, we have the following bound on the dimension of the variety of distinguished tangents.

Proposition 4.2 $\dim(\mathcal{D}_y(l)) \leq \dim(Y) - 1 - h^0(l, N_l^*)$.

Proof. Given a tangent vector v to $\mathcal{D}_y(l)$, we can find a deformation l_t of $l_0 = l$ fixing y so that the derivative of their tangent directions gives v by **d2**. The initial velocity of this deformation is given by an element $\kappa \in H^0(l, N_l)$ with $\kappa_y = 0$. Given any $w \in H^0(l, N_l^*)$, the pairing $< w, \kappa >$ must be a constant holomorphic function on l. Thus

$$\begin{aligned} 0 &= d < w, \kappa > (T_y(l)) \\ &= < dw(T_y(l)), \kappa_y > + < w, d\kappa(T_y(l)) > \\ &= < w, v >_y . \end{aligned}$$

So sections of N_l^* give constraints on tangent vectors of $\mathcal{D}_y(l)$. If y is sufficiently general, there must be $h^0(l, N_l^*)$ independent constraints. □

As a matter of fact, even when l is not smooth, an analogue of Proposition 4.2 holds if we replace N_l^* by a suitable sheaf (section 1 of [HM4]). An irreducible component of the variety of minimal rational tangents on a Fano manifold is an example of the variety of distinguished tangents and Proposition 4.2 is an analogue of the fact that the dimension of $\mathcal{C}_x$ is $p = n - 1 - h^0(C, N_C^*)$ because

$$N_C^* \cong \mathcal{O}^{n-1-p}$$

for a standard minimal rational curve C. The next proposition provides a good supply of examples of varieties of distinguished tangents.

Proposition 4.3 *Let $f : Y \to X$ be a generically finite morphism from a smooth variety Y onto a Fano manifold X of Picard number 1. For a sufficiently general point $x \in X$ outside the branch locus, let $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ be the variety of minimal rational tangents for a choice of the minimal rational component on X. Then for $y \in f^{-1}(x)$, each irreducible component of $df_y^{-1}(\mathcal{C}_x) \subset \mathbf{P}T_y(Y)$, the inverse image of the variety of minimal rational tangents under the differential $df_y : T_y(Y) \to T_x(X)$, is a variety of distinguished tangents.*

Proof. For simplicity, we will assume that all curves involved are smooth. Consider the irreducible family Z of rational curves defining an irreducible component A of $\mathcal{C}_x$. Their inverse images have components containing y, defining an irreducible family of curves through y. By Proposition 4.1, $df_y^{-1}(A) \subset \mathcal{D}_y(l)$ for some curve l where $f(l)$ is a generic member of Z. We know that $df_y^{-1}(A)$ has dimension p. On the other hand, since $h^0(f(l), N_{f(l)}^*) = n-1-p$, $h^0(l, N_l^*) \geq n-1-p$. Thus $df_y^{-1}(A)$ and $\mathcal{D}_y(l)$ must have the same dimension by Proposition 4.2, and we are done. □

4.3 Rigidity of generically finite morphisms

We are ready for the following rigidity theorem ([HM7]).

Theorem 4.4 *Let Y be a smooth variety and $X_t, t \in \Delta := \{t \in \mathbf{C}, |t| < 1\}$ be a family of Fano manifolds of Picard number 1 with minimal rational components $\mathcal{K}_t$ for which Cartan-Fubini type extension (Theorem 3.2) holds. Suppose $f_t : Y \to S$ is a family of generically finite morphisms parametrized by $t \in \Delta$. Then there exists a family g_t of biholomorphic morphisms from X_0 to X_t with $g_0 = id$ such that $f_t = g_t \circ f_0$.*

Proof. Choose a connected analytic open subset $U \subset Y$ so that $f_t|_U$ is biholomorphic for all $t \in \Delta$ and let $U_t = f_t(U)$. For a generic point $y \in U$ and $x_t = f_t(y)$, components of $df_t^{-1}(\mathcal{C}_{x_t})$ form a family of varieties of distinguished tangents in $\mathbf{P}T_y(Y)$ by Proposition 4.3. From **d1**, $df_t^{-1}(\mathcal{C}_{x_t}) = df_0^{-1}(\mathcal{C}_{x_0})$ at y for all $t \in \Delta$. It follows that $\varphi : U_0 \to U_t$ defined by

$$\varphi_t := f_t \circ (f_0|_U)^{-1}$$

preserves the varieties of minimal rational tangents. By Cartan-Fubini type extension, we see that φ_t extends to a biholomorphic morphism $g_t : X_0 \to X_t$ with the desired property. □

Theorem 4.4 is certainly not true when X_t is the projective space. It can be applied to all the examples we have seen in Section 1 other than the projective space as long as $p > 0$. It is new even for Fano hypersurfaces or Hermitian symmetric spaces. An immediate corollary is

Corollary 4.5 *For a given smooth variety Y, there are only countably many smooth hypersurfaces of degree $\leq n-1$ in $\mathbf{P}_{n+1}$ which can be the image of a generically finite morphism from Y.*

4.4 The case of $p = 0$

It is natural to ask whether Corollary 4.5 holds for smooth hypersurfaces of arbitrary degrees. It is well-known that there can be only finitely many hypersurfaces of degree $\geq n+3$ which can be the image of a rational map from Y (e.g. [KO]). E. Viehweg had told me that using the results on the semi-positivity of the direct images of powers of dualizing sheaves (e.g. Section 7 in [Mo2]), one can show that only countably many hypersurfaces of degree $n+2$ can be the image of a morphism from Y. Thus only the degrees n and $n+1$ are open. In a recent work ([HM8]), Mok and I settled these cases by proving an analogue of Theorem 4.4 for the case of $p = 0$, although we were not able to establish Cartan-Fubini type extension for that case:

Theorem 4.6 *Let Y be a smooth variety and $X_t, t \in \Delta$ be a family of Fano manifolds of Picard number 1 which have minimal rational components $\mathcal{K}_t$ with $p = 0$. Given a family of generically finite morphisms $f_t : Y \to X_t$, there exists a family of biholomorphic morphisms $g_t : X_0 \to X_t$ so that $f_t = g_t \circ f_0$.*

The proof is quite different from that of Theorem 4.4. This is an example of a general phenomenon that the geometry of the case of $p = 0$ has essential difference from the geometry of the cases of $p > 0$. Of course, this may be due to our lack of proper understanding of the situation. When $p = 0$, there are finitely many minimal rational curves through a generic point. The geometric structure it defines in a neighborhood of a generic point is that of a multi-valued foliation by curves. Such a geometric structure is called a web. The local differential geometry of webs is rather complicated and so far, we were not able to use it. Instead, we exploit the existence of discriminantal locus of the multi-valued foliation.

The condition $p = 0$ is equivalent to the triviality of the normal bundles of standard minimal rational curves and we need to study curves with trivial normal bundles in general smooth variety to prove Theorem 4.6. Let Y be a smooth variety. Following the name of the local geometric structure it defines, a projective variety $\mathcal{M}$ with finitely many components in the reduction of the Chow scheme of Y is called a **web**, if (a) generic members of each component of $\mathcal{M}$ are curves with only nodal singularities and with trivial normal bundles, and (b) members of each component of $\mathcal{M}$ cover a Zariski open subset in Y.

Let $\mathcal{M}$ be a web and $\rho : \mathcal{U} \to \mathcal{M}, \mu : \mathcal{U} \to Y$ be the universal family morphisms. In an analytic neighborhood of a fiber of ρ corresponding to a curve with nodal singularity and trivial normal bundle, the morphism μ is an immersion. In particular, μ is generically finite.

The degree of μ is called the **degree** of the web $\mathcal{M}$. As before, we can define the tangent map $\tau : \mathcal{U} \to \mathbf{P}T(Y)$. Let $\mathcal{C} \subset \mathbf{P}T(Y)$ be the closure of the image $\tau(\mathcal{U})$ and $\pi : \mathcal{C} \to Y$ be the natural projection, which is generically finite. An irreducible hypersurface $M \subset Y$ is called a **discriminantal divisor** of the web $\mathcal{M}$ if π is not etale over a generic point of M. (This definition is different from that of [HM8], but suffices for our purpose here.) Many examples of webs with non-empty discriminantal divisors are provided by Fano manifolds:

Proposition 4.7 *For a Fano manifold X of Picard number 1 which has a minimal rational component $\mathcal{K}$ with $p = 0$, the set $\mathbf{H}$ of discriminantal divisors of the web $\mathcal{K}$ is non-empty. Moreover a member of $\mathcal{K}$ intersects $\mathbf{H}$ at least at two distinct points on the normalization $\mathbf{P}_1$.*

Proof. Suppose $\mathbf{H}$ is empty. Then $\mu : \mathcal{U} \to X$ is etale outside a set of codimension ≥ 2. A generic minimal rational curve is disjoint from the set of codimension ≥ 2 by Lemma 2.1, so its inverse image in $\mathcal{U}$ must have d distinct components from the simply-connectedness of $\mathbf{P}_1$ where d is the degree of $\mathcal{M}$. Thus $\mu : \mathcal{U} \to X$ is a birational morphism from Lemma 3.5. Since μ is unramified in a neighborhood of a generic fiber of $\rho : \mathcal{U} \to \mathcal{K}$, this is a contradiction to the Picard number of X. Now for the last statement, apply the same argument to $\mathbf{C}$, the complement of one point on $\mathbf{P}_1$. □

A key property of webs is the following simple Lemma which follows from the unramifiedness of μ in a neighborhood of a generic fiber of ρ. We will leave the proof as an exercise.

Lemma 4.8 *Given a web $\mathcal{M}$ on Y and an irreducible hypersurface $H \subset Y$, a component C of a member of $\mathcal{M}$ passing through a generic point $h \in H$ is either transversal to H at every point of $H \cap C$ or contained in H.*

The following Proposition provides many examples of webs whose members are not necessarily rational curves.

Proposition 4.9 *Let $f : Y' \to Y$ be a generically finite morphism between smooth varieties. Suppose Y has a web $\mathcal{M}$. Then for a generic member C of $\mathcal{M}$, each component of $f^{-1}(C)$ is a curve with nodal singularity whose normal bundle is trivial.*

Proof. A generic member of each component of the web $\mathcal{M}$ intersects the branch locus of f transversally from Lemma 4.8. From this we see that each component of $f^{-1}(C)$ has only nodal singularities. Now the $n-1$ independent sections of the conormal bundle of C can be pulled back to those of components of $f^{-1}(C)$, which gives the triviality of the normal bundle of each component of $f^{-1}(C)$. □

From Proposition 4.9, we see that the components of $f^{-1}(C)$ as C varies over $\mathcal{M}$ define a web on Y', which we call the **inverse image web** and denote by $f^{-1}(\mathcal{M})$. The degree of $f^{-1}(\mathcal{M})$ is the same as the degree of $\mathcal{M}$. A key property of the inverse image web is

Proposition 4.10 *Let the notation be as in Proposition 4.9. For a discriminantal divisor $M \subset Y$ of the web $\mathcal{K}$, each component of $f^{-1}(M)$ on which f is generically finite, is a discriminantal divisor of $f^{-1}(\mathcal{K})$.*

Proof. It suffices to show that if a hypersurface H of Y is not a discriminantal divisor of $f^{-1}(\mathcal{K})$, then $f(H)$ is not a discriminantal divisor of $\mathcal{K}$. We may assume that H is a ramification divisor of f. Let d be the degree of $\mathcal{K}$. Through a generic point h of H, there are d distinct curves $C_1, \ldots, C_d$, belonging to $\mathcal{M}$ which has d distinct tangent vectors. We claim that at most one of $C_1, \ldots, C_d$ is not contained in H. In fact, if C_1, C_2 are not contained in H, $f(C_1)$ and $f(C_2)$ are transversal to $f(H)$ by Lemma 4.8. This implies that C_1 and C_2 are tangent to the kernel of df_h, so they are tangent to each other at h, a contradiction. Since $f|_H$ is unramified at h, d or $d-1$ members among $C_1, \ldots, C_d$, which are contained in H, are sent to curves in $f(H)$ with distinct tangents in at $f(h)$. Thus $f(C_1), \ldots, f(C_d)$ have d distinct tangents at $f(h)$. Thus $f(H)$ is not a discriminantal divisor. □

Proof of Theorem 4.6. The key point is that the inverse image web $f_t^{-1}(\mathcal{K}_t)$ is independent of t. This is because there are only countably many webs on Y from the countability of the number of components of the Chow scheme. Let M_t be the union of all discriminantal divisors of $\mathcal{K}_t$. Then $f_t^{-1}(M_t)$ is also independent of t from Proposition 4.10. Fix a generic member C of any component of $f^{-1}(\mathcal{K}_t)$. By a general argument, which we will skip, we can reduce the proof to showing that any two points on C which have the same image under f_0 have the same image under f_t for any t. Since $f_t^{-1}(M_t)$ is independent of t, we know that any two points which are sent to the same point in M_0 are sent to the same point in M_t. But by Proposition 4.7, at least two points of $f_t(C)$ are in M_t. Thus $f_t|_C$ can be regarded as meromorphic functions on the curve C with the same zeroes and poles, and so they are constant multiple of each other, which implies that any two points with the same value of f_0 must have the same value of f_t. □

4.5 Morphisms from G/P onto smooth varieties

Another application of Proposition 4.3 is the following theorem proved in [HM4] which gives an affirmative answer to a problem of Lazarsfeld ([La]). As a matter of fact, this problem was the main motivation for introducing the concept of varieties of distinguished tangents.

Theorem 4.11 *Let X be a smooth variety and G/P be a homogeneous Fano manifold of Picard number 1. If there exists a surjective morphism $f : G/P \to X$, then either X is the projective space or f is an isomorphism.*

It is easy to see that X is a Fano manifold of Picard number 1 and f is a finite morphism. So we can apply Proposition 4.3. For the proof, we need to use the structure of the isotropy representation of G/P in detail. To give the essence of the idea without using too much Lie theory, I will just discuss the case when G/P is the Grassmannian.

Sketch of the proof for the Grassmannian. For the Grassmannian $Gr(s, V)$ of s-dimensional subspaces in a complex vector space V of dimension $\geq 2s$, the tangent space at $[W]$ is naturally isomorphic to $Hom(W, V/W)$. The isotropy subgroup at $[W]$ is the group of linear automorphisms of V preserving W. Under the action of this group, $\mathbf{P}Hom(W, V/W)$ has orbits $S^1, \ldots, S^s$ where

$$S^k := \{\zeta \in Hom(W, V/W), \text{ (the rank of } \zeta) = k\}.$$

The variety of minimal rational tangents $\mathcal{C}_{[W]} \subset \mathbf{P}Hom(W, V/W)$ corresponds to S^1. It is well-known that the closure of S^k is an irreducible subvariety of $\mathbf{P}Hom(W, V/W)$ whose singular locus is precisely the closure of S^{k-1}, for $1 \leq k \leq s$, with $S^0 = \emptyset$. Consider the fiber subbundle $\mathcal{S}^k \subset \mathbf{P}T(Gr(s, V))$ whose fiber at $[W]$ is the closure of S_k.

Given a finite morphism $f : Gr(s, V) \to X$ with X different from the projective space, let $U \subset X$ be a small connected open set disjoint from the branch locus. Let U_1, U_2 be

two components of $f^{-1}(U)$ and $\varphi : U_1 \to U_2$ be the biholomorphism induced by f. Since X is different from $\mathbf{P}_n$, the variety of minimal rational tangents is a proper subvariety of $\mathbf{P}T_x(X)$ for $x \in U$ by [CM]. Thus $df_y^{-1}(\mathcal{C}_x) = \mathcal{S}_y^l$ for some $l < s$ by Proposition 4.3 because a variety of distinguished tangents must be $\mathcal{S}_y^k$ for some k by **d3**. It means that φ preserves $\mathcal{S}^l$ and hence $\mathcal{S}^1$ because $\mathcal{S}^{k-1}$ is precisely the singular locus of $\mathcal{S}^k$ for $1 \leq k \leq s$. From the Cartan-Fubini type extension applied to the Grassmannian, φ can be extended to an automorphism of $Gr(s, V)$.

Since U_1, U_2 can be chosen as any components of $f^{-1}(U)$, we see that f is a Galois covering outside the ramification locus. Moreover one can show that an automorphism extending φ must fix the ramification locus of f pointwise. Thus there exists a finite group G acting on $Gr(s, V)$ fixing an effective divisor H pointwise. But one can show that if a homogeneous Fano manifold of Picard number 1 has a finite group action fixing a hypersurface pointwise, the Fano manifold must be either the projective space or the hyperquadric and the quotient by the group must be the projective space, a contradiction to the assumption that X is not the projective space. □

The above proof can be applied to other Hermitian symmetric spaces directly. The orbit structure of the isotropy representation for these cases is quite similar to that of the Grassmannian. For general G/P, this is no longer true. Especially, the number of orbits of the isotropy representation can be infinite for exceptional groups. Nonetheless, essentially the same proof works with minor modification. Also let me point out that one can avoid using the result of [CM] by using a much weaker result in [Mk1]. See [HM4] for details.

Some special cases of Theorem 4.11 were obtained previously by different methods. For the case when G/P is a hyperquadric, it was proved by Paranjape-Srinivas ([PS]) and Cho-Sato ([CS]) independently. The special case when G/P is a Hermitian symmetric space, in particular a Grassmannian, was proved by Tsai ([Ts]). In another direction, Paranjape-Srinivas studied self-maps of G/P of arbitrary Picard number and showed that a ramified self-map factors through a self-map of some projective space. They asked the following analogue of Theorem 4.11 for abelian varieties, which was proved by Debarre ([De]):

Theorem 4.12 *Let A be a simple abelian variety and $f : A \to Y$ be a finite morphism onto a smooth variety Y with non-empty ramification. Then Y is a projective space.*

Recently, Mok and I generalized it to arbitrary abelian varieties ([HM6]) using some of the ideas presented in Section 2:

Theorem 4.13 *Let A be an abelian variety and $f : A \to Y$ be a finite morphism onto a smooth variety Y with non-empty ramification. Then Y is a bundle of projective spaces over a smooth variety Y' of strictly smaller dimension, namely, there exists a smooth morphism $\rho : Y \to Y'$ whose fibers are projective spaces. Moreover, there exists an abelian variety A' with a (not necessarily ramified) finite morphism $f' : A' \to Y'$.*

The essential point of Theorem 4.13 is when Y is a Fano manifold of Picard number 1. For Theorem 4.12, Debarre showed that curves in a simple abelian variety have ample normal bundles, which implies that minimal rational curves in Y have ample normal bundles. For a general abelian variety A, one can show easily that the variety of minimal rational tangents of Y consists of finitely many linear subspaces. The main point of the proof of Theorem 4.13 is to exclude this possibility, combining deformation theory of rational curves and some topological considerations.

5 Open questions

There are many important questions about Fano manifolds for which the methods discussed in this lecture series may have some applications. Here I will list some questions which are directly concerned with the theory of the variety of minimal rational tangents itself. Most of these are about the structure of the variety of minimal rational tangents. From the examples that we have examined, we can see some general patterns about the variety of minimal rational tangents. Of course, it is dangerous to make a guess about general Fano manifolds of Picard number 1 from the data of a handful of examples. But some speculations can serve as guides for further study.

Question 1 *For any Fano manifold of Picard number 1 and any choice of a minimal rational component, is the tangent map at a generic point an immersion, or even an embedding?*

This was true for all the examples we have seen. By Proposition 1.4, τ_x is an immersion if all the minimal rational curves through a generic point are standard rational curves. To show this in general seems to be very difficult. However this question is open even for many specific cases, for example, the low genus cases of the moduli of bundles on curves.

Question 2 *Is $\mathcal{C}_x$ irreducible if it has positive dimension? Is it at least true for prime Fano manifolds?*

Of course, if $\mathcal{C}_x$ is smooth and $2p \geq n$, then $\mathcal{C}_x$ is irreducible because any two subvarieties of dimension p in $\mathbf{P}_{n-1}$ intersect if $2p \geq n$. Other than this trivial one, I do not know of any general result.

Question 3 *Assume that τ_x is an embedding and $\mathcal{C}_x$ is irreducible, is $\mathcal{C}_x$ linearly normal in its linear span in $\mathbf{P}T_x(X)$? Is it true at least for prime Fano manifolds?*

Of course, this is the case if $2p \geq 3n$ by Zak's theorem on linear normality. It is remarkable that in all the examples we have seen with $p > 0$, the variety of minimal rational tangents is linearly normal. For Hermitian symmetric spaces and homogeneous contact manifolds, this fact was crucially used in the proof of their deformation rigidity ([HM2], [Hw1]).

Question 4 *Are there examples where $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ is degenerate, which does not come from linear sections of homogeneous contact manifolds? Especially when $2p = n$, is there an example of X with degenerate $\mathcal{C}_x$, different from the homogeneous contact manifolds? Is there one among prime Fano manifolds?*

When $\mathcal{C}_x$ is smooth and degenerate with $p = 2n$, the distribution W defined by the linear span of $\mathcal{C}_x$ must have corank 1 by Theorem 2.5 . Moreover one can show that its Frobenius bracket gives a symplectic form, thus W defines a contact structure at generic points. It is not clear whether this contact structure can be extended to the whole X. If that is the case, then we can say that X is a homogeneous contact manifold at least when it is a prime Fano manifold.

Question 5 *Does Cartan-Fubini type extension holds for the case of $p = 0$?*

Step 1 of the proof of Theorem 3.2 is automatic for $p = 0$. The main difficulty is with the analytic continuation in Step 2 and the univalence in Step 3. It is very likely that the answer is affirmative. One evidence is Theorem 4.6, which is an analogue of Theorem 4.4. Another evidence is that the following weaker extension holds in some cases.

Let U, U' be connected open subsets in X and $\varphi : U \to U'$ be a biholomorphic map preserving the varieties of minimal rational tangents. Then φ can be extended to an automorphism of X.

For example, we can check by direct calculation that Mukai-Umemura threefolds ([MU]) has this extension property. Also this holds for hypersurfaces of degree n in $\mathbf{P}_{n+1}$ from the work of Jensen-Musso ([JM]). I do not know of other examples.

Question 6 *Is the Gauss map of the variety of minimal rational tangents generically finite for a Fano manifold of Picard number 1 different from the projective space?*

This is true for all the examples we know of. If the answer to Question 1 is yes, Question 6 is equivalent to the following by Zak's theorem on the finiteness of the Gauss map for a nonlinear submanifold ([Za]).

Question 7 *Can the components of the variety of minimal rational tangents be proper linear subspaces of positive dimension in $\mathbf{P}T_x(X)$?*

This is related to the main difficulty encountered in the proof of Theorem 4.13 ([HM6]). If a component is linear, the local distribution defined by that component is integrable by the arguments in the proof of Proposition 2.4. This implies, by Proposition 2.2, that the variety of minimal rational tangents cannot be irreducible. The leaf of this local foliation is very close to a projective space in the sense that it is covered by rational curves of minimal degree through a generic point. In this sense, Question 7 is related to the following question.

Question 8 *Can there exist a submanifold $Z \subset X$ isomorphic to $\mathbf{P}_k$ for some $1 < k < n$ so that the normal bundle of Z is trivial and lines of $\mathbf{P}_k$ correspond to standard minimal rational curves on X?*

Combining Question 5 and Question 6, we can ask

Question 9 *Does Cartan-Fubini type extension holds for all Fano manifolds of Picard number 1, excepting the projective space?*

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References

[Be] Berger, M. : Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés Riemanniennes. Bull. Soc. Math. France **83** (1955) 279-330

[Bo] Boothby, W.: Homogeneous complex contact manifolds. Proc. symp. pure math. **3** *Differential Geometry* (1961) 144-154

[BV] Brivio, S and Verra, A.: On the theta divisor of $SU(2,1)$. Intern. J. Math. **10** (1999) 925-942

[CS] Cho, K. and Sato, E.: Smooth varieties dominated by G/P. preprint (1989) and Smooth projective varieties dominated by smooth quadric hypersurfaces in any characteristic. Math. Zeit. **217** (1994) 553-565

[CM] Cho, K. and Miyaoka, Y.: A characterization of the projective spaces in terms of the minimal degrees of rational curves. preprint (1998)

[CP] Campana, F. and Peternell, T.: Projective manifolds whose tangent bundles are numerically effective. Math. Annalen **289** (1991) 169-187

[De] Debarre, O.: Images lises d'une variété abélienne simple. C. R. Acad. Sci. Paris **309** (1989) 119-122

[GH] Griffiths, P. and Harris, J.: Algebraic geometry and local differential geometry. Ann. scient. Éc. Norm. Sup. **12** (1979) 355-432

[Gu] Guillemin, V.: The integrability problem for G-structures. Trans. A. M. S. **116** (1965) 544-560

[HM1] Hwang, J.-M. and Mok, N.: Uniruled projective manifolds with irreducible reductive G-structures. J. reine angew. Math. **490** (1997) 55-64

[HM2] Hwang, J.-M. and Mok, N.: Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. Invent. math. **131** (1998) 393-418

[HM3] Hwang, J.-M. and Mok, N.: Characterization and deformation rigidity of compact irreducible Hermitian symmetric spaces of rank ≥ 2 among Fano manifolds. in *Algebra and Geometry* edited by M.-C. Kang, International Press (1998)

[HM4] Hwang, J.-M. and Mok, N.: Holomorphic maps from rational homogeneous spaces of Picard number 1 onto projective manifolds. Invent. math. **136** (1999) 209-231

[HM5] Hwang, J.-M. and Mok, N.: Varieties of minimal rational tangents on uniruled manifolds. in *Several Complex Variables*, edited by M. Schneider and Y.-T. Siu, MSRI publications **37** Cambridge University Press (2000) 351-389

[HM6] Hwang, J.-M. and Mok, N.: Projective manifolds dominated by abelian varieties. to appear in Math. Zeit.

[HM7] Hwang, J.-M. and Mok, N.: Cartan-Fubini type extension of holomorphic maps for Fano manifolds with Picard number 1. preprint (1998)

[HM8] Hwang, J.-M. and Mok, N.: Finite morphisms over Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles. preprint (2000)

[HM9] Hwang, J.-M. and Mok, N., in preparation.

[Hw1] Hwang, J.-M.: Rigidity of homogeneous contact manifolds under Fano deformation. J. reine angew. Math. **486** (1997) 153-163

[Hw2] Hwang, J.-M.: Stability of tangent bundles of low dimensional Fano manifolds with Picard number 1. Math. Annalen **312** (1998) 599-606

[Hw3] Hwang, J.-M.: Tangent vectors to Hecke curves on the moduli space of rank 2 bundles over an algebraic curve. Duke Math. J. **101** (2000) 179-187

[IP] Iskovskikh, V. A. and Prokhorov, Yu. G.: *Algebraic Geometry V, Fano varieties.* Encycl. Math. Sci. **47** Springer (1999)

[Is] Iskovskikh, V. A.: Anticanonical models of 3-dimensional algebraic varieties. J. Soviet Math. **13** (1980) 745-814

[JM] Jensen, G. and Musso, E.: Rigidity of hypersurfaces in complex projective space. Ann. scient. E. N. S. **27** (1994) 227-248

[Kl] Kollár, J.: *Rational curves on algebraic varieties.* Ergebnisse Math. 3 Folge **32** Springer, (1996)

[KM] Kollár, J. and Mori, S.: *Birational geometry of algebraic varieties.* Cambridge University Press (1998)

[KO] Kobayashi, S. and Ochiai, T.: Meromorphic mappings into compact complex spaces of general type. Invent. math. **31** (1975) 7-16

[La] Lazarsfeld, R.: Some applications of the theory of positive vector bundles. Lect. Notes Math. **1092** *Complete intersections.* (1984) 29-61

[Le] LeBrun, C.: Spaces of complex null geodesics in complex-Riemannian geometry. Trans. A.M.S. **278** (1983) 209-231

[Ma] Manin, Y.: *Gauge field theory and complex geometry.* Grundl. Math. Wiss. **289** Springer, 1988

[Mi] Miyaoka, Y.: Geometry of rational curves on varieties. DMV Seminar **26** *Geometry of higher dimensional algebraic varieties.* Birkhäuser (1997) 1-127

[Mk1] Mok, N.: The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. J. Diff. Geom. **27** (1988) 179-214

[Mk2] Mok, N.: *Metric rigidity theorems on Hermitian locally symmetric manifolds.* Series in Pure Math. **6** World Scientific (1989)

[Mk3] Mok, N.: G-structures on irreducible Hermitian symmetric spaces of rank ≥ 2 and deformation rigidity. Contemp. Math. **222** *Complex geometric analysis in Pohang.* (1999) 81-107

[Mo1] Mori, S.: Projective manifolds with ample tangent bundles. Annals Math. **110** (1979) 593-606

[Mo2] Mori, S.: Classification of higher-dimensional varieties. Proc. symp. pure math. **46-1** *Algebraic Geometry Bowdoin 1985* (1987) 269-331

[MU] Mukai, S. and Umemura, H.: Minimal rational threefolds. Lect. Notes Math. **1016** *Algebraic Geometry, Tokyo/Kyoto.* (1983) 490-518

[NR] Narasimhan, M. S. and Ramanan, S. : Geometry of Hecke cycles I. in C. P. Ramanujam-a tribute. Springer (1978) 291-345

[Oc] Ochiai, T.: Geometry associated with semisimple flat homogeneous spaces. Trans. A. M. S. **152** (1970) 159-193

[Pe] Peternell, T.: Manifolds of semipositive curvature. Lect. Notes Math. **1646** *Transcendental methods in algebraic geometry* (1996) 98-142

[PS] Paranjape, K. H. and Srinivas, V.: Self maps of homogeneous spaces. Invent. math. **98** (1989) 425-444

[Si] Simons, J.: On the transitivity of holonomy systems. Annals Math. **76** (1962) 213-234

[SY] Siu, Y.-T. and Yau, S.-T.: Compact Kähler manifolds with positive bisectional curvature. Invent. math. **59** (1980) 189-204

[Ts] Tsai, I. H.: Rigidity of holomorphic maps from compact Hermitian symmetric spaces to smooth projective varieties. J. Alg. Geom. **2** (1993) 603-633

[Ya] Yamaguchi, K. : Differential systems associated with simple graded Lie algebras. Adv. Stud. Pure Math. **22** *Progress in differential geometry.* (1993) 413-494

[Za] Zak, F. L.: *Tangents and secants of algebraic varieties.* Transl. Math. Monographs **127** A.M.S. (1993)