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Analytic methods and multiplier ideal sheaves

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Analytic methods and multiplier ideal sheaves

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0. Introduction

Transcendental methods of algebraic geometry have been extensively studied since a very long time, starting with the work of Abel, Jacobi and Riemann in the nineteenth century. More recently, in the period 1940-1970, the work of Hodge, Hirzebruch, Kodaira, Atiyah revealed still deeper relations between complex analysis, topology, PDE theory and algebraic geometry. In the last ten years, gauge theory has proved to be a very efficient tool for the study of many important questions: moduli spaces, stable sheaves, non abelian Hodge theory, low dimensional topology ...

Our main purpose here is to describe a few analytic tools which are useful to study questions such as linear series and vanishing theorems for algebraic vector bundles. One of the early success of analytic methods in this context is Kodaira's use of the Bochner technique in relation with the theory of harmonic forms, during the decade 1950-60. The idea is to represent cohomology classes by harmonic forms and to prove vanishing theorems by means of suitable a priori curvature estimates.

The prototype of such results is the Akizuki-Kodaira-Nakano theorem (1954): if X is a nonsingular projective algebraic variety and L is a holomorphic line bundle on X with positive curvature, then $H^q(X, \Omega^p_X \otimes L) = 0$ for $p+q > \dim X$ (throughout the paper we set $\Omega^p_X = \Lambda^p T^*_X$ and $K_X = \Lambda^n T^*_X$, $n = \dim X$, viewing these objects either as holomorphic bundles or as locally free $\mathcal{O}_X$-modules). It is only much later that an algebraic proof of this result has been proposed by Deligne-Illusie, via characteristic p methods, in 1986.

A refinement of the Bochner technique used by Kodaira led about ten years later to fundamental L^2 estimates due to Hörmander [Hör65], concerning solutions of the Cauchy-Riemann operator. Not only vanishing theorems are proved, but more precise information of a quantitative nature is obtained about solutions of $\overline{\partial}$-equations. The best way of expressing these L^2 estimates is to use a geometric setting first considered by Andreotti-Vesentini [AV65]. More explicitly, suppose that we have a holomorphic line bundle L equipped with a hermitian metric of weight $e^{-2\varphi}$, where φ is a (locally defined) plurisubharmonic function; then explicit bounds on the L^2 norm $\int_X |f|^2 e^{-2\varphi}$ of solutions is obtained. The result is still more useful if the plurisubharmonic weight φ is allowed to have singularities. Following Nadel [Nad89], we define the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally summable. Then $\mathcal{I}(\varphi)$ is a coherent algebraic sheaf over X and $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geq 1$ if the curvature of L is positive (as a current). This important result can be seen as a generalization of the Kawamata-Viehweg vanishing theorem ([Kaw82], [Vie82]), which is one of the cornerstones of higher dimensional algebraic geometry (especially of Mori's minimal model program).

In the dictionary between analytic geometry and algebraic geometry, the ideal $\mathcal{I}(\varphi)$ plays a very important role, since it directly converts an analytic object into an algebraic one, and, simultaneously, takes care of the singularities in a very efficient way. Another analytic tool used to deal with singularities is the theory of positive currents introduced by Lelong [Lel57]. Currents can be seen as generalizations of algebraic cycles, and many classical results of intersection theory still apply to currents. The concept of Lelong number of a current is the analytic analogue of the concept of multiplicity of a germ of algebraic variety. Intersections of cycles correspond to wedge products of currents (whenever these products are defined).

Besides the Kodaira-Nakano vanishing theorem, one of the most basic "effective result" expected to hold in algebraic geometry is expressed in the following conjecture of Fujita [Fuj87]: if L is an ample (i.e. positive) line bundle on a projective n-dimensional algebraic variety X, then $K_X + (n+1)L$ is generated by sections and $K_X + (n+2)L$ is very ample. In the last decade, a lot of efforts has been brought for the solution of this conjecture - and it seems indeed that a solution might finally emerge in the first years or the third millenium - hopefully during this Summer School! The first major results are the proof of the Fujita conjecture in the case of surfaces by Reider [Rei88] (the case of curves is easy and has been known since a very long time), and the numerical criterion for the very ampleness of $2K_X + L$ given in [Dem93b], obtained by means of analytic techniques and Monge-Ampère equations with isolated singularities. Alternative algebraic techniques have been developed slightly later by Kollár [Kol92], Ein-Lazarsfeld [EL93], Fujita [Fuj93], Siu [Siu95, 96], Kawamata [Kaw97a] and Helmke [Hel97]. We will explain here Siu's method

because it is technically the simplest method; one of results obtained by this method is the following effective result: $2K_X + mL$ is very ample for $m \geq 2 + \binom{3n+1}{n}$. The basic idea is to apply the Kawamata-Viehweg vanishing theorem, and to combine this with the Riemann-Roch formula in order to produce sections through a clever induction procedure on the dimension of the base loci of the linear systems involved.

Although Siu's result is certainly not optimal, it is sufficient to obtain a nice constructive proof of *Matsusaka's big theorem* ([Siu93], [Dem96]). The result states that there is an effective value m_0 depending only on the intersection numbers L^n and $L^{n-1} \cdot K_X$, such that mL is very ample for $m \geq m_0$. The basic idea is to combine results on the very ampleness of $2K_X + mL$ together with the theory of holomorphic Morse inequalities ([Dem85b]). The Morse inequalities are used to construct sections of $m'L - K_X$ for m' large. Again this step can be made algebraic (following suggestions by F. Catanese and R. Lazarsfeld), but the analytic formulation apparently has a wider range of applicability.

In the next sections, we pursue the study of L^2 estimates, in relation with the Nullstellenstatz and with the extension problem. Skoda [Sko72b, Sko78] showed that the division problem $f = \sum g_j h_j$ can be solved holomorphically with very precise L^2 estimates, provided that the L^2 norm of $|f|\,|g|^{-p}$ is finite for some sufficiently large exponent p ($p > n = \dim X$ is enough). Skoda's estimates have a nice interpretation in terms of local algebra, and they lead to precise qualitative and quantitative estimates in connection with the Bzout problem. Another very important result is the L^2 extension theorem by Ohsawa-Takegoshi [OT87, Ohs88], which has also been generalized later by Manivel [Man93]. The main statement is that every L^2 section f of a suitably positive line bundle defined on a subavariety $Y \subset X$ can be extended to a L^2 section $\tilde{f}$ defined over the whole of X. The positivity condition can be understood in terms of the canonical sheaf and normal bundle to the subvariety. The extension theorem turns out to have an incredible amount of important consequences: among them, let us mention for instance Siu's theorem [Siu74] on the analyticity of Lelong numbers, the basic approximation theorem of closed positive $(1,1)$-currents by divisors, the subadditivity property $\mathcal{I}(\varphi+\psi) \subset \mathcal{I}(\varphi)\mathcal{I}(\psi)$ of multiplier ideals [DEL00], the restriction formula $\mathcal{I}(\varphi_{|Y}) \subset \mathcal{I}(\varphi)_{|Y}$, ... A suitable combination of these results can be used to reprove Fujita's result [Fuj94] on approximate Zariski decomposition, as we show in the vary last section.

These notes are essentially written with the idea of serving as an analytic toolbox for algebraic geometry. Although efficient algebraic techniques exist, our feeling is that the analytic techniques are very flexible and offer a large variety of guidelines for more algebraic questions (including applications to number theory which are not discussed here). We made a special effort to use as little prerequisites and to be as self-contained as possible; hence the rather long preliminary sections dealing with basic facts of complex differential geometry. I am indebted to L. Ein, J. Kollár, R. Lazarsfeld, Th. Peternell, M. Schneider and Y.T. Siu for many discussions on these subjects over a period of time of at least one decade. These discussions have certainly had a great influence on my research work and therefore on the contents of the present notes.

1. Preliminary Material

1.A. Dolbeault Cohomology and Sheaf Cohomology

Let X be a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T^*_X$ the bundle of differential forms of bidegree (p,q) on X, i.e., differential forms which can be written as

$$u = \sum_{|I|=p,\,|J|=q} u_{I,J}\, dz_I \wedge d\bar{z}_J.$$

Here $(z_1, \ldots, z_n)$ denote arbitrary local holomorphic coordinates, $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ are multiindices (increasing sequences of integers in the range $[1, \ldots, n]$, of lengths $|I| = p$, $|J| = q$), and

$$dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p}, \qquad d\bar{z}_J := d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

Let $\mathcal{E}^{p,q}$ be the sheaf of germs of complex valued differential (p,q)-forms with C^∞ coefficients. Recall that the exterior derivative d splits as $d = d' + d''$ where

$$d'u = \sum_{|I|=p,\,|J|=q,1\leq k\leq n} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J,$$

$$d''u = \sum_{|I|=p,\,|J|=q,1\leq k\leq n} \frac{\partial u_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. The well-known Dolbeault-Grothendieck lemma asserts that any d''-closed form of type (p,q) with $q > 0$ is locally d''-exact (this is the analogue for d'' of the usual Poincaré lemma for d, see e.g. Hörmander 1966). In other words, the complex of sheaves $(\mathcal{E}^{p,\bullet}, d'')$ is exact in degree $q > 0$; in degree $q = 0$, $\operatorname{Ker} d''$ is the sheaf Ω^p_X of germs of holomorphic forms of degree p on X.

More generally, if F is a holomorphic vector bundle of rank r over X, there is a natural d'' operator acting on the space $C^\infty(X, \Lambda^{p,q}T^*_X \otimes F)$ of smooth (p,q)-forms with values in F; if $s = \sum_{1\leq\lambda\leq r} s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of F, we simply define $d''s := \sum d''s_\lambda \otimes e_\lambda$, observing that the holomorphic transition matrices involved in changes of holomorphic frames do not affect the computation of d''. It is then clear that the Dolbeault-Grothendieck lemma still holds for F-valued forms. For every integer $p = 0, 1, \ldots, n$, the *Dolbeault Cohomology* groups $H^{p,q}(X,F)$ are defined to be the cohomology groups of the complex of global (p,q) forms (graded by q):

$$H^{p,q}(X,F) = H^q\big(C^\infty(X, \Lambda^{p,\bullet}T^*_X \otimes F)\big). \tag{1.1}$$

Now, let us recall the following fundamental result from sheaf theory (De Rham-Weil isomorphism theorem): let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, i.e. a complex of sheaves $(\mathcal{L}^\bullet, \delta)$ such that there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{A} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{\delta^0} \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{\delta^q} \mathcal{L}^{q+1} \longrightarrow \cdots,$$

and $H^s(X, \mathcal{L}^q) = 0$ for all $q \geq 0$ and $s \geq 1$. Then there is a fonctorial isomorphism

(1.2) $$H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big) \longrightarrow H^q(X,\mathcal{A}).$$

We apply this to the following situation: let $\mathcal{E}(F)^{p,q}$ be the sheaf of germs of C^∞ sections of $\Lambda^{p,q}T^*_X \otimes F$, Then $(\mathcal{E}(F)^{p,\bullet}, d'')$ is a resolution of the locally free $\mathcal{O}_X$-module $\Omega^p_X \otimes \mathcal{O}(F)$ (Dolbeault-Grothendieck lemma), and the sheaves $\mathcal{E}(F)^{p,q}$ are acyclic as modules over the soft sheaf of rings $\mathcal{C}^\infty$. Hence by (1.2) we get

(1.3) Dolbeault Isomorphism Theorem (1953). *For every holomorphic vector bundle F on X, there is a canonical isomorphism*

$$H^{p,q}(X,F) \simeq H^q(X,\Omega^p_X \otimes \mathcal{O}(F)).$$ □

If X is projective algebraic and F is an algebraic vector bundle, Serre's GAGA theorem [Ser56] shows that the algebraic sheaf cohomology group $H^q(X,\Omega^p_X \otimes \mathcal{O}(F))$ computed with algebraic sections over Zariski open sets is actually isomorphic to the analytic cohomology group. These results are the most basic tools to attack algebraic problems via analytic methods. Another important tool is the theory of plurisubharmonic functions and positive currents originated by K. Oka and P. Lelong in the decades 1940-1960.

1.B. Plurisubharmonic Functions

Plurisubharmonic functions have been introduced independently by Lelong and Oka in the study of holomorphic convexity. We refer to [Lel67, 69] for more details.

(1.4) Definition. *A function $u : \Omega \longrightarrow [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if*

a) *u is upper semicontinuous ;*

b) *for every complex line $L \subset \mathbb{C}^n$, $u_{|\Omega\cap L}$ is subharmonic on $\Omega \cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \complement\Omega)$, the function u satisfies the mean value inequality*

$$u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a + e^{i\theta}\,\xi)\,d\theta.$$

The set of psh functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

We list below the most basic properties of psh functions. They all follow easily from the definition.

(1.5) Basic properties.

a) Every function $u \in \mathrm{Psh}(\Omega)$ is subharmonic, namely it satisfies the mean value inequality on euclidean balls or spheres:

$$u(a) \le \frac{1}{\pi^n r^{2n}/n!}\int_{B(a,r)} u(z)\,d\lambda(z)$$

for every $a \in \Omega$ and $r < d(a, \complement\Omega)$. Either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}$ on every connected component of Ω.

b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

c) Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon$ is C^∞ and psh on

$$\Omega_\varepsilon = \big\{x \in \Omega\,;\, d(x, \complement\Omega) > \varepsilon\big\},$$

the family $(u \star \rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$.

d) Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are psh on Ω. □

(1.6) Lemma. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh on Ω if and only if the hermitian form $Hu(a)(\xi) = \sum_{1\le j,k\le n} \partial^2 u/\partial z_j \partial \overline{z}_k(a)\,\xi_j\overline{\xi}_k$ is semipositive at every point $a \in \Omega$.*

Proof. This is an easy consequence of the following standard formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + e^{i\theta}\,\xi)\,d\theta - u(a) = \frac{2}{\pi}\int_0^1 \frac{dt}{t}\int_{|\zeta|<t} Hu(a + \zeta\xi)(\xi)\,d\lambda(\zeta),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Lemma 1.6 is a strong evidence that plurisubharmonicity is the natural complex analogue of linear convexity. □

For non smooth functions, a similar characterization of plurisubharmonicity can be obtained by means of a regularization process.

(1.7) Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1\le j,k\le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\,\xi_j\overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u. □

In order to get a better geometric insight of this notion, we assume more generally that u is a function on a complex n-dimensional manifold X. If $\Phi : X \to Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ \Phi) = \Phi^* d'd'' v$, hence

$$H(v \circ \Phi)(a, \xi) = Hv\big(\Phi(a), \Phi'(a).\xi\big).$$

In particular Hu, viewed as a hermitian form on T_X, does not depend on the choice of coordinates $(z_1, \ldots, z_n)$. Therefore, the notion of psh function makes sense on any complex manifold. More generally, we have

(1.8) Proposition. *If $\Phi : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ \Phi \in \mathrm{Psh}(X)$.* □

(1.9) Example. It is a standard fact that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in H^0(X, \mathcal{O}_X)$. More generally

$$\log\left(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\right) \in \mathrm{Psh}(X)$$

for every $f_j \in H^0(X, \mathcal{O}_X)$ and $\alpha_j \geq 0$ (apply Property 1.5 d with $u_j = \alpha_j \log|f_j|$). We will be especially interested in the singularities obtained at points of the zero variety $f_1 = \ldots = f_q = 0$, when the α_j are rational numbers. □

(1.10) Definition. *A psh function $u \in \mathrm{Psh}(X)$ will be said to have analytic singularities if u can be written locally as*

$$u = \frac{\alpha}{2}\log\left(|f_1|^2 + \cdots + |f_N|^2\right) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say that u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

We then introduce the ideal $\mathcal{J} = \mathcal{J}(u/\alpha)$ of germs of holomorphic functions h such that $|h| \leq Ce^{u/\alpha}$ for some constant C, i.e.

$$|h| \leq C\big(|f_1| + \cdots + |f_N|\big).$$

This is a globally defined ideal sheaf on X, locally equal to the integral closure $\overline{\mathcal{I}}$ of the ideal sheaf $\mathcal{I} = (f_1, \ldots, f_N)$, thus $\mathcal{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are local generators of $\mathcal{J}$, we still have

$$u = \frac{\alpha}{2}\log\left(|g_1|^2 + \cdots + |g_{N'}|^2\right) + O(1).$$

If X is projective algebraic and u has analytic singularities with $\alpha \in \mathbb{Q}_+$, then u automatically has algebraic singularities. From an algebraic point of view, the singularities of u are in 1:1 correspondence with the "algebraic data" $(\mathcal{J}, \alpha)$. Later on, we will see another important method for associating an ideal sheaf to a psh function.

(1.11) Exercise. Show that the above definition of the integral closure of an ideal $\mathcal{I}$ is equivalent to the following more algebraic definition: $\overline{\mathcal{I}}$ consists of all germs h satisfying an integral equation

$$h^d + a_1 h^{d-1} + \ldots + a_{d-1}h + a_d = 0, \qquad a_k \in \mathcal{I}^k.$$

Hint. One inclusion is clear. To prove the other inclusion, consider the normalization of the blow-up of X along the (non necessarily reduced) zero variety $V(\mathcal{I})$. □

1.C. Positive Currents

The reader can consult [Fed69] for a more thorough treatment of current theory. Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element Θ in the dual space $\mathcal{D}'_p(M) := \big(\mathcal{D}^p(M)\big)'$ of the space $\mathcal{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$\langle \Theta, \alpha\rangle = \int_M \Theta \wedge \alpha, \qquad \alpha \in \mathcal{D}^p(M). \tag{1.12}$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M:

$$\langle [S], \alpha\rangle = \int_S \alpha, \qquad \deg\alpha = p = \dim_{\mathbb{R}} S. \tag{1.13}$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S] = 0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta \in \mathcal{D}'_p(M)$. The current Θ is said to be *closed* if $d\Theta = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X), \qquad n = \dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p, n-q)$ on X. According to [Lel57], a current Θ of bidimension (p,p) is said to be *(weakly) positive* if for every choice of smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$ on X the distribution

$$\Theta \wedge \mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p \quad \text{is a positive measure.} \tag{1.14}$$

(1.15) Exercise. If Θ is positive, show that the coefficients $\Theta_{I,J}$ of Θ are complex measures, and that, up to constants, they are dominated by the trace measure

$$\sigma_\Theta = \Theta \wedge \frac{1}{p!}\beta^p = 2^{-p}\sum \Theta_{I,I}, \qquad \beta = \frac{\mathrm{i}}{2}d'd''|z|^2 = \frac{\mathrm{i}}{2}\sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j,$$

which is a positive measure.
Hint. Observe that $\sum \Theta_{I,I}$ is invariant by unitary changes of coordinates and that the (p,p)-forms $\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p$ generate $\Lambda^{p,p}T^*_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

A current $\Theta = \mathrm{i}\sum_{1\leq j,k\leq n} \Theta_{jk} dz_j \wedge dz_k$ of bidegree $(1,1)$ is easily seen to be positive if and only if the complex measure $\sum \lambda_j \overline{\lambda}_k \Theta_{jk}$ is a positive measure for every n-tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$.

(1.16) Example. If u is a (not identically $-\infty$) psh function on X, we can associate with u a (closed) positive current $\Theta = \mathrm{i}\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, every

closed positive current of bidegree $(1,1)$ can be written under this form on any open subset $\Omega \subset X$ such that $H^2_{DR}(\Omega,\mathbb{R}) = H^1(\Omega,\mathcal{O}) = 0$, e.g. on small coordinate balls (exercise to the reader). □

It is not difficult to show that a product $\Theta_1 \wedge \ldots \wedge \Theta_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined (this is certainly the case if all Θ_j but one at most are smooth; much finer conditions will be discussed in Section 2).

We now discuss another very important example of closed positive current. In fact, with every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration

$$\langle [A], \alpha \rangle = \int_{A_{\mathrm{reg}}} \alpha, \quad \alpha \in \mathcal{D}^{p,p}(X), \tag{1.17}$$

obtained by integrating over the regular points of A. In order to show that (1.17) is a correct definition of a current on X, one must show that A_{reg} has locally finite area in a neighborhood of A_{sing}. This result, due to [Lel57] is shown as follows. Suppose that 0 is a singular point of A By the local parametrization theorem for analytic sets, there is a linear change of coordinates on $\mathbb{C}^n$ such that all projections

$$\pi_I : (z_1,\ldots,z_n) \mapsto (z_{i_1},\ldots,z_{i_p})$$

define a finite ramified covering of the intersection $A \cap \Delta$ with a small polydisk Δ in $\mathbb{C}^n$ onto a small polydisk Δ_I in $\mathbb{C}^p$. Let n_I be the sheet number. Then the p-dimensional area of $A \cap \Delta$ is bounded above by the sum of the areas of its projections counted with multiplicities, i.e.

$$\mathrm{Area}(A \cap \Delta) \leq \sum n_I \mathrm{Vol}(\Delta_I).$$

The fact that $[A]$ is positive is also easy. In fact

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2 \, \mathrm{i}w_1 \wedge \overline{w}_1 \wedge \ldots \wedge \mathrm{i}w_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$ in terms of local coordinates $(w_1,\ldots,w_p)$ on A_{reg}. This shows that all such forms are ≥ 0 in the canonical orientation defined by $\mathrm{i}w_1 \wedge \overline{w}_1 \wedge \ldots \wedge \mathrm{i}w_p \wedge \overline{w}_p$. More importantly, Lelong [Lel57] has shown that $[A]$ is d-closed in X, even at points of A_{sing}. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem. For this we need the following definition: a *complete pluripolar* set is a set E such that there is an open covering (Ω_j) of X and psh functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar (take u_j as in Example 1.9).

(1.18) Theorem (Skoda [Sko81], El Mir [EM84], Sibony [Sib85]). *Let E be a closed complete pluripolar set in X, and let Θ be a closed positive current on $X \setminus E$ such that the coefficients $\Theta_{I,J}$ of Θ are measures with locally finite mass near E. Then the trivial extension $\tilde{\Theta}$ obtained by extending the measures $\Theta_{I,J}$ by 0 on E is still closed on X.*

Lelong's result $d[A] = 0$ is obtained by applying the Skoda-El Mir theorem to $\Theta = [A_{\mathrm{reg}}]$ on $X \setminus A_{\mathrm{sing}}$.

Proof of Theorem 1.18. The statement is local on X, so we may work on a small open set Ω such that $E \cap \Omega = v^{-1}(-\infty)$, $v \in \mathrm{Psh}(\Omega)$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function such that $\chi(t) = 0$ for $t \leq -1$ and $\chi(0) = 1$. By shrinking Ω and putting $v_k = \chi(k^{-1}v \star \rho_{\varepsilon_k})$ with $\varepsilon_k \to 0$ fast, we get a sequence of functions $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$ such that $0 \leq v_k \leq 1$, $v_k = 0$ in a neighborhood of $E \cap \Omega$ and $\lim v_k(x) = 1$ at every point of $\Omega \setminus E$. Let $\theta \in C^\infty([0,1])$ be a function such that $\theta = 0$ on $[0,1/3]$, $\theta = 1$ on $[2/3,1]$ and $0 \leq \theta \leq 1$. Then $\theta \circ v_k = 0$ near $E \cap \Omega$ and $\theta \circ v_k \to 1$ on $\Omega \setminus E$. Therefore $\tilde{\Theta} = \lim_{k\to+\infty}(\theta \circ v_k)\Theta$ and

$$d'\tilde{\Theta} = \lim_{k\to+\infty} \Theta \wedge d'(\theta \circ v_k)$$

in the weak topology of currents. It is therefore sufficient to verify that $\Theta \wedge d'(\theta \circ v_k)$ converges weakly to 0 (note that $d''\tilde{\Theta}$ is conjugate to $d'\tilde{\Theta}$, thus $d''\tilde{\Theta}$ will also vanish).

Assume first that $\Theta \in \mathcal{D}'^{n-1,n-1}(X)$. Then $\Theta \wedge d'(\theta \circ v_k) \in \mathcal{D}'^{n,n-1}(\Omega)$, and we have to show that

$$\langle \Theta \wedge d'(\theta \circ v_k), \overline{\alpha} \rangle = \langle \Theta, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle \underset{k\to+\infty}{\longrightarrow} 0, \quad \forall \alpha \in \mathcal{D}^{1,0}(\Omega).$$

As $\gamma \mapsto \langle \Theta, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle$ is a non-negative hermitian form on $\mathcal{D}^{1,0}(\Omega)$, the Cauchy-Schwarz inequality yields

$$\left|\langle \Theta, \mathrm{i}\beta \wedge \overline{\gamma} \rangle\right|^2 \leq \langle \Theta, \mathrm{i}\beta \wedge \overline{\beta} \rangle \, \langle \Theta, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle, \quad \forall \beta, \gamma \in \mathcal{D}^{1,0}(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$, $0 \leq \psi \leq 1$, be equal to 1 in a neighborhood of $\mathrm{Supp}\,\alpha$. We find

$$\left|\langle \Theta, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle\right|^2 \leq \langle \Theta, \psi \mathrm{i}d'v_k \wedge d''v_k \rangle \, \langle \Theta, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle.$$

By hypothesis $\int_{\Omega\setminus E} \Theta \wedge \mathrm{i}\alpha \wedge \overline{\alpha} < +\infty$ and $\theta'(v_k)$ converges everywhere to 0 on Ω, thus $\langle \Theta, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle$ converges to 0 by Lebesgue's dominated convergence theorem. On the other hand

$$\mathrm{i}d'd''v_k^2 = 2v_k \, \mathrm{i}d'd''v_k + 2\mathrm{i}d'v_k \wedge d''v_k \geq 2\mathrm{i}d'v_k \wedge d''v_k,$$
$$2\langle \Theta, \psi \mathrm{i}d'v_k \wedge d''v_k \rangle \leq \langle \Theta, \psi \mathrm{i}d'd''v_k^2 \rangle.$$

As $\psi \in \mathcal{D}(\Omega)$, $v_k = 0$ near E and $d\Theta = 0$ on $\Omega \setminus E$, an integration by parts yields

$$\langle \Theta, \psi \mathrm{i}d'd''v_k^2 \rangle = \langle \Theta, v_k^2 \mathrm{i}d'd''\psi \rangle \leq C \int_{\Omega\setminus E} \|\Theta\| < +\infty$$

where C is a bound for the coefficients of $\mathrm{i}d'd''\psi$. Thus $\langle \Theta, \psi \mathrm{i}d'v_k \wedge d''v_k \rangle$ is bounded, and the proof is complete when $\Theta \in \mathcal{D}'^{n-1,n-1}$.

In the general case $\Theta \in \mathcal{D}'^{p,p}$, $p < n$, we simply apply the result already proved to all positive currents $\Theta \wedge \gamma \in \mathcal{D}'^{n-1,n-1}$ where $\gamma = \mathrm{i}\gamma_1 \wedge \overline{\gamma}_1 \wedge \ldots \wedge \mathrm{i}\gamma_{n-p-1} \wedge \overline{\gamma}_{n-p-1}$ runs over a basis of forms of $\Lambda^{n-p-1,n-p-1}T^\star_\Omega$ with constant coefficients (Lemma 1.4). Then we get $d(\tilde{\Theta} \wedge \gamma) = d\tilde{\Theta} \wedge \gamma = 0$ for all such γ, hence $d\tilde{\Theta} = 0$. □

(1.19) Corollary. *Let Θ be a closed positive current on X and let E be a complete pluripolar set. Then $\mathbf{1}_E\Theta$ and $\mathbf{1}_{X\smallsetminus E}\Theta$ are closed positive currents. In fact, $\tilde{\Theta} = \mathbf{1}_{X\smallsetminus E}\Theta$ is the trivial extension of $\Theta_{|X\smallsetminus E}$ to X, and $\mathbf{1}_E\Theta = \Theta - \tilde{\Theta}$.* □

As mentioned above, any current $\Theta = \mathrm{i}d'd''u$ associated with a psh function u is a closed positive $(1,1)$-current. In the special case $u = \log|f|$ where $f \in H^0(X,\mathcal{O}_X)$ is a non zero holomorphic function, we have the important

(1.20) Lelong-Poincaré equation. *Let $f \in H^0(X,\mathcal{O}_X)$ be a non zero holomorphic function, $Z_f = \sum m_j Z_j$, $m_j \in \mathbb{N}$, the zero divisor of f and $[Z_f] = \sum m_j[Z_j]$ the associated current of integration. Then*

$$\frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log|f| = [Z_f].$$

Proof (sketch). It is clear that $\mathrm{i}d'd''\log|f| = 0$ in a neighborhood of every point $x \notin \mathrm{Supp}(Z_f) = \bigcup Z_j$, so it is enough to check the equation in a neighborhood of every point of $\mathrm{Supp}(Z_f)$. Let A be the set of singular points of $\mathrm{Supp}(Z_f)$, i.e. the union of the pairwise intersections $Z_j \cap Z_k$ and of the singular loci $Z_{j,\mathrm{sing}}$; we thus have $\dim A \leq n-2$. In a neighborhood of any point $x \in \mathrm{Supp}(Z_f) \smallsetminus A$ there are local coordinates $(z_1,\ldots,z_n)$ such that $f(z) = z_1^{m_j}$ where m_j is the multiplicity of f along the component Z_j which contains x and $z_1 = 0$ is an equation for Z_j near x. Hence

$$\frac{\mathrm{i}}{\pi}d'd''\log|f| = m_j\frac{\mathrm{i}}{\pi}d'd''\log|z_1| = m_j[Z_j]$$

in a neighborhood of x, as desired (the identity comes from the standard formula $\frac{\mathrm{i}}{\pi}d'd''\log|z| =$ Dirac measure δ_0 in $\mathbb{C}$). This shows that the equation holds on $X \smallsetminus A$. Hence the difference $\frac{\mathrm{i}}{\pi}d'd''\log|f| - [Z_f]$ is a closed current of degree 2 with measure coefficients, whose support is contained in A. By Exercise 1.21, this current must be 0, for A has too small dimension to carry its support (A is stratified by submanifolds of real codimension ≥ 4). □

(1.21) Exercise. Let Θ be a current of degree q on a real manifold M, such that both Θ and $d\Theta$ have measure coefficients ("normal current"). Suppose that $\mathrm{Supp}\,\Theta$ is contained in a real submanifold A with $\mathrm{codim}_{\mathbb{R}} A > q$. Show that $\Theta = 0$.
Hint: Let $m = \dim_{\mathbb{R}} M$ and let $(x_1,\ldots,x_m)$ be a coordinate system in a neighborhood Ω of a point $a \in A$ such that $A \cap \Omega = \{x_1 = \ldots = x_k = 0\}$, $k > q$. Observe that $x_j\Theta = x_j d\Theta = 0$ for $1 \leq j \leq k$, thanks to the hypothesis on supports and on the normality of Θ, hence $dx_j \wedge \Theta = d(x_j\Theta) - x_j d\Theta = 0$, $1 \leq j \leq k$. Infer from this that all coefficients in $\Theta = \sum_{|I|=q}\Theta_I dx_I$ vanish. □

We now recall a few basic facts of slicing theory (the reader will profitably consult [Fed69] and [Siu74] for further developments). Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is, a current which can be written locally as $\Theta = U + dV$ where U, V have L^1_{loc} coefficients. It is a standard fact (see Federer) that every current Θ such that

both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flat. Then, for almost every $x' \in M'$, there is a well defined *slice* $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{|\sigma^{-1}(x')} + dV_{|\sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. The slices $\Theta_{x'}$ are currents on the fibers with the same degree as Θ (thus of dimension $\dim\Theta - \dim$ (fibers)). Of course, every slice $\Theta_{x'}$ coincides with the usual restriction of Θ to the fiber if Θ has smooth coefficients. By using a regularization $\Theta_\varepsilon = \Theta \star \rho_\varepsilon$, it is easy to show that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in $L^1_{\mathrm{loc}}(\sigma^{-1}(x'))$, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. Now, the basic slicing formula is

$$\int_M \Theta\wedge\alpha\wedge\sigma^*\beta = \int_{x'\in M'}\Big(\int_{x''\in\sigma^{-1}(x')}\Theta_{x'}(x'')\wedge\alpha_{|\sigma^{-1}(x')}(x'')\Big)\beta(x') \tag{1.22}$$

for every smooth form α on M and β on M', such that α has compact support and $\deg\alpha = \dim M - \dim M' - \deg\Theta$, $\deg\beta = \dim M'$. This is an easy consequence of the usual Fubini theorem applied to U and V in the decomposition $\Theta = U + dV$, if we identify locally σ with a projection map $M = M'\times M'' \to M'$, $x = (x',x'') \mapsto x'$, and use a partitition of unity on the support of α.

To conclude this section, we discuss De Rham and Dolbeault cohomology theory in the context of currents. A basic observation is that the Poincaré and Dolbeault-Grothendieck lemma still hold for currents. Namely, if $(\mathcal{D}'^q, d)$ and $(\mathcal{D}'(F)^{p,q}, d'')$ denote the complex of sheaves of degree q currents (resp. of (p,q)-currents with values in a holomorphic vector bundle F), we still have De Rham and Dolbeault sheaf resolutions

$$0 \to \mathbb{R} \to \mathcal{D}'^\bullet, \qquad 0 \to \Omega^p_X\otimes\mathcal{O}(F) \to \mathcal{D}'(F)^{p,\bullet}.$$

Hence we get canonical isomorphisms

$$\begin{aligned} H^q_{\mathrm{DR}}(M,\mathbb{R}) &= H^q\big((\Gamma(M,\mathcal{D}'^\bullet),d)\big),\\ H^{p,q}(X,F) &= H^q\big((\Gamma(X,\mathcal{D}'(F)^{p,\bullet}),d'')\big). \end{aligned} \tag{1.23}$$

In other words, we can attach a cohomology class $\{\Theta\} \in H^q_{\mathrm{DR}}(M,\mathbb{R})$ to any closed current Θ of degree q, resp. a cohomology class $\{\Theta\} \in H^{p,q}(X,F)$ to any d''-closed current of bidegree (p,q). Replacing if necessary every current by a smooth representative in the same cohomology class, we see that there is a well defined cup product given by the wedge product of differential forms

$$\begin{aligned} H^{q_1}(M,\mathbb{R})\times\ldots\times H^{q_m}(M,\mathbb{R}) &\longrightarrow H^{q_1+\ldots+q_m}(M,\mathbb{R}),\\ (\{\Theta_1\},\ldots,\{\Theta_1\}) &\longmapsto \{\Theta_1\}\wedge\ldots\wedge\{\Theta_m\}. \end{aligned}$$

In particular, if M is a compact oriented variety and $q_1+\ldots+q_m = \dim M$, there is a well defined intersection number

$$\{\Theta_1\}\cdot\{\Theta_2\}\cdot\cdots\cdot\{\Theta_m\} = \int_M \{\Theta_1\}\wedge\ldots\wedge\{\Theta_m\}.$$

However, as we will see in the next section, the pointwise product $\Theta_1 \wedge \ldots \wedge \Theta_m$ need not exist in general.

2. Lelong Numbers and Intersection Theory

2.A. Multiplication of Currents and Monge-Ampère Operators

Let X be a n-dimensional complex manifold. We set

$$d^c = \frac{1}{2i\pi}(d' - d'').$$

It follows in particular that d^c is a real operator, i.e. $\overline{d^c u} = d^c \overline{u}$, and that $dd^c = \frac{i}{\pi} d'd''$. Although not quite standard, the $1/2i\pi$ normalization is very convenient for many purposes, since we may then forget the factor π or 2π almost everywhere (e.g. in the Lelong-Poincaré equation (1.20)).

Let u be a psh function and let Θ be a closed positive current on X. Our desire is to define the wedge product $dd^c u \wedge \Theta$ even when neither u nor Θ are smooth. In general, this product does not make sense because $dd^c u$ and Θ have measure coefficients and measures cannot be multiplied; see Kiselman [Kis84] for interesting counterexamples. Even in the algebraic setting considered here, multiplication of currents is not always possible: suppose e.g. that $\Theta = [D]$ is the exceptional divisor of a blow-up in a surface; then $D \cdot D = -1$ cannot be the cohomology class of a closed positive current $[D]^2$. Assume however that u is a *locally bounded* psh function. Then the current $u\Theta$ is well defined since u is a locally bounded Borel function and Θ has measure coefficients. According to Bedford-Taylor [BT82] we define

$$dd^c u \wedge \Theta = dd^c(u\Theta)$$

where $dd^c(\ )$ is taken in the sense of distribution theory.

(2.1) Proposition. *If u is a locally bounded psh function, the wedge product $dd^c u \wedge \Theta$ is again a closed positive current.*

Proof. The result is local. Use a convolution $u_\nu = u \star \rho_{1/\nu}$ to get a decreasing sequence of smooth psh functions converging to u. Then write

$$dd^c(u\Theta) = \lim_{\nu \to +\infty} dd^c(u_\nu \Theta) = dd^c u_\nu \wedge \Theta$$

as a weak limit of closed positive currents. Observe that $u_\nu \Theta$ converges weakly to $u\Theta$ by Lebesgue's monotone convergence theorem. □

More generally, if $u_1, \ldots, u_m$ are locally bounded psh functions, we can define

$$dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \Theta = dd^c\big(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_m \wedge \Theta\big)$$

by induction on m. (Chern, Levine and Nirenberg, 1969) noticed the following useful inequality. Define the *mass* of a current Θ on a compact set K to be

$$\|\Theta\|_K = \int_K \sum_{I,J} |\Theta_{I,J}|$$

whenever K is contained in a coordinate patch and $\Theta = \sum \Theta_{I,J} dz_I \wedge d\overline{z}_J$. Up to seminorm equivalence, this does not depend on the choice of coordinates. If K is not contained in a coordinate patch, we use a partition of unity to define a suitable seminorm $\|\Theta\|_K$. If $\Theta \geq 0$, Exercise 1.15 shows that the mass is controlled by the trace measure, i.e. $\|\Theta\|_K \leq C \int_K \Theta \wedge \beta^p$.

(2.2) Chern-Levine-Nirenberg inequality. *For all compact subsets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$\|dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \Theta\|_L \leq C_{K,L} \, \|u_1\|_{L^\infty(K)} \ldots \|u_m\|_{L^\infty(K)} \, \|\Theta\|_K$$

Proof. By induction, it is sufficient to prove the result for $m = 1$ and $u_1 = u$. There is a covering of L by a family of open balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X. Let (p,p) be the bidimension of Θ, let $\beta = \frac{i}{2} d'd''|z|^2$, and let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$\|dd^c u \wedge \Theta\|_{L \cap \overline{B}'_j} \leq C \int_{\overline{B}'_j} dd^c u \wedge \Theta \wedge \beta^{p-1} \leq C \int_{B_j} \chi \, dd^c u \wedge \Theta \wedge \beta^{p-1}.$$

As Θ and β are closed, an integration by parts yields

$$\|dd^c u \wedge \Theta\|_{L \cap \overline{B}'_j} \leq C \int_{B_j} u \, \Theta \wedge dd^c \chi \wedge \beta^{p-1} \leq C' \|u\|_{L^\infty(K)} \|\Theta\|_K$$

where C' is equal to C multiplied by a bound for the coefficients of the smooth form $dd^c \chi \wedge \beta^{p-1}$. □

Various examples (cf. [Kis84]) show however that products of $(1,1)$-currents $dd^c u_j$ cannot be defined in a reasonable way for arbitrary psh functions u_j. However, functions u_j with $-\infty$ poles can be admitted if the polar sets are sufficiently small.

(2.3) Proposition. *Let u be a psh function on X, and let Θ be a closed positive current of bidimension (p,p). Suppose that u is locally bounded on $X \smallsetminus A$, where A is an analytic subset of X of dimension $< p$ at each point. Then $dd^c u \wedge \Theta$ can be defined in such a way that $dd^c u \wedge \Theta = \lim_{\nu \to +\infty} dd^c u_\nu \wedge \Theta$ in the weak topology of currents, for any decreasing sequence $(u_\nu)_{\nu \geq 0}$ of psh functions converging to u.*

Proof. When u is locally bounded everywhere, we have $\lim u_\nu \Theta = u\Theta$ by the monotone convergence theorem and the result follows from the continuity of dd^c with respect to the weak topology.

First assume that A is discrete. Since our results are local, we may suppose that X is a ball $B(0,R) \subset \mathbb{C}^n$ and that $A = \{0\}$. For every $s \leq 0$, the function $u^{\geq s} = \max(u, s)$ is locally bounded on X, so the product $\Theta \wedge dd^c u^{\geq s}$ is well defined. For $|s|$ large, the function $u^{\geq s}$ differs from u only in a small neighborhood of the origin, at which u may have a $-\infty$ pole. Let γ be a $(p-1, p-1)$-form with constant

coefficients and set $s(r) = \liminf_{|z|\to r-0} u(z)$. By Stokes' formula, we see that the integral

$$I(s) := \int_{B(0,r)} dd^c u^{\geq s} \wedge \Theta \wedge \gamma \tag{2.4}$$

does not depend on s when $s < s(r)$, for the difference $I(s) - I(s')$ of two such integrals involves the dd^c of a current $(u^{\geq s} - u^{\geq s'}) \wedge \Theta \wedge \gamma$ with compact support in $B(0,r)$. Taking $\gamma = (dd^c|z|^2)^{p-1}$, we see that the current $dd^c u \wedge \Theta$ has finite mass on $B(0,r) \smallsetminus \{0\}$ and we can define $\langle 1_{\{0\}}(dd^c u \wedge \Theta), \gamma\rangle$ to be the limit of the integrals (2.4) as r tends to zero and $s < s(r)$. In this case, the weak convergence statement is easily deduced from the locally bounded case discussed above.

In the case where $0 < \dim A < p$, we use a slicing technique to reduce the situation to the discrete case. Set $q = p-1$. There are linear coordinates $(z_1, \ldots, z_n)$ centered at any point of A, such that 0 is an isolated point of $A \cap \big(\{0\} \times \mathbb{C}^{n-q}\big)$. Then there are small balls $B' = B(0,r')$ in $\mathbb{C}^q$, $B'' = B(0,r'')$ in $\mathbb{C}^{n-q}$ such that $A \cap (\overline{B}' \times \partial B'') = \emptyset$, and the projection map

$$\pi : \mathbb{C}^n \to \mathbb{C}^q, \quad z = (z_1, \ldots, z_n) \mapsto z' = (z_1, \ldots, z_q)$$

defines a finite proper mapping $A \cap (B' \times B'') \to B'$. These properties are preserved if we slightly change the direction of projection. Take sufficiently many projections π_m associated to coordinate systems $(z_1^m, \ldots, z_n^m)$, $1 \leq m \leq N$, in such a way that the family of (q,q)-forms

$$i\, dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge i\, dz_q^m \wedge d\overline{z}_q^m$$

defines a basis of the space of (q,q)-forms. Expressing any compactly supported smooth (q,q)-form in such a basis, we see that we need only define

$$\int_{B'\times B''} dd^c u \wedge \Theta \wedge f(z',z'')\, i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_q \wedge d\overline{z}_q = \int_{B'} \Big\{ \int_{B''} f(z',\bullet)\, dd^c u(z',\bullet) \wedge \Theta(z',\bullet) \Big\} i\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i\, dz_q \wedge d\overline{z}_q \tag{2.5}$$

where f is a test function with compact support in $B' \times B''$, and $\Theta(z',\bullet)$ denotes the slice of Θ on the fiber $\{z'\} \times B''$ of the projection $\pi : \mathbb{C}^n \to \mathbb{C}^q$. Each integral $\int_{B''}$ in the right hand side of (2.5) makes sense since the slices $(\{z'\} \times B'') \cap A$ are discrete. Moreover, the double integral $\int_{B'} \int_{B''}$ is convergent. Indeed, observe that u is bounded on any compact cylinder

$$K_{\delta,\varepsilon} = \overline{B}\big((1-\delta)r'\big) \times \Big(\overline{B}(r'') \smallsetminus \overline{B}\big((1-\varepsilon)r''\big)\Big)$$

disjoint from A. Take $\varepsilon \ll \delta \ll 1$ so small that

$$\operatorname{Supp} f \subset \overline{B}\big((1-\delta)r'\big) \times \overline{B}\big((1-\varepsilon)r''\big).$$

For all $z' \in \overline{B}((1-\delta)r')$, the proof of the Chern-Levine-Nirenberg inequality 2.2 with a cut-off function $\chi(z'')$ equal to 1 on $B((1-\varepsilon)r'')$ and with support in $B((1-\varepsilon/2)r'')$ shows that

$$\int_{B((1-\epsilon)r'')} dd^c u(z',\bullet) \wedge \Theta(z',\bullet) \leq C_\varepsilon \|u\|_{L^\infty(K_{\delta,\varepsilon})} \int_{z''\in B((1-\varepsilon/2)r'')} \Theta(z',z'') \wedge dd^c|z''|^2.$$

This implies that the double integral is convergent. Now replace u everywhere by u_ν and observe that $\lim_{\nu\to+\infty} \int_{B''}$ is the expected integral for every z' such that $\Theta(z',\bullet)$ exists (apply the discrete case already proven). Moreover, the Chern-Levine-Nirenberg inequality yields uniform bounds for all functions u_ν, hence Lebesgue's dominated convergence theorem can be applied to $\int_{B'}$. We conclude from this that the sequence of integrals (2.5) converges when $u_\nu \downarrow u$, as expected. □

(2.6) Remark. In the above proof, the fact that A is an analytic set does not play an essential role. The main point is just that the slices $(\{z'\} \times B'') \cap A$ consist of isolated points for generic choices of coordinates (z', z''). In fact, the proof even works if the slices are totally discontinuous, in particular if they are of zero Hausdorff measure $\mathcal{H}_1$. It follows that Proposition 2.3 still holds whenever A is a closed set such that $\mathcal{H}_{2p-1}(A) = 0$. □

2.B. Lelong Numbers

The concept of Lelong number is an analytic analogue of the algebraic notion of multiplicity. It is a very useful technique to extend results in the intersection theory of algebraic cycles to currents. Lelong numbers have been introduced for the first time by Lelong in [Lel57]. See also [Lel69], [Siu74], [Dem82a, 85a, 87] for further developments.

Let us first recall a few definitions. Let Θ be a closed positive current of bidimension (p,p) on a coordinate open set $\Omega \subset \mathbb{C}^n$ of a complex manifold X. The Lelong number of Θ at a point $x \in \Omega$ is defined to be the limit

$$\nu(\Theta, x) = \lim_{r\to 0+} \nu(\Theta, x, r), \qquad \text{where } \nu(\Theta,x,r) = \frac{\sigma_\Theta(B(x,r))}{\pi^p r^{2p}/p!}$$

measures the ratio of the area of Θ in the ball $B(x,r)$ to the area of the ball of radius r in $\mathbb{C}^p$. As $\sigma_\Theta = \Theta \wedge \frac{1}{p!}(\pi dd^c|z|^2)^p$ by 1.15, we also get

$$\nu(\Theta,x,r) = \frac{1}{r^{2p}} \int_{B(x,r)} \Theta(z) \wedge (dd^c|z|^2)^p. \tag{2.7}$$

The main results concerning Lelong numbers are summarized in the following theorems, due respectively to Lelong, Thie and Siu.

(2.8) Theorem ([Lel57]).

a) *For every positive current Θ, the ratio $\nu(\Theta,x,r)$ is a nonnegative increasing function of r, in particular the limit $\nu(\Theta,x)$ as $r \to 0+$ always exists.*

b) *If $\Theta = dd^c u$ is the bidegree $(1,1)$-current associated with a psh function u, then*

$$\nu(\Theta,x)=\sup\big\{\gamma\geq 0\,;\,u(z)\leq\gamma\log|z-x|+O(1)\quad\text{at } x\big\}.$$

In particular, if $u=\log|f|$ with $f\in H^0(X,\mathcal{O}_X)$ and $\Theta=dd^c u=[Z_f]$, we have

$$\nu([Z_f],x)=\mathrm{ord}_x(f)=\max\{m\in\mathbb{N};\ D^\alpha f(x)=0,\ |\alpha|<m\}.$$

(2.9) Theorem ([Thi67]). *In the case where Θ is a current of integration $[A]$ over an analytic subvariety A, the Lelong number $\nu([A],x)$ coincides with the multiplicity of A at x (defined e.g. as the sheet number in the ramified covering obtained by taking a generic linear projection of the germ (A,x) onto a p-dimensional linear subspace through x in any coordinate patch Ω).*

(2.10) Theorem ([Siu74]). *Let Θ be a closed positive current of bidimension (p,p) on the complex manifold X.*

a) *The Lelong number $\nu(\Theta,x)$ is invariant by holomorphic changes of local coordinates.*

b) *For every $c>0$, the set $E_c(\Theta)=\big\{x\in X\,;\,\nu(\Theta,x)\geq c\big\}$ is a closed analytic subset of X of dimension $\leq p$.*

The most important result is 2.10 b), which is a deep application of Hörmander L^2 estimates (see Section 5). The earlier proofs of all other results were rather intricate in spite of their rather simple nature. We reproduce below a sketch of elementary arguments based on the use of a more general and more flexible notion of Lelong number introduced in [Dem87]. Let φ be a continuous psh function with an isolated $-\infty$ pole at x, e.g. a function of the form $\varphi(z)=\log\sum_{1\leq j\leq N}|g_j(z)|^{\gamma_j}$, $\gamma_j>0$, where $(g_1,\ldots,g_N)$ is an ideal of germs of holomorphic functions in $\mathcal{O}_x$ with $g^{-1}(0)=\{x\}$. The *generalized Lelong number* $\nu(\Theta,\varphi)$ of Θ with respect to the weight φ is simply defined to be the mass of the measure $\Theta\wedge(dd^c\varphi)^p$ carried by the point x (the measure $\Theta\wedge(dd^c\varphi)^p$ is always well defined thanks to Prop. 2.3). This number can also be seen as the limit $\nu(\Theta,\varphi)=\lim_{t\to-\infty}\nu(\Theta,\varphi,t)$, where

$$\nu(\Theta,\varphi,t)=\int_{\varphi(z)<t}\Theta\wedge(dd^c\varphi)^p. \tag{2.11}$$

The relation with our earlier definition of Lelong numbers (as well as part a) of Theorem 2.8) comes from the identity

$$\nu(\Theta,x,r)=\nu(\Theta,\varphi,\log r),\qquad \varphi(z)=\log|z-x|, \tag{2.12}$$

in particular $\nu(\Theta,x)=\nu(\Theta,\log|\bullet-x|)$. This equality is in turn a consequence of the following general formula, applied to $\chi(t)=e^{2t}$ and $t=\log r$:

$$\int_{\varphi(z)<t}\Theta\wedge(dd^c\chi\circ\varphi)^p=\chi'(t-0)^p\int_{\varphi(z)<t}\Theta\wedge(dd^c\varphi)^p, \tag{2.13}$$

where χ is an arbitrary convex increasing function. To prove the formula, we use a regularization and thus suppose that Θ, φ and χ are smooth, and that t is a non critical value of φ. Then Stokes' formula shows that the integrals on the left and right hand side of (2.13) are equal respectively to

$$\int_{\varphi(z)=t}\Theta\wedge(dd^c\chi\circ\varphi)^{p-1}\wedge d^c(\chi\circ\varphi),\qquad \int_{\varphi(z)=t}\Theta\wedge(dd^c\varphi)^{p-1}\wedge d^c\varphi,$$

and the differential form of bidegree $(p-1,p)$ appearing in the integrand of the first integral is equal to $(\chi'\circ\varphi)^p\,(dd^c\varphi)^{p-1}\wedge d^c\varphi$. The expected formula follows. Part b) of Theorem 2.8 is a consequence of the Jensen-Lelong formula, whose proof is left as an exercise to the reader.

(2.14) Jensen-Lelong formula. *Let u be any psh function on X. Then u is integrable with respect to the measure $\mu_r=(dd^c\varphi)^{n-1}\wedge d^c\varphi$ supported by the pseudo-sphere $\{\varphi(z)=r\}$ and*

$$\mu_r(u)=\int_{\{\varphi<r\}}u(dd^c\varphi)^n+\int_{-\infty}^{r}\nu(dd^c u,\varphi,t)\,dt. \qquad\square$$

In our case, we set $\varphi(z)=\log|z-x|$. Then $(dd^c\varphi)^n=\delta_x$ and μ_r is just the unitary invariant mean value measure on the sphere $S(x,e^r)$. For $r<r_0$, Formula 2.14 implies

$$\mu_r(u)-\mu_{r_0}(u)=\int_{r_0}^{r}\nu(dd^c u,x,t)\sim(r-r_0)\nu(dd^c u,x)\qquad\text{as } r\to-\infty.$$

From this, using the Harnack inequality for subharmonic functions, we get

$$\liminf_{z\to x}\frac{u(z)}{\log|z-x|}=\lim_{r\to-\infty}\frac{\mu_r(u)}{r}=\nu(dd^c u,x).$$

These equalities imply statement 2.8 b).

Next, we show that the Lelong numbers $\nu(T,\varphi)$ only depend on the asymptotic behaviour of φ near the polar set $\varphi^{-1}(-\infty)$. In a precise way:

(2.15) Comparison theorem. *Let Θ be a closed positive current on X, and let $\varphi,\psi:X\to[-\infty,+\infty[$ be continuous psh functions with isolated poles at some point $x\in X$. Assume that*

$$\ell:=\limsup_{z\to x}\frac{\psi(z)}{\varphi(z)}<+\infty.$$

Then $\nu(\Theta,\psi)\leq\ell^p\nu(\Theta,\varphi)$, and the equality holds if $\ell=\lim\psi/\varphi$.

Proof. (2.12) shows that $\nu(\Theta,\lambda\varphi)=\lambda^p\nu(\Theta,\varphi)$ for every positive constant λ. It is thus sufficient to verify the inequality $\nu(\Theta,\psi)\leq\nu(\Theta,\varphi)$ under the hypothesis $\limsup\psi/\varphi<1$. For any $c>0$, consider the psh function

$$u_c=\max(\psi-c,\varphi).$$

Fix $r\ll 0$. For $c>0$ large enough, we have $u_c=\varphi$ on a neighborhood of $\varphi^{-1}(r)$ and Stokes' formula gives

$$\nu(\Theta,\varphi,r)=\nu(\Theta,u_c,r)\geq\nu(\Theta,u_c).$$

The hypothesis $\limsup \psi/\varphi < 1$ implies on the other hand that there exists $t_0 < 0$ such that $u_c = \psi - c$ on $\{u_c < t_0\}$. We thus get

$$\nu(\Theta, u_c) = \nu(\Theta, \psi - c) = \nu(\Theta, \psi),$$

hence $\nu(\Theta, \psi) \le \nu(\Theta, \varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim \varphi/\psi = 1/l$. □

Part a) of Theorem 2.10 follows immediately from 2.15 by considering the weights $\varphi(z) = \log|\tau(z) - \tau(x)|$, $\psi(z) = \log|\tau'(z) - \tau'(x)|$ associated to coordinates systems $\tau(z) = (z_1, \ldots, z_n)$, $\tau'(z) = (z'_1, \ldots, z'_n)$ in a neighborhood of x. Another application is a direct simple proof of Thie's Theorem 2.9 when $\Theta = [A]$ is the current of integration over an analytic set $A \subset X$ of pure dimension p. For this, we have to observe that Theorem 2.15 still holds provided that x is an isolated point in $\mathrm{Supp}(\Theta) \cap \varphi^{-1}(-\infty)$ and $\mathrm{Supp}(\Theta) \cap \psi^{-1}(-\infty)$ (even though x is not isolated in $\varphi^{-1}(-\infty)$ or $\psi^{-1}(-\infty)$), under the weaker assumption that $\limsup_{\mathrm{Supp}\{\Theta\} \ni z \to x} \psi(z)/\varphi(z) = \ell$. The reason for this is that all integrals involve currents supported on $\mathrm{Supp}(\Theta)$. Now, by a generic choice of local coordinates $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$ on (X, x), the germ (A, x) is contained in a cone $|z''| \le C|z'|$. If $B' \subset \mathbb{C}^p$ is a ball of center 0 and radius r' small, and $B'' \subset \mathbb{C}^{n-p}$ is the ball of center 0 and radius $r'' = Cr'$, the projection

$$\mathrm{pr} : A \cap (B' \times B'') \longrightarrow B'$$

is a ramified covering with finite sheet number m. When $z \in A$ tends to $x = 0$, the functions

$$\varphi(z) = \log|z| = \log(|z'|^2 + |z''|^2)^{1/2}, \qquad \psi(z) = \log|z'|.$$

satisfy $\lim_{z \to x} \psi(z)/\varphi(z) = 1$. Hence Theorem 2.15 implies

$$\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi).$$

Now, Formula 2.13 with $\chi(t) = e^{2t}$ yields

$$\begin{aligned}
\nu([A], \psi, \log t) &= t^{-2p} \int_{\{\psi < \log t\}} [A] \wedge \Big(\frac{1}{2} dd^c e^{2\psi}\Big)^p \\
&= t^{-2p} \int_{A \cap \{|z'| < t\}} \Big(\frac{1}{2} \mathrm{pr}^* \, dd^c |z'|^2\Big)^p \\
&= m\, t^{-2p} \int_{\mathbb{C}^p \cap \{|z'| < t\}} \Big(\frac{1}{2} dd^c |z'|^2\Big)^p = m,
\end{aligned}$$

hence $\nu([A], \psi) = m$. Here, we have used the fact that pr is an étale covering with m sheets over the complement of the ramification locus $S \subset B'$, and the fact that S is of zero Lebesgue measure in B'.

(2.16) Proposition. *Under the assumptions of Proposition 2.3, we have*

$$\nu(dd^c u \wedge \Theta, x) \ge \nu(u, x)\, \nu(\Theta, x)$$

at every point $x \in X$.

Proof. Assume that $X = B(0, r)$ and $x = 0$. By definition

$$\nu(dd^c u \wedge \Theta, x) = \lim_{r \to 0} \int_{|z| \le r} dd^c u \wedge \Theta \wedge (dd^c \log|z|)^{p-1}.$$

Set $\gamma = \nu(u, x)$ and

$$u_\nu(z) = \max\big(u(z), (\gamma - \varepsilon)\log|z| - \nu\big)$$

with $0 < \varepsilon < \gamma$ (if $\gamma = 0$, there is nothing to prove). Then u_ν decreases to u and

$$\int_{|z| \le r} dd^c u \wedge \Theta \wedge (dd^c \log|z|)^{p-1} \ge \limsup_{\nu \to +\infty} \int_{|z| \le r} dd^c u_\nu \wedge \Theta \wedge (dd^c \log|z|)^{p-1}$$

by the weak convergence of $dd^c u_\nu \wedge \Theta$; here $(dd^c \log|z|)^{p-1}$ is not smooth on $\overline{B}(0, r)$, but the integrals remain unchanged if we replace $\log|z|$ by $\chi(\log|z|/r)$ with a smooth convex function χ such that $\chi(t) = t$ for $t \ge -1$ and $\chi(t) = 0$ for $t \le -2$. Now, we have $u(z) \le \gamma \log|z| + C$ near 0, so $u_\nu(z)$ coincides with $(\gamma - \varepsilon)\log|z| - \nu$ on a small ball $B(0, r_\nu) \subset B(0, r)$ and we infer

$$\begin{aligned}
\int_{|z| \le r} dd^c u_\nu \wedge \Theta \wedge (dd^c \log|z|)^{p-1} &\ge (\gamma - \varepsilon) \int_{|z| \le r_\nu} \Theta \wedge (dd^c \log|z|)^p \\
&\ge (\gamma - \varepsilon)\nu(\Theta, x).
\end{aligned}$$

As $r \in]0, R[$ and $\varepsilon \in]0, \gamma[$ were arbitrary, the desired inequality follows. □

We will later need an important decomposition formula of [Siu74]. We start with the following lemma.

(2.17) Lemma. *If Θ is a closed positive current of bidimension (p, p) and Z is an irreducible analytic set in X, we set*

$$m_Z = \inf\{x \in Z\,;\, \nu(\Theta, x)\}.$$

a) *There is a countable family of proper analytic subsets (Z'_j) of Z such that $\nu(\Theta, x) = m_Z$ for all $x \in Z \setminus \bigcup Z'_j$. We say that m_Z is the generic Lelong number of Θ along Z.*

b) *If $\dim Z = p$, then $\Theta \ge m_Z[Z]$ and $\mathbf{1}_Z \Theta = m_Z[Z]$.*

Proof. a) By definition of m_Z and $E_c(\Theta)$, we have $\nu(\Theta, x) \ge m_Z$ for every $x \in Z$ and

$$\nu(\Theta, x) = m_Z \quad \text{on} \quad Z \setminus \bigcup_{c \in \mathbb{Q},\, c > m_Z} Z \cap E_c(\Theta).$$

However, for $c > m_Z$, the intersection $Z \cap E_c(\Theta)$ is a proper analytic subset of A.

b) Left as an exercise to the reader. It is enough to prove that $\Theta \ge m_Z[Z_{\mathrm{reg}}]$ at regular points of Z, so one may assume that Z is a p-dimensional linear subspace in $\mathbb{C}^n$. Show that the measure $(\Theta - m_Z[Z]) \wedge (dd^c|z|^2)^p$ has nonnegative mass on every ball $|z - a| < r$ with center $a \in Z$. Conclude by using arbitrary affine changes of coordinates that $\Theta - m_Z[Z] \ge 0$. □

(2.18) Decomposition formula ([Siu74]). *Let Θ be a closed positive current of bidimension (p,p). Then Θ can be written as a convergent series of closed positive currents*

$$\Theta = \sum_{k=1}^{+\infty} \lambda_k [Z_k] + R,$$

where $[Z_k]$ is a current of integration over an irreducible analytic set of dimension p, and R is a residual current with the property that $\dim E_c(R) < p$ for every $c > 0$. This decomposition is locally and globally unique: the sets Z_k are precisely the p-dimensional components occurring in the sublevel sets $E_c(\Theta)$, and $\lambda_k = \min_{x \in Z_k} \nu(\Theta, x)$ is the generic Lelong number of Θ along Z_k.

Proof of uniqueness. If Θ has such a decomposition, the p-dimensional components of $E_c(\Theta)$ are $(Z_j)_{\lambda_j \geq c}$, for $\nu(\Theta, x) = \sum \lambda_j \nu([Z_j], x) + \nu(R, x)$ is non zero only on $\bigcup Z_j \cup \bigcup E_c(R)$, and is equal to λ_j generically on Z_j (more precisely, $\nu(\Theta, x) = \lambda_j$ at every regular point of Z_j which does not belong to any intersection $Z_j \cup Z_k$, $k \neq j$ or to $\bigcup E_c(R)$). In particular Z_j and λ_j are unique.

Proof of existence. Let $(Z_j)_{j \geq 1}$ be the countable collection of p-dimensional components occurring in one of the sets $E_c(\Theta)$, $c \in \mathbb{Q}_+^*$, and let $\lambda_j > 0$ be the generic Lelong number of Θ along Z_j. Then Lemma 2.17 shows by induction on N that $R_N = \Theta - \sum_{1 \leq j \leq N} \lambda_j [Z_j]$ is positive. As R_N is a decreasing sequence, there must be a limit $R = \lim_{N \to +\infty} R_N$ in the weak topology. Thus we have the asserted decomposition. By construction, R has zero generic Lelong number along Z_j, so $\dim E_c(R) < p$ for every $c > 0$. □

It is very important to note that some components of lower dimension can actually occur in $E_c(R)$, but they cannot be subtracted because R has bidimension (p,p). A typical case is the case of a bidimension $(n-1, n-1)$ current $\Theta = dd^c u$ with $u = \log(|f_j|^{\gamma_1} + \ldots |f_N|^{\gamma_N})$ and $f_j \in H^0(X, \mathcal{O}_X)$. In general $\bigcup E_c(\Theta) = \bigcap f_j^{-1}(0)$ has dimension $< n-1$.

Corollary 2.19. *Let $\Theta_j = dd^c u_j$, $1 \leq j \leq p$, be closed positive $(1,1)$-currents on a complex manifold X. Suppose that there are analytic sets $A_2 \supset \ldots \supset A_p$ in X with $\operatorname{codim} A_j \geq j$ at every point such that each u_j, $j \geq 2$, is locally bounded on $X \smallsetminus A_j$. Let $\{A_{p,k}\}_{k \geq 1}$ be the irreducible components of A_p of codimension p exactly and let $\nu_{j,k} = \min_{x \in A_{p,k}} \nu(\Theta_j, x)$ be the generic Lelong number of Θ_j along $A_{p,k}$. Then $\Theta_1 \wedge \ldots \wedge \Theta_p$ is well-defined and*

$$\Theta_1 \wedge \ldots \wedge \Theta_p \geq \sum_{k=1}^{+\infty} \nu_{1,k} \ldots \nu_{p,k} [A_{p,k}].$$

Proof. By induction on p, Prop. 2.3 shows that $\Theta_1 \wedge \ldots \wedge \Theta_p$ is well defined. Moreover, Prop. 2.16 implies

$$\nu(\Theta_1 \wedge \ldots \wedge \Theta_p, x) \geq \nu(\Theta_1, x) \ldots \nu(\Theta_p, x) \geq \nu_{1,k} \ldots \nu_{p,k}$$

at every point $x \in A_{p,k}$. The desired inequality is then a consequence of Siu's decomposition theorem. □

3. Hermitian Vector Bundles, Connections and Curvature

The goal of this section is to recall the most basic definitions of hemitian differential geometry related to the concepts of connection, curvature and first Chern class of a line bundle.

Let F be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on F is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T_M^* \otimes F) \to C^\infty(M, \Lambda^{q+1} T_M^* \otimes F)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{3.1}$$

for all forms $f \in C^\infty(M, \Lambda^p T_M^*)$, $u \in C^\infty(X, \Lambda^q T_M^* \otimes F)$. On an open set $\Omega \subset M$ where F admits a trivialization $\theta : F_{|\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T_M^* \otimes \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2 u \simeq_\theta (d\Gamma + \Gamma \wedge \Gamma) \wedge u \quad \text{on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$\Theta(D) \in C^\infty(M, \Lambda^2 T_M^* \otimes \operatorname{Hom}(F, F)) \tag{3.2}$$

such that $D^2 u = \Theta(D) \wedge u$ for every form u with values in F.

Assume now that F is endowed with a C^∞ hermitian metric along the fibers and that the isomorphism $F_{|\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing

$$\begin{aligned} C^\infty(M, \Lambda^p T_M^* \otimes F) \times C^\infty(M, \Lambda^q T_M^* \otimes F) &\longrightarrow C^\infty(M, \Lambda^{p+q} T_M^* \otimes \mathbb{C}) \\ (u, v) &\longmapsto \{u, v\} \end{aligned} \tag{3.3}$$

given by

$$\{u, v\} = \sum_{\lambda, \mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu \rangle, \qquad u = \sum u_\lambda \otimes e_\lambda, \quad v = \sum v_\mu \otimes e_\mu.$$

The connection D is said to be *hermitian* if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u} \{u, Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^* = -\Gamma$. In this case $\Theta(D)^* = -\Theta(D)$, thus

$$i\Theta(D) \in C^\infty(M, \Lambda^2 T^*_M \otimes \mathrm{Herm}(F,F)).$$

(3.4) Special case. For a bundle F of rank 1, the connection form Γ of a hermitian connection D can be seen as a 1-form with purely imaginary coefficients $\Gamma = iA$ (A real). Then we have $\Theta(D) = d\Gamma = i dA$. In particular $i\Theta(F)$ is a closed 2-form. The *First Chern class* of F is defined to be the cohomology class

$$c_1(F)_{\mathbb{R}} = \Big\{ \frac{i}{2\pi}\Theta(D) \Big\} \in H^2_{\mathrm{DR}}(M, \mathbb{R}).$$

The cohomology class is actually independent of the connection, since any other connection D_1 differs by a global 1-form, $D_1 u = Du + B \wedge u$, so that $\Theta(D_1) = \Theta(D) + dB$. It is well-known that $c_1(F)_{\mathbb{R}}$ is the image in $H^2(M, \mathbb{R})$ of an integral class $c_1(F) \in H^2(M, \mathbb{Z})$; by using the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^* \to 0,$$

$c_1(F)$ can be defined in Čech cohomology theory as the image by the coboundary map $H^1(M, \mathcal{E}^*) \to H^2(M, \mathbb{Z})$ of the cocycle $\{g_{jk}\} \in H^1(M, \mathcal{E}^*)$ defining F; see e.g. [GH78] for details. □

We now concentrate ourselves on the complex analytic case. If $M = X$ is a complex manifold X, every connection D on a complex C^∞ vector bundle F can be splitted in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D = D' + D''$. In a local trivialization θ given by a C^∞ frame, one can write

$$D'u \simeq_\theta d'u + \Gamma' \wedge u, \tag{3.5'}$$

$$D''u \simeq_\theta d''u + \Gamma'' \wedge u, \tag{3.5''}$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^*$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0,1)$ part D''.

Assume now that the bundle F itself has a *holomorphic* structure. The unique hermitian connection for which D'' is the d'' operator defined in § 1 is called the *Chern connection* of F. In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric is given by the hermitian matrix $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu \rangle$. We have

$$\{u, v\} = \sum_{\lambda,\mu} h_{\lambda\mu} u_\lambda \wedge \overline{v}_\mu = u^\dagger \wedge H\overline{v},$$

where $u^\dagger$ is the transposed matrix of u, and easy computations yield

$$\begin{aligned} d\{u,v\} &= (du)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge (dH \wedge \overline{v} + H\overline{dv}) \\ &= \big(du + \overline{H}^{-1} d'\overline{H} \wedge u\big)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge \overline{(dv + \overline{H}^{-1} d'\overline{H} \wedge v)} \end{aligned}$$

using the fact that $dH = d'H + \overline{d'\overline{H}}$ and $\overline{H}^\dagger = H$. Therefore the Chern connection D coincides with the hermitian connection defined by

$$\begin{cases} Du \simeq_\theta du + \overline{H}^{-1} d'\overline{H} \wedge u, \\ D' \simeq_\theta d' + \overline{H}^{-1} d'\overline{H} \wedge \bullet = \overline{H}^{-1} d'(\overline{H}\bullet), \quad D'' = d''. \end{cases} \tag{3.6}$$

It is clear from this relations that $D'^2 = D''^2 = 0$. Consequently D^2 is given by to $D^2 = D'D'' + D''D'$, and the curvature tensor $\Theta(D)$ is of type $(1,1)$. Since $d'd'' + d''d' = 0$, we get

$$\begin{aligned} (D'D'' + D''D')u &\simeq_\theta \overline{H}^{-1} d'\overline{H} \wedge d''u + d''(\overline{H}^{-1} d'\overline{H} \wedge u) \\ &= d''(\overline{H}^{-1} d'\overline{H}) \wedge u. \end{aligned}$$

(3.7) Proposition. *The Chern curvature tensor $\Theta(F) := \Theta(D)$ is such that*

$$i\Theta(F) \in C^\infty(X, \Lambda^{1,1} T^*_X \otimes \mathrm{Herm}(F,F)).$$

If $\theta : E_{|\Omega} \to \Omega \times \mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $F_{|\Omega}$, then

$$i\Theta(F) \simeq_\theta i\, d''(\overline{H}^{-1} d'\overline{H}) \quad \text{on } \Omega. \qquad \square$$

Let $(z_1, \ldots, z_n)$ be holomorphic coordinates on X and let $(e_\lambda)_{1 \le \lambda \le r}$ be an orthonormal frame of F. Writing

$$i\Theta(F) = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu,$$

we can identify the curvature tensor to a hermitian form

$$\widetilde{\Theta}(F)(\xi \otimes v) = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu \tag{3.8}$$

on $T_X \otimes F$. This leads in a natural way to positivity concepts, following definitions introduced by Kodaira [Kod53], Nakano [Nak55] and Griffiths [Gri66].

(3.9) Definition. *The hermitian vector bundle F is said to be*

a) *positive in the sense of Nakano if $\widetilde{\Theta}(F)(\tau) > 0$ for all non zero tensors $\tau = \sum \tau_{j\lambda} \partial/\partial z_j \otimes e_\lambda \in T_X \otimes F$.*

b) *positive in the sense of Griffiths if $\widetilde{\Theta}(F)(\xi \otimes v) > 0$ for all non zero decomposable tensors $\xi \otimes v \in T_X \otimes F$;*

Corresponding semipositivity concepts are defined by relaxing the strict inequalities.

(3.10) Special case of rank 1 bundles. Assume that F is a line bundle. The hermitian matrix $H = (h_{11})$ associated to a trivialization $\theta : F_{|\Omega} \simeq \Omega \times \mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-2\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. In this case the curvature form $\Theta(F)$ can be identified to the $(1,1)$-form $2d'd''\varphi$, and

$$\frac{i}{2\pi}\Theta(F) = \frac{i}{\pi} d'd''\varphi = dd^c\varphi$$

is a real $(1,1)$-form. Hence F is semipositive (in either Nakano or Griffiths sense) if and only if φ is psh, resp. positive if and only if φ is *strictly psh.* In this setting, the Lelong-Poincaré equation can be generalized as follows: let $\sigma \in H^0(X,F)$ be a non zero holomorphic section. Then

$$dd^c \log\|\sigma\| = [Z_\sigma] - \frac{\mathrm{i}}{2\pi}\Theta(F). \tag{3.11}$$

Formula (3.11) is immediate if we write $\|\sigma\| = |\theta(\sigma)|e^{-\varphi}$ and if we apply (1.20) to the holomorphic function $f = \theta(\sigma)$. As we shall see later, it is very important for applications to consider also singular hermitian metrics.

(3.12) Definition. *A singular (hermitian) metric on a line bundle F is a metric which is given in any trivialization $\theta : F_{|\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}$ by*

$$\|\xi\| = |\theta(\xi)|\, e^{-\varphi(x)}, \qquad x \in \Omega,\ \xi \in F_x$$

where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function, called the weight of the metric with respect to the trivialization θ.

If $\theta' : F_{|\Omega'} \longrightarrow \Omega' \times \mathbb{C}$ is another trivialization, φ' the associated weight and $g \in \mathcal{O}^*(\Omega \cap \Omega')$ the transition function, then $\theta'(\xi) = g(x)\,\theta(\xi)$ for $\xi \in F_x$, and so $\varphi' = \varphi + \log|g|$ on $\Omega \cap \Omega'$. The curvature form of F is then given formally by the closed $(1,1)$-current $\frac{\mathrm{i}}{2\pi}\Theta(F) = dd^c\varphi$ on Ω; our assumption $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ guarantees that $\Theta(F)$ exists in the sense of distribution theory. As in the smooth case, $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is globally defined on X and independent of the choice of trivializations, and its De Rham cohomology class is the image of the first Chern class $c_1(F) \in H^2(X,\mathbb{Z})$ in $H^2_{DR}(X,\mathbb{R})$. Before going further, we discuss two basic examples.

(3.13) Example. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $F = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions u such that $\mathrm{div}(u) + D \geq 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|u\| = |u|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$ then $\theta(u) = u\prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω, thus our singular metric is associated to the weight $\varphi = \sum \alpha_j \log|g_j|$. By the Lelong-Poincaré equation, we find

$$\frac{\mathrm{i}}{2\pi}\Theta\big(\mathcal{O}(D)\big) = dd^c\varphi = [D],$$

where $[D] = \sum \alpha_j [D_j]$ denotes the current of integration over D. □

(3.14) Example. Assume that $\sigma_1, \ldots, \sigma_N$ are non zero holomorphic sections of F. Then we can define a natural (possibly singular) hermitian metric on F^* by

$$\|\xi^*\|^2 = \sum_{1\leq j\leq n} \big|\xi^*.\sigma_j(x)\big|^2 \quad \text{for } \xi^* \in F_x^*.$$

The dual metric on F is given by

$$\|\xi\|^2 = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2 + \ldots + |\theta(\sigma_N(x))|^2}$$

with respect to any trivialization θ. The associated weight function is thus given by $\varphi(x) = \log\big(\sum_{1\leq j\leq N} |\theta(\sigma_j(x))|^2\big)^{1/2}$. In this case φ is a psh function, thus $\mathrm{i}\Theta(F)$ is a closed positive current. Let us denote by Σ the linear system defined by $\sigma_1, \ldots, \sigma_N$ and by $B_\Sigma = \bigcap \sigma_j^{-1}(0)$ its base locus. We have a meromorphic map

$$\Phi_\Sigma : X \smallsetminus B_\Sigma \to \mathbb{P}^{N-1}, \qquad x \mapsto (\sigma_1(x) : \sigma_2(x) : \ldots : \sigma_N(x)).$$

Then $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is equal to the pull-back over $X \smallsetminus B_\Sigma$ of the Fubini-Study metric $\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\log(|z_1|^2 + \ldots + |z_N|^2)$ of $\mathbb{P}^{N-1}$ by Φ_Σ. □

(3.15) Ample and very ample line bundles. *A holomorphic line bundle F over a compact complex manifold X is said to be*

a) *very ample if the map $\Phi_{|F|} : X \to \mathbb{P}^{N-1}$ associated to the complete linear system $|F| = P(H^0(X,F))$ is a regular embedding (by this we mean in particular that the base locus is empty, i.e. $B_{|F|} = \emptyset$).*

b) *ample if some multiple mF, $m > 0$, is very ample.*

Here we use an additive notation for $\mathrm{Pic}(X) = H^1(X,\mathcal{O}^*)$, hence the symbol mF denotes the line bundle $F^{\otimes m}$. By Example 3.15, every ample line bundle F has a smooth hermitian metric with positive definite curvature form: indeed, if the linear system $|mF|$ gives an embedding in projective space, then we get a smooth hermitian metric on $F^{\otimes m}$, and the m-th root yields a metric on F such that $\frac{\mathrm{i}}{2\pi}\Theta(F) = \frac{1}{m}\Phi^*_{|mF|}\omega_{\mathrm{FS}}$. Conversely, the Kodaira embedding theorem [Kod54] tells us that every positive line bundle F is ample (see Exercise 5.14 for a straightforward analytic proof of the Kodaira embedding theorem).

4. Bochner Technique and Vanishing Theorems

We first recall briefly a few basic facts of Hodge theory. Assume for the moment that M is a differentiable manifold equipped with a riemannian metric $g = \sum g_{ij}dx_i \otimes dx_j$. Given a q-form u on M with values in F, we consider the global L^2 norm

$$\|u\|^2 = \int_M |u(x)|^2 dV_g(x)$$

where $|u|$ is the pointwise hermitian norm and dV_g is the riemannian volume form. The Laplace-Beltrami operator associated to the connection D is

$$\Delta = DD^* + D^*D$$

where

$$D^* : C^\infty(M, \Lambda^q T^*_M \otimes F) \to C^\infty(M, \Lambda^{q-1} T^*_M \otimes F)$$

is the (formal) adjoint of D with respect to the L^2 inner product. Assume that M is *compact.* Since

$$\Delta : C^\infty(M, \Lambda^q T^*_M \otimes F) \to C^\infty(M, \Lambda^q T^*_M \otimes F)$$

is a self-adjoint elliptic operator in each degree, standard results of PDE theory show that there is an orthogonal decomposition

$$C^\infty(M, \Lambda^q T^*_M \otimes F) = \mathcal{H}^q(M, F) \oplus \operatorname{Im} \Delta$$

where $\mathcal{H}^q(M,F) = \operatorname{Ker}\Delta$ is the space of harmonic forms of degree q; $\mathcal{H}^q(M,F)$ is a finite dimensional space. Assume moreover that the connection D is *integrable*, i.e. that $D^2 = 0$. It is then easy to check that there is an orthogonal direct sum

$$\operatorname{Im}\Delta = \operatorname{Im} D \oplus \operatorname{Im} D^*,$$

indeed $\langle Du, D^*v\rangle = \langle D^2u, v\rangle = 0$ for all u, v. Hence we get an orthogonal decomposition

$$C^\infty(M, \Lambda^q T^*_M \otimes F) = \mathcal{H}^q(M,F) \oplus \operatorname{Im} D \oplus \operatorname{Im} D^*,$$

and $\operatorname{Ker}\Delta$ is precisely equal to $\mathcal{H}^q(M,F) \oplus \operatorname{Im} D$. Especially, the q-th cohomology group $\operatorname{Ker}\Delta/\operatorname{Im}\Delta$ is isomorphic to $\mathcal{H}^q(M,F)$. All this can be applied for example in the case of the De Rham groups $H^q_{\mathrm{DR}}(M,\mathbb{C})$, taking F to be the trivial bundle $F = M \times \mathbb{C}$ (notice, however, that a nontrivial bundle F usually does not admit any integrable connection):

(4.1) Hodge Fundamental Theorem. *If M is a compact riemannian manifold, there is an isomorphism*

$$H^q_{\mathrm{DR}}(M,\mathbb{C}) \simeq \mathcal{H}^q(M,\mathbb{C})$$

from De Rham cohomology groups onto spaces of harmonic forms. □

A rather important consequence of the Hodge fundamental theorem is a proof of the *Poincaré duality theorem*. Assume that the Riemannian manifold (M,g) is oriented. Then there is a (conjugate linear) Hodge star operator

$$\star : \Lambda^q T^*_M \otimes \mathbb{C} \to \Lambda^{m-q} T^*_M \otimes \mathbb{C}, \qquad m = \dim_{\mathbb{R}} M$$

defined by $u \wedge \star v = \langle u, v\rangle dV_g$ for any two complex valued q-forms u, v. A standard computation shows that $\star$ commutes with Δ, hence $\star u$ is harmonic if and only if u is. This implies that the natural pairing

$$H^q_{\mathrm{DR}}(M,\mathbb{C}) \times H^{m-q}_{\mathrm{DR}}(M,\mathbb{C}), \qquad (\{u\},\{v\}) \mapsto \int_M u \wedge v \tag{4.2}$$

is a nondegenerate duality, the dual of a class $\{u\}$ represented by a harmonic form being $\{\star u\}$.

Let us now suppose that X is a compact complex manifold equipped with a hermitian metric $\omega = \sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Let F be a holomorphic vector bundle on X equipped with a hermitian metric, and let $D = D' + D''$ be its Chern curvature form. All that we said above for the Laplace-Beltrami operator Δ still applies to the complex Laplace operators

$$\Delta' = D'D'^* + D'^*D', \qquad \Delta'' = D''D''^* + D''^*D'',$$

with the great advantage that we always have $D'^2 = D''^2 = 0$. Especially, if X is a compact complex manifold, there are isomorphisms

$$H^{p,q}(X,F) \simeq \mathcal{H}^{p,q}(X,F) \tag{4.3}$$

between Dolbeault cohomology groups $H^{p,q}(X,F)$ and spaces $\mathcal{H}^{p,q}(X,F)$ of Δ''-harmonic forms of bidegree (p,q) with values in F. Now, there is a generalized Hodge star operator

$$\star : \Lambda^{p,q} T^*_X \otimes F \to \Lambda^{n-p,n-q} T^*_X \otimes F^*, \qquad n = \dim_{\mathbb{C}} X,$$

such that $u \wedge \star v = \langle u, v\rangle dV_g$, when the for any two F-valued (p,q)-forms, when the wedge product $u \wedge \star v$ is combined with the pairing $F \times F^* \to \mathbb{C}$. This leads to the *Serre duality theorem* [Ser55]: the bilinear pairing

$$H^{p,q}(X,F) \times H^{n-p,n-q}(X,F^*), \qquad (\{u\},\{v\}) \mapsto \int_X u \wedge v \tag{4.4}$$

is a nondegenerate duality. Combining this with the Dolbeault isomorphism, we may restate the result in the form of the duality formula

$$H^q(X, \Omega^p_X \otimes \mathcal{O}(F))^* \simeq H^{n-q}(X, \Omega^{n-p}_X \otimes \mathcal{O}(F^*)). \tag{4.4'}$$

We now proceed to explain the basic ideas of the Bochner technique used to prove vanishing theorems. Great simplifications occur in the computations if the hermitian metric on X is supposed to be *Kähler*, i.e. if the associated *fundamental* $(1,1)$*-form*

$$\omega = \mathrm{i} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$$

satisfies $d\omega = 0$. It can be easily shown that ω is Kähler if and only if there are holomorphic coordinates $(z_1, \ldots, z_n)$ centered at any point $x_0 \in X$ such that the matrix of coefficients (ω_{jk}) is tangent to identity at order 2, i.e.

$$\omega_{jk}(z) = \delta_{jk} + O(|z|^2) \quad \text{at } x_0.$$

It follows that all order 1 operators D, D', D'' and their adjoints D^*, D'^*, D''^* admit at x_0 the same expansion as the analogous operators obtained when all hermitian metrics on X or F are constant. From this, the basic commutation relations of Kähler geometry can be checked. If A, B are differential operators acting on the algebra $C^\infty(X, \Lambda^{\bullet,\bullet} T^*_X \otimes F)$, their graded commutator (or graded Lie bracket) is defined by

$$[A,B] = AB - (-1)^{ab} BA$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following purely formal *Jacobi identity* holds:

$$(-1)^{ca}[A,[B,C]] + (-1)^{ab}[B,[C,A]] + (-1)^{bc}[C,[A,B]] = 0.$$

(4.5) Basic commutation relations. *Let (X,ω) be a Kähler manifold and let L be the operators defined by $Lu = \omega \wedge u$ and $\Lambda = L^*$. Then*

$$[D''^*, L] = \mathrm{i}D', \qquad [D'^*, L] = -\mathrm{i}D'',$$
$$[\Lambda, D''] = -\mathrm{i}D'^*, \qquad [\Lambda, D'] = \mathrm{i}D''^*.$$

Proof (sketch). The first step is to check the identity $[d''^{\star}, L] = \mathrm{i}d'$ for constant metrics on $X = \mathbb{C}^n$ and $F = X \times \mathbb{C}$, by a brute force calculation. All three other identities follow by taking conjugates or adjoints. The case of variable metrics follows by looking at Taylor expansions up to order 1. □

(4.6) Bochner-Kodaira-Nakano identity. *If (X, ω) is Kähler, the complex Laplace operators Δ' and Δ'' acting on F-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\mathrm{i}\Theta(F), \Lambda].$$

Proof. The last equality in (4.5) yields $D''^{\star} = -\mathrm{i}[\Lambda, D']$, hence

$$\Delta'' = [D'', \delta''] = -\mathrm{i}[D'', [\Lambda, D']].$$

By the Jacobi identity we get

$$[D'', [\Lambda, D']] = [\Lambda, [D', D'']] + [D', [D'', \Lambda]] = [\Lambda, \Theta(F)] + \mathrm{i}[D', D'^{\star}],$$

taking into account that $[D', D''] = D^2 = \Theta(F)$. The formula follows. □

Assume that X is compact and that $u \in C^\infty(X, \Lambda^{p,q}T^\star X \otimes F)$ is an arbitrary (p,q)-form. An integration by parts yields

$$\langle \Delta' u, u\rangle = \|D'u\|^2 + \|D'^{\star}u\|^2 \geq 0$$

and similarly for Δ'', hence we get the basic a priori inequality

$$\|D''u\|^2 + \|D''^{\star}u\|^2 \geq \int_X \langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle dV_\omega. \tag{4.7}$$

This inequality is known as the *Bochner-Kodaira-Nakano* inequality (see [Boc48], [Kod53], [Nak55]). When u is Δ''-harmonic, we get

$$\int_X \big(\langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle\big) dV \leq 0.$$

If the hermitian operator $[\mathrm{i}\Theta(F), \Lambda]$ acting on $\Lambda^{p,q}T^\star_X \otimes F$ is positive on each fiber, we infer that u must be zero, hence

$$H^{p,q}(X, F) = \mathcal{H}^{p,q}(X, F) = 0$$

by Hodge theory. The main point is thus to compute the curvature form $\Theta(F)$ and find sufficient conditions under which the operator $[\mathrm{i}\Theta(F), \Lambda]$ is positive definite. Elementary (but somewhat tedious) calculations yield the following formulae: if the curvature of F is written as in (3.8) and $u = \sum u_{J,K,\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda$, $|J| = p$, $|K| = q$, $1 \leq \lambda \leq r$ is a (p,q)-form with values in F, then

$$\begin{aligned}\langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle = &\sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu}\, u_{J,jS,\lambda}\, \overline{u_{J,kS,\mu}} \\ &+ \sum_{j,k,\lambda,\mu,R,K} c_{jk\lambda\mu}\, u_{kR,K,\lambda}\, \overline{u_{jR,K,\mu}} \\ &- \sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu}\, u_{J,K,\lambda}\, \overline{u_{J,K,\mu}},\end{aligned} \tag{4.8}$$

where the sum is extended to all indices $1 \leq j, k \leq n$, $1 \leq \lambda, \mu \leq r$ and multiindices $|R| = p-1$, $|S| = q-1$ (here the notation $u_{JK\lambda}$ is extended to non necessarily increasing multiindices by making it alternate with respect to permutations). It is usually hard to decide the sign of the curvature term (4.8), except in some special cases.

The easiest case is when $p = n$. Then all terms in the second summation of (4.8) must have $j = k$ and $R = \{1, \ldots, n\} \smallsetminus \{j\}$, therefore the second and third summations are equal. It follows that $[\mathrm{i}\Theta(F), \Lambda]$ is positive on (n,q)-forms under the assumption that F is positive in the sense of Nakano. In this case X is automatically Kähler since

$$\omega = \mathrm{Tr}_F(\mathrm{i}\Theta(F)) = \mathrm{i}\sum_{j,k,\lambda} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k = \mathrm{i}\Theta(\det F)$$

is a Kähler metric.

(4.9) Nakano vanishing theorem (1955). *Let X be a compact complex manifold and let F be a Nakano positive vector bundle on X. Then*

$$H^{n,q}(X, F) = H^q(X, K_X \otimes F) = 0 \qquad \textit{for every } q \geq 1.$$ □

Another tractable case is the case where F is a line bundle ($r = 1$). Indeed, at each point $x \in X$, we may then choose a coordinate system which diagonalizes simultaneously the hermitians forms $\omega(x)$ and $\mathrm{i}\Theta(F)(x)$, in such a way that

$$\omega(x) = \mathrm{i}\sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j, \qquad \mathrm{i}\Theta(F)(x) = \mathrm{i}\sum_{1\leq j\leq n} \gamma_j dz_j \wedge d\overline{z}_j$$

with $\gamma_1 \leq \ldots \leq \gamma_n$. The curvature eigenvalues $\gamma_j = \gamma_j(x)$ are then uniquely defined and depend continuously on x. With our previous notation, we have $\gamma_j = c_{jj11}$ and all other coefficients $c_{jk\lambda\mu}$ are zero. For any (p,q)-form $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \otimes e_1$, this gives

$$\begin{aligned}\langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle &= \sum_{|J|=p, |K|=q} \Big(\sum_{j\in J}\gamma_j + \sum_{j\in K}\gamma_j - \sum_{1\leq j\leq n}\gamma_j\Big)|u_{JK}|^2 \\ &\geq (\gamma_1 + \ldots + \gamma_q - \gamma_{n-p+1} - \ldots - \gamma_n)|u|^2.\end{aligned} \tag{4.10}$$

Assume that $\mathrm{i}\Theta(F)$ is positive. It is then natural to make the special choice $\omega = \mathrm{i}\Theta(F)$ for the Kähler metric. Then $\gamma_j = 1$ for $j = 1, 2, \ldots, n$ and we obtain $\langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle = (p+q-n)|u|^2$. As a consequence:

(4.11) Akizuki-Kodaira-Nakano vanishing theorem ([AN54]). *If F is a positive line bundle on a compact complex manifold X, then*

$$H^{p,q}(X, F) = H^q(X, \Omega^p_X \otimes F) = 0 \qquad \textit{for } p+q \geq n+1.$$ □

More generally, if F is a Griffiths positive (or ample) vector bundle of rank $r \geq 1$, Le Potier [LP75] proved that $H^{p,q}(X, F) = 0$ for $p + q \geq n + r$. The proof is

not a direct consequence of the Bochner technique. A rather easy proof has been found by M. Schneider [Sch74], using the Leray spectral sequence associated to the projectivized bundle projection $\mathbb{P}(F) \to X$.

(4.12) Exercise. It is important for various applications to obtain vanishing theorems which are also valid in the case of semipositive line bundles. The easiest case is the following result of Girbau [Gir76]: let (X,ω) be compact Kähler; assume that F is a line bundle and that $i\Theta(F) \geq 0$ has at least $n-k$ positive eigenvalues at each point, for some integer $k \geq 0$; show that $H^{p,q}(X,F) = 0$ for $p+q \geq n+k+1$.
Hint: use the Kähler metric $\omega_\varepsilon = i\Theta(F) + \varepsilon\omega$ with $\varepsilon > 0$ small.

A stronger and more natural "algebraic version" of this result has been obtained by Sommese [Som78]: define F to be k-ample if some multiple mF is such that the canonical map

$$\Phi_{|mF|} : X \smallsetminus B_{|mF|} \to \mathbb{P}^{N-1}$$

has at most k-dimensional fibers and $\dim B_{|mF|} \leq k$. If X is projective and F is k-ample, show that $H^{p,q}(X,F) = 0$ for $p+q \geq n+k+1$.
Hint: prove the dual result $H^{p,q}(X,F^{-1}) = 0$ for $p+q \leq n-k-1$ by induction on k. First show that F 0-ample $\Rightarrow F$ positive; then use hyperplane sections $Y \subset X$ to prove the induction step, thanks to the exact sequences

$$0 \longrightarrow \Omega^p_X \otimes F^{-1} \otimes \mathcal{O}(-Y) \longrightarrow \Omega^p_X \otimes F^{-1} \longrightarrow \big(\Omega^p_X \otimes F^{-1}\big)_{|Y} \longrightarrow 0,$$
$$0 \longrightarrow \Omega^{p-1}_Y \otimes F^{-1}_{|Y} \longrightarrow \big(\Omega^p_X \otimes F^{-1}\big)_{|Y} \longrightarrow \Omega^p_Y \otimes F^{-1}_{|Y} \longrightarrow 0. \qquad \square$$

5. L^2 Estimates and Existence Theorems

The starting point is the following L^2 existence theorem, which is essentially due to Hörmander [Hör65, 66], and Andreotti-Vesentini [AV65]. We will only outline the main ideas, referring e.g. to [Dem82b] for a detailed exposition of the technical situation considered here.

(5.1) Theorem. *Let (X,ω) be a Kähler manifold. Here X is not necessarily compact, but we assume that the geodesic distance δ_ω is complete on X. Let F be a hermitian vector bundle of rank r over X, and assume that the curvature operator $A = A^{p,q}_{F,\omega} = [i\Theta(F), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^*_X \otimes F$, $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^*_X \otimes F)$ satisfying $D''g = 0$ and $\int_X \langle A^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes F)$ such that $D''f = g$ and*

$$\int_X |f|^2\, dV_\omega \leq \int_X \langle A^{-1}g, g\rangle\, dV_\omega.$$

Proof. The assumption that δ_ω is complete implies the existence of cut-off functions ψ_ν with arbitrarily large compact support such that $|d\psi_\nu| \leq 1$ (take ψ_ν to be a function of the distance $x \mapsto \delta_\omega(x_0,x)$, which is an almost everywhere differentiable 1-Lipschitz function, and regularize if necessary). From this, it follows that very

form $u \in L^2(X, \Lambda^{p,q}T^*_X \otimes F)$ such that $D''u \in L^2$ and $D''^*u \in L^2$ in the sense of distribution theory is a limit of a sequence of smooth forms u_ν with compact support, in such a way that $u_\nu \to u$, $D''u_\nu \to D''u$ and $D''^*u_\nu \to D''^*u$ in L^2 (just take u_ν to be a regularization of $\psi_\nu u$). As a consequence, the basic a priori inequality (4.7) extends to arbitrary forms u such that $u,\ D''u, D''^*u \in L^2$. Now, consider the Hilbert space orthogonal decomposition

$$L^2(X, \Lambda^{p,q}T^*_X \otimes F) = \operatorname{Ker} D'' \oplus (\operatorname{Ker} D'')^\perp,$$

observing that $\operatorname{Ker} D''$ is weakly (hence strongly) closed. Let $v = v_1 + v_2$ be the decomposition of a smooth form $v \in \mathcal{D}^{p,q}(X,F)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general!). Since $(\operatorname{Ker} D'')^\perp \subset \operatorname{Ker} D''^*$ by duality and $g, v_1 \in \operatorname{Ker} D''$ by hypothesis, we get $D''^*v_2 = 0$ and

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 \leq \int_X \langle A^{-1}g, g\rangle\, dV_\omega \int_X \langle Av_1, v_1\rangle\, dV_\omega$$

thanks to the Cauchy-Schwarz inequality. The a priori inequality (4.7) applied to $u = v_1$ yields

$$\int_X \langle Av_1, v_1\rangle\, dV_\omega \leq \|D''v_1\|^2 + \|D''^*v_1\|^2 = \|D''^*v_1\|^2 = \|D''^*v\|^2.$$

Combining both inequalities, we find

$$|\langle g, v\rangle|^2 \leq \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big)\|D''^*v\|^2$$

for every smooth (p,q)-form v with compact support. This shows that we have a well defined linear form

$$w = D''^*v \longmapsto \langle v, g\rangle, \qquad L^2(X, \Lambda^{p,q-1}T^*_X \otimes F) \supset D''^*(\mathcal{D}^{p,q}(F)) \longrightarrow \mathbb{C}$$

on the range of D''^*. This linear form is continuous in L^2 norm and has norm $\leq C$ with

$$C = \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big)^{1/2}.$$

By the Hahn-Banach theorem, there is an element $f \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes F)$ with $\|f\| \leq C$, such that $\langle v, g\rangle = \langle D''^*v, f\rangle$ for every v, hence $D''f = g$ in the sense of distributions. The inequality $\|f\| \leq C$ is equivalent to the last estimate in the theorem. $\square$

The above L^2 existence theorem can be applied in the fairly general context of *weakly pseudoconvex* manifolds. By this, we mean a complex manifold X such that there exists a smooth psh exhaustion function ψ on X (ψ is said to be an exhaustion if for every $c > 0$ the sublevel set $X_c = \psi^{-1}(c)$ is relatively compact, i.e. $\psi(z)$ tends to $+\infty$ when z is taken outside larger and larger compact subsets of X). In particular, every compact complex manifold X is weakly pseudoconvex (take $\psi = 0$), as well as every Stein manifold, e.g. affine algebraic submanifolds of $\mathbb{C}^N$ (take $\psi(z) = |z|^2$), open balls $X = B(z_0,r)$ (take $\psi(z) = 1/(r - |z - z_0|^2)$), convex open subsets, etc.

Now, a basic observation is that every weakly pseudoconvex Kähler manifold (X,ω) carries a *complete* Kähler metric: let $\psi \geq 0$ be a psh exhaustion function and set

$$\omega_\varepsilon = \omega + \varepsilon\, \mathrm{i}d'd''\psi^2 = \omega + 2\varepsilon(2\mathrm{i}\psi d'd''\psi + \mathrm{i}d'\psi \wedge d''\psi).$$

Then $|d\psi|_{\omega_\varepsilon} \leq 1/\varepsilon$ and $|\psi(x) - \psi(y)| \leq \varepsilon^{-1}\delta_{\omega_\varepsilon}(x,y)$. It follows easily from this estimate that the geodesic balls are relatively compact, hence $\delta_{\omega_\varepsilon}$ is complete for every $\varepsilon > 0$. Therefore, the L^2 existence theorem can be applied to each Kähler metric ω_ε, and by passing to the limit it can even be applied to the non necessarily complete metric ω. An important special case is the following

(5.2) Theorem. *Let (X,ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let F be a hermitian line bundle and let*

$$\gamma_1(x) \leq \ldots \leq \gamma_n(x)$$

be the curvature eigenvalues (i.e. the eigenvalues of $\mathrm{i}\Theta(F)$ with respect to the metric ω) at every point. Assume that the curvature is positive, i.e. $\gamma_1 > 0$ everywhere. Then for any form $g \in L^2(X, \Lambda^{n,q}T^*_X \otimes F)$ satisfying $D''g = 0$ and $\int_X ((\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes F)$ such that $D''f = g$ and

$$\int_X |f|^2\, dV_\omega \leq \int_X (\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega.$$

Proof. Indeed, for $p = n$, Formula 4.10 shows that

$$\langle Au, u\rangle \geq (\gamma_1 + \ldots + \gamma_q)|u|^2,$$

hence $\langle A^{-1}u, u\rangle \geq (\gamma_1 + \ldots + \gamma_q)^{-1}|u|^2$. □

An important observation is that the above theorem still applies when the hermitian metric on F is a singular metric with positive curvature in the sense of currents. In fact, by standard regularization techniques (convolution of psh functions by smoothing kernels), the metric can be made smooth and the solutions obtained by (5.1) or (5.2) for the smooth metrics have limits satisfying the desired estimates. Especially, we get the following

(5.3) Corollary. *Let (X,ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let F be a holomorphic line bundle equipped with a singular metric whose local weights are denoted $\varphi \in L^1_{\mathrm{loc}}$. Suppose that*

$$\mathrm{i}\Theta(F) = 2\mathrm{i}d'd''\varphi \geq \varepsilon\omega$$

for some $\varepsilon > 0$. Then for any form $g \in L^2(X, \Lambda^{n,q}T^*_X \otimes F)$ satisfying $D''g = 0$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes F)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-2\varphi}\, dV_\omega \leq \frac{1}{q\varepsilon}\int_X |g|^2 e^{-2\varphi}\, dV_\omega.$$

□

Here we denoted somewhat incorrectly the metric by $|f|^2e^{-2\varphi}$, as if the weight φ was globally defined on X (of course, this is so only if F is globally trivial). We will use this notation anyway, because it clearly describes the dependence of the L^2 norm on the psh weights.

We now introduce the concept of *multiplier ideal sheaf*, following A. Nadel [Nad89]. The main idea actually goes back to the fundamental works of Bombieri [Bom70] and H. Skoda [Sko72a].

(5.4) Definition. *Let φ be a psh function on an open subset $\Omega \subset X$; to φ is associated the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.*

The zero variety $V(\mathcal{I}(\varphi))$ is thus the set of points in a neighborhood of which $e^{-2\varphi}$ is non integrable. Of course, such points occur only if φ has logarithmic poles. This is made precise as follows.

(5.5) Definition. *A psh function φ is said to have a logarithmic pole of coefficient γ at a point $x \in X$ if the Lelong number*

$$\nu(\varphi, x) := \liminf_{z\to x} \frac{\varphi(z)}{\log|z-x|}$$

is non zero and if $\nu(\varphi,x) = \gamma$.

(5.6) Lemma (Skoda [Sko72a]). *Let φ be a psh function on an open set Ω and let $x \in \Omega$.*

a) *If $\nu(\varphi,x) < 1$, then $e^{-2\varphi}$ is integrable in a neighborhood of x, in particular $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$.*

b) *If $\nu(\varphi,x) \geq n + s$ for some integer $s \geq 0$, then $e^{-2\varphi} \geq C|z-x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\varphi)_x \subset \mathfrak{m}^{s+1}_{\Omega,x}$, where $\mathfrak{m}_{\Omega,x}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.*

c) *The zero variety $V(\mathcal{I}(\varphi))$ of $\mathcal{I}(\varphi)$ satisfies*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

where $E_c(\varphi) = \{x \in X\,;\, \nu(\varphi,x) \geq c\}$ is the c-sublevel set of Lelong numbers of φ.

Proof. a) Set $\Theta = dd^c\varphi$ and $\gamma = \nu(\Theta,x) = \nu(\varphi,x)$. Let χ be a cut-off function will support in a small ball $B(x,r)$, equal to 1 in $B(x,r/2)$. As $(dd^c\log|z|)^n = \delta_0$, we get

$$\begin{aligned}\varphi(z) &= \int_{B(x,r)} \chi(\zeta)\varphi(\zeta)(dd^c\log|\zeta - z|)^n \\ &= \int_{B(x,r)} dd^c(\chi(\zeta)\varphi(\zeta)) \wedge \log|\zeta - z|(dd^c\log|\zeta - z|)^{n-1}\end{aligned}$$

for $z \in B(x,r/2)$. Expanding $dd^c(\chi\varphi)$ and observing that $d\chi = dd^c\chi = 0$ on $B(x,r/2)$, we find

$$\varphi(z) = \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1} + \text{smooth terms}$$

on $B(x, r/2)$. Fix r so small that

$$\int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - x|)^{n-1} \leq \nu(\Theta, x, r) < 1.$$

By continuity, there exists $\delta, \varepsilon > 0$ such that

$$I(z) := \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1} \leq 1 - \delta$$

for all $z \in B(x, \varepsilon)$. Applying Jensen's convexity inequality to the probability measure

$$d\mu_z(\zeta) = I(z)^{-1}\chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1},$$

we find

$$-\varphi(z) = \int_{B(x,r)} I(z) \log|\zeta - z|^{-1}\, d\mu_z(\zeta) + O(1) \quad \Longrightarrow$$

$$e^{-2\varphi(z)} \leq C \int_{B(x,r)} |\zeta - z|^{-2I(z)}\, d\mu_z(\zeta).$$

As

$$d\mu_z(\zeta) \leq C_1|\zeta - z|^{-(2n-2)}\Theta(\zeta) \wedge (dd^c|\zeta|^2)^{n-1} = C_2|\zeta - z|^{-(2n-2)}d\sigma_\Theta(\zeta),$$

we get

$$e^{-2\varphi(z)} \leq C_3 \int_{B(x,r)} |\zeta - z|^{-2(1-\delta)-(2n-2)} d\sigma_\Theta(\zeta),$$

and the Fubini theorem implies that $e^{-2\varphi(z)}$ is integrable on a neighborhood of x.

b) If $\nu(\varphi, x) = \gamma$, the convexity properties of psh functions, namely, the convexity of $\log r \mapsto \sup_{|z-x|=r} \varphi(z)$ implies that

$$\varphi(z) \leq \gamma \log|z - x|/r_0 + M,$$

where M is the supremum on $B(x, r_0)$. Hence there exists a constant $C > 0$ such that $e^{-2\varphi(z)} \geq C|z - x|^{-2\gamma}$ in a neighborhood of x. The desired result follows from the identity

$$\int_{B(0,r_0)} \frac{\left|\sum a_\alpha z^\alpha\right|^2}{|z|^{2\gamma}} dV(z) = \text{Const} \int_0^{r_0} \Big(\sum |a_\alpha|^2 r^{2|\alpha|}\Big) r^{2n-1-2\gamma}\, dr,$$

which is an easy consequence of Parseval's formula. In fact, if γ has integral part $[\gamma] = n + s$, the integral converges if and only if $a_\alpha = 0$ for $|\alpha| \leq s$.

c) is just a simple formal consequence of a) and b). □

(5.7) Proposition ([Nad89]). *For any psh function φ on $\Omega \subset X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω. Moreover, if Ω is a bounded Stein open set, the sheaf $\mathcal{I}(\varphi)$ is generated by any Hilbert basis of the L^2 space $\mathcal{H}^2(\Omega, \varphi)$ of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2\varphi}\, d\lambda < +\infty$.*

Proof. Since the result is local, we may assume that Ω is a bounded pseudoconvex open set in $\mathbb{C}^n$. By the strong noetherian property of coherent sheaves, the family of sheaves generated by finite subsets of $\mathcal{H}^2(\Omega, \varphi)$ has a maximal element on each compact subset of Ω, hence $\mathcal{H}^2(\Omega, \varphi)$ generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $d''u = g := d''(\theta f)$ by means of Hörmander's L^2 estimates 5.3, where F is the trivial line bundle $\Omega \times \mathbb{C}$ equipped with the strictly psh weight

$$\widetilde{\varphi}(z) = \varphi(z) + (n + s)\log|z - x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-2\varphi}|z - x|^{-2(n+s)} d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in \mathcal{H}^2(\Omega, \varphi)$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves the coherence. Now, $\mathcal{J}$ is generated by any Hilbert basis of $\mathcal{H}^2(\Omega, \varphi)$, because it is well-known that the space of sections of any coherent sheaf is a Fréchet space, therefore closed under local L^2 convergence. □

The multiplier ideal sheaves satisfy the following basic functoriality property with respect to direct images of sheaves by modifications.

(5.8) Proposition. *Let $\mu : X' \to X$ be a modification of non singular complex manifolds (i.e. a proper generically 1:1 holomorphic map), and let φ be a psh function on X. Then*

$$\mu_*\big(\mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi).$$

Proof. Let $n = \dim X = \dim X'$ and let $S \subset X$ be an analytic set such that $\mu : X' \setminus S' \to X \setminus S$ is a biholomorphism. By definition of multiplier ideal sheaves, $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$ is just the sheaf of holomorphic n-forms f on open sets $U \subset X$ such that $\mathrm{i}^{n^2} f \wedge \overline{f} e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)$. Since φ is locally bounded from above, we may even consider forms f which are a priori defined only on $U \setminus S$, because f will be in $L^2_{\mathrm{loc}}(U)$ and therefore will automatically extend through S. The change of variable formula yields

$$\int_U \mathrm{i}^{n^2} f \wedge \overline{f} e^{-2\varphi} = \int_{\mu^{-1}(U)} \mathrm{i}^{n^2} \mu^* f \wedge \overline{\mu^* f}\, e^{-2\varphi\circ\mu},$$

hence $f \in \Gamma(U, \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi))$ iff $\mu^* f \in \Gamma(\mu^{-1}(U), \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu))$. Proposition 5.8 is proved. □

(5.9) Remark. If φ has analytic singularities (according to Definition 1.10), the computation of $\mathcal{I}(\varphi)$ can be reduced to a purely algebraic problem.

The first observation is that $\mathcal{I}(\varphi)$ can be computed easily if φ has the form $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are nonsingular irreducible divisors with normal crossings. Then $\mathcal{I}(\varphi)$ is the sheaf of functions h on open sets $U \subset X$ such that

$$\int_U |h|^2 \prod |g_j|^{-2\alpha_j} dV < +\infty.$$

Since locally the g_j can be taken to be coordinate functions from a local coordinate system $(z_1, \ldots, z_n)$, the condition is that h is divisible by $\prod g_j^{m_j}$ where $m_j - \alpha_j > -1$ for each j, i.e. $m_j \geq \lfloor \alpha_j \rfloor$ (integer part). Hence

$$\mathcal{I}(\varphi) = \mathcal{O}(-\lfloor D \rfloor) = \mathcal{O}(-\sum \lfloor \alpha_j \rfloor D_j)$$

where $\lfloor D \rfloor$ denotes the integral part of the $\mathbb{Q}$-divisor $D = \sum \alpha_j D_j$.

Now, consider the general case of analytic singularities and suppose that $\varphi \sim \frac{\alpha}{2} \log\left(|f_1|^2 + \cdots + |f_N|^2\right)$ near the poles. By the remarks after Definition 1.10, we may assume that the (f_j) are generators of the integrally closed ideal sheaf $\mathcal{J} = \mathcal{J}(\varphi/\alpha)$, defined as the sheaf of holomorphic functions h such that $|h| \leq C \exp(\varphi/\alpha)$. In this case, the computation is made as follows (see also L. Bonavero's work [Bon93], where similar ideas are used in connection with "singular" holomorphic Morse inequalities).

First, one computes a smooth modification $\mu : \widetilde{X} \to X$ of X such that $\mu^* \mathcal{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D = \sum \lambda_j D_j$, where (D_j) are the components of the exceptional divisor of $\widetilde{X}$ (take the blow-up X' of X with respect to the ideal $\mathcal{J}$ so that the pull-back of $\mathcal{J}$ to X' becomes an invertible sheaf $\mathcal{O}(-D')$. then blow up again by Hironaka [Hir64] to make X' smooth and D' have normal crossings). Now, we have $K_{\widetilde{X}} = \mu^* K_X + R$ where $R = \sum \rho_j D_j$ is the zero divisor of the Jacobian function J_μ of the blow-up map. By the direct image formula 5.8, we get

$$\mathcal{I}(\varphi) = \mu_* \big(\mathcal{O}(K_{\widetilde{X}} - \mu^* K_X) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mu_* \big(\mathcal{O}(R) \otimes \mathcal{I}(\varphi \circ \mu)\big).$$

Now, $(f_j \circ \mu)$ are generators of the ideal $\mathcal{O}(-D)$, hence

$$\varphi \circ \mu \sim \alpha \sum \lambda_j \log |g_j|$$

where g_j are local generators of $\mathcal{O}(-D_j)$. We are thus reduced to computing multiplier ideal sheaves in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum \alpha \lambda_j D_j$. We obtain $\mathcal{I}(\varphi \circ \mu) = \mathcal{O}(-\sum \lfloor \alpha \lambda_j \rfloor D_j)$, hence

$$\mathcal{I}(\varphi) = \mu_* \mathcal{O}_{\widetilde{X}}\big(\sum (\rho_j - \lfloor \alpha \lambda_j \rfloor) D_j\big). \qquad \square$$

(5.10) Exercise. Compute the multiplier ideal sheaf $\mathcal{I}(\varphi)$ associated with $\varphi = \log(|z_1|^{\alpha_1} + \ldots + |z_p|^{\alpha_p})$ for arbitrary real numbers $\alpha_j > 0$.
Hint: using Parseval's formula and polar coordinates $z_j = r_j e^{i\theta_j}$, show that the problem is equivalent to determining for which p-tuples $(\beta_1, \ldots, \beta_p) \in \mathbb{N}^p$ the integral

$$\int_{[0,1]^p} \frac{r_1^{2\beta_1} \ldots r_p^{2\beta_p} r_1 dr_1 \ldots r_p dr_p}{r_1^{2\alpha_1} + \ldots + r_p^{2\alpha_p}} = \int_{[0,1]^p} \frac{t_1^{(\beta_1+1)/\alpha_1} \ldots t_p^{(\beta_p+1)/\alpha_p}}{t_1 + \ldots + t_p} \frac{dt_1}{t_1} \cdots \frac{dt_p}{t_p}$$

is convergent. Conclude from this that $\mathcal{I}(\varphi)$ is generated by the monomials $z_1^{\beta_1} \ldots z_p^{\beta_p}$ such that $\sum (\beta_p + 1)/\alpha_p > 1$. (This exercise shows that the analytic definition of $\mathcal{I}(\varphi)$ is sometimes also quite convenient for computations). $\square$

Let F be a line bundle over X with a singular metric h of curvature current $\Theta_h(F)$. If φ is the weight representing the metric in an open set $\Omega \subset X$, the ideal sheaf $\mathcal{I}(\varphi)$ is independent of the choice of the trivialization and so it is the restriction to Ω of a global coherent sheaf $\mathcal{I}(h)$ on X. We will sometimes still write $\mathcal{I}(h) = \mathcal{I}(\varphi)$ by abuse of notation. In this context, we have the following fundamental vanishing theorem, which is probably one of the most central results of analytic and algebraic geometry (as we will see later, it contains the Kawamata-Viehweg vanishing theorem as a special case).

(5.11) Nadel vanishing theorem ([Nad89], [Dem93b]). *Let (X, ω) be a Kähler weakly pseudoconvex manifold, and let F be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight φ. Assume that $i\Theta_h(F) \geq \varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(h)\big) = 0 \qquad \text{for all } q \geq 1.$$

Proof. Let $\mathcal{L}^q$ be the sheaf of germs of (n,q)-forms u with values in F and with measurable coefficients, such that both $|u|^2 e^{-2\varphi}$ and $|d''u|^2 e^{-2\varphi}$ are locally integrable. The d'' operator defines a complex of sheaves $(\mathcal{L}^\bullet, d'')$ which is a resolution of the sheaf $\mathcal{O}(K_X + F) \otimes \mathcal{I}(\varphi)$: indeed, the kernel of d'' in degree 0 consists of all germs of holomorphic n-forms with values in F which satisfy the integrability condition; hence the coefficient function lies in $\mathcal{I}(\varphi)$; the exactness in degree $q \geq 1$ follows from Corollary 5.3 applied on arbitrary small balls. Each sheaf $\mathcal{L}^q$ is a $\mathcal{C}^\infty$-module, so $\mathcal{L}^\bullet$ is a resolution by acyclic sheaves. Let ψ be a smooth psh exhaustion function on X. Let us apply Corollary 5.3 globally on X, with the original metric of F multiplied by the factor $e^{-\chi \circ \psi}$, where χ is a convex increasing function of arbitrary fast growth at infinity. This factor can be used to ensure the convergence of integrals at infinity. By Corollary 5.3, we conclude that $H^q\big(\Gamma(X, \mathcal{L}^\bullet)\big) = 0$ for $q \geq 1$. The theorem follows. $\square$

(5.12) Corollary. *Let (X, ω), F and φ be as in Theorem 5.11 and let $x_1, \ldots, x_N$ be isolated points in the zero variety $V(\mathcal{I}(\varphi))$. Then there is a surjective map*

$$H^0(X, K_X + F) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X + L)_{x_j} \otimes \big(\mathcal{O}_X / \mathcal{I}(\varphi)\big)_{x_j}.$$

Proof. Consider the long exact sequence of cohomology associated to the short exact sequence $0 \to \mathcal{I}(\varphi) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(\varphi) \to 0$ twisted by $\mathcal{O}(K_X + F)$, and apply Theorem 5.11 to obtain the vanishing of the first H^1 group. The asserted surjectivity property follows. $\square$

(5.13) Corollary. *Let (X, ω), F and φ be as in Theorem 5.11 and suppose that the weight function φ is such that $\nu(\varphi, x) \geq n + s$ at some point $x \in X$ which is an isolated point of $E_1(\varphi)$. Then $H^0(X, K_X + F)$ generates all s-jets at x.*

Proof. The assumption is that $\nu(\varphi, y) < 1$ for y near x, $y \neq x$. By Skoda's lemma 5.6 b), we conclude that $e^{-2\varphi}$ is integrable at all such points y, hence $\mathcal{I}(\varphi)_y = \mathcal{O}_{X,y}$, whilst $\mathcal{I}(\varphi)_x \subset \mathfrak{m}_{X,x}^{s+1}$ by 5.6 a). Corollary 5.13 is thus a special case of 5.12. □

The philosophy of these results (which can be seen as generalizations of the Hörmander-Bombieri-Skoda theorem [Bom70], [Sko72a, 75]) is that the problem of constructing holomorphic sections of $K_X + F$ can be solved by constructing suitable hermitian metrics on F such that the weight φ has isolated poles at given points x_j.

(5.14) Exercise. Assume that X is compact and that L is a positive line bundle on X. Let $\{x_1, \ldots, x_N\}$ be a finite set. Show that there are constants $a, b \geq 0$ depending only on L and N such that $H^0(X, mL)$ generates jets of any order s at all points x_j for $m \geq as + b$.
Hint: Apply Corollary 5.12 to $F = -K_X + mL$, with a singular metric on L of the form $h = h_0 e^{-\epsilon\psi}$, where h_0 is smooth of positive curvature, $\epsilon > 0$ small and $\psi(z) \sim \log|z - x_j|$ in a neighborhood of x_j.
Derive the Kodaira embedding theorem from the above result:

(5.15) Theorem (Kodaira). *If L is a line bundle on a compact complex manifold, then L is ample if and only if L is positive.* □

(5.16) Exercise (solution of the Levi problem). Show that the following two properties are equivalent.

a) X is strongly pseudoconvex, i.e. X admits a strongly psh exhaustion function.

b) X is Stein, i.e. the global holomorphic functions $H^0(X, \mathcal{O}_X)$ separate points and yield local coordinates at any point, and X is holomorphically convex (this means that for any discrete sequence z_ν there is a function $f \in H^0(X, \mathcal{O}_X)$ such that $|f(z_\nu)| \to \infty$). □

(5.17) Remark. As long as forms of bidegree (n, q) are considered, the L^2 estimates can be extended to complex spaces with arbitrary singularities. In fact, if X is a complex space and φ is a psh weight function on X, we may still define a sheaf $K_X(\varphi)$ on X, such that the sections on an open set U are the holomorphic n-forms f on the regular part $U \cap X_{\text{reg}}$, satisfying the integrability condition $i^{n^2} f \wedge \overline{f}\, e^{-2\varphi} \in L^1_{\text{loc}}(U)$. In this setting, the fonctoriality property 5.8 becomes

$$\mu_\star\big(K_{X'}(\varphi \circ \mu)\big) = K_X(\varphi)$$

for arbitrary complex spaces X, X' such that $\mu : X' \to X$ is a modification. If X is nonsingular we have $K_X(\varphi) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$, however, if X is singular, the symbols K_X and $\mathcal{I}(\varphi)$ must not be dissociated. The statement of the Nadel vanishing theorem becomes $H^q(X, \mathcal{O}(F) \otimes K_X(\varphi)) = 0$ for $q \geq 1$, under the same assumptions (X Kähler and weakly pseudoconvex, curvature $\geq \varepsilon\omega$). The proof can be obtained by restricting everything to X_{reg}. Although in general X_{reg} is not weakly pseudoconvex (e.g. in case $\operatorname{codim} X_{\text{sing}} \geq 2$), X_{reg} is always Kähler complete (the complement of a proper analytic subset in a Kähler weakly pseudoconvex space is

complete Kähler, see e.g. [Dem82a]). As a consequence, Nadel's vanishing theorem is essentially insensitive to the presence of singularities. □

6. Numerically Effective Line Bundles

Many problems of algebraic geometry (e.g. problems of classification of algebraic surfaces or higher dimensional varieties) lead in a natural way to the study of line bundles satisfying semipositivity conditions. It turns out that semipositivity in the sense of curvature (at least, as far as smooth metrics are considered) is not a very satisfactory notion. A more flexible notion perfectly suitable for algebraic purposes is the notion of *numerical effectivity*. The goal of this section is to give a few fundamental algebraic definitions and to discuss their differential geometric counterparts. We first suppose that X is a projective algebraic manifold, $\dim X = n$.

(6.1) Definition. *A holomorphic line bundle L over a projective manifold X is said to be numerically effective, nef for short, if $L \cdot C = \int_C c_1(L) \geq 0$ for every curve $C \subset X$.*

If L is nef, it can be shown that $L^p \cdot Y = \int_Y c_1(L)^p \geq 0$ for any p-dimensional subvariety $Y \subset X$ (see e.g. [Har70]). In relation with this, let us recall the Nakai-Moishezon ampleness criterion: a line bundle L is ample if and only if $L^p \cdot Y > 0$ for every p-dimensional subvariety Y. From this, we easily infer

(6.2) Proposition. *Let L be a line bundle on a projective algebraic manifold X, on which an ample line bundle A and a hermitian metric ω are given. The following properties are equivalent:*

a) *L is nef;*

b) *for any integer $k \geq 1$, the line bundle $kL + A$ is ample;*

c) *for every $\varepsilon > 0$, there is a smooth metric h_ε on L such that $i\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$.*

Proof. a) ⇒ b). If L is nef and A is ample then clearly $kL + A$ satisfies the Nakai-Moishezon criterion, hence $kL + A$ is ample.

b) ⇒ c). Condition c) is independent of the choice of the hermitian metric, so we may select a metric h_A on A with positive curvature and set $\omega = i\Theta(A)$. If $kL + A$ is ample, this bundle has a metric h_{kL+A} of positive curvature. Then the metric $h_L = (h_{kL+A} \otimes h_A^{-1})^{1/k}$ has curvature

$$i\Theta(L) = \frac{1}{k}\big(i\Theta(kL + A) - i\Theta(A)\big) \geq -\frac{1}{k} i\Theta(A)\,;$$

in this way the negative part can be made smaller than $\varepsilon\,\omega$ by taking k large enough.

c) ⇒ a). Under hypothesis c), we get $L \cdot C = \int_C \frac{i}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\frac{\varepsilon}{2\pi}\int_C \omega$ for every curve C and every $\varepsilon > 0$, hence $L \cdot C \geq 0$ and L is nef. □

Let now X be an arbitrary compact complex manifold. Since there need not exist any curve in X, Property 6.2 c) is simply taken as a definition of nefness ([DPS94]):

(6.3) Definition. *A line bundle L on a compact complex manifold X is said to be nef if for every $\varepsilon > 0$, there is a smooth hermitian metric h_ε on L such that $\mathrm{i}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$.*

In general, it is not possible to extract a smooth limit h_0 such that $\mathrm{i}\Theta_{h_0}(L) \geq 0$. The following simple example is given in [DPS94] (Example 1.7). Let E be a non trivial extension $0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$ over an elliptic curve C and let $X = P(E)$ be the corresponding ruled surface over C. Then $L = \mathcal{O}_{P(E)}(1)$ is nef but does not admit any smooth metric of nonnegative curvature. This example answers negatively a question raised by [Fuj83].

Let us now introduce the important concept of *Kodaira-Iitaka dimension* of a line bundle.

(6.4) Definition. *If L is a line bundle, the Kodaira-Iitaka dimension $\kappa(L)$ is the supremum of the rank of the canonical maps*

$$\Phi_m : X \smallsetminus B_m \longrightarrow P(V_m^*), \qquad x \longmapsto H_x = \{\sigma \in V_m\,;\, \sigma(x) = 0\}, \qquad m \geq 1$$

with $V_m = H^0(X, mL)$ and $B_m = \bigcap_{\sigma \in V_m} \sigma^{-1}(0)$ = base locus of V_m. In case $V_m = \{0\}$ for all $m \geq 1$, we set $\kappa(L) = -\infty$.
A line bundle is said to be big if $\kappa(L) = \dim X$.

The following lemma is well-known (the proof is a rather elementary consequence of the Schwarz lemma).

(6.5) Serre-Siegel lemma ([Ser54], [Sie55]). *Let L be any line bundle on a compact complex manifold. Then we have*

$$h^0(X, mL) \leq O(m^{\kappa(L)}) \qquad \text{for } m \geq 1,$$

and $\kappa(L)$ is the smallest constant for which this estimate holds. □

We now discuss the various concepts of positive cones in the space of numerical classes of line bundles, and establish a simple dictionary relating these concepts to corresponding concepts in the context of differential geometry.

Let us recall that an integral cohomology class in $H^2(X, \mathbb{Z})$ is the first Chern class of a holomorphic (or algebraic) line bundle if and only if it lies in the *Neron-Severi* group

$$\mathrm{NS}(X) = \mathrm{Ker}\left(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)\right)$$

(this fact is just an elementary consequence of the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$). If X is compact Kähler, as we will suppose from now on in this section, this is the same as saying that the class is of type $(1,1)$ with respect to Hodge decomposition.

Let $\mathrm{NS}_{\mathbb{R}}(X)$ be the real vector space $\mathrm{NS}(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$. We define four convex cones

$$K_{\mathrm{amp}}(X) \subset K_{\mathrm{eff}}(X) \subset \mathrm{NS}_{\mathbb{R}}(X),$$
$$K_{\mathrm{nef}}(X) \subset K_{\mathrm{psef}}(X) \subset \mathrm{NS}_{\mathbb{R}}(X)$$

which are, respectively, the *convex cones* generated by Chern classes $c_1(L)$ of ample and effective line bundles, resp. the *closure of the convex cones* generated by numerically effective and pseudo-effective line bundles; we say that L is effective if mL has a section for some $m > 0$, i.e. if $\mathcal{O}(mL) \simeq \mathcal{O}(D)$ for some effective divisor D; and we say that L pseudo-effective if $c_1(L)$ is the cohomology class of some closed positive current T, i.e. if L can be equipped with a singular hermitian metric h with $T = \frac{\mathrm{i}}{2\pi}\Theta_h(L) \geq 0$ as a current. For each of the ample, effective, nef and pseudo-effective cones, the first Chern class $c_1(L)$ of a line bundle L lies in the cone if and only if L has the corresponding property (for K_{psef} use the fact that the space of positive currents of mass 1 is weakly compact; the case of all other cones is obvious).

(6.6) Proposition. *Let (X, ω) be a compact Kähler manifold. The numerical cones satisfy the following properties.*

a) $K_{\mathrm{amp}} = K_{\mathrm{amp}}^\circ \subset K_{\mathrm{nef}}^\circ$, $K_{\mathrm{nef}} \subset K_{\mathrm{psef}}$.

b) *If moreover X is projective algebraic, we have $K_{\mathrm{amp}} = K_{\mathrm{nef}}^\circ$ (therefore $\overline{K}_{\mathrm{amp}} = K_{\mathrm{nef}}$), and $\overline{K}_{\mathrm{eff}} = K_{\mathrm{psef}}$.*

If L is a line bundle on X and h denotes a hermitian metric on L, the following properties are equivalent:

c) $c_1(L) \in K_{\mathrm{amp}} \Leftrightarrow \exists \varepsilon > 0$, $\exists h$ *smooth such that* $\mathrm{i}\Theta_h(L) \geq \varepsilon\omega$.

d) $c_1(L) \in K_{\mathrm{nef}} \Leftrightarrow \forall \varepsilon > 0$, $\exists h$ *smooth such that* $\mathrm{i}\Theta_h(L) \geq -\varepsilon\omega$.

e) $c_1(L) \in K_{\mathrm{psef}} \Leftrightarrow \exists h$ *possibly singular such that* $\mathrm{i}\Theta_h(L) \geq 0$.

f) *If moreover X is projective algebraic, then*
$c_1(L) \in K_{\mathrm{eff}}^\circ \Leftrightarrow \kappa(L) = \dim X$
$\Leftrightarrow \exists \varepsilon > 0$, $\exists h$ *possibly singular such that* $\mathrm{i}\Theta_h(L) \geq \varepsilon\omega$.

Proof. c) and d) are already known and e) is a definition.

a) The ample cone K_{amp} is always open by definition and contained in K_{nef}, so the first inclusion is obvious (K_{amp} is of course empty if X is not projective algebraic). Let us now prove that $K_{\mathrm{nef}} \subset K_{\mathrm{psef}}$. Let L be a line bundle with $c_1(L) \in K_{\mathrm{nef}}$. Then for every $\varepsilon > 0$, there is a current $T_\varepsilon = \frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$. Then $T_\varepsilon + \varepsilon\omega$ is a closed positive current and the family is uniformly bounded in mass for $\varepsilon \in]0, 1]$, since

$$\int_X (T_\varepsilon + \varepsilon\omega) \wedge \omega^{n-1} = \int_X c_1(L) \wedge \omega^{n-1} + \varepsilon \int_X \omega^n.$$

By weak compactness, some subsequence converges to a weak limit $T \geq 0$ and $T \in c_1(L)$ (the cohomology class $\{T\}$ of a current is easily shown to depend continuously on T with respect to the weak topology; use e.g. Poincaré duality to check this).

b) If X is projective, the equality $K_{\mathrm{amp}} = K_{\mathrm{nef}}^\circ$ is a simple consequence of 6.2 b) and of the fact that ampleness (or positivity) is an open property. It remains to

show that $K_{\text{psef}} \subset \overline{K}_{\text{eff}}$. Let L be a line bundle with $c_1(L) \in K_{\text{psef}}$ and let h_L be a singular hermitian on L such that $T = \frac{\mathrm{i}}{2\pi}\Theta(L) \geq 0$. Fix a point $x_0 \in X$ such that the Lelong number of T at x_0 is zero, and take a sufficiently positive line bundle A (replacing A by a multiple if necessary), such that $A - K_X$ has a singular metric h_{A-K_X} of curvature $\geq \varepsilon\omega$ and such that h_{A-K_X} is smooth on $X \smallsetminus \{x_0\}$ and has an isolated logarithmic pole of Lelong number $\geq n$ at x_0. Then apply Corollary 5.13 to $F = mL + A - K_X$ equipped with the metric $h_L^{\otimes m} \otimes h_{A-K_X}$. Since the weight φ of this metric has a Lelong number $\geq n$ at x_0 and a Lelong number equal to the Lelong number of $T = \frac{\mathrm{i}}{2\pi}\Theta(L)$ at nearby points, $\limsup_{x\to x_0} \nu(T,x) = \nu(T,x_0) = 0$, Corollary 5.13 implies that $H^0(X, K_X + F) = H^0(X, mL + A)$ has a section which does not vanish at x_0. Hence there is an effective divisor D_m such that $\mathcal{O}(mL+A) = \mathcal{O}(D_m)$ and $c_1(L) = \frac{1}{m}\{D_m\} - \frac{1}{m}c_1(A) = \lim \frac{1}{m}\{D_m\}$ is in $\overline{K}_{\text{eff}}$. □

f) Fix a nonsingular ample divisor A. If $c_1(L) \in K^\circ_{\text{eff}}$, there is an integer $m > 0$ such that $c_1(L) - \frac{1}{m}c_1(A)$ is still effective, hence for m, p large we have $mpL - pA = D + F$ with an effective divisor D and a numerically trivial line bundle F. This implies $\mathcal{O}(kmpL) = \mathcal{O}(kpA + kD + kF) \supset \mathcal{O}(kpA + kF)$, hence $h^0(X, kmpL) \geq h^0(X, kpA + kF) \sim (kp)^n A^n/n!$ by the Riemann-Roch formula. Therefore $\kappa(L) = n$.

If $\kappa(L) = n$, then $h^0(X, kL) \geq ck^n$ for $k \geq k_0$ and $c > 0$. The exact cohomology sequence

$$0 \longrightarrow H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{|A})$$

where $h^0(A, kL_{|A}) = O(k^{n-1})$ shows that $kL - A$ has non zero sections for k large. If D is the divisor of such a section, then $kL \simeq \mathcal{O}(A + D)$. Select a smooth metric on A such that $\frac{\mathrm{i}}{2\pi}\Theta(A) \geq \varepsilon_0\omega$ for some $\varepsilon_0 > 0$, and take the singular metric on $\mathcal{O}(D)$ with weight function $\varphi_D = \sum \alpha_j \log|g_j|$ described in Example 3.13. Then the metric with weight $\varphi_L = \frac{1}{k}(\varphi_A + \varphi_D)$ on L yields

$$\frac{\mathrm{i}}{2\pi}\Theta(L) = \frac{1}{k}\Big(\frac{\mathrm{i}}{2\pi}\Theta(A) + [D]\Big) \geq (\varepsilon_0/k)\,\omega,$$

as desired.

Finally, the curvature condition $\mathrm{i}\Theta_h(L) \geq \varepsilon\omega$ in the sense of currents yields by definition $c_1(L) \in K^\circ_{\text{psef}}$. Moreover, b) implies $K^\circ_{\text{psef}} = K^\circ_{\text{eff}}$. □

Before going further, we need a lemma.

(6.7) Lemma. *Let X be a compact Kähler n-dimensional manifold, let L be a nef line bundle on X, and let E be an arbitrary holomorphic vector bundle. Then $h^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) = o(k^n)$ as $k \to +\infty$, for every $q \geq 1$. If X is projective algebraic, the following more precise bound holds:*

$$h^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) = O(k^{n-q}), \qquad \forall q \geq 0.$$

Proof. The Kähler case will be proved in Section 12, as a consequence of the holomorphic Morse inequalities. In the projective algebraic case, we proceed by induction on $n = \dim X$. If $n = 1$ the result is clear, as well as if $q = 0$. Now let A be a nonsingular ample divisor such that $E \otimes \mathcal{O}(A - K_X)$ is Nakano positive. Then the Nakano

vanishing theorem applied to the vector bundle $F = E \otimes \mathcal{O}(kL + A - K_X)$ shows that $H^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL + A)) = 0$ for all $q \geq 1$. The exact sequence

$$0 \to \mathcal{O}(kL) \to \mathcal{O}(kL + A) \to \mathcal{O}(kL + A)_{|A} \to 0$$

twisted by E implies

$$H^q(X, \mathcal{O}(E) \otimes \mathcal{O}(kL)) \simeq H^{q-1}(A, \mathcal{O}(E_{|A} \otimes \mathcal{O}(kL + A)_{|A}),$$

and we easily conclude by induction since $\dim A = n - 1$. Observe that the argument does not work any more if X is not algebraic. It seems to be unknown whether the $O(k^{n-q})$ bound still holds in that case. □

(6.8) Corollary. *If L is nef, then L is big (i.e. $\kappa(L) = n$) if and only if $L^n > 0$. Moreover, if L is nef and big, then for every $\delta > 0$, L has a singular metric $h = e^{-2\varphi}$ such that $\max_{x\in X} \nu(\varphi, x) \leq \delta$ and $\mathrm{i}\Theta_h(L) \geq \varepsilon\,\omega$ for some $\varepsilon > 0$. The metric h can be chosen to be smooth on the complement of a fixed divisor D, with logarithmic poles along D.*

Proof. By Lemma 6.7 and the Riemann-Roch formula, we have $h^0(X, kL) = \chi(X, kL) + o(k^n) = k^n L^n/n! + o(k^n)$, whence the first statement. If L is big, the proof made in 6.5 f) shows that there is a singular metric h_1 on L such that

$$\frac{\mathrm{i}}{2\pi}\Theta_{h_1}(L) = \frac{1}{k}\Big(\frac{\mathrm{i}}{2\pi}\Theta(A) + [D]\Big)$$

with a positive line bundle A and an effective divisor D. Now, for every $\varepsilon > 0$, there is a smooth metric h_ε on L such that $\frac{\mathrm{i}}{2\pi}\Theta_{h_\varepsilon}(L) \geq -\varepsilon\omega$, where $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A)$. The convex combination of metrics $h'_\varepsilon = h_1^{k\varepsilon} h_\varepsilon^{1-k\varepsilon}$ is a singular metric with poles along D which satisfies

$$\frac{\mathrm{i}}{2\pi}\Theta_{h'_\varepsilon}(L) \geq \varepsilon(\omega + [D]) - (1 - k\varepsilon)\varepsilon\omega \geq k\varepsilon^2\omega.$$

Its Lelong numbers are $\varepsilon\nu(D, x)$ and they can be made smaller than δ by choosing $\varepsilon > 0$ small. □

We still need a few elementary facts about the numerical dimension of nef line bundles.

(6.9) Definition. *Let L be a nef line bundle on a compact Kähler manifold X. One defines the numerical dimension of L to be*

$$\nu(L) = \max\big\{k = 0, \ldots, n\,;\, c_1(L)^k \neq 0 \text{ in } H^{2k}(X, \mathbb{R})\big\}.$$

By Corollary 6.8, we have $\kappa(L) = n$ if and only if $\nu(L) = n$. In general, we merely have an inequality.

(6.10) Proposition. *If L is a nef line bundle on a compact Kähler manifold, then $\kappa(L) \leq \nu(L)$.*

Proof. By induction on $n = \dim X$. If $\nu(L) = n$ or $\kappa(L) = n$ the result is true, so we may assume $r := \kappa(L) \leq n-1$ and $k := \nu(L) \leq n-1$. Fix $m > 0$ so that $\Phi = \Phi_{|mL|}$ has generic rank r. Select a nonsingular ample divisor A in X such that the restriction of $\Phi_{|mL|}$ to A still has rank r (for this, just take A passing through a point $x \notin B_{|mL|}$ at which $\operatorname{rank}(d\Phi_x) = r < n$, in such a way that the tangent linear map $d\Phi_{x|T_{A,x}}$ still has rank r). Then $\kappa(L_{|A}) \geq r = \kappa(L)$ (we just have an equality because there might exist sections in $H^0(A, mL_{|A})$ xhich do not extend to X). On the other hand, we claim that $\nu(L_{|A}) = k = \nu(L)$. The inequality $\nu(L_{|A}) \geq \nu(L)$ is clear. Conversely, if we set $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A) > 0$, the cohomology class $c_1(L)^k$ can be represented by a closed positive current of bidegree (k,k)

$$T = \lim_{\varepsilon \to 0} \Big(\frac{\mathrm{i}}{2\pi} \Theta_{h_\varepsilon}(L) + \varepsilon\omega \Big)^k$$

after passing to some subsequence (there is a uniform bound for the mass thanks to the Kähler assumption, taking wedge products with ω^{n-k}). The current T must be non zero since $c_1(L)^k \neq 0$ by definition of $k = \nu(L)$. Then $\{[A]\} = \{\omega\}$ as cohomology classes, and

$$\int_A c_1(L_{|A})^k \wedge \omega^{n-1-k} = \int_X c_1(L)^k \wedge [A] \wedge \omega^{n-1-k} = \int_X T \wedge \omega^{n-k} > 0.$$

This implies $\nu(L_{|A}) \geq k$, as desired. The induction hypothesis with X replaced by A yields

$$\kappa(L) \leq \kappa(L_{|A}) \leq \nu(L_{|A}) \leq \nu(L). \qquad \square$$

(6.11) Remark. It may happen that $\kappa(L) < \nu(L)$: take e.g.

$$L \to X = X_1 \times X_2$$

equal to the total tensor product of an ample line bundle L_1 on a projective manifold X_1 and of a unitary flat line bundle L_2 on an elliptic curve X_2 given by a representation $\pi_1(X_2) \to U(1)$ such that no multiple kL_2 with $k \neq 0$ is trivial. Then $H^0(X, kL) = H^0(X_1, kL_1) \otimes H^0(X_2, kL_2) = 0$ for $k > 0$, and thus $\kappa(L) = -\infty$. However $c_1(L) = \mathrm{pr}_1^* c_1(L_1)$ has numerical dimension equal to $\dim X_1$. The same example shows that the Kodaira dimension may increase by restriction to a subvariety (if $Y = X_1 \times \{\text{point}\}$, then $\kappa(L_{|Y}) = \dim Y$). $\square$

We now derive an algebraic version of the Nadel vanishing theorem in the context of nef line bundles. This algebraic vanishing theorem has been obtained independently by Kawamata [Kaw82] and Viehweg [Vie82], who both reduced it to the Kodaira-Nakano vanishing theorem by cyclic covering constructions. Since then, a number of other proofs have been given, one based on connections with logarithmic singularities [EV86], another on Hodge theory for twisted coefficient systems [Kol85], a third one on the Bochner technique [Dem89] (see also [EV92] for a general survey, and [Eno93] for an extension to the compact Kähler case). Since the result is best expressed in terms of multiplier ideal sheaves (avoiding then any unnecessary desingularization in the statement), we feel that the direct approach via Nadel's vanishing theorem is probably the most natural one.

If $D = \sum \alpha_j D_j \geq 0$ is an effective $\mathbb{Q}$-divisor, we define the *multiplier ideal sheaf* $\mathcal{I}(D)$ to be equal to $\mathcal{I}(\varphi)$ where $\varphi = \sum \alpha_j \log|g_j|$ is the corresponding psh function defined by generators g_j of $\mathcal{O}(-D_j)$; as we saw in Remark 5.9, the computation of $\mathcal{I}(D)$ can be made algebraically by using desingularizations $\mu : \tilde{X} \to X$ such that $\mu^* D$ becomes a divisor with normal crossings on $\tilde{X}$.

(6.12) Kawamata-Viehweg vanishing theorem. *Let X be a projective algebraic manifold and let F be a line bundle over X such that some positive multiple mF can be written $mF = L + D$ where L is a nef line bundle and D an effective divisor. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(m^{-1}D)\big) = 0 \quad \text{for} \quad q > n - \nu(L).$$

(6.13) Special case. *If F is a nef line bundle, then*

$$H^q\big(X, \mathcal{O}(K_X + F)\big) = 0 \quad \text{for} \quad q > n - \nu(F).$$

Proof of Theorem 6.12. First suppose that $\nu(L) = n$, i.e. that L is big. By the proof of 6.5 f), there is a singular hermitian metric on L such that the corresponding weight $\varphi_{L,0}$ has algebraic singularities and

$$\mathrm{i}\Theta_0(L) = 2\mathrm{i}d'd''\varphi_L \geq \varepsilon_0\omega$$

for some $\varepsilon_0 > 0$. On the other hand, since L is nef, there are metrics given by weights $\varphi_{L,\varepsilon}$ such that $\frac{\mathrm{i}}{2\pi}\Theta_\varepsilon(L) \geq \varepsilon\omega$ for every $\varepsilon > 0$, ω being a Kähler metric. Let $\varphi_D = \sum \alpha_j \log|g_j|$ be the weight of the singular metric on $\mathcal{O}(D)$ described in Example 3.13. We define a singular metric on F by

$$\varphi_F = \frac{1}{m}\big((1-\delta)\varphi_{L,\varepsilon} + \delta\varphi_{L,0} + \varphi_D\big)$$

with $\varepsilon \ll \delta \ll 1$, δ rational. Then φ_F has algebraic singularities, and by taking δ small enough we find $\mathcal{I}(\varphi_F) = \mathcal{I}(\frac{1}{m}\varphi_D) = \mathcal{I}(\frac{1}{m}D)$. In fact, $\mathcal{I}(\varphi_F)$ can be computed by taking integer parts of $\mathbb{Q}$-divisors (as explained in Remark 5.9), and adding $\delta\varphi_{L,0}$ does not change the integer part of the rational numbers involved when δ is small. Now

$$\begin{aligned} dd^c\varphi_F &= \frac{1}{m}\big((1-\delta)dd^c\varphi_{L,\varepsilon} + \delta dd^c\varphi_{L,0} + dd^c\varphi_D\big) \\ &\geq \frac{1}{m}\big(-(1-\delta)\varepsilon\omega + \delta\varepsilon_0\omega + [D] \geq \frac{\delta\varepsilon}{m}\omega, \end{aligned}$$

if we choose $\varepsilon \leq \delta\varepsilon_0$. Nadel's theorem thus implies the desired vanishing result for all $q \geq 1$.

Now, if $\nu(L) < n$, we use hyperplane sections and argue by induction on $n = \dim X$. Since the sheaf $\mathcal{O}(K_X) \otimes \mathcal{I}(m^{-1}D)$ behaves fonctorially with respect to modifications (and since the L^2 cohomology complex is "the same" upstairs and downstairs), we may assume after blowing-up that D is a divisor with normal crossings. By Remark 5.9, the multiplier ideal sheaf $\mathcal{I}(m^{-1}D) = \mathcal{O}(-\lfloor m^{-1}D \rfloor)$ is locally free. By Serre duality, the expected vanishing is equivalent to

$$H^q(X, \mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1} D \rfloor)) = 0 \qquad \text{for } q < \nu(L).$$

Then select a nonsingular ample divisor A such that A meets all components D_j transversally. Select A positive enough so that $\mathcal{O}(A + F - \lfloor m^{-1} D \rfloor)$ is ample. Then $H^q(X, \mathcal{O}(-A-F) \otimes \mathcal{O}(\lfloor m^{-1} D \rfloor)) = 0$ for $q < n$ by Kodaira vanishing, and the exact sequence $0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to (i_A)_* \mathcal{O}_A \to 0$ twisted by $\mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1} D \rfloor)$ yields an isomorphism

$$H^q(X, \mathcal{O}(-F) \otimes \mathcal{O}(\lfloor m^{-1} D \rfloor)) \simeq H^q(A, \mathcal{O}(-F_{\restriction A}) \otimes \mathcal{O}(\lfloor m^{-1} D_{\restriction A} \rfloor)).$$

The proof of 5.8 showed that $\nu(L_{\restriction A}) = \nu(L)$, hence the induction hypothesis implies that the cohomology group on A in the right hand side is zero for $q < \nu(L)$. □

7. A Simple Algebraic Approach to Fujita's Conjecture

This section is devoted to a proof of various results related to the Fujita conjecture. The main ideas occuring here are inspired by a recent work of Y.T. Siu [Siu96]. His method, which is algebraic in nature and quite elementary, consists in a combination of the Riemann-Roch formula together with Nadel's vanishing theorem (in fact, only the algebraic case is needed, thus the original Kawamata-Viehweg vanishing theorem would be sufficient). Slightly later, Angehrn and Siu [AS94], [Siu95] introduced other closely related methods, producing better bounds for the global generation question; since their method is rather delicate, we can only refer the ready to the above references. In the sequel, X denotes a projective algebraic n-dimensional manifold. The first observation is the following well-known consequence of the Riemann-Roch formula.

(7.1) Special case of Riemann-Roch. *Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal sheaf on X such that the subscheme $Y = V(\mathcal{J})$ has dimension d (with possibly some lower dimensional components). Let $[Y] = \sum \lambda_j [Y_j]$ be the effective algebraic cycle of dimension d associated to the d dimensional components of Y (taking into account multiplicities λ_j given by the ideal $\mathcal{J}$). Then for any line bundle F, the Euler characteristic*

$$\chi(Y, \mathcal{O}(F + mL)_{\restriction Y}) = \chi(X, \mathcal{O}(F + mL) \otimes \mathcal{O}_X/\mathcal{J})$$

is a polynomial $P(m)$ of degree d and leading coefficient $L^d \cdot [Y]/d!$

The second fact is an elementary lemma about numerical polynomials (polynomials with rational coefficients, mapping $\mathbb{Z}$ into $\mathbb{Z}$).

(7.2) Lemma. *Let $P(m)$ be a numerical polynomial of degree $d > 0$ and leading coefficient $a_d/d!$, $a_d \in \mathbb{Z}$, $a_d > 0$. Suppose that $P(m) \geq 0$ for $m \geq m_0$. Then*

a) *For every integer $N \geq 0$, there exists $m \in [m_0, m_0 + Nd]$ such that $P(m) \geq N$.*

b) *For every $k \in \mathbb{N}$, there exists $m \in [m_0, m_0 + kd]$ such that $P(m) \geq a_d k^d / 2^{d-1}$.*

c) *For every integer $N \geq 2d^2$, there exists $m \in [m_0, m_0 + N]$ such that $P(m) \geq N$.*

Proof. a) Each of the N equations $P(m) = 0$, $P(m) = 1$, ..., $P(m) = N - 1$ has at most d roots, so there must be an integer $m \in [m_0, m_0 + dN]$ which is not a root of these.

b) By Newton's formula for iterated differences $\Delta P(m) = P(m+1) - P(m)$, we get

$$\Delta^d P(m) = \sum_{1 \leq j \leq d} (-1)^j \binom{d}{j} P(m + d - j) = a_d, \qquad \forall m \in \mathbb{Z}.$$

Hence if $j \in \{0, 2, 4, \ldots, 2\lfloor d/2 \rfloor\} \subset [0, d]$ is the even integer achieving the maximum of $P(m_0 + d - j)$ over this finite set, we find

$$2^{d-1} P(m_0 + d - j) = \left(\binom{d}{0} + \binom{d}{2} + \ldots \right) P(m_0 + d - j) \geq a_d,$$

whence the existence of an integer $m \in [m_0, m_0 + d]$ with $P(m) \geq a_d / 2^{d-1}$. The case $k = 1$ is thus proved. In general, we apply the above case to the polynomial $Q(m) = P(km - (k-1)m_0)$, which has leading coefficient $a_d k^d / d!$

c) If $d = 1$, part a) already yields the result. If $d = 2$, a look at the parabola shows that

$$\max_{m \in [m_0, m_0 + N]} P(m) \geq \begin{cases} a_2 N^2/8 & \text{if } N \text{ is even,} \\ a_2 (N^2 - 1)/8 & \text{if } N \text{ is odd;} \end{cases}$$

thus $\max_{m \in [m_0, m_0+N]} P(m) \geq N$ whenever $N \geq 8$. If $d \geq 3$, we apply b) with k equal to the smallest integer such that $k^d / 2^{d-1} \geq N$, i.e. $k = \lceil 2(N/2)^{1/d} \rceil$, where $\lceil x \rceil \in \mathbb{Z}$ denotes the round-up of $x \in \mathbf{R}$. Then $kd \leq (2(N/2)^{1/d} + 1)d \leq N$ whenever $N \geq 2d^2$, as a short computation shows. □

We now apply Nadel's vanishing theorem pretty much in the same way as Siu [Siu96], but with substantial simplifications in the technique and improvements in the bounds. Our method yields simultaneously a simple proof of the following basic result.

(7.3) Theorem. *If L is an ample line bundle over a projective n-fold X, then the adjoint line bundle $K_X + (n+1)L$ is nef.*

By using Mori theory and the base point free theorem ([Mor82], [Kaw84]), one can even show that $K_X + (n+1)L$ is semiample, i.e., there exists a positive integer m such that $m(K_X + (n+1)L)$ is generated by sections (see [Kaw85] and [Fuj87]). The proof rests on the observation that $n + 1$ is the maximal length of extremal rays of smooth projective n-folds. Our proof of (7.3) is different and will be given simultaneously with the proof of Th. (7.4) below.

(7.4) Theorem. *Let L be an ample line bundle and let G be a nef line bundle on a projective n-fold X. Then the following properties hold.*

a) *$2K_X + mL + G$ generates simultaneous jets of order $s_1, \ldots, s_p \in \mathbb{N}$ at arbitrary points $x_1, \ldots, x_p \in X$, i.e., there is a surjective map*

$$H^0(X, 2K_X + mL + G) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{1\le j\le p} \mathcal{O}(2K_X + mL + G) \otimes \mathcal{O}_{X,x_j}/\mathfrak{m}_{X,x_j}^{s_j+1},$$

provided that $m \ge 2 + \sum_{1\le j\le p} \binom{3n+2s_j-1}{n}$.

In particular $2K_X + mL + G$ *is very ample for* $m \ge 2 + \binom{3n+1}{n}$.

b) $2K_X + (n+1)L + G$ *generates simultaneous jets of order* $s_1, \ldots, s_p$ *at arbitrary points* $x_1, \ldots, x_p \in X$ *provided that the intersection numbers* $L^d \cdot Y$ *of* L *over all* d*-dimensional algebraic subsets* Y *of* X *satisfy*

$$L^d \cdot Y > \frac{2^{d-1}}{\lfloor n/d \rfloor^d} \sum_{1\le j\le p} \binom{3n+2s_j-1}{n}.$$

Proof. The proofs of (7.3) and (7.4 a, b) go along the same lines, so we deal with them simultaneously (in the case of (7.3), we simply agree that $\{x_1, \ldots, x_p\} = \emptyset$). The idea is to find an integer (or rational number) m_0 and a singular hermitian metric h_0 on $K_X + m_0 L$ with strictly positive curvature current $i\Theta_{h_0} \ge \varepsilon\omega$, such that $V(\mathcal{I}(h_0))$ is 0-dimensional and the weight φ_0 of h_0 satisfies $\nu(\varphi_0, x_j) \ge n + s_j$ for all j. As L and G are nef, $(m - m_0)L + G$ has for all $m \ge m_0$ a metric h' whose curvature $i\Theta_{h'}$ has arbitrary small negative part (see [Dem90]), e.g., $i\Theta_{h'} \ge -\frac{\varepsilon}{2}\omega$. Then $i\Theta_{h_0} + i\Theta_{h'} \ge \frac{\varepsilon}{2}\omega$ is again positive definite. An application of Cor (1.5) to $F = K_X + mL + G = (K_X + m_0 L) + ((m - m_0)L + G)$ equipped with the metric $h_0 \otimes h'$ implies the existence of the desired sections in $K_X + F = 2K_X + mL + G$ for $m \ge m_0$.

Let us fix an embedding $\Phi_{|\mu L|} : X \to \mathbb{P}^N$, $\mu \gg 0$, given by sections $\lambda_0, \ldots, \lambda_N \in H^0(X, \mu L)$, and let h_L be the associated metric on L of positive definite curvature form $\omega = \frac{i}{2\pi}\Theta(L)$. In order to obtain the desired metric h_0 on $K_X + m_0 L$, we fix $a \in \mathbb{N}^*$ and use a double induction process to construct singular metrics $(h_{k,\nu})_{\nu\ge 1}$ on $aK_X + b_k L$ for a non increasing sequence of positive integers $b_1 \ge b_2 \ge \ldots \ge b_k \ge \ldots$. Such a sequence much be stationary and m_0 will just be the stationary limit $m_0 = \lim b_k/a$. The metrics $h_{k,\nu}$ are taken to satisfy the following properties:

α) $h_{k,\nu}$ is an algebraic metric of the form

$$\|\xi\|^2_{h_{k,\nu}} = \frac{|\tau_k(\xi)|^2}{\left(\sum_{1\le i\le\nu, 0\le j\le N} \left|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu} \cdot \lambda_j^{(a+1)b_k - am_i})\right|^2\right)^{1/(a+1)\mu}},$$

defined by sections $\sigma_i \in H^0(X, (a+1)K_X + m_i L)$, $m_i < \frac{a+1}{a} b_k$, $1 \le i \le \nu$, where $\xi \mapsto \tau_k(\xi)$ is an arbitrary local trivialization of $aK_X + b_k L$; note that $\sigma_i^{a\mu} \cdot \lambda_j^{(a+1)b_k - am_i}$ is a section of

$$a\mu((a+1)K_X + m_i L) + ((a+1)b_k - am_i)\mu L = (a+1)\mu(aK_X + b_k L).$$

β) $\operatorname{ord}_{x_j}(\sigma_i) \ge (a+1)(n+s_j)$ for all i, j;

γ) $\mathcal{I}(h_{k,\nu+1}) \supset \mathcal{I}(h_{k,\nu})$ and $\mathcal{I}(h_{k,\nu+1}) \ne \mathcal{I}(h_{k,\nu})$ whenever the zero variety $V(\mathcal{I}(h_{k,\nu}))$ has positive dimension.

The weight $\varphi_{k,\nu} = \frac{1}{2(a+1)\mu} \log \sum \left|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu} \cdot \lambda_j^{(a+1)b_k - am_i})\right|^2$ of $h_{k,\nu}$ is plurisubharmonic and the condition $m_i < \frac{a+1}{a} b_k$ implies $(a+1)b_k - am_i \ge 1$, thus the difference $\varphi_{k,\nu} - \frac{1}{2(a+1)\mu} \log \sum |\tau(\lambda_j)|^2$ is also plurisubharmonic. Hence $\frac{i}{2\pi}\Theta_{h_{k,\nu}}(aK_X + b_k L) = \frac{i}{\pi} d'd'' \varphi_{k,\nu} \ge \frac{1}{(a+1)}\omega$. Moreover, condition β) clearly implies $\nu(\varphi_{k,\nu}, x_j) \ge a(n + s_j)$. Finally, condition γ) combined with the strong Noetherian property of coherent sheaves ensures that the sequence $(h_{k,\nu})_{\nu\ge 1}$ will finally produce a zero dimensional subscheme $V(\mathcal{I}(h_{k,\nu}))$. We agree that the sequence $(h_{k,\nu})_{\nu\ge 1}$ stops at this point, and we denote by $h_k = h_{k,\nu}$ the final metric, such that $\dim V(\mathcal{I}(h_k)) = 0$.

For $k = 1$, it is clear that the desired metrics $(h_{1,\nu})_{\nu\ge 1}$ exist if b_1 is taken large enough (so large, say, that $(a+1)K_X + (b_1 - 1)L$ generates jets of order $(a+1)(n + \max s_j)$ at every point; then the sections $\sigma_1, \ldots, \sigma_\nu$ can be chosen with $m_1 = \ldots = m_\nu = b_1 - 1$). Suppose that the metrics $(h_{k,\nu})_{\nu\ge 1}$ and h_k have been constructed and let us proceed with the construction of $(h_{k+1,\nu})_{\nu\ge 1}$. We do this again by induction on ν, assuming that $h_{k+1,\nu}$ is already constructed and that $\dim V(\mathcal{I}(h_{k+1,\nu})) > 0$. We start in fact the induction with $\nu = 0$, and agree in this case that $\mathcal{I}(h_{k+1,0}) = 0$ (this would correspond to an infinite metric of weight identically equal to $-\infty$). By Nadel's vanishing theorem applied to

$$F_m = aK_X + mL = (aK_X + b_k L) + (m - b_k)L$$

with the metric $h_k \otimes (h_L)^{\otimes m - b_k}$, we get

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{I}(h_k)) = 0 \qquad \text{for } q \ge 1,\ m \ge b_k.$$

As $V(\mathcal{I}(h_k))$ is 0-dimensional, the sheaf $\mathcal{O}_X/\mathcal{I}(h_k)$ is a skyscraper sheaf, and the exact sequence $0 \to \mathcal{I}(h_k) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(h_k) \to 0$ twisted with the invertible sheaf $\mathcal{O}((a+1)K_X + mL)$ shows that

$$H^q(X, \mathcal{O}((a+1)K_X + mL)) = 0 \qquad \text{for } q \ge 1,\ m \ge b_k.$$

Similarly, we find

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{I}(h_{k+1,\nu})) = 0 \qquad \text{for } q \ge 1,\ m \ge b_{k+1}$$

(also true for $\nu = 0$, since $\mathcal{I}(h_{k+1,0}) = 0$), and when $m \ge \max(b_k, b_{k+1}) = b_k$, the exact sequence $0 \to \mathcal{I}(h_{k+1,\nu}) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu}) \to 0$ implies

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) = 0 \qquad \text{for } q \ge 1,\ m \ge b_k.$$

In particular, since the H^1 group vanishes, every section u' of $(a+1)K_X + mL$ on the subscheme $V(\mathcal{I}(h_{k+1,\nu}))$ has an extension u to X. Fix a basis $u'_1, \ldots, u'_N$ of the sections on $V(\mathcal{I}(h_{k+1,\nu}))$ and take arbitrary extensions $u_1, \ldots, u_N$ to X. Look at the linear map assigning the collection of jets of order $(a+1)(n+s_j) - 1$ at all points x_j

$$u = \sum_{1\le j\le N} a_j u_j \longmapsto \bigoplus J_{x_j}^{(a+1)(n+s_j)-1}(u).$$

Since the rank of the bundle of s-jets is $\binom{n+s}{n}$, the target space has dimension

$$\delta = \sum_{1\le j\le p} \binom{n + (a+1)(n+s_j) - 1}{n}.$$

In order to get a section $\sigma_{\nu+1} = u$ satisfying condition β) with non trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{I}(h_{k+1,\nu}))$, we need at least $N = \delta + 1$ independent sections $u'_1, \ldots, u'_N$. This condition is achieved by applying Lemma (7.2) to the numerical polynomial

$$\begin{aligned} P(m) &= \chi(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) \\ &= h^0(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{I}(h_{k+1,\nu})) \geq 0, \qquad m \geq b_k. \end{aligned}$$

The polynomial P has degree $d = \dim V(\mathcal{I}(h_{k+1,\nu})) > 0$. We get the existence of an integer $m \in [b_k, b_k + \eta]$ such that $N = P(m) \geq \delta + 1$ with some explicit integer $\eta \in \mathbb{N}$ (for instance $\eta = n(\delta+1)$ always works by (7.2 a), but we will also use the other possibilities to find an optimal choice in each case). Then we find a section $\sigma_{\nu+1} \in H^0(X, (a+1)K_X + mL)$ with non trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{I}(h_{k+1,\nu}))$, vanishing at order $\geq (a+1)(n+s_j)$ at each point x_j. We just set $m_{\nu+1} = m$, and the condition $m_{\nu+1} < \frac{a+1}{a} b_{k+1}$ is satisfied if $b_k + \eta < \frac{a+1}{a} b_{k+1}$. This shows that we can take inductively

$$b_{k+1} = \left\lfloor \frac{a}{a+1}(b_k + \eta) \right\rfloor + 1.$$

By definition, $h_{k+1,\nu+1} \leq h_{k+1,\nu}$, hence $\mathcal{I}(h_{k+1,\nu+1}) \supset \mathcal{I}(h_{k+1,\nu})$. We necessarily have $\mathcal{I}(h_{k+1,\nu+1}) \neq \mathcal{I}(h_{k+1,\nu})$, for $\mathcal{I}(h_{k+1,\nu+1})$ contains the ideal sheaf associated with the zero divisor of $\sigma_{\nu+1}$, whilst $\sigma_{\nu+1}$ does not vanish identically on $V(\mathcal{I}(h_{k+1,\nu}))$. Now, an easy computation shows that the iteration of $b_{k+1} = \lfloor \frac{a}{a+1}(b_k + \eta) \rfloor + 1$ stops at $b_k = a(\eta+1)+1$ for any large initial value b_1. In this way, we obtain a metric h_∞ of positive definite curvature on $aK_X + (a(\eta+1)+1)L$, with $\dim V(\mathcal{I}(h_\infty)) = 0$ and $\nu(\varphi_\infty, x_j) \geq a(n+s_j)$ at each point x_j.

Proof of (7.3). In this case, the set $\{x_j\}$ is taken to be empty, thus $\delta = 0$. By (7.2 a), the condition $P(m) \geq 1$ is achieved for some $m \in [b_k, b_k + n]$ and we can take $\eta = n$. As μL is very ample, there exists on μL a metric with an isolated logarithmic pole of Lelong number 1 at any given point x_0 (e.g., the algebraic metric defined with all sections of μL vanishing at x_0). Hence

$$F'_a = aK_X + (a(n+1)+1)L + n\mu L$$

has a metric h'_a such that $V(\mathcal{I}(h'_a))$ is zero dimensional and contains $\{x_0\}$. By Cor (1.5), we conclude that

$$K_X + F'_a = (a+1)K_X + (a(n+1)+1+n\mu)L$$

is generated by sections, in particular $K_X + \frac{a(n+1)+1+n\mu}{a+1} L$ is nef. As a tends to $+\infty$, we infer that $K_X + (n+1)L$ is nef. □

Proof of (7.4 a). Here, the choice $a = 1$ is sufficient for our purposes. Then

$$\delta = \sum_{1 \leq j \leq p} \binom{3n + 2s_j - 1}{n}.$$

If $\{x_j\} \neq \emptyset$, we have $\delta + 1 \geq \binom{3n-1}{n} + 1 \geq 2n^2$ for $n \geq 2$. Lemma (7.2 c) shows that $P(m) \geq \delta + 1$ for some $m \in [b_k, b_k + \eta]$ with $\eta = \delta + 1$. We can start in fact the

induction procedure $k \mapsto k+1$ with $b_1 = \eta + 1 = \delta + 2$, because the only property needed for the induction step is the vanishing property

$$H^0(X, 2K_X + mL) = 0 \qquad \text{for } q \geq 1,\ m \geq b_1,$$

which is true by the Kodaira vanishing theorem and the ampleness of $K_X + b_1 L$ (here we use Fujita's result (7.3), observing that $b_1 > n+1$). Then the recursion formula $b_{k+1} = \lfloor \frac{1}{2}(b_k + \eta) \rfloor + 1$ yields $b_k = \eta + 1 = \delta + 2$ for all k, and (7.4 a) follows. □

Proof of (7.4 b). Quite similar to (7.4 a), except that we take $\eta = n$, $a = 1$ and $b_k = n+1$ for all k. By Lemma (7.2 b), we have $P(m) \geq a_d k^d / 2^{d-1}$ for some integer $m \in [m_0, m_0 + kd]$, where $a_d > 0$ is the coefficient of highest degree in P. By Lemma (7.1) we have $a_d \geq \inf_{\dim Y = d} L^d \cdot Y$. We take $k = \lfloor n/d \rfloor$. The condition $P(m) \geq \delta + 1$ can thus be realized for some $m \in [m_0, m_0 + kd] \subset [m_0, m_0 + n]$ as soon as

$$\inf_{\dim Y = d} L^d \cdot Y \ \lfloor n/d \rfloor^d / 2^{d-1} > \delta,$$

which is equivalent to the condition given in (7.4 b). □

(7.5) Corollary. *Let X be a smooth projective n-fold, let L be an ample line bundle and G a nef line bundle over X. Then $m(K_X + (n+2)L) + G$ is very ample for $m \geq \binom{3n+1}{n} - 2n$.*

Proof. Apply Th. (7.4 a) with $G' = a(K_X + (n+1)L) + G$, so that

$$2K_X + mL + G' = (a+2)(K_X + (n+2)L) + (m - 2n - 4 - a)L + G,$$

and take $m = a + 2n + 4 \geq 2 + \binom{3n+1}{n}$. □

The main drawback of the above technique is that multiples of L at least equal to $(n+1)L$ are required to avoid zeroes of the Hilbert polynomial. In particular, it is not possible to obtain directly a very ampleness criterion for $2K_X + L$ in the statement of (7.4 b). Nevertheless, using different ideas from Angehrn-Siu [AS94], [Siu96] has obtained such a criterion. We derive here a slightly weaker version, thanks to the following elementary Lemma.

(7.6) Lemma. *Assume that for some integer $\mu \in \mathbb{N}^*$ the line bundle μF generates simultaneously all jets of order $\mu(n+s_j)+1$ at any point x_j in a subset $\{x_1, \ldots, x_p\}$ of X. Then $K_X + F$ generates simultaneously all jets of order s_j at x_j.*

Proof. Take the algebraic metric on F defined by a basis of sections $\sigma_1, \ldots, \sigma_N$ of μF which vanish at order $\mu(n+s_j)+1$ at all points x_j. Since we are still free to choose the homogeneous term of degree $\mu(n+s_j)+1$ in the Taylor expansion at x_j, we find that $x_1, \ldots, x_p$ are isolated zeroes of $\bigcap \sigma_j^{-1}(0)$. If φ is the weight of the metric of F near x_j, we thus have $\varphi(z) \sim (n + s_j + \frac{1}{\mu}) \log|z - x_j|$ in suitable coordinates. We replace φ in a neighborhood of x_j by

$$\varphi'(z) = \max\left(\varphi(z),\ |z|^2 - C + (n+s_j)\log|z - x_j|\right)$$

and leave φ elsewhere unchanged (this is possible by taking $C > 0$ very large). Then $\varphi'(z) = |z|^2 - C + (n+s_j)\log|z-x_j|$ near x_j, in particular φ' is strictly plurisubharmonic near x_j. In this way, we get a metric h' on F with semipositive curvature everywhere on X, and with positive definite curvature on a neighborhood of $\{x_1,\dots,x_p\}$. The conclusion then follows directly from Hörmander's L^2 estimates (5.1) and (5.2). □

(7.7) Theorem. *Let X be a smooth projective n-fold, and let L be an ample line bundle over X. Then $2K_X + L$ generates simultaneous jets of order $s_1,\dots,s_p$ at arbitrary points $x_1,\dots,x_p \in X$ provided that the intersection numbers $L^d \cdot Y$ of L over all d-dimensional algebraic subsets Y of X satisfy*

$$L^d \cdot Y > \frac{2^{d-1}}{[n/d]^d} \sum_{1\le j\le p} \binom{(n+1)(4n+2s_j+1)-2}{n}, \qquad 1 \le d \le n.$$

Proof. By Lemma (7.6) applied with $F = K_X + L$ and $\mu = n+1$, the desired jet generation of $2K_X + L$ occurs if $(n+1)(K_X+L)$ generates jets of order $(n+1)(n+s_j)+1$ at x_j. By Lemma (7.5) again with $F = aK_X + (n+1)L$ and $\mu = 1$, we see by backward induction on a that we need the simultaneous generation of jets of order $(n+1)(n+s_j)+1+(n+1-a)(n+1)$ at x_j. In particular, for $2K_X + (n+1)L$ we need the generation of jets of order $(n+1)(2n+s_j-1)+1$. Theorem (7.4 b) yields the desired condition. □

We now list a few immediate consequences of Theorem 7.4, in connection with some classical questions of algebraic geometry.

(7.8) Corollary. *Let X be a projective n-fold of general type with K_X ample. Then mK_X is very ample for $m \ge m_0 = \binom{3n+1}{n} + 4$.*

(7.9) Corollary. *Let X be a Fano n-fold, that is, a n-fold such that $-K_X$ is ample. Then $-mK_X$ is very ample for $m \ge m_0 = \binom{3n+1}{n}$.*

Proof. Corollaries 7.8, 7.9 follow easily from Theorem 7.4 a) applied to $L = \pm K_X$. Hence we get pluricanonical embeddings $\Phi : X \to \mathbb{P}^N$ such that $\Phi^*\mathcal{O}(1) = \pm m_0 K_X$. The image $Y = \Phi(X)$ has degree

$$\deg(Y) = \int_Y c_1(\mathcal{O}(1))^n = \int_X c_1(\pm m_0 K_X)^n = m_0^n |K_X^n|.$$

It can be easily reproved from this that there are only finitely many deformation types of Fano n-folds, as well as of n-folds of general type with K_X ample, corresponding to a given discriminant $|K_X^n|$ (from a theoretical viewpoint, this result is a consequence of Matsusaka's big theorem [Mat72] and [KoM72], but the bounds which can be obtained from it are probably extremely huge). In the Fano case, a fundamental result obtained indepently by Kollár-Miyaoka-Mori [KoMM92] and Campana [Cam92] shows that the discriminant K_X^n is in fact bounded by a constant C_n depending only on n. Therefore, one can find an explicit bound C_n' for the degree

of the embedding Φ, and it follows that there are only finitely many families of Fano manifolds in each dimension. □

In the case of surfaces, much more is known. We will content ourselves with a brief account of recent results. If X is a surface, the failure of an adjoint bundle $K_X + L$ to be globally generated or very ample is described in a very precise way by the following result of I. Reider [Rei88].

(7.10) Reider's Theorem. *Let X be a smooth projective surface and let L be a nef line bundle on X.*

a) *Assume that $L^2 \ge 5$ and let $x \in X$ be a given point. Then $K_X + L$ has a section which does not vanish at x, unless there is an effective divisor $D \subset X$ passing through x such that either*

$$L\cdot D = 0 \quad \text{and} \quad D^2 = -1; \qquad \text{or}$$
$$L\cdot D = 1 \quad \text{and} \quad D^2 = 0.$$

b) *Assume that $L^2 \ge 10$. Then any two points $x, y \in X$ (possibly infinitely near) are separated by sections of $K_X + L$, unless there is an effective divisor $D \subset X$ passing through x and y such that either*

$$L\cdot D = 0 \quad \text{and} \quad D^2 = -1 \text{ or } -2; \qquad \text{or}$$
$$L\cdot D = 1 \quad \text{and} \quad D^2 = 0 \text{ or } -1; \qquad \text{or}$$
$$L\cdot D = 2 \quad \text{and} \quad D^2 = 0.$$

□

(7.11) Corollary. *Let L be an ample line bundle on a smooth projective surface X. Then $K_X + 3L$ is globally generated and $K_X + 4L$ is very ample. If $L^2 \ge 2$ then $K_X + 2L$ is globally generated and $K_X + 3L$ is very ample.* □

The case of higher order jets can be treated similarly. The most general result in this direction has been obtained by Beltrametti and Sommese [BeS93].

(7.12) Theorem ([BeS93]). *Let X be a smooth projective surface and let L be a nef line bundle on X. Let p be a positive integer such that $L^2 > 4p$. Then for every 0-dimensional subscheme $Z \subset X$ of length $h^0(Z,\mathcal{O}_Z) \le p$ the restriction*

$$\rho_Z : H^0(X,\mathcal{O}_X(K_X+L)) \longrightarrow H^0(Z,\mathcal{O}_Z(K_X+L))$$

is surjective, unless there is an effective divisor $D \subset X$ intersecting the support $|Z|$ such that

$$L\cdot D - p \le D^2 < \frac{1}{2} L\cdot D.$$

□

Proof (Sketch). The proof the above theorems rests in an essential way on the construction of rank 2 vector bundles sitting in an exact sequence

$$0 \to \mathcal{O}_X \to E \to L \otimes \mathcal{I}_Z \to 0.$$

Arguing by induction on the length of Z, we may assume that Z is a 0-dimensional subscheme such that ρ_Z is not surjective, but such that $\rho_{Z'}$ is surjective for every proper subscheme $Z' \subset Z$. The existence of E is obtained by a classical construction of Serre (unfortunately, this construction only works in dimension 2). The numerical condition on L^2 in the hypotheses ensures that $c_1(E)^2 - 4c_2(E) > 0$, hence E is unstable in the sense of Bogomolov. The existence of the effective divisor D asserted in 7.10 or 7.12 follows. We refer to [Rei88], [BeS93] and [Laz97] for details. The reader will find in [FdB93] a proof of the Bogomolov inequality depending only on the Kawamata-Viehweg vanishing theorem. □

(7.13) Exercise. The goal of the exercise is to prove the following weaker form of Theorems 7.10 and 7.12, by a simple direct method based on Nadel's vanishing theorem:

> *Let L be a nef line bundle on a smooth projective surface X. Fix points $x_1, \dots, x_N$ and corresponding multiplicities $s_1, \dots, s_N$, and set $p = \sum (2+s_j)^2$. Then $H^0(X, K_X + L)$ generates simultaneously jets of order s_j at all points x_j provided that $L^2 > p$ and $L \cdot C > p$ for all curves C passing through one of the points x_j.*

a) Using the Riemann-Roch formula, show that the condition $L^2 > p$ implies the existence of a section of a large multiple mL vanishing at order $> m(2+s_j)$ at each of the points.

b) Construct a sequence of singular hermitian metrics on L with positive definite curvature, such that the weights φ_ν have algebraic singularities, $\nu(\varphi_\nu, x_j) \geq 2 + s_j$ at each point, and such that for some integer $m_1 > 0$ the multiplier ideal sheaves satisfy $\mathcal{I}(m_1\varphi_{\nu+1}) \supsetneq \mathcal{I}(m_1\varphi_\nu)$ if $V(\mathcal{I}(\varphi_\nu))$ is not 0-dimensional near some x_j.

Hint: a) starts the procedure. Fix $m_0 > 0$ such that $m_0 L - K_X$ is ample. Use Nadel's vanishing theorem to show that

$$H^q(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{I}(\lambda m \varphi_\nu)) = 0 \qquad \text{for all } q \geq 1,\ m \geq 0,\ \lambda \in [0,1].$$

Let D_ν be the effective $\mathbb{Q}$-divisor describing the 1-dimensional singularities of φ_ν. Then $\mathcal{I}(\lambda m \varphi_\nu) \subset \mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)$ and the quotient has 0-dimensional support, hence

$$H^q(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)) = 0 \qquad \text{for all } q \geq 1,\ m \geq 0,\ \lambda \in [0,1].$$

By Riemann-Roch again prove that

$$(\star) \quad h^0(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}/\mathcal{O}(-\lfloor \lambda m D_\nu \rfloor)) = \frac{m^2}{2}(2\lambda\, L \cdot D_\nu - \lambda^2 D_\nu^2) + O(m).$$

As the left hand side of $(\star)$ is increasing with λ, one must have $D_\nu^2 \leq L \cdot D_\nu$. If $V(\mathcal{I}(\varphi_\nu))$ is not 0-dimensional at x_j, then the coefficient of some component of D_ν passing through x_j is at least 1, hence

$$2\, L \cdot D_\nu - D_\nu^2 \geq L \cdot D_\nu \geq p + 1.$$

Show the existence of an integer $m_1 > 0$ independent of ν such that

$$h^0(X, \mathcal{O}((m+m_0)L) \otimes \mathcal{O}/\mathcal{O}(-\lfloor mD_\nu \rfloor)) > \sum_{1 \leq j \leq N} \binom{(m+m_0)(2+s_j)+2}{2}$$

for $m \geq m_1$, and infer the existence of a suitable section of $(m_1+m_0)L$ which is not in $H^0(X, \mathcal{O}((m_1+m_0)L - \lfloor m_1 D_\nu \rfloor))$. Use this section to construct $\varphi_{\nu+1}$ such that $\mathcal{I}(m_1\varphi_{\nu+1}) \supsetneq \mathcal{I}(m_1\varphi_\nu)$.

8. Holomorphic Morse Inequalities

Let X be a compact Kähler manifold, E a holomorphic vector bundle of rank r and L a line bundle over X. If L is equipped with a smooth metric of curvature form $\Theta(L)$, we define the q-index set of L to be the open subset

$$X(q,L) = \left\{ x \in X\ ;\ i\Theta(L)_x \text{ has } \begin{matrix} q & \text{negative eigenvalues} \\ n-q & \text{positive eigenvalues} \end{matrix} \right\} \tag{8.1}$$

for $0 \leq q \leq n$. Hence X admits a partition $X = \Delta \cup \bigcup_q X(q,L)$ where $\Delta = \{x \in X\,;\, \det(\Theta(L)_x) = 0\}$ is the degeneracy set. We also introduce

$$X(\leq q, L) = \bigcup_{0 \leq j \leq q} X(j,L). \tag{8.1'}$$

It is shown in [Dem85b] that the cohomology groups $H^q\big(X, E \otimes \mathcal{O}(kL)\big)$ satisfy the following asymptotic *weak Morse inequalities* as $k \to +\infty$

$$h^q\big(X, E \otimes \mathcal{O}(kL)\big) \leq r\frac{k^n}{n!} \int_{X(q,L)} (-1)^q \Big(\frac{i}{2\pi}\Theta(L)\Big)^n + o(k^n). \tag{8.2}$$

A sharper form is given by the *strong Morse inequalities*

$$\sum_{0 \leq j \leq q} (-1)^{q-j} h^j\big(X, E \otimes \mathcal{O}(kL)\big) \leq r\frac{k^n}{n!} \int_{X(\leq q,L)} (-1)^q \Big(\frac{i}{2\pi}\Theta(L)\Big)^n + o(k^n). \tag{8.2'}$$

These inequalities are a useful complement to the Riemann-Roch formula when information is needed about individual cohomology groups, and not just about the Euler-Poincaré characteristic.

One difficulty in the application of these inequalities is that the curvature integral is in general quite uneasy to compute, since it is neither a topological nor an algebraic invariant. However, the Morse inequalities can be reformulated in a more algebraic setting in which only algebraic invariants are involved. We give here two such reformulations.

(8.3) Theorem. *Let $L = F - G$ be a holomorphic line bundle over a compact Kähler manifold X, where F and G are numerically effective line bundles. Then for every $q = 0, 1, \dots, n = \dim X$, there is an asymptotic strong Morse inequality*

$$\sum_{0\le j\le q}(-1)^{q-j}h^j(X,kL)\le\frac{k^n}{n!}\sum_{0\le j\le q}(-1)^{q-j}\binom{n}{j}F^{n-j}\cdot G^j+o(k^n).$$

Proof. By adding ε times a Kähler metric ω to the curvature forms of F and G, $\varepsilon>0$ one can write $\frac{i}{2\pi}\Theta(L)=\theta_\varepsilon(F)-\theta_\varepsilon(G)$ where $\theta_\varepsilon(F)=\frac{i}{2\pi}\Theta(F)+\varepsilon\omega$ and $\theta_\varepsilon(G)=\frac{i}{2\pi}\Theta(G)+\varepsilon\omega$ are positive definite. Let $\lambda_1\ge\ldots\ge\lambda_n>0$ be the eigenvalues of $\theta_\varepsilon(G)$ with respect to $\theta_\varepsilon(F)$. Then the eigenvalues of $\frac{i}{2\pi}\Theta(L)$ with respect to $\theta_\varepsilon(F)$ are the real numbers $1-\lambda_j$ and the set $X(\le q,L)$ is the set $\{\lambda_{q+1}<1\}$ of points $x\in X$ such that $\lambda_{q+1}(x)<1$. The strong Morse inequalities yield

$$\sum_{0\le j\le q}(-1)^{q-j}h^j(X,kL)\le\frac{k^n}{n!}\int_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\le j\le n}(1-\lambda_j)\theta_\varepsilon(F)^n+o(k^n).$$

On the other hand we have

$$\binom{n}{j}\theta_\varepsilon(F)^{n-j}\wedge\theta_\varepsilon(G)^j=\sigma_n^j(\lambda)\,\theta_\varepsilon(F)^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in $\lambda_1,\ldots,\lambda_n$, hence

$$\sum_{0\le j\le q}(-1)^{q-j}\binom{n}{j}F^{n-j}\cdot G^j=\lim_{\varepsilon\to0}\int_X\sum_{0\le j\le q}(-1)^{q-j}\sigma_n^j(\lambda)\,\theta_\varepsilon(F)^n.$$

Thus, to prove the Lemma, we only have to check that

$$\sum_{0\le j\le n}(-1)^{q-j}\sigma_n^j(\lambda)-\mathbf{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\le j\le n}(1-\lambda_j)\ge0$$

for all $\lambda_1\ge\ldots\ge\lambda_n\ge0$, where $\mathbf{1}_{\{\ldots\}}$ denotes the characteristic function of a set. This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda)=\sigma_{n-1}^j(\lambda)+\sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

In the case $q=1$, we get an especially interesting lower bound (this bound has been observed and used by S. Trapani [Tra95] in a similar context).

(8.4) Consequence. $h^0(X,kL)-h^1(X,kL)\ge\frac{k^n}{n!}(F^n-nF^{n-1}\cdot G)-o(k^n)$. *Therefore some multiple kL has a section as soon as $F^n-nF^{n-1}\cdot G>0$.*

(8.5) Remark. The weaker inequality

$$h^0(X,kL)\ge\frac{k^n}{n!}(F^n-nF^{n-1}\cdot G)-o(k^n)$$

is easy to prove if X is projective algebraic. Indeed, by adding a small ample $\mathbb{Q}$-divisor to F and G, we may assume that F, G are ample. Let m_0G be very ample and let k' be the smallest integer $\ge k/m_0$. Then $h^0(X,kL)\ge h^0(X,kF-k'm_0G)$. We select k' smooth members G_j, $1\le j\le k'$ in the linear system $|m_0G|$ and use the exact sequence

$$0\to H^0(X,kF-\sum G_j)\to H^0(X,kF)\to\bigoplus H^0(G_j,kF_{|G_j}).$$

Kodaira's vanishing theorem yields $H^q(X,kF)=0$ and $H^q(G_j,kF_{|G_j})=0$ for $q\ge1$ and $k\ge k_0$. By the exact sequence combined with Riemann-Roch, we get

$$\begin{aligned}h^0(X,kL)&\ge h^0(X,kF-\sum G_j)\\&\ge\frac{k^n}{n!}F^n-O(k^{n-1})-\sum\Big(\frac{k^{n-1}}{(n-1)!}F^{n-1}\cdot G_j-O(k^{n-2})\Big)\\&\ge\frac{k^n}{n!}\Big(F^n-n\frac{k'm_0}{k}F^{n-1}\cdot G\Big)-O(k^{n-1})\\&\ge\frac{k^n}{n!}\Big(F^n-n\,F^{n-1}\cdot G\Big)-O(k^{n-1}).\end{aligned}$$

(This simple proof is due to F. Catanese.) □

(8.6) Corollary. *Suppose that F and G are nef and that F is big. Some multiple of $mF-G$ has a section as soon as*

$$m>n\,\frac{F^{n-1}\cdot G}{F^n}.$$

In the last condition, the factor n is sharp: this is easily seen by taking $X=\mathbb{P}_1^n$ and $F=\mathcal{O}(a,\ldots,a)$ and $G=\mathcal{O}(b_1,\ldots,b_n)$ over $\mathbb{P}_1^n$; the condition of the Corollary is then $m>\sum b_j/a$, whereas $k(mF-G)$ has a section if and only if $m\ge\sup b_j/a$; this shows that we cannot replace n by $n(1-\varepsilon)$. □

9. Effective Version of Matsusaka's Big Theorem

An important problem of algebraic geometry is to find effective bounds m_0 such that multiples mL of an ample line bundle become very ample for $m\ge m_0$. From a theoretical point of view, this problem has been solved by Matsusaka [Mat72] and Kollár-Matsusaka [KoM83]. Their result is that there is a bound $m_0=m_0(n,L^n,L^{n-1}\cdot K_X)$ depending only on the dimension and on the first two coefficients L^n and $L^{n-1}\cdot K_X$ in the Hilbert polynomial of L. Unfortunately, the original proof does not tell much on the actual dependence of m_0 in terms of these coefficients. The goal of this section is to find effective bounds for such an integer m_0, along the lines of [Siu93]. However, one of the technical lemmas used in [Siu93] to deal with dualizing sheaves can be sharpened. Using this sharpening of the lemma, Siu's bound will be here substantially improved. We first start with the simpler problem of obtaining merely a nontrivial section in mL. The idea, more generally, is to obtain a criterion for the ampleness of $mL-B$ when B is nef. In this way, one is able to subtract from mL any multiple of K_X which happens to get added by the application of Nadel's vanishing theorem (for this, replace B by B plus a multiple of $K_X+(n+1)L$).

(9.1) Proposition. *Let L be an ample line bundle over a projective n-fold X and let B be a nef line bundle over X. Then K_X+mL-B has a nonzero section for some integer m such that*

$$m \le n\,\frac{L^{n-1}\cdot B}{L^n} + n + 1.$$

Proof. Let m_0 be the smallest integer $> n\,\frac{L^{n-1}\cdot B}{L^n}$. Then $m_0 L - B$ can be equipped with a singular hermitian metric of positive definite curvature. Let φ be the weight of this metric. By Nadel's vanishing theorem, we have

$$H^q(X, \mathcal{O}(K_X + mL - B)\otimes \mathcal{I}(\varphi)) = 0 \qquad \text{for } q \ge 1,$$

thus $P(m) = h^0(X, \mathcal{O}(K_X + mL - B)\otimes \mathcal{I}(\varphi))$ is a polynomial for $m \ge m_0$. Since P is a polynomial of degree n and is not identically zero, there must be an integer $m \in [m_0, m_0 + n]$ which is not a root. Hence there is a nontrivial section in

$$H^0(X, \mathcal{O}(K_X + mL - B)) \supset H^0(X, \mathcal{O}(K_X + mL - B)\otimes \mathcal{I}(\varphi))$$

for some $m \in [m_0, m_0 + n]$, as desired. □

(9.2) Corollary. *If L is ample and B is nef, then $mL - B$ has a nonzero section for some integer*

$$m \le n\Big(\frac{L^{n-1}\cdot B + L^{n-1}\cdot K_X}{L^n} + n + 1\Big).$$

Proof. By Fujita's result 8.3 a), $K_X + (n+1)L$ is nef. We can thus replace B by $B + K_X + (n+1)L$ in the result of Prop. 9.1. Corollary 9.2 follows. □

(9.3) Remark. We do not know if the above Corollary is sharp, but it is certainly not far from being so. Indeed, for $B = 0$, the initial constant n cannot be replaced by anything smaller than $n/2$: take X to be a product of curves C_j of large genus g_j and $B = 0$; our bound for $L = \mathcal{O}(a_1[p_1])\otimes\ldots\otimes\mathcal{O}(a_n[p_n])$ to have $|mL| \ne \emptyset$ becomes $m \le \sum(2g_j - 2)/a_j + n(n+1)$, which fails to be sharp only by a factor 2 when $a_1 = \ldots = a_n = 1$ and $g_1 \gg g_2 \gg \ldots \gg g_n \to +\infty$. On the other hand, the additive constant $n + 1$ is already best possible when $B = 0$ and $X = \mathbb{P}^n$. □

So far, the method is not really sensitive to singularities (the Morse inequalities are indeed still true in the singular case as is easily seen by using desingularizations of the ambient variety). The same is true with Nadel's vanishing theorem, provided that K_X is replaced by the L^2 dualizing sheaf ω_X (according to the notation introduced in Remark 5.17, $\omega_X = K_X(0)$ is the sheaf of holomorphic n-forms u on $X_{\rm reg}$ such that $i^{n^2}u\wedge\overline{u}$ is integrable in a neighborhood of the singular set). Then Prop. 9.1 can be generalized as

(9.4) Proposition. *Let L be an ample line bundle over a projective n-fold X and let B be a nef line bundle over X. For every p-dimensional (reduced) algebraic subvariety Y of X, there is an integer*

$$m \le p\,\frac{L^{p-1}\cdot B\cdot Y}{L^p\cdot Y} + p + 1$$

such that the sheaf $\omega_Y\otimes\mathcal{O}_Y(mL - B)$ has a nonzero section. □

To proceed further, we need the following useful "upper estimate" about L^2 dualizing sheaves (this is one of the crucial steps in Siu's approach: unfortunately, it has the effect of producing rather large final bounds when the dimension increases).

(9.5) Proposition. *Let H be a very ample line bundle on a projective algebraic manifold X, and let $Y \subset X$ be a p-dimensional irreducible algebraic subvariety. If $\delta = H^p\cdot Y$ is the degree of Y with respect to H, the sheaf*

$$\mathcal{H}om\big(\omega_Y, \mathcal{O}_Y((\delta - p - 2)H)\big)$$

has a nontrivial section.

Observe that if Y is a smooth hypersurface of degree δ in $(X, H) = (\mathbb{P}^{p+1}, \mathcal{O}(1))$, then $\omega_Y = \mathcal{O}_Y(\delta - p - 2)$ and the estimate is optimal. On the other hand, if Y is a smooth complete intersection of multidegree $(\delta_1, \ldots, \delta_r)$ in $\mathbb{P}^{p+r}$, then $\delta = \delta_1\ldots\delta_r$ whilst $\omega_Y = \mathcal{O}_Y(\delta_1 + \ldots + \delta_r - p - r - 1)$; in this case, Prop. (9.5) is thus very far from being sharp.

Proof. Let $X \subset \mathbb{P}^N$ be the embedding given by H, so that $H = \mathcal{O}_X(1)$. There is a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{p+1}$ whose restriction $\pi : Y \to \mathbb{P}^{p+1}$ to Y is a finite and regular birational map of Y onto an algebraic hypersurface Y' of degree δ in $\mathbb{P}^{p+1}$. Let $s \in H^0(\mathbb{P}^{p+1}, \mathcal{O}(\delta))$ be the polynomial of degree δ defining Y'. We claim that for any small Stein open set $W \subset \mathbb{P}^{p+1}$ and any L^2 holomorphic p-form u on $Y'\cap W$, there is a L^2 holomorphic $(p+1)$-form $\widetilde{u}$ on W with values in $\mathcal{O}(\delta)$ such that $\widetilde{u}_{|Y'\cap W} = u\wedge ds$. In fact, this is precisely the conclusion of the Ohsawa-Takegoshi extension theorem [OT87], [Ohs88] (see also [Man93] for a more general version); one can also invoke more standard local algebra arguments (see Hartshorne [Har77], Th. III-7.11). As $K_{\mathbb{P}^{p+1}} = \mathcal{O}(-p-2)$, the form $\widetilde{u}$ can be seen as a section of $\mathcal{O}(\delta - p - 2)$ on W, thus the sheaf morphism $u \mapsto u\wedge ds$ extends into a global section of $\mathcal{H}om\big(\omega_{Y'}, \mathcal{O}_{Y'}(\delta - p - 2)\big)$. The pull-back by π^* yields a section of $\mathcal{H}om\big(\pi^*\omega_{Y'}, \mathcal{O}_Y((\delta - p - 2)H)\big)$. Since π is finite and generically $1:1$, it is easy to see that $\pi^*\omega_{Y'} = \omega_Y$. The Proposition follows. □

By an appropriate induction process based on the above results, we can now improve Siu's effective version of the Big Matsusaka Theorem [Siu93]. Our version depends on a constant λ_n such that $m(K_X + (n+2)L) + G$ is very ample for $m \ge \lambda_n$ and every nef line bundle G. Corollary (8.5) shows that $\lambda_n \le \binom{3n+1}{n} - 2n$, and a similar argument involving the recent results of Angehrn-Siu [AS94] implies $\lambda_n \le n^3 - n^2 - n - 1$ for $n \ge 2$. Of course, it is expected that $\lambda_n = 1$ in view of the Fujita conjecture.

(9.6) Effective version of the Big Matsusaka Theorem. *Let L and B be nef line bundles on a projective n-fold X. Assume that L is ample and let H be the very ample line bundle $H = \lambda_n(K_X + (n+2)L)$. Then $mL - B$ is very ample for*

$$m \ge (2n)^{(3^{n-1}-1)/2}\,\frac{(L^{n-1}\cdot(B+H))^{(3^{n-1}+1)/2}(L^{n-1}\cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}}.$$

In particular mL is very ample for

$$m \geq C_n\ (L^n)^{3^{n-2}} \left(n+2+\frac{L^{n-1}\cdot K_X}{L^n}\right)^{3^{n-2}(n/2+3/4)+1/4}$$

with $C_n = (2n)^{(3^{n-1}-1)/2}(\lambda_n)^{3^{n-2}(n/2+3/4)+1/4}$.

Proof. We use Prop. (9.4) and Prop. (9.5) to construct inductively a sequence of (non necessarily irreducible) algebraic subvarieties $X = Y_n \supset Y_{n-1} \supset \ldots \supset Y_2 \supset Y_1$ such that $Y_p = \bigcup_j Y_{p,j}$ is p-dimensional, and Y_{p-1} is obtained for each $p \geq 2$ as the union of zero sets of sections

$$\sigma_{p,j} \in H^0(Y_{p,j}, \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B))$$

with suitable integers $m_{p,j} \geq 1$. We proceed by induction on decreasing values of the dimension p, and find inductively upper bounds m_p for the integers $m_{p,j}$.

By Cor. (9.2), an integer m_n for $m_nL - B$ to have a section σ_n can be found with

$$m_n \leq n\,\frac{L^{n-1}\cdot(B+K_X+(n+1)L)}{L^n} \leq n\,\frac{L^{n-1}\cdot(B+H)}{L^n}.$$

Now suppose that the sections $\sigma_n, \ldots, \sigma_{p+1,j}$ have been constructed. Then we get inductively a p-cycle $\widetilde{Y}_p = \sum \mu_{p,j} Y_{p,j}$ defined by $\widetilde{Y}_p =$ sum of zero divisors of sections $\sigma_{p+1,j}$ in $\widetilde{Y}_{p+1,j}$, where the mutiplicity $\mu_{p,j}$ on $Y_{p,j} \subset Y_{p+1,k}$ is obtained by multiplying the corresponding multiplicity $\mu_{p+1,k}$ with the vanishing order of $\sigma_{p+1,k}$ along $Y_{p,j}$. As cohomology classes, we find

$$\widetilde{Y}_p \equiv \sum (m_{p+1,k}L - B)\cdot(\mu_{p+1,k}Y_{p+1,k}) \leq m_{p+1}L\cdot\widetilde{Y}_{p+1}.$$

Inductively, we thus have the numerical inequality

$$\widetilde{Y}_p \leq m_{p+1}\ldots m_n L^{n-p}.$$

Now, for each component $Y_{p,j}$, Prop. (9.4) shows that there exists a section of $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ for some integer

$$m_{p,j} \leq p\frac{L^{p-1}\cdot B\cdot Y_{p,j}}{L^p\cdot Y_{p,j}} + p + 1 \leq p m_{p+1}\ldots m_n\, L^{n-1}\cdot B + p + 1.$$

Here, we have used the obvious lower bound $L^{p-1}\cdot Y_{p,j} \geq 1$ (this is of course a rather weak point in the argument). The degree of $Y_{p,j}$ with respect to H admits the upper bound

$$\delta_{p,j} := H^p\cdot Y_{p,j} \leq m_{p+1}\ldots m_n H^p\cdot L^{n-p}.$$

We use the Hovanski-Teissier concavity inequality (10.2 b)

$$(L^{n-p}\cdot H^p)^{\frac{1}{p}}(L^n)^{1-\frac{1}{p}} \leq L^{n-1}\cdot H$$

to express our bounds in terms of the intersection numbers L^n and $L^{n-1}\cdot H$ only. We then get

$$\delta_{p,j} \leq m_{p+1}\ldots m_n\,\frac{(L^{n-1}\cdot H)^p}{(L^n)^{p-1}}.$$

By Prop. (9.5), there is a nontrivial section in

$$\mathcal{H}om\big(\omega_{Y_{p,j}}, \mathcal{O}_{Y_{p,j}}((\delta_{p,j}-p-2)H)\big).$$

Combining this section with the section in $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ already constructed, we get a section of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B + (\delta_{p,j}-p-2)H)$ on $Y_{p,j}$. Since we do not want H to appear at this point, we replace B with $B + (\delta_{p,j}-p-2)H$ and thus get a section $\sigma_{p,j}$ of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ with some integer $m_{p,j}$ such that

$$\begin{aligned} m_{p,j} &\leq p m_{p+1}\ldots m_n\, L^{n-1}\cdot(B + (\delta_{p,j}-p-2)H) + p + 1 \\ &\leq p\, m_{p+1}\ldots m_n\, \delta_{p,j}\, L^{n-1}\cdot(B+H) \\ &\leq p\,(m_{p+1}\ldots m_n)^2\frac{(L^{n-1}\cdot H)^p}{(L^n)^{p-1}}\, L^{n-1}\cdot(B+H). \end{aligned}$$

Therefore, by putting $M = n\,L^{n-1}\cdot(B+H)$, we get the recursion relation

$$m_p \leq M\,\frac{(L^{n-1}\cdot H)^p}{(L^n)^{p-1}}\,(m_{p+1}\ldots m_n)^2 \qquad \text{for } 2 \leq p \leq n-1,$$

with initial value $m_n \leq M/L^n$. If we let $(\overline{m}_p)$ be the sequence obtained by the same recursion formula with equalities instead of inequalities, we get $m_p \leq \overline{m}_p$ with $\overline{m}_{n-1} = M^3(L^{n-1}\cdot H)^{n-1}/(L^n)^n$ and

$$\overline{m}_p = \frac{L^n}{L^{n-1}\cdot H}\,\overline{m}_{p+1}^2\overline{m}_{p+1}$$

for $2 \leq p \leq n-2$. We then find inductively

$$m_p \leq \overline{m}_p = M^{3^{n-p}}\frac{(L^{n-1}\cdot H)^{3^{n-p-1}(n-3/2)+1/2}}{(L^n)^{3^{n-p-1}(n-1/2)+1/2}}.$$

We next show that $m_0L - B$ is nef for

$$m_0 = \max\big(m_2\,,\, m_3, \ldots, m_n\,,\, m_2\ldots m_n\, L^{n-1}\cdot B\big).$$

In fact, let $C \subset X$ be an arbitrary irreducible curve. Either $C = Y_{1,j}$ for some j or there exists an integer $p = 2, \ldots, n$ such that C is contained in Y_p but not in Y_{p-1}. If $C \subset Y_{p,j} \setminus Y_{p-1}$, then $\sigma_{p,j}$ does not vanish identically on C. Hence $(m_{p,j}L - B)_{|C}$ has nonnegative degree and

$$(m_0L - B)\cdot C \geq (m_{p,j}L - B)\cdot C \geq 0.$$

On the other hand, if $C = Y_{1,j}$, then

$$(m_0L - B)\cdot C \geq m_0 - B\cdot\widetilde{Y}_1 \geq m_0 - m_2\ldots m_n\, L^{n-1}\cdot B \geq 0.$$

By the definition of λ_n (and by Cor. (8.5) showing that such a constant exists), $H + G$ is very ample for every nef line bundle G, in particular $H + m_0L - B$ is very ample. We thus replace again B with $B + H$. This has the effect of replacing M with $M = n\,(L^{n-1}\cdot(B+2H))$ and m_0 with

$$m_0 = \max\big(m_n\,,\ m_{n-1}, \ldots, m_2\,,\ m_2 \ldots m_n\, L^{n-1}\cdot(B+H)\big).$$

The last term is the largest one, and from the estimate on $\overline{m}_p$, we get

$$m_0 \le M^{(3^{n-1}-1)/2}\frac{(L^{n-1}\cdot H)^{(3^{n-2}-1)(n-3/2)/2+(n-2)/2}(L^{n-1}\cdot(B+H))}{(L^n)^{(3^{n-2}-1)(n-1/2)/2+(n-2)/2+1}}$$
$$\le (2n)^{(3^{n-1}-1)/2}\frac{(L^{n-1}\cdot(B+H))^{(3^{n-1}+1)/2}(L^{n-1}\cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}}$$

□

(9.7) Remark. In the surface case $n = 2$, one can take $\lambda_n = 1$ and our bound yields mL very ample for

$$m \ge 4\,\frac{(L\cdot(K_X+4L))^2}{L^2}.$$

If one looks more carefully at the proof, the initial constant 4 can be replaced by 2. In fact, it has been shown recently by Fernández del Busto that mL is very ample for

$$m > \frac{1}{2}\left[\frac{(L\cdot(K_X+4L)+1)^2}{L^2}+3\right],$$

and an example of G. Xiao shows that this bound is essentially optimal (see [FdB94]).

10. Positivity concepts for vector bundles

In the course of the proof of Skoda's L^2 estimates, we will have to deal with dual bundles and exact sequences of hermitian vector bundles. The following fundamental differential geometric lemma will be needed.

(10.1) Lemma. *Let E be a hermitian holomorphic vector bundle of rank r on a complex n-dimensional manifold X. Then the Chern connections of E and E^* are related by $\Theta(E^*) = -{}^t\Theta(E)$ where ^{t} denotes transposition. In other words, the associated hermitian forms $\widetilde{\Theta}(E)$ and $\widetilde{\Theta}(E^*)$ are related by*

$$\widetilde{\Theta}(E)(\tau,\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \qquad \tau = \sum_{j,\lambda}\tau_{j,\lambda}\frac{\partial}{\partial z_j}\otimes e_\lambda,$$
$$\widetilde{\Theta}(E^*)(\tau,\tau) = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\tau^*_{j\lambda}\overline{\tau}^*_{k\mu}, \qquad \tau^* = \sum_{j,\lambda}\tau^*_{j,\lambda}\frac{\partial}{\partial z_j}\otimes e^*_\lambda.$$

In particular $E >_{\mathrm{Grif}} 0$ if and only if $E^* <_{\mathrm{Grif}} 0$.

Notice that the corresponding duality statement for Nakano positivity is wrong (because of the twist of indices, which is fortunately irrelevant in the case of decomposable tensors).

Proof. The Chern connections of E and E^* are related by the Leibnitz rule

$$d(\sigma\wedge s) = (D_{E^*}\sigma)\wedge s + (-1)^{\deg\sigma}\sigma\wedge D_E s$$

whenever s, σ are forms with values in E, E^* respectively, and $\sigma\wedge s$ is computed using the pairing $E^*\otimes E\to\mathbb{C}$. If we differentiate a second time, this yields the identity

$$0 = (D^2_{E^*}\sigma)\wedge s + \sigma\wedge D^2_E s,$$

which is equivalent to the formula $\Theta(E^*) = -{}^t\Theta(E)$. All other assertions follow. □

(10.2) Lemma. *Let*

$$0\longrightarrow S\xrightarrow{\ j\ } E\xrightarrow{\ g\ } Q\longrightarrow 0$$

be an exact sequence of holomorphic vector bundles. Assume that E is equipped with a smooth hermitian metric, and that S and Q are endowed with the metrics (restriction-metric and quotient-metric) induced by that of E. Then

$$j^*\oplus g : E\to S\oplus Q, \qquad j\oplus g^* : S\oplus Q\to E \tag{10.3}$$

are C^∞ isomorphisms of bundles, which are inverse of each other. In the C^∞-splitting $E\simeq S\oplus Q$, the Chern connection of E admits a matrix decomposition

$$D_E = \begin{pmatrix} D_S & -\beta^* \\ \beta & D_Q\end{pmatrix} \tag{10.4}$$

in terms of the Chern connections of S and Q, where

$$\beta\in C^\infty\big(X,\Lambda^{1,0}T^*_X\otimes\mathrm{Hom}(S,Q)\big), \qquad \beta^*\in C^\infty\big(X,\Lambda^{0,1}T^*_X\otimes\mathrm{Hom}(Q,S)\big).$$

The form β is called the second fundamental form associated with the exact sequence. It is uniquely defined by each of the two formulas

$$D'_{\mathrm{Hom}(S,E)}j = g^*\circ\beta, \qquad j\circ\beta^* = -D''_{\mathrm{Hom}(Q,E)}g^*. \tag{10.5}$$

We have $D'_{\mathrm{Hom}(S,Q)}\beta = 0$, $D''_{\mathrm{Hom}(Q,S)}\beta^* = 0$, and the curvature form of E splits as

$$\Theta(E) = \begin{pmatrix}\Theta(S)-\beta^*\wedge\beta & -D'_{\mathrm{Hom}(Q,S)}\beta^* \\ D''_{\mathrm{Hom}(S,Q)}\beta & \Theta(Q)-\beta\wedge\beta^*\end{pmatrix}, \tag{10.6}$$

and the curvature forms of S and Q can be expressed as

$$\Theta(S) = \Theta(E)_{|S}+\beta^*\wedge\beta, \qquad \Theta(Q) = \Theta(E)_{|Q}+\beta\wedge\beta^*, \tag{10.7}$$

where $\Theta(E)_{|S}$, $\Theta(E)_{|Q}$ stand for $j^*\circ\Theta(E)\circ j$ and $g\circ\Theta(E)\circ g^*$.

Proof. Because of the uniqueness property of Chern connections, it is easy to see that we have a Leibnitz formula

$$D_F(f\wedge u) = (D_{\mathrm{Hom}(E,F)}f)\wedge u + (-1)^{\deg f}f\wedge D_E u$$

whenever u, f are forms with values in hermitian vector bundles E and $\mathrm{Hom}(E,F)$ (where $\mathrm{Hom}(E,F) = E^*\otimes F$ is equipped with the tensor product metric and $f\wedge u$ incorporates the evaluation mapping $\mathrm{Hom}(E,F)\otimes E\to F$). In our case, given a

form u with values in E, we write $u = ju_S + g^*u_Q$ where $u_S = j^*u$ and $u_Q = gu$ are the projections of u on S and Q. We then get

$$\begin{aligned} D_E u &= D_E(ju_S + g^*u_Q) \\ &= (D_{\mathrm{Hom}(S,E)}j) \wedge u_S + j \cdot D_S u_S + (D_{\mathrm{Hom}(Q,E)}g^*) \wedge u_Q + g^* \cdot D_Q u_Q. \end{aligned}$$

Since j is holomorphic as well as $j^* \circ j = \mathrm{Id}_S$, we find $D''_{Hom(S,E)}j = 0$ and

$$D''_{Hom(S,S)}\,\mathrm{Id}_S = 0 = D''_{\mathrm{Hom}(E,S)}j^* \circ j.$$

By taking the adjoint, we see that $j^* \circ D'_{\mathrm{Hom}(S,E)}j = 0$, hence $D'_{\mathrm{Hom}(S,E)}j$ takes values in g^*Q and we thus have a unique form β as in the Lemma such that $D'_{\mathrm{Hom}(S,E)}j = g^* \circ \beta$. Similarly, g and $g \circ g^* = \mathrm{Id}_Q$ are holomorphic, thus

$$D''_{Hom(Q,Q)}\,\mathrm{Id}_Q = 0 = g \circ D''_{\mathrm{Hom}(Q,E)}g^*$$

and there is a form $\gamma \in C^\infty\big(X, \Lambda^{0,1}T^*_X \otimes \mathrm{Hom}(Q,S)\big)$ such that $D''_{\mathrm{Hom}(Q,E)}g^* = j \circ \gamma$. By adjunction, we get $D'_{\mathrm{Hom}(E,Q)}g = \gamma^* \circ j^*$ and $D''_{Hom(E,Q)}g = 0$ implies $D'_{\mathrm{Hom}(Q,E)}g^* = 0$. If we differentiate $g \circ j = 0$ we then get

$$0 = D'_{\mathrm{Hom}(E,Q)}g \circ j + g \circ D'_{\mathrm{Hom}(S,E)}j = \gamma^* \circ j^* \circ j + g \circ g^* \circ \beta = \gamma^* + \beta,$$

thus $\gamma = -\beta^*$ and $D''_{\mathrm{Hom}(Q,E)}g^* = -j \circ \beta^*$. Combining all this, we get

$$\begin{aligned} D_E u &= g^*\beta \wedge u_S + j \cdot D_S u_S - j\beta^* \wedge u_Q + g^* \cdot D_Q u_Q \\ &= j\big(D_S u_S - \beta^* \wedge u_Q\big) + g^*\big(\beta \wedge u_S + D_Q u_Q\big), \end{aligned}$$

and the asserted matrix decomposition formula follows. By squaring the matrix, we get

$$D_E^2 = \begin{pmatrix} D_S^2 - \beta^* \wedge \beta & -D_S \circ \beta^* - \beta^* \circ D_Q \\ D_Q \circ \beta + \beta \circ D_S & D_Q^2 - \beta \wedge \beta^* \end{pmatrix}.$$

As $D_Q \circ \beta + \beta \circ D_S = D_{\mathrm{Hom}(S,Q)}\beta$ and $D_S \circ \beta^* + \beta^* \circ D_Q = D_{\mathrm{Hom}(Q,S)}\beta^*$ by the Leibnitz rule, the curvature formulas follow (observe, since the Chern curvature form is of type $(1,1)$, that we must have $D'_{\mathrm{Hom}(S,Q)}\beta = 0$, $D''_{\mathrm{Hom}(Q,S)}\beta^* = 0$). □

(10.8) Corollary. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then*

a) $E \geq_{\mathrm{Grif}} 0 \implies Q \geq_{\mathrm{Grif}} 0$,

b) $E \leq_{\mathrm{Grif}} 0 \implies S \leq_{\mathrm{Grif}} 0$,

c) $E \leq_{\mathrm{Nak}} 0 \implies S \leq_{\mathrm{Nak}} 0$,

and analogous implications hold true for strict positivity.

Proof. If β is written $\sum dz_j \otimes \beta_j$, $\beta_j \in \mathrm{Hom}(S,Q)$, then formulas (10.7) yield

$$\begin{aligned} \mathrm{i}\Theta(S) &= \mathrm{i}\Theta(E)_{|S} - \sum dz_j \wedge d\bar{z}_k \otimes \beta_k^*\beta_j, \\ \mathrm{i}\Theta(Q) &= \mathrm{i}\Theta(E)_{|Q} + \sum dz_j \wedge d\bar{z}_k \otimes \beta_j\beta_k^*. \end{aligned}$$

Since $\beta \cdot (\xi \otimes s) = \sum \xi_j \beta_j \cdot s$ and $\beta^* \cdot (\xi \otimes s) = \sum \overline{\xi}_k \beta_k^* \cdot s$ we get

$$\widetilde{\Theta}(S)(\xi \otimes s, \xi' \otimes s') = \widetilde{\Theta}(E)(\xi \otimes s, \xi' \otimes s') - \sum_{j,k} \xi_j \overline{\xi}'_k \langle \beta_j \cdot s, \beta_k \cdot s' \rangle,$$

$$\widetilde{\Theta}(S)(u,u) = \widetilde{\Theta}(E)(u,u) - |\beta \cdot u|^2,$$

$$\widetilde{\Theta}(Q)(\xi \otimes s, \xi' \otimes s') = \widetilde{\Theta}(E)(\xi \otimes s, \xi' \otimes s') + \sum_{j,k} \xi_j \overline{\xi}'_k \langle \beta_k^* \cdot s, \beta_j^* \cdot s' \rangle,$$

$$\widetilde{\Theta}(Q)(\xi \otimes s, \xi \otimes s) = \widetilde{\Theta}(E)(\xi \otimes s, \xi \otimes s) = |\beta^* \cdot (\xi \otimes s)|^2. \qquad \square$$

Next, we need positivity properties which somehow interpolate between Griffiths and Nakano positivity. This leads to the concept of m-tensor positivity.

(10.9) Definition. *Let T and E be complex vector spaces of dimensions n, r respectively, and let Θ be a hermitian form on $T \otimes E$.*

a) *A tensor $u \in T \otimes E$ is said to be of rank m if m is the smallest ≥ 0 integer such that u can be written*

$$u = \sum_{j=1}^{m} \xi_j \otimes s_j, \qquad \xi_j \in T,\ s_j \in E.$$

b) *Θ is said to be m-tensor positive (resp. m-tensor semi-positive) if $\Theta(u,u) > 0$ (resp. $\Theta(u,u) \geq 0$) for every tensor $u \in T \otimes E$ of rank $\leq m$, $u \neq 0$. In this case, we write*

$$\Theta >_m 0 \quad (\text{resp. } \Theta \geq_m 0).$$

We say that a hermitian vector bundle E is m-tensor positive if $\widetilde{\Theta}(E) >_m 0$. Griffiths positivity corresponds to $m = 1$ and Nakano positivity to $m \geq \min(n,r)$. Recall from (4.8) that we have

$$\langle [\mathrm{i}\Theta(E), \Lambda]u, u \rangle = \sum_{|S|=q-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, u_{jS,\lambda} \overline{u}_{kS,\mu}$$

for every (n,q)-form $u = \sum u_{K,\lambda}\, dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_K \otimes e_\lambda$ with values in E. Since $u_{jS,\lambda} = 0$ for $j \in S$, the rank of the tensor $(u_{jS,\lambda})_{j,\lambda} \in \mathbb{C}^n \otimes \mathbb{C}^r$ is in fact $\leq \min\{n-q+1, r\}$. We obtain therefore:

(10.10) Lemma. *Assume that $E \geq_m 0$ (resp. $E >_m 0$). Then the hermitian operator $[\mathrm{i}\Theta(E), \Lambda]$ is semipositive (resp. positive definite) on $\Lambda^{n,q}T^*X \otimes E$ for $q \geq 1$ and $m \geq \min\{n-q+1, r\}$.*

The Nakano vanishing theorem can then be improved as follows.

(10.11) Theorem. *Let X be a weakly pseudoconvex Kähler manifold of dimension n and let E a hermitian vector bundle of rank r such that $\widetilde{\Theta}(E) >_m 0$ over X. Then*

$$H^{n,q}(X,E)=0 \quad \text{for } q\geq 1 \text{ and } m\geq \min\{n-q+1,r\}.$$

We next study some important relations which exist between the various positivity concepts. Our starting point is the following result of [DSk79].

(10.12) Theorem. *For any hermitian vector bundle E,*

$$E>_{\mathrm{Grif}}0 \implies E\otimes\det E>_{\mathrm{Nak}}0.$$

To prove this result, we use the fact that

$$\Theta(\det E)=\mathrm{Tr}_E\,\Theta(E) \tag{10.13}$$

where $\mathrm{Tr}_E:\mathrm{Hom}(E,E)\to\mathbb{C}$ is the trace map, together with the identity

$$\Theta(E\otimes\det E)=\Theta(E)+\mathrm{Tr}_E(\Theta(E))\otimes\mathrm{Id}_E,$$

which is itself a consequence of (10.13) and of the standard formula

$$\Theta(E\otimes F)=\Theta(E)\otimes\mathrm{Id}_F+\mathrm{Id}_E\otimes\Theta(F).$$

In order to prove (10.13), for instance, we differentiate twice a wedge product, according to the formula

$$D_{\Lambda^p E}(s_1\wedge\cdots\wedge s_p)=\sum_{j=1}^{p}(-1)^{\deg s_1+\cdots+\deg s_{j-1}}s_1\wedge\cdots\wedge s_{j-1}\wedge D_E s_j\wedge\cdots\wedge s_p.$$

The corresponding hermitian forms on $T_X\otimes E$ are thus related by

$$\widetilde{\Theta}(E\otimes\det E)=\widetilde{\Theta}(E)+\mathrm{Tr}_E\,\widetilde{\Theta}(E)\otimes h,$$

where h denotes the hermitian metric on E and $\mathrm{Tr}_E\,\widetilde{\Theta}(E)$ is the hermitian form on T_X defined by

$$\mathrm{Tr}_E\,\widetilde{\Theta}(E)(\xi,\xi)=\sum_{1\leq\lambda\leq r}\widetilde{\Theta}(E)(\xi\otimes e_\lambda,\xi\otimes e_\lambda),\qquad \xi\in T_X,$$

for any orthonormal frame $(e_1,\ldots,e_r)$ of E. Theorem 10.12 is now a consequence of the following simple property of hermitian forms on a tensor product of complex vector spaces.

(10.14) Proposition. *Let T,E be complex vector spaces of respective dimensions n,r, and h a hermitian metric on E. Then for every hermitian form Θ on $T\otimes E$*

$$\Theta>_{\mathrm{Grif}}0 \implies \Theta+\mathrm{Tr}_E\,\Theta\otimes h>_{\mathrm{Nak}}0.$$

We first need a lemma analogous to Fourier inversion formula for discrete Fourier transforms.

(10.15) Lemma. *Let q be an integer ≥ 3, and x_λ, y_μ, $1\leq\lambda,\mu\leq r$, be complex numbers. Let σ describe the set U_q^r of r-tuples of q-th roots of unity and put*

$$x'_\sigma=\sum_{1\leq\lambda\leq r}x_\lambda\overline{\sigma}_\lambda,\qquad y'_\sigma=\sum_{1\leq\mu\leq r}y_\mu\overline{\sigma}_\mu,\qquad \sigma\in U_q^r.$$

Then for every pair (α,β), $1\leq\alpha,\beta\leq r$, the following identity holds:

$$q^{-r}\sum_{\sigma\in U_q^r}x'_\sigma\overline{y}'_\sigma\sigma_\alpha\overline{\sigma}_\beta=\begin{cases}x_\alpha\overline{y}_\beta & \text{if } \alpha\neq\beta,\\ \displaystyle\sum_{1\leq\mu\leq r}x_\mu\overline{y}_\mu & \text{if } \alpha=\beta.\end{cases}$$

Proof. The coefficient of $x_\lambda\overline{y}_\mu$ in the summation $q^{-r}\sum_{\sigma\in U_q^r}x'_\sigma\overline{y}'_\sigma\sigma_\alpha\overline{\sigma}_\beta$ is given by

$$q^{-r}\sum_{\sigma\in U_q^r}\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu.$$

This coefficient equals 1 when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are equal (in which case $\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu=1$ for any one of the q^r elements of U_q^r). Hence, it is sufficient to prove that

$$\sum_{\sigma\in U_q^r}\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu=0$$

when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are distinct.

If $\{\alpha,\mu\}\neq\{\beta,\lambda\}$, then one of the elements of one of the pairs does not belong to the other pair. As the four indices α,β,λ,μ play the same role, we may suppose for example that $\alpha\notin\{\beta,\lambda\}$. Let us apply to σ the substitution $\sigma\mapsto\tau$, where τ is defined by

$$\tau_\alpha=e^{2\pi i/q}\sigma_\alpha,\ \tau_\nu=\sigma_\nu \quad \text{for} \quad \nu\neq\alpha.$$

We get

$$\sum_\sigma\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu=\sum_\tau=\begin{cases}e^{2\pi i/q}\displaystyle\sum_\sigma & \text{if } \alpha\neq\mu,\\ e^{4\pi i/q}\displaystyle\sum_\sigma & \text{if } \alpha=\mu,\end{cases}$$

Since $q\geq 3$ by hypothesis, it follows that

$$\sum_\sigma\sigma_\alpha\overline{\sigma}_\beta\overline{\sigma}_\lambda\sigma_\mu=0.$$

Proof of Proposition 10.14. Let $(t_j)_{1\leq j\leq n}$ be a basis of T, $(e_\lambda)_{1\leq\lambda\leq r}$ an orthonormal basis of E and $\xi=\sum_j\xi_j t_j\in T$, $u=\sum_{j,\lambda}u_{j\lambda}\,t_j\otimes e_\lambda\in T\otimes E$. The coefficients $c_{jk\lambda\mu}$ of Θ with respect to the basis $t_j\otimes e_\lambda$ satisfy the symmetry relation $\overline{c}_{jk\lambda\mu}=c_{kj\mu\lambda}$, and we have the formulas

$$\Theta(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu},$$

$$\mathrm{Tr}_E\,\Theta(\xi,\xi) = \sum_{j,k,\lambda} c_{jk\lambda\lambda} \xi_j \overline{\xi}_k,$$

$$(\Theta + \mathrm{Tr}_E\,\Theta \otimes h)(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu} + c_{jk\lambda\lambda} u_{j\mu} \overline{u}_{k\mu}.$$

For every $\sigma \in U_q^r$ (cf. Lemma 10.15), put

$$u'_{j\sigma} = \sum_{1\le\lambda\le r} u_{j\lambda}\overline{\sigma}_\lambda \in \mathbb{C},$$

$$\widehat{u}_\sigma = \sum_j u'_{j\sigma} t_j \in T \quad , \quad \widehat{e}_\sigma = \sum_\lambda \sigma_\lambda e_\lambda \in E.$$

Lemma 10.15 implies

$$q^{-r} \sum_{\sigma\in U_q^r} \Theta(\widehat{u}_\sigma \otimes \widehat{e}_\sigma, \widehat{u}_\sigma \otimes \widehat{e}_\sigma) = q^{-r} \sum_{\sigma\in U_q^r} c_{jk\lambda\mu} u'_{j\sigma} \overline{u}'_{k\sigma} \sigma_\lambda \overline{\sigma}_\mu$$

$$= \sum_{j,k,\lambda\neq\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda} u_{j\mu} \overline{u}_{k\mu}.$$

The Griffiths positivity assumption shows that the left hand side is ≥ 0, hence

$$(\Theta + \mathrm{Tr}_E\,\Theta \otimes h)(u,u) \ge \sum_{j,k,\lambda} c_{jk\lambda\lambda} u_{j\lambda} \overline{u}_{k\lambda} \ge 0$$

with strict positivity if $\Theta >_{\mathrm{Grif}} 0$ and $u \neq 0$. □

We now relate Griffiths positivity to m-tensor positivity. The most useful result is the following

(10.16) Proposition. *Let T be a complex vector space and (E,h) a hermitian vector space of respective dimensions n,r with $r \ge 2$. Then for any hermitian form Θ on $T \otimes E$ and any integer $m \ge 1$*

$$\Theta >_{\mathrm{Grif}} 0 \quad \Longrightarrow \quad m\,\mathrm{Tr}_E\,\Theta \otimes h - \Theta >_m 0.$$

Proof. Let us distinguish two cases.

a) $m = 1$. Let $u \in T \otimes E$ be a tensor of rank 1. Then u can be written $u = \xi_1 \otimes e_1$ with $\xi_1 \in T$, $\xi_1 \neq 0$, and $e_1 \in E$, $|e_1| = 1$. Complete e_1 into an orthonormal basis $(e_1,\dots,e_r)$ of E. One gets immediately

$$(\mathrm{Tr}_E\,\Theta \otimes h)(u,u) = \mathrm{Tr}_E\,\Theta(\xi_1,\xi_1) = \sum_{1\le\lambda\le r} \Theta(\xi_1\otimes e_\lambda, \xi_1 \otimes e_\lambda)$$

$$> \Theta(\xi_1 \otimes e_1, \xi_1 \otimes e_1) = \Theta(u,u).$$

b) $m \ge 2$. Every tensor $u \in T \otimes E$ of rank $\le m$ can be written

$$u = \sum_{1\le\lambda\le q} \xi_\lambda \otimes e_\lambda \quad , \quad \xi_\lambda \in T,$$

with $q = \min(m,r)$ and $(e_\lambda)_{1\le\lambda\le r}$ an orthonormal basis of E. Let F be the vector subspace of E generated by $(e_1,\dots,e_q)$ and Θ_F the restriction of Θ to $T \otimes F$. The first part shows that

$$\Theta' := \mathrm{Tr}_F\,\Theta_F \otimes h - \Theta_F >_{\mathrm{Grif}} 0.$$

Proposition 10.14 applied to Θ' on $T \otimes F$ yields

$$\Theta' + \mathrm{Tr}_F\,\Theta' \otimes h = q\,\mathrm{Tr}_F\,\Theta_F \otimes h - \Theta_F >_q 0.$$

Since $u \in T \otimes F$ is of rank $\le q \le m$, we get (for $u \neq 0$)

$$\Theta(u,u) = \Theta_F(u,u) < q(\mathrm{Tr}_F\,\Theta_F \otimes h)(u,u)$$

$$= q \sum_{1\le j,\lambda\le q} \Theta(\xi_j\otimes e_\lambda, \xi_j \otimes e_\lambda) \le m\,\mathrm{Tr}_E\,\Theta \otimes h(u,u). \qquad \square$$

Proposition 10.16 is of course also true in the semi-positive case. From these facts, we deduce

(10.17) Theorem. *Let E be a Griffiths (semi-)positive bundle of rank $r \ge 2$. Then for any integer $m \ge 1$*

$$E^* \otimes (\det E)^m >_m 0 \quad (\textit{resp.} \quad \ge_m 0).$$

Proof. We apply Prop. 10.16 to $\Theta = -\Theta(E^*) = {}^t\Theta(E) \ge_{\mathrm{Grif}} 0$ on $T_X \otimes E^*$ and observe that

$$\Theta(\det E) = \mathrm{Tr}_E\,\Theta(E) = \mathrm{Tr}_{E^*}\,\Theta.$$

(10.18) Theorem. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then for any $m \ge 1$*

$$E >_m 0 \quad \Longrightarrow \quad S \otimes (\det Q)^m >_m 0.$$

Proof. Formulas (10.7) imply

$$\mathrm{i}\Theta(S) >_m \mathrm{i}\beta^* \wedge \beta \quad , \quad \mathrm{i}\Theta(Q) >_m \mathrm{i}\beta \wedge \beta^*,$$

$$\mathrm{i}\Theta(\det Q) = \mathrm{Tr}_Q(\mathrm{i}\Theta(Q)) > \mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^*).$$

If we write $\beta = \sum dz_j \otimes \beta_j$ as in the proof of Corollary 10.8, then

$$\mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^*) = \sum i dz_j \wedge d\overline{z}_k\,\mathrm{Tr}_Q(\beta_j\beta_k^*)$$

$$= \sum i dz_j \wedge d\overline{z}_k\,\mathrm{Tr}_S(\beta_k^*\beta_j) = \mathrm{Tr}_S(-\mathrm{i}\beta^*\wedge\beta).$$

Furthermore, it has been already proved that $-\mathrm{i}\beta^*\wedge\beta \ge_{\mathrm{Nak}} 0$. By Prop. 8.1 applied to the corresponding hermitian form Θ on $T_X \otimes S$, we get

$$m\,\mathrm{Tr}_S(-\mathrm{i}\beta^*\wedge\beta)\otimes \mathrm{Id}_S + \mathrm{i}\beta^*\wedge\beta \geq_m 0,$$

and Theorem 10.18 follows. □

(10.19) Corollary. *Let X be a weakly pseudoconvex Kähler n-dimensional manifold, E a holomorphic vector bundle of rank $r \geq 2$ and $m \geq 1$ an integer. Then*

a) $E >_{\mathrm{Grif}} 0 \Rightarrow H^{n,q}(X, E\otimes \det E) = 0$ *for* $q \geq 1$;

b) $E >_{\mathrm{Grif}} 0 \Rightarrow H^{n,q}\big(X, E^*\otimes(\det E)^m\big) = 0$ *for* $q \geq 1$ *and* $m \geq \min\{n-q+1, r\}$;

c) *Let* $0 \to S \to E \to Q \to 0$ *be an exact sequence of vector bundles and* $m = \min\{n-q+1, \mathrm{rk}\, S\}$, $q \geq 1$. *If* $E >_m 0$ *and if* L *is a line bundle such that* $L\otimes(\det Q)^{-m} \geq 0$, *then*

$$H^{n,q}(X, S\otimes L) = 0.$$

Proof. Immediate consequence of Theorem 10.11, in combination with 10.12 for a), 10.17 for b) and 10.18 for c). □

11. Skoda's L^2 Estimates for Surjective Bundle Morphisms

Let (X,ω) be a Kähler manifold, $\dim X = n$, and let $g : E \to Q$ a holomorphic morphism of hermitian vector bundles over X. Assume in the first instance that g is *surjective*. We are interested in conditions insuring that the induced morphisms $g : H^{n,k}(X,E) \longrightarrow H^{n,k}(X,Q)$ are also surjective (dealing with $(n,\bullet)$ bidegrees is always easier, since we have to understand positivity conditions for the curvature term). For that purpose, it is natural to consider the subbundle $S = \mathrm{Ker}\, g \subset E$ and the exact sequence

$$0 \longrightarrow S \xrightarrow{j} E \xrightarrow{g} Q \longrightarrow 0 \tag{11.1}$$

where $j : S \to E$ is the inclusion. In fact, we need a little more flexibility to handle the curvature terms, so we take the tensor product of the exact sequence by a holomorphic line bundle L (whose properties will be specified later):

$$0 \longrightarrow S\otimes L \longrightarrow E\otimes L \xrightarrow{g} Q\otimes L \longrightarrow 0. \tag{11.2}$$

(11.3) Theorem. *Let k be an integer such that $0 \leq k \leq n$. Set $r = \mathrm{rk}\, E$, $q = \mathrm{rkQ}$, $s = \mathrm{rk}\, S = r - q$ and*

$$m = \min\{n-k, s\} = \min\{n-k, r-q\}.$$

Assume that (X,ω) possesses also a complete Kähler metric $\widehat{\omega}$, that $E \geq_m 0$, and that $L \longrightarrow X$ is a hermitian holomorphic line bundle such that

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \geq 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q\otimes L$ such that $\|f\| < +\infty$, there exists a D''-closed form h of type (n,k) with values in $E\otimes L$ such that $f = g\cdot h$ and

$$\|h\|^2 \leq (1 + m/\varepsilon)\,\|f\|^2.$$

The idea of the proof is essentially due to [Sko78], who actually proved the special case $k = 0$. The general case appeared in [Dem82b].

Proof. Let $j : S \to E$ be the inclusion morphism, $g^* : Q \to E$ and $j^* : E \to S$ the adjoints of g, j, and the matrix of D_E with respect to the orthogonal splitting $E \simeq S\oplus Q$ (cf. Lemma 6.2). Then g^*f is a lifting of f in $E\otimes L$. We will try to find h under the form

$$h = g^*f + ju, \qquad u \in L^2(X, \Lambda^{n,k}T^*_X\otimes S\otimes L).$$

As the images of S and Q in E are orthogonal, we have $|h|^2 = |f|^2 + |u|^2$ at every point of X. On the other hand $D''_{Q\otimes L}f = 0$ by hypothesis and $D''g^* = -j\circ\beta^*$ by (6.5), hence

$$D''_{E\otimes L}h = -j(\beta^*\wedge f) + j\,D''_{S\otimes L} = j(D''_{S\otimes L} - \beta^*\wedge f).$$

We are thus led to solve the equation

$$D''_{S\otimes L}u = \beta^*\wedge f, \tag{11.4}$$

and for that, we apply Th. 5.1 to the $(n,k+1)$-form $\beta^*\wedge f$. One now observes that the curvature of $S\otimes L$ can be expressed in terms of β. This remark will be used to prove:

(11.5) Lemma. *Let $A_k = [\mathrm{i}\Theta(S\otimes L), \Lambda]$ be the curvature operator acting as an hermitian operator on the bundle of $(n,k+1)$-forms. Then*

$$\langle A_k^{-1}(\beta^*\wedge f), (\beta^*\wedge f)\rangle \leq (m/\varepsilon)\,|f|^2.$$

If the Lemma is taken for granted, Th. 8.4 yields a solution u of (11.4) in $L^2(X, \Lambda^{n,q}T^*_X\otimes S\otimes L)$ such that $\|u\|^2 \leq (m/\varepsilon)\,\|f\|^2$. As $\|h\|^2 = \|f\|^2 + \|u\|^2$, the proof of Th. 11.3 is complete. □

Proof of Lemma 11.5. Exactly as in the proof of Th. 6.18 formulas (6.7) yield

$$\mathrm{i}\Theta(S) \geq_m \mathrm{i}\beta^*\wedge\beta, \qquad \mathrm{i}\Theta(\det Q) \geq \mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^*) = \mathrm{Tr}_S(-\mathrm{i}\beta^*\wedge\beta).$$

Since $C^\infty(X, \Lambda^{1,1}T^*_X\otimes \mathrm{Herm}\, S) \ni \Theta := -\mathrm{i}\beta^*\wedge\beta \geq_{\mathrm{Grif}} 0$, Prop. 6.16 implies

$$m\,\mathrm{Tr}_S(-\mathrm{i}\beta^*\wedge\beta)\otimes \mathrm{Id}_S + \mathrm{i}\beta^*\wedge\beta \geq_m 0.$$

From the hypothesis on the curvature of L we get

$$\begin{aligned}\mathrm{i}\Theta(S\otimes L) &\geq_m \mathrm{i}\Theta(S)\otimes \mathrm{Id}_L + (m+\varepsilon)\,\mathrm{i}\Theta(\det Q)\otimes \mathrm{Id}_{S\otimes L}\\ &\geq_m \big(\mathrm{i}\beta^*\wedge\beta + (m+\varepsilon)\,\mathrm{Tr}_S(-\mathrm{i}\beta^*\wedge\beta)\otimes \mathrm{Id}_S\big)\otimes \mathrm{Id}_L\\ &\geq_m (\varepsilon/m)\,(-\mathrm{i}\beta^*\wedge\beta)\otimes \mathrm{Id}_S\otimes \mathrm{Id}_L\,.\end{aligned}$$

For any $v\in \Lambda^{n,k+1}T^*_X\otimes S\otimes L$, Lemma 6.10 implies

$$\langle A_k v, v\rangle \geq (\varepsilon/m)\,\langle -\mathrm{i}\beta^*\wedge\beta\wedge\Lambda v, v\rangle, \tag{11.6}$$

because $\mathrm{rk}(S\otimes L)=s$ and $m=\min\{n-k,s\}$. Let $(dz_1,\ldots,dz_n)$ be an orthonormal basis of T^*_X at a given point $x_0\in X$ and set

$$\beta=\sum_{1\leq j\leq n} dz_j\otimes\beta_j,\qquad \beta_j\in \mathrm{Hom}(S,Q).$$

The adjoint of the operator $\beta^*\wedge\bullet=\sum d\overline{z}_j\wedge\beta_j^*\bullet$ is the contraction operator $\beta\lrcorner\bullet$ defined by

$$\beta\lrcorner v=\sum\frac{\partial}{\partial\overline{z}_j}\lrcorner(\beta_j v)=\sum -\mathrm{i}dz_j\wedge\Lambda(\beta_j v)=-\mathrm{i}\beta\wedge\Lambda v.$$

We get consequently $\langle -\mathrm{i}\beta^*\wedge\beta\wedge\Lambda v,v\rangle=|\beta\lrcorner v|^2$ and (11.6) implies

$$|\langle\beta^*\wedge f,v\rangle|^2=|\langle f,\beta\lrcorner v\rangle|^2\leq |f|^2\,|\beta\lrcorner v|^2\leq (m/\varepsilon)\langle A_k v,v\rangle\,|f|^2.$$

This is equivalent to the estimate asserted in the lemma. □

If X has a plurisubharmonic exhaustion function ψ, we can select a convex increasing function $\chi\in C^\infty(\mathbb{R},\mathbb{R})$ and multiply the metric of L by the weight $\exp(-\chi\circ\psi)$ in order to make the L^2 norm of f converge. Theorem 11.3 implies therefore:

(11.7) Corollary. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, $g:E\to Q$ a surjective bundle morphism with $r=\mathrm{rk}\,E$, $q=\mathrm{rk}\,Q$, and $L\to X$ a hermitian holomorphic line bundle. We set $m=\min\{n-k,r-q\}$ and assume that $E\geq_m 0$ and*

$$\mathrm{i}\Theta(L)-(m+\varepsilon)\,\mathrm{i}\Theta(\det Q)\geq 0$$

for some $\varepsilon>0$. Then g induces a surjective map

$$H^{n,k}(X,E\otimes L)\longrightarrow H^{n,k}(X,Q\otimes L).$$

The most remarkable feature of this result is that it does not require any strict positivity assumption on the curvature (for instance E can be a flat bundle). A careful examination of the proof shows that it amounts to verify that the image of the coboundary morphism

$$-\beta^*\wedge\bullet\ :\ H^{n,k}(X,Q\otimes L)\longrightarrow H^{n,k+1}(X,S\otimes L)$$

vanishes; however the cohomology group $H^{n,k+1}(X,S\otimes L)$ itself does not necessarily vanish, as it would do under a strict positivity assumption.

We want now to get also estimates when Q is endowed with a metric given a priori, that can be distinct from the quotient metric of E by g. Then the map $g^*(gg^*)^{-1}\ :\ Q\longrightarrow E$ is the lifting of Q orthogonal to $S=\mathrm{Ker}\,g$. The quotient metric $|\bullet|'$ on Q is therefore defined in terms of the original metric $|\bullet|$ by

$$|v|'^2=|g^*(gg^*)^{-1}v|^2=\langle(gg^*)^{-1}v,v\rangle=\det(gg^*)^{-1}\,\langle\widetilde{gg^*}v,v\rangle$$

where $\widetilde{gg^*}\in\mathrm{End}(Q)$ denotes the endomorphism of Q whose matrix is the transposed of the comatrix of gg^*. For every $w\in\det Q$, we find

$$|w|'^2=\det(gg^*)^{-1}\,|w|^2.$$

If Q' denotes the bundle Q with the quotient metric, we get

$$\mathrm{i}\Theta(\det Q')=\mathrm{i}\Theta(\det Q)+\mathrm{i}d'd''\log\det(gg^*).$$

In order that the hypotheses of Th. 11.3 be satisfied, we are led to define a new metric $|\bullet|'$ on L by $|u|'^2=|u|^2\,\big(\det(gg^*)\big)^{-m-\varepsilon}$. Then

$$\mathrm{i}\Theta(L')=\mathrm{i}\Theta(L)+(m+\varepsilon)\,\mathrm{i}d'd''\log\det(gg^*)\geq(m+\varepsilon)\,\mathrm{i}\Theta(\det Q').$$

Theorem 11.3 applied to (E,Q',L') can now be reformulated:

(11.8) Theorem. *Let X be a complete Kähler manifold equipped with a Kähler metric ω on X, let $E\to Q$ be a surjective morphism of hermitian vector bundles and let $L\to X$ be a hermitian holomorphic line bundle. Set $r=\mathrm{rk}\,E$, $q=\mathrm{rk}\,Q$ and $m=\min\{n-k,r-q\}$, and assume that $E\geq_m 0$ and*

$$\mathrm{i}\Theta(L)-(m+\varepsilon)\mathrm{i}\Theta(\det Q)\geq 0$$

for some $\varepsilon>0$. Then for every D''-closed form f of type (n,k) with values in $Q\otimes L$ such that

$$I=\int_X\langle\widetilde{gg^*}f,f\rangle\,(\det gg^*)^{-m-1-\varepsilon}\,dV<+\infty,$$

there exists a D''-closed form h of type (n,k) with values in $E\otimes L$ such that $f=g\cdot h$ and

$$\int_X|h|^2\,(\det gg^*)^{-m-\varepsilon}\,dV\leq(1+m/\varepsilon)\,I.$$ □

Our next goal is to extend Th. 11.8 in the case when $g\ :\ E\longrightarrow Q$ is only *generically* surjective; this means that the analytic set

$$Y=\{x\in X\ ;\ g_x\ :\ E_x\longrightarrow Q_x\text{ is not surjective }\}$$

defined by the equation $\Lambda^q g=0$ is nowhere dense in X. Here $\Lambda^q g$ is a section of the bundle $\mathrm{Hom}(\Lambda^q E,\det Q)$. The idea is to apply the above Theorem 11.8 to $X\setminus Y$. For this, we have to know whether $X\setminus Y$ has a complete Kähler metric.

(11.9) Lemma. *Let (X,ω) be a Kähler manifold, and $Y=\sigma^{-1}(0)$ an analytic subset defined by a section of a hermitian vector bundle $E\to X$. If X is weakly pseudoconvex and exhausted by $X_c=\{x\in X\ ;\ \psi(x)<c\}$, then $X_c\setminus Y$ has a complete Kähler*

metric for all $c \in \mathbb{R}$. The same conclusion holds for $X \setminus Y$ if (X, ω) is complete and if for some constant $C \geq 0$ we have $\Theta_E \leq_{\text{Grif}} C\omega \otimes \langle\ ,\ \rangle_E$ on X.

Proof. Set $\tau = \log|\sigma|^2$. Then $d'\tau = \{D'\sigma, \sigma\}/|\sigma|^2$ and $D''D'\sigma = D^2\sigma = \Theta(E)\sigma$, thus

$$\mathrm{i}d'd''\tau = \mathrm{i}\frac{\{D'\sigma, D'\sigma\}}{|\sigma|^2} - \mathrm{i}\frac{\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\}}{|\sigma|^4} - \frac{\{\mathrm{i}\Theta(E)\sigma, \sigma\}}{|\sigma|^2}.$$

For every $\xi \in T_X$, we find therefore

$$\begin{aligned} H\tau(\xi) &= \frac{|\sigma|^2\,|D'\sigma \cdot \xi|^2 - |\langle D'\sigma \cdot \xi, \sigma\rangle|^2}{|\sigma|^4} - \frac{\widetilde{\Theta}(E)(\xi \otimes \sigma, \xi \otimes \sigma)}{|\sigma|^2} \\ &\geq -\frac{\widetilde{\Theta}(E)(\xi \otimes \sigma, \xi \otimes \sigma)}{|\sigma|^2} \end{aligned}$$

by the Cauchy-Schwarz inequality. If C is a bound for the coefficients of $\widetilde{\Theta}(E)$ on the compact subset $\overline{X}_c$, we get $\mathrm{i}d'd''\tau \geq -C\omega$ on X_c. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a convex increasing function. We set

$$\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi \circ \tau) = \omega + \mathrm{i}\big(\chi' \circ \tau\ d'd''\tau + \chi'' \circ \tau\ d'\tau \wedge d''\tau\big).$$

We thus see that $\widehat{\omega}$ is positive definite if $\chi' \leq 1/2C$, and by a computation similar to that in proposition 8.7, we check that $\widehat{\omega}$ is complete near $Y = \tau^{-1}(-\infty)$ as soon as

$$\int_{-\infty}^{0} \sqrt{\chi''(t)}\,dt = +\infty.$$

One can choose for example χ such that $\chi(t) = \frac{1}{5C}(t - \log|t|)$ for $t \leq -1$. In order to obtain a complete Kähler metric on $X_c \setminus Y$, we also need the metric to be complete near ∂X_c. If $\widehat{\omega}$ is not, such a metric can be defined by

$$\begin{aligned} \widetilde{\omega} &= \widehat{\omega} + \mathrm{i}d'd''\log(c - \psi)^{-1} = \widehat{\omega} + \frac{\mathrm{i}d'd''\psi}{c - \psi} + \frac{\mathrm{i}d'\psi \wedge d''\psi}{(c - \psi)^2} \\ &\geq \mathrm{i}d'\log(c - \psi)^{-1} \wedge d''\log(c - \psi)^{-1}\ ; \end{aligned}$$

$\widetilde{\omega}$ is complete on $X_c \setminus \Omega$ because $\log(c - \psi)^{-1}$ tends to $+\infty$ on ∂X_c. □

We also need another elementary lemma dealing with the extension of partial differential equalities across analytic sets.

(11.10) Lemma. *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Assume that v is a $(p, q-1)$-form with L^2_{loc} coefficients and w a (p, q)-form with L^1_{loc} coefficients such that $d''v = w$ on $\Omega \setminus Y$ (in the sense of distribution theory). Then $d''v = w$ on Ω.*

Proof. An induction on the dimension of Y shows that it is sufficient to prove the result in a neighborhood of a regular point $a \in Y$. By using a local analytic isomorphism, the proof is reduced to the case where Y is contained in the hyperplane $z_1 = 0$, with $a = 0$. Let $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function such that $\lambda(t) = 0$ for $t \leq \frac{1}{2}$ and $\lambda(t) = 1$ for $t \geq 1$. We must show that

$$\int_\Omega w \wedge \alpha = (-1)^{p+q} \int_\Omega v \wedge d''\alpha \tag{11.11}$$

for all $\alpha \in \mathcal{D}(\Omega, \Lambda^{n-p,n-q}T^*_\Omega)$. Set $\lambda_\varepsilon(z) = \lambda(|z_1|/\varepsilon)$ and replace α in the integral by $\lambda_\varepsilon\alpha$. Then $\lambda_\varepsilon\alpha \in \mathcal{D}(\Omega \setminus Y, \Lambda^{n-p,n-q}T^*_\Omega)$ and the hypotheses imply

$$\int_\Omega w \wedge \lambda_\varepsilon\alpha = (-1)^{p+q} \int_\Omega v \wedge d''(\lambda_\varepsilon\alpha) = (-1)^{p+q} \int_\Omega v \wedge (d''\lambda_\varepsilon \wedge \alpha + \lambda_\varepsilon d''\alpha).$$

As w and v have L^1_{loc} coefficients on Ω, the integrals of $w \wedge \lambda_\varepsilon\alpha$ and $v \wedge \lambda_\varepsilon d''\alpha$ converge respectively to the integrals of $w \wedge \alpha$ and $v \wedge d''\alpha$ as ε tends to 0. The remaining term can be estimated by means of the Cauchy-Schwarz inequality:

$$\Big|\int_\Omega v \wedge d''\lambda_\varepsilon \wedge \alpha\Big|^2 \leq \int_{|z_1| \leq \varepsilon} |v \wedge \alpha|^2\,dV \cdot \int_{\text{Supp}\,\alpha} |d''\lambda_\varepsilon|^2\,dV\ ;$$

as $v \in L^2_{\text{loc}}(\Omega)$, the integral $\int_{|z_1| \leq \varepsilon} |v \wedge \alpha|^2\,dV$ converges to 0 with ε, whereas

$$\int_{\text{Supp}\,\alpha} |d''\lambda_\varepsilon|^2\,dV \leq \frac{C}{\varepsilon^2}\,\text{Vol}\big(\text{Supp}\,\alpha \cap \{|z_1| \leq \varepsilon\}\big) \leq C'.$$

Equality (11.11) follows when ε tends to 0. □

(11.12) Theorem. *The existence statement and the estimates of Th. 11.8 remain true for a generically surjective morphism $g : E \to Q$, provided that X is weakly pseudoconvex.*

Proof. Apply Th. 11.8 to each relatively compact domain $X_c \setminus Y$ (these domains are complete Kähler by Lemma 11.9). From a sequence of solutions on $X_c \setminus Y$ we can extract a subsequence converging weakly on $X \setminus Y$ as c tends to $+\infty$. One gets a form h satisfying the estimates, such that $D''h = 0$ on $X \setminus Y$ and $f = g \cdot h$. In order to see that $D''h = 0$ on X, it suffices to apply Lemma 11.10 and to observe that h has L^2_{loc} coefficients on X by our estimates. □

A very special but interesting case is obtained for the trivial bundles $E = \Omega \times \mathbb{C}^r$, $Q = \Omega \times \mathbb{C}$ over a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Then the morphism g is given by a r-tuple $(g_1, \ldots, g_r)$ of holomorphic functions on Ω. Let us take $k = 0$ and $L = \Omega \times \mathbb{C}$ with the metric given by a weight $e^{-\varphi}$. If we observe that $\widetilde{gg^*} = \mathrm{Id}$ when $\mathrm{rk}\,Q = 1$, Th. 11.8 applied on $X = \Omega \setminus g^{-1}(0)$ and Lemmas 11.9, 11.10 give:

(11.13) Theorem (Skoda [Sko72b]). *Let Ω be a complete Kähler open subset of $\mathbb{C}^n$ and φ a plurisubharmonic function on Ω. Set $m = \min\{n, r-1\}$. Then for every holomorphic function f on Ω such that*

$$I = \int_{\Omega \setminus Z} |f|^2\,|g|^{-2(m+1+\varepsilon)}e^{-\varphi}\,dV < +\infty,$$

where $Z = g^{-1}(0)$, there exist holomorphic functions $(h_1, \ldots, h_r)$ on Ω such that $f = \sum g_j h_j$ and

$$\int_{\Omega\setminus Y} |h|^2\,|g|^{-2(m+\varepsilon)}e^{-\varphi}\,dV \le (1+m/\varepsilon)I. \qquad \square$$

We now show that Theorem 11.13 can be applied to get deep results concerning ideals of the local ring $\mathcal{O}_n = \mathbb{C}\{z_1,\dots,z_n\}$ of germs of holomorphic functions on $(\mathbb{C}^n,0)$. Let $\mathcal{I} = (g_1,\dots,g_r) \neq (0)$ be an ideal of $\mathcal{O}_n$.

(11.14) Definition. *Let $k \in \mathbb{R}_+$. We associate to $\mathcal{I}$ the following ideals:*

a) *the ideal $\overline{\mathcal{I}}^{(k)}$ of germs $u \in \mathcal{O}_n$ such that $|u| \le C|g|^k$ for some constant $C \ge 0$, where $|g|^2 = |g_1|^2 + \cdots + |g_r|^2$.*

b) *the ideal $\widehat{\mathcal{I}}^{(k)}$ of germs $u \in \mathcal{O}_n$ such that*

$$\int_\Omega |u|^2\,|g|^{-2(k+\varepsilon)}\,dV < +\infty$$

on a small ball Ω centered at 0, if $\varepsilon > 0$ is small enough.

(11.15) Proposition. *For all $k,l \in \mathbb{R}_+$ we have*

a) $\overline{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k)}$;

b) $\mathcal{I}^k \subset \overline{\mathcal{I}}^{(k)}$ *if* $k \in \mathbb{N}$;

c) $\overline{\mathcal{I}}^{(k)}\,\overline{\mathcal{I}}^{(l)} \subset \overline{\mathcal{I}}^{(k+l)}$;

d) $\overline{\mathcal{I}}^{(k)}\,\widehat{\mathcal{I}}^{(l)} \subset \widehat{\mathcal{I}}^{(k+l)}$.

All properties are immediate from the definitions except a) which is a consequence of the integrability of $|g|^{-\epsilon}$ for $\varepsilon > 0$ small (exercise to the reader!). Before stating the main result, we need a simple lemma.

(11.16) Lemma. *If $\mathcal{I} = (g_1,\dots,g_r)$ and $r > n$, we can find elements $\widetilde{g}_1,\dots,\widetilde{g}_n \in \mathcal{I}$ such that $C^{-1}|g| \le |\widetilde{g}| \le C|g|$ on a neighborhood of 0. Each $\widetilde{g}_j$ can be taken to be a linear combination*

$$\widetilde{g}_j = a_j.g = \sum_{1\le k\le r} a_{jk}g_k, \qquad a_j \in \mathbb{C}^r \setminus \{0\}$$

where the coefficients $([a_1],\dots,[a_n])$ are chosen in the complement of a proper analytic subset of $(\mathbb{P}^{r-1})^n$.

It follows from the Lemma that the ideal $\mathcal{J} = (\widetilde{g}_1,\dots,\widetilde{g}_n) \subset \mathcal{I}$ satisfies $\overline{\mathcal{J}}^{(k)} = \overline{\mathcal{I}}^{(k)}$ and $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$ for all k.

Proof. Assume that $g \in \mathcal{O}(\Omega)^r$. Consider the analytic subsets in $\Omega\times(\mathbb{P}^{r-1})^n$ defined by

$$A = \big\{(z,[w_1],\dots,[w_n])\,;\,w_j.g(z) = 0\big\},$$

$$A^* = \bigcup \text{irreducible components of } A \text{ not contained in } g^{-1}(0)\times(\mathbb{P}^{r-1})^n.$$

For $z \notin g^{-1}(0)$ the fiber $A_z = \{([w_1],\dots,[w_n])\,;\,w_j.g(z) = 0\} = A^*_z$ is a product of n hyperplanes in $\mathbb{P}^{r-1}$, hence $A\cap(\Omega\setminus g^{-1}(0))\times(\mathbb{P}^{r-1})^n$ is a fiber bundle with base $\Omega\setminus g^{-1}(0)$ and fiber $(\mathbb{P}^{r-2})^n$. As A^* is the closure of this set in $\Omega\times(\mathbb{P}^{r-1})^n$, we have

$$\dim A^* = n + n(r-2) = n(r-1) = \dim(\mathbb{P}^{r-1})^n.$$

It follows that the zero fiber

$$A^*_0 = A^* \cap \big(\{0\}\times(\mathbb{P}^{r-1})^n\big)$$

is a proper subset of $\{0\}\times(\mathbb{P}^{r-1})^n$. Choose $(a_1,\dots,a_n) \in (\mathbb{C}^r\setminus\{0\})^n$ such that $(0,[a_1],\dots,[a_n])$ is not in A^*_0. By an easy compactness argument the set $A^*\cap\big(\overline{B}(0,\varepsilon)\times(\mathbb{P}^{r-1})^n\big)$ is disjoint from the neighborhood $B(0,\varepsilon)\times\prod[B(a_j,\varepsilon)]$ of $(0,[a_1],\dots,[a_n])$ for ε small enough. For $z \in B(0,\varepsilon)$ we have $|a_j.g(z)| \ge \varepsilon|g(z)|$ for some j, otherwise the inequality $|a_j.g(z)| < \varepsilon|g(z)|$ would imply the existence of $h_j \in \mathbb{C}^r$ with $|h_j| < \varepsilon$ and $a_j.g(z) = h_j.g(z)$. Since $g(z) \neq 0$, we would have

$$(z,[a_1-h_1],\dots,[a_n-h_n]) \in A^*\cap\big(B(0,\varepsilon)\times(\mathbb{P}^{r-1})^n\big),$$

a contradiction. We obtain therefore

$$\varepsilon|g(z)| \le \max|a_j.g(z)| \le (\max|a_j|)\,|g(z)| \quad \text{on } B(0,\varepsilon). \qquad \square$$

(11.17) Theorem (Briançon-Skoda [BSk74]). *Set $p = \min\{n-1, r-1\}$. Then*

a) $\widehat{\mathcal{I}}^{(k+1)} = \mathcal{I}\,\widehat{\mathcal{I}}^{(k)} = \overline{\mathcal{I}}\,\widehat{\mathcal{I}}^{(k)}$ *for* $k \ge p$.

b) $\overline{\mathcal{I}}^{(k+p)} \subset \widehat{\mathcal{I}}^{(k+p)} \subset \mathcal{I}^k$ *for all* $k \in \mathbb{N}$.

Proof. a) The inclusions $\mathcal{I}\,\widehat{\mathcal{I}}^{(k)} \subset \overline{\mathcal{I}}\,\widehat{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k+1)}$ are obvious thanks to Prop. 11.15, so we only have to prove that $\widehat{\mathcal{I}}^{(k+1)} \subset \mathcal{I}\,\widehat{\mathcal{I}}^{(k)}$. Assume first that $r \le n$. Let $f \in \widehat{\mathcal{I}}^{(k+1)}$ be such that

$$\int_\Omega |f|^2\,|g|^{-2(k+1+\varepsilon)}\,dV < +\infty.$$

For $k \ge p-1$, we can apply Th. 11.13 with $m = r-1$ and with the weight $\varphi = (k-m)\log|g|^2$. Hence f can be written $f = \sum g_j h_j$ with

$$\int_\Omega |h|^2\,|g|^{-2(k+\varepsilon)}\,dV < +\infty,$$

thus $h_j \in \widehat{\mathcal{I}}^{(k)}$ and $f \in \mathcal{I}\,\widehat{\mathcal{I}}^{(k)}$. When $r > n$, Lemma 11.16 shows that there is an ideal $\mathcal{J} \subset \mathcal{I}$ with n generators such that $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{I}}^{(k)}$. We find

$$\widehat{\mathcal{I}}^{(k+1)} = \widehat{\mathcal{J}}^{(k+1)} \subset \mathcal{J}\,\widehat{\mathcal{J}}^{(k)} \subset \mathcal{I}\,\widehat{\mathcal{I}}^{(k)} \quad \text{for } k \ge n-1.$$

b) Property a) implies inductively $\widehat{\mathcal{I}}^{(k+p)} = \mathcal{I}^k\,\widehat{\mathcal{I}}^{(p)}$ for all $k \in \mathbb{N}$. This gives in particular $\widehat{\mathcal{I}}^{(k+p)} \subset \mathcal{I}^k$. $\square$

(11.18) Corollary.

a) *The ideal $\overline{\mathcal{I}}$ is the integral closure of $\mathcal{I}$, i.e. by definition the set of germs $u \in \mathcal{O}_n$ which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \quad a_s \in \mathcal{I}^s, \quad 1 \le s \le d.$$

b) *Similarly, $\overline{\mathcal{I}}^{(k)}$ is the set of germs $u \in \mathcal{O}_n$ which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \quad a_s \in \mathcal{I}^{\lceil ks \rceil}, \quad 1 \le s \le d,$$

where $\lceil t \rceil$ denotes the smallest integer $\ge t$.

As the ideal $\overline{\mathcal{I}}^{(k)}$ is finitely generated, property b) shows that there always exists a rational number $l \ge k$ such that $\overline{\mathcal{I}}^{(l)} = \overline{\mathcal{I}}^{(k)}$.

Proof. a) If $u \in \mathcal{O}_n$ satisfies a polynomial equation with coefficients $a_s \in \mathcal{I}^s$, then clearly $|a_s| \le C_s |g|^s$ and the usual elementary bound

$$|\text{roots}| \le 2 \max_{1 \le s \le d} |a_s|^{1/s}$$

for the roots of a monic polynomial implies $|u| \le C |g|$.

Conversely, assume that $u \in \overline{\mathcal{I}}$. The ring $\mathcal{O}_n$ is Noetherian, so the ideal $\widehat{\mathcal{I}}^{(p)}$ has a finite number of generators $v_1, \ldots, v_N$. For every j we have $uv_j \in \overline{\mathcal{I}}\,\widehat{\mathcal{I}}^{(p)} = \mathcal{I}\,\widehat{\mathcal{I}}^{(p)}$, hence there exist elements $b_{jk} \in \mathcal{I}$ such that

$$uv_j = \sum_{1 \le k \le N} b_{jk} v_k.$$

The matrix $(u\delta_{jk} - b_{jk})$ has the non zero vector (v_j) in its kernel, thus u satisfies the equation $\det(u\delta_{jk} - b_{jk}) = 0$, which is of the required type.

b) Observe that $v_1, \ldots, v_N$ satisfy simultaneously some integrability condition $\int_\Omega |v_j|^{-2(p+\epsilon)} < +\infty$, thus $\widehat{\mathcal{I}}^{(p)} = \widehat{\mathcal{I}}^{(p+\eta)}$ for $\eta \in [0, \epsilon[$. Let $u \in \overline{\mathcal{I}}^{(k)}$. For every integer $m \in \mathbb{N}$ we have

$$u^m v_j \in \overline{\mathcal{I}}^{(km)}\, \widehat{\mathcal{I}}^{(p+\eta)} \subset \widehat{\mathcal{I}}^{(km+\eta+p)}.$$

If $k \notin \mathbb{Q}$, we can find m such that $d(km + \epsilon/2, \mathbb{Z}) < \epsilon/2$, thus $km + \eta \in \mathbb{N}$ for some $\eta \in]0, \epsilon[$. If $k \in \mathbb{Q}$, we take m such that $km \in \mathbb{N}$ and $\eta = 0$. Then

$$u^m v_j \in \widehat{\mathcal{I}}^{(N+p)} = \mathcal{I}^N\, \widehat{\mathcal{I}}^{(p)} \quad \text{with} \quad N = km + \eta \in \mathbb{N},$$

and the reasoning made in a) gives $\det(u^m \delta_{jk} - b_{jk}) = 0$ for some $b_{jk} \in \mathcal{I}^N$. This is an equation of the type described in b), where the coefficients a_s vanish when s is not a multiple of m and $a_{ms} \in \mathcal{I}^{Ns} \subset \mathcal{I}^{\lceil kms \rceil}$. □

Let us mention that Briançon and Skoda's result 11.17 b) is optimal for $k = 1$. Take for example $\mathcal{I} = (g_1, \ldots, g_r)$ with $g_j(z) = z_j^r$, $1 \le j \le r$, and $f(z) = z_1 \ldots z_r$. Then $|f| \le C|g|$ and 11.17 b) yields $f^r \in \mathcal{I}$; however, it is easy to verify that $f^{r-1} \notin \mathcal{I}$. The theorem also gives an answer to the following conjecture made by J. Mather.

(11.19) Corollary. *Let $f \in \mathcal{O}_n$ and $\mathcal{I}_f = (z_1 \partial f/\partial z_1, \ldots, z_n \partial f/\partial z_n)$. Then $f \in \overline{\mathcal{I}}_f$, and for every integer $k \ge 0$, $f^{k+n-1} \in \mathcal{I}_f^k$.*

The Corollary is also optimal for $k = 1$: for example, one can verify that the function $f(z) = (z_1 \ldots z_n)^3 + z_1^{3n-1} + \cdots + z_n^{3n-1}$ is such that $f^{n-1} \notin \mathcal{I}_f$.

Proof. Set $g_j(z) = z_j\, \partial f/\partial z_j$, $1 \le j \le n$. By 11.17 b), it suffices to show that $|f| \le C|g|$. For every germ of analytic curve $\mathbb{C} \ni t \longmapsto \gamma(t)$, $\gamma \not\equiv 0$, the vanishing order of $f \circ \gamma(t)$ at $t = 0$ is the same as that of

$$t\, \frac{d(f \circ \gamma)}{dt} = \sum_{1 \le j \le n} t\, \gamma_j'(t)\, \frac{\partial f}{\partial z_j}(\gamma(t)).$$

We thus obtain

$$|f \circ \gamma(t)| \le C_1 |t| \left| \frac{d(f \circ \gamma)}{dt} \right| \le C_2 \sum_{1 \le j \le n} |t\, \gamma_j'(t)| \left| \frac{\partial f}{\partial z_j}(\gamma(t)) \right| \le C_3 |g \circ \gamma(t)|$$

and conclude by the following elementary lemma. □

(11.20) Curve selection lemma. *Let $f, g_1, \ldots, g_r \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at 0. Then we have $|f| \le C|g|$ for some constant C if and only if for every germ of analytic curve γ through 0 there exists a constant C_γ such that $|f \circ \gamma| \le C_\gamma |g \circ \gamma|$.*

Proof. If the inequality $|f| \le C|g|$ does not hold on any neighborhood of 0, the germ of analytic set $(A, 0) \subset (\mathbb{C}^{n+r}, 0)$ defined by

$$g_j(z) = f(z) z_{n+j}, \quad 1 \le j \le r,$$

contains a sequence of points $\big(z_\nu, g_j(z_\nu)/f(z_\nu)\big)$ converging to 0 as ν tends to $+\infty$, with $f(z_\nu) \ne 0$. Hence $(A, 0)$ contains an irreducible component on which $f \not\equiv 0$ and there is a germ of curve $\widetilde{\gamma} = (\gamma, \gamma_{n+j}) : (\mathbb{C}, 0) \to (\mathbb{C}^{n+r}, 0)$ contained in $(A, 0)$ such that $f \circ \gamma \not\equiv 0$. We get $g_j \circ \gamma = (f \circ \gamma)\gamma_{n+j}$, hence $|g \circ \gamma(t)| \le C|t|\, |f \circ \gamma(t)|$ and the inequality $|f \circ \gamma| \le C_\gamma |g \circ \gamma|$ does not hold. □

12. The Ohsawa-Takegoshi L^2 Extension Theorem

We address here the following extension problem: let Y be a complex analytic submanifold of a complex manifold X ; given a holomorphic function f on Y satisfying suitable L^2 conditions on Y, find a holomorphic extension F of f to X, together with a good L^2 estimate for F on X. The first satisfactory solution has been obtained only rather recently by Ohsawa-Takegoshi [OT87, Ohs88]. We follow here a more

geometric approach due to Manivel [Man93], which provides a generalized extension theorem in the general framework of vector bundles. As in Ohsawa-Takegoshi's fundamental paper, the main idea is to use a modified Bochner-Kodaira-Nakano inequality. Such inequalities were originally introduced in the work of Donnelly-Fefferman [DF83] and Donnelly-Xavier [DX84]. The main a priori inequality we are going to use is a simplified (and slightly extended) version of the original Ohsawa-Takegoshi a priori inequality, as proposed recently by Ohsawa [Ohs95].

(12.1) Lemma (Ohsawa [Ohs95]). *Let E be a hermitian vector bundle on a complex manifold X equipped with a Kähler metric ω. Let $\eta, \lambda > 0$ be smooth functions on X. Then for every form $u \in \mathcal{D}(X, \Lambda^{p,q}T^*_X \otimes E)$ with compact support we have*

$$\|(\eta^{\frac12} + \lambda^{\frac12})D''^*u\|^2 + \|\eta^{\frac12} D''u\|^2 + \|\lambda^{\frac12} D'u\|^2 + 2\|\lambda^{-\frac12} d'\eta \wedge u\|^2 \\ \geq \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda]u, u\rangle\!\rangle.$$

Proof. Let us consider the "twisted" Laplace-Beltrami operators

$$\begin{aligned} D'\eta D'^* + D'^*\eta D' &= \eta[D', D'^*] + [D', \eta]D'^* + [D'^*, \eta]D' \\ &= \eta\Delta' + (d'\eta)D'^* - (d'\eta)^* D', \\ D''\eta D''^* + D''^*\eta D'' &= \eta[D'', D''^*] + [D'', \eta]D''^* + [D''^*, \eta]D'' \\ &= \eta\Delta'' + (d''\eta)D''^* - (d''\eta)^* D'', \end{aligned}$$

where η, $(d'\eta)$, $(d''\eta)$ are abbreviated notations for the multiplication operators $\eta\bullet$, $(d'\eta)\wedge\bullet$, $(d''\eta)\wedge\bullet$. By subtracting the above equalities and taking into account the Bochner-Kodaira-Nakano identity $\Delta'' - \Delta' = [\mathrm{i}\Theta(E), \Lambda]$, we get

$$\begin{aligned} &D''\eta D''^* + D''^*\eta D'' - D'\eta D'^* - D'^*\eta D' \\ &\quad = \eta[\mathrm{i}\Theta(E), \Lambda] + (d''\eta)D''^* - (d''\eta)^* D'' + (d'\eta)^* D' - (d'\eta)D'^*. \end{aligned} \tag{12.2}$$

Moreover, the Jacobi identity yields

$$[D'', [d'\eta, \Lambda]] - [d'\eta, [\Lambda, D'']] + [\Lambda, [D'', d'\eta]] = 0,$$

whilst $[\Lambda, D''] = -\mathrm{i}D'^*$ by the basic commutation relations 7.2. A straightforward computation shows that $[D'', d'\eta] = -(d'd''\eta)$ and $[d'\eta, \Lambda] = \mathrm{i}(d''\eta)^*$. Therefore we get

$$\mathrm{i}[D'', (d''\eta)^*] + \mathrm{i}[d'\eta, D'^*] - [\Lambda, (d'd''\eta)] = 0,$$

that is,

$$[\mathrm{i}\, d'd''\eta, \Lambda] = [D'', (d''\eta)^*] + [D'^*, d'\eta] = D''(d''\eta)^* + (d''\eta)^* D'' + D'^*(d'\eta) + (d'\eta)D'^*.$$

After adding this to (12.2), we find

$$\begin{aligned} &D''\eta D''^* + D''^*\eta D'' - D'\eta D'^* - D'^*\eta D' + [\mathrm{i}\, d'd''\eta, \Lambda] \\ &\quad = \eta[\mathrm{i}\Theta(E), \Lambda] + (d''\eta)D''^* + D''(d''\eta)^* + (d'\eta)^* D' + D'^*(d'\eta). \end{aligned}$$

We apply this identity to a form $u \in \mathcal{D}(X, \Lambda^{p,q}T^*_X \otimes E)$ and take the inner bracket with u. Then

$$\langle\!\langle (D''\eta D''^*)u, u\rangle\!\rangle = \langle\!\langle \eta D''^*u, D''^*u\rangle\!\rangle = \|\eta^{\frac12} D''^*u\|^2,$$

and likewise for the other similar terms. The above equalities imply

$$\|\eta^{\frac12} D''^*u\|^2 + \|\eta^{\frac12} D''u\|^2 - \|\eta^{\frac12} D'u\|^2 - \|\eta^{\frac12} D'^*u\|^2 = \\ \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta, \Lambda]u, u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D''^*u, (d''\eta)^*u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta \wedge u\rangle\!\rangle.$$

By neglecting the negative terms $-\|\eta^{\frac12} D'u\|^2 - \|\eta^{\frac12} D'^*u\|^2$ and adding the squares

$$\|\lambda^{\frac12} D''^*u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D''^*u, (d''\eta)^*u\rangle\!\rangle + \|\lambda^{-\frac12}(d''\eta)^*u\|^2 \geq 0, \\ \|\lambda^{\frac12} D'u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta \wedge u\rangle\!\rangle + \|\lambda^{-\frac12} d'\eta \wedge u\|^2 \geq 0$$

we get

$$\|(\eta^{\frac12} + \lambda^{\frac12})D''^*u\|^2 + \|\eta^{\frac12} D''u\|^2 + \|\lambda^{\frac12} D'u\|^2 + \|\lambda^{-\frac12} d'\eta \wedge u\|^2 + \|\lambda^{-\frac12}(d''\eta)^*u\|^2 \\ \geq \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta, \Lambda]u, u\rangle\!\rangle.$$

Finally, we use the identities

$$(d'\eta)^*(d'\eta) - (d''\eta)(d''\eta)^* = \mathrm{i}[d''\eta, \Lambda](d'\eta) + \mathrm{i}(d''\eta)[d'\eta, \Lambda] = [\mathrm{i}d''\eta \wedge d'\eta, \Lambda],$$
$$\|\lambda^{-\frac12} d'\eta \wedge u\|^2 - \|\lambda^{-\frac12}(d''\eta)^*u\|^2 = -\langle\!\langle [\mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda]u, u\rangle\!\rangle,$$

The inequality asserted in Lemma 12.1 follows by adding the second identity to our last inequality. □

In the special case of (n,q)-forms, the forms $D'u$ and $d'\eta \wedge u$ are of bidegree $(n+1, q)$, hence the estimate takes the simpler form

$$\|(\eta^{\frac12} + \lambda^{\frac12})D''^*u\|^2 + \|\eta^{\frac12} D''u\|^2 \geq \langle\!\langle [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda]u, u\rangle\!\rangle. \tag{12.3}$$

(12.4) Proposition. *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let E be a hermitian vector bundle over X. Assume that there are smooth and bounded functions $\eta, \lambda > 0$ on X such that the (hermitian) curvature operator $B = B^{n,q}_{E,\omega,\eta} = [\eta\, \mathrm{i}\Theta(E) - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1} d'\eta \wedge d''\eta, \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{n,q}T^*_X \otimes E$, for some $q \geq 1$. Then for every form $g \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that $D''g = 0$ and $\int_X \langle B^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{n,q-1}T^*_X \otimes E)$ such that $D''f = g$ and*

$$\int_X (\eta + \lambda)^{-1} |f|^2\, dV_\omega \leq 2 \int_X \langle B^{-1}g, g\rangle\, dV_\omega.$$

Proof. The proof is almost identical to the proof of Theorem 8.4, except that we use (12.4) instead of (7.4). Assume first that ω is complete. With the same notation as in 7.4, we get for every $v = v_1 + v_2 \in (\mathrm{Ker}\, D'') \oplus (\mathrm{Ker}\, D'')^\perp$ the inequalities

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 \leq \int_X \langle B^{-1}g, g\rangle\, dV_\omega \int_X \langle Bv_1, v_1\rangle\, dV_\omega,$$

and

$$\int_X \langle Bv_1, v_1\rangle\, dV_\omega \le \|(\eta^{\frac12}+\lambda^{\frac12})D''^*v_1\|^2 + \|\eta^{\frac12}D''v_1\|^2 = \|(\eta^{\frac12}+\lambda^{\frac12})D''^*v\|^2$$

provided that $v \in \mathrm{Dom}\, D''^*$. Combining both, we find

$$|\langle g, v\rangle|^2 \le \Big(\int_X \langle B^{-1}g, g\rangle\, dV_\omega\Big)\|(\eta^{\frac12}+\lambda^{\frac12})D''^*v\|^2.$$

This shows the existence of an element $w \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that

$$\|w\|^2 \le \int_X \langle B^{-1}g, g\rangle\, dV_\omega \qquad \text{and}$$
$$\langle\!\langle v, g\rangle\!\rangle = \langle\!\langle (\eta^{\frac12}+\lambda^{\frac12})D''^*v, w\rangle\!\rangle \qquad \forall g \in \mathrm{Dom}\, D'' \cap \mathrm{Dom}\, D''^*.$$

As $(\eta^{1/2}+\lambda^{\frac12})^2 \le 2(\eta+\lambda)$, it follows that $f = (\eta^{1/2}+\lambda^{\frac12})w$ satisfies $D''f = g$ as well as the desired L^2 estimate. If ω is not complete, we set $\omega_\varepsilon = \omega + \varepsilon\widehat{\omega}$ with some complete Kähler metric $\widehat{\omega}$. The final conclusion is then obtained by passing to the limit and using a monotonicity argument (the integrals are monotonic with respect to ε). □

(12.5) Remark. We will also need a variant of the L^2-estimate, so as to obtain approximate solutions with weaker requirements on the data: given $\delta > 0$ and $g \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that $D''g = 0$ and $\int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists an approximate solution $f \in L^2(X, \Lambda^{n,q-1}T^*_X \otimes E)$ and a correcting term $h \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that $D''f + \delta^{1/2}h = g$ and

$$\int_X (\eta+\lambda)^{-1}|f|^2\, dV_\omega + \int_X |h|^2\, dV_\omega \le 2\int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega.$$

The proof is almost unchanged, we rely instead on the estimates

$$|\langle g, v_1\rangle|^2 \le \int_X \langle (B+\delta I)^{-1}g, g\rangle\, dV_\omega \int_X \langle (B+\delta I)v_1, v_1\rangle\, dV_\omega,$$

and

$$\int_X \langle (B+\delta I)v_1, v_1\rangle\, dV_\omega \le \|(\eta^{\frac12}+\lambda^{\frac12})D''^*v\|^2 + \delta\|v\|^2. \qquad \square$$

(12.6) Theorem. *Let X be a weakly pseudoconvex n-dimensional complex manifold equipped with a Kähler metric ω, let L (resp. E) be a hermitian holomorphic line bundle (resp. a hermitian holomorphic vector bundle of rank r over X), and s a global holomorphic section of E. Assume that s is generically transverse to the zero section, and let*

$$Y = \big\{x \in X;\ s(x) = 0, \Lambda^r ds(x) \neq 0\big\}, \qquad p = \dim Y = n - r.$$

Moreover, assume that the $(1,1)$-form $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2$ is semipositive and that there is a continuous function $\alpha \ge 1$ such that the following two inequalities hold everywhere on X:

a) $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2 \ge \alpha^{-1}\dfrac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2}$,

b) $|s| \le e^{-\alpha}$.

Then for every smooth D''-closed $(0,q)$-form f over Y with values in the line bundle $\Lambda^n T^*_X \otimes L$ (restricted to Y), such that $\int_Y |f|^2|\Lambda^r(ds)|^{-2}dV_\omega < +\infty$, there exists a D''-closed $(0,q)$-form F over X with values in $\Lambda^n T^*_X \otimes L$, such that F is smooth over $X \smallsetminus \{s = \Lambda^r(ds) = 0\}$, satisfies $F_{|Y} = f$ and

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \le C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega},$$

where C_r is a numerical constant depending only on r.

Observe that the differential ds (which is intrinsically defined only at points where s vanishes) induces a vector bundle isomorphism $ds : T_X/T_Y \to E$ along Y, hence a non vanishing section $\Lambda^r(ds)$, taking values in

$$\Lambda^r(T_X/T_Y)^* \otimes \det E \subset \Lambda^r T^*_X \otimes \det E.$$

The norm $|\Lambda^r(ds)|$ is computed here with respect to the metrics on $\Lambda^r T^*_X$ and $\det E$ induced by the Kähler metric ω and by the given metric on E. Also notice that if hypothesis a) is satisfied for some α, one can always achieve b) by multiplying the metric of E with a sufficiently small weight $e^{-\chi\circ\psi}$ (with ψ a psh cxhaustion on X and χ a convex increasing function; property a) remains valid after we multiply the metric of L by $e^{-(r+\alpha_0^{-1})\chi\circ\psi}$, where $\alpha_0 = \inf_{x\in X}\alpha(x)$.

Proof. Let us first assume that the singularity set $\Sigma = \{s = 0\} \cap \{\Lambda^r(ds) = 0\}$ is empty, so that Y is closed and nonsingular. We claim that there exists a smooth section

$$F_\infty \in C^\infty(X, \Lambda^{n,q}T^*_X \otimes L) = C^\infty(X, \Lambda^{0,q}T^*_X \otimes \Lambda^n T^*_X \otimes L)$$

such that F_∞ coincides with f on Y and $D''F_\infty = 0$ at every point of Y. In fact, let us consider a covering of Y by open coordinates patches $U_j \subset X$ such that there are holomorphic retractions $\rho_j : U_j \to Y \cap U_j$ of U_j onto $Y \cap Y_j$, and let e_j be a non vanishing holomorphic section of $\Lambda^n T^*_X \otimes L_{|U_j}$. On $Y \cap U_j$, we can write the form f as $f = w_j \otimes e_{j|Y\cap U_j}$ with a d''-closed $(0,q)$-form w_j on $Y \cap U_j$, hence $\widetilde{f}_j = (\rho_j^* w_j)\otimes e_j$ is a d''-closed extension of f to U_j. Let $\theta_j \in \mathcal{D}(U_j)$ be a partition of unity such that $\sum\theta_j = 1$ on a neighborhood of Y. Then

$$F_\infty = \sum \theta_j \widetilde{f}_j$$

satisfies all requirements, for

$$D''F_\infty = \sum d''\theta_j \wedge \widetilde{f}_j, \qquad D''F_\infty = \sum d''\theta_j \wedge f = 0 \quad \text{on } Y.$$

Since we do not know about F_∞ far from Y, we will consider a truncation F_ε of F_∞ with support in a small tubular neighborhood $|s| < \varepsilon$ of Y, and solve the equation $D''u_\varepsilon = D''F_\varepsilon$ with the constraint that u_ε should be 0 on Y. As $\mathrm{codim}\, Y = r$, this will be the case if we can guarantee that $|u_\varepsilon|^2|s|^{-2r}$ is locally integrable near Y. For this, we will apply Proposition 12.4 with a suitable choice of the functions η and λ, and an additional weight $|s|^{-2r}$ in the metric of L.

Let us consider the smooth strictly convex function $\chi_0 :]-\infty,0] \to]-\infty,0]$ defined by $\chi_0(t) = t - \log(1-t)$ for $t \le 0$, which is such that $\chi_0(t) \le t$, $1 \le \chi_0' \le 2$ and $\chi_0''(t) = 1/(1-t)^2$. We set

$$\sigma_\varepsilon = \log(|s|^2 + \varepsilon^2), \qquad \eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon).$$

As $|s| \le e^{-\alpha} \le e^{-1}$, we have $\sigma_\varepsilon \le 0$ for ε small, and

$$\eta_\varepsilon \ge \varepsilon - \sigma_\varepsilon \ge \varepsilon - \log(e^{-2\alpha} + \varepsilon^2).$$

Given a relatively compact subset $X_c = \{\psi < c\} \subset\subset X$, we thus have $\eta_\varepsilon \ge 2\alpha$ for $\varepsilon < \varepsilon(c)$ small enough. Simple calculations yield

$$\begin{aligned}
\mathrm{i}\, d'\sigma_\varepsilon &= \frac{\mathrm{i}\{D's,s\}}{|s|^2+\varepsilon^2},\\
\mathrm{i}\, d'd''\sigma_\varepsilon &= \frac{\mathrm{i}\{D's,D's\}}{|s|^2+\varepsilon^2} - \frac{\mathrm{i}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2}\\
&\ge \frac{\varepsilon^2}{|s|^2}\frac{\mathrm{i}\{D's,s\}\wedge\{s,D's\}}{(|s|^2+\varepsilon^2)^2} - \frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2}\\
&\ge \frac{\varepsilon^2}{|s|^2}\mathrm{i}d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon - \frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2},
\end{aligned}$$

thanks to Lagrange's inequality $\mathrm{i}\{D's,s\}\wedge\{s,D's\} \le |s|^2\mathrm{i}\{D's,D's\}$. On the other hand, we have $d'\eta_\varepsilon = -\chi_0'(\sigma_\varepsilon)d\sigma_\varepsilon$ with $1 \le \chi_0'(\sigma_\varepsilon) \le 2$, hence

$$\begin{aligned}
-\mathrm{i}d'd''\eta_\varepsilon &= \chi_0'(\sigma_\varepsilon)\mathrm{i}d'd''\sigma_\varepsilon + \chi_0''(\sigma_\varepsilon)\mathrm{i}d'\sigma_\varepsilon\wedge d''\sigma_\varepsilon\\
&\ge \Big(\frac{\varepsilon^2}{2|s|^2} + \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\Big)\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon - 2\frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2}.
\end{aligned}$$

We consider the original metric of L multiplied by the weight $|s|^{-2r}$. In this way, we get a curvature form

$$\mathrm{i}\Theta(L) + r\,\mathrm{i}d'd''\log|s|^2 \ge \alpha^{-1}\frac{\{\mathrm{i}\Theta(E)s,s\}}{|s|^2+\varepsilon^2}$$

by hypothesis a) [the inequality is still valid with $|s|^2+\varepsilon^2$ in the denominator in place of $|s|^2$, thanks to the semipositivity of the left hand side]. As $\eta_\varepsilon \ge 2\alpha$ on X_c for ε small, we infer

$$\eta_\varepsilon(\mathrm{i}\Theta(L) + \mathrm{i}d'd''\log|s|^2) - \mathrm{i}d'd''\eta_\varepsilon - \frac{\chi_0''(\sigma_\varepsilon)}{\chi_0'(\sigma_\varepsilon)^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon \ge \frac{\varepsilon^2}{2|s|^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon$$

on X_c. Hence, if $\lambda_\varepsilon = \chi_0'(\sigma_\varepsilon)^2/\chi_0''(\sigma_\varepsilon)$, we obtain

$$\begin{aligned}
B_\varepsilon &:= \big[\eta_\varepsilon(\mathrm{i}\Theta(L) + \mathrm{i}d'd''\log|s|^2) - \mathrm{i}d'd''\eta_\varepsilon - \lambda_\varepsilon^{-1}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon\,,\,\Lambda\big]\\
&\ge \Big[\frac{\varepsilon^2}{2|s|^2}\mathrm{i}d'\eta_\varepsilon\wedge d''\eta_\varepsilon\,,\,\Lambda\Big] = \frac{\varepsilon^2}{2|s|^2}(d''\eta_\varepsilon)(d''\eta_\varepsilon)^*
\end{aligned}$$

as an operator on (n,q)-forms (see the proof of Lemma 12.1).

Let $\theta : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\theta(t) = 1$ on $]-\infty,1/2]$, $\mathrm{Supp}\,\theta \subset\,]-\infty,1[$ and $|\theta'| \le 3$. For $\varepsilon > 0$ small, we consider the (n,q)-form $F_\varepsilon = \theta(\varepsilon^{-2}|s|^2)F_\infty$ and its D''-derivative

$$g_\varepsilon = D''F_\varepsilon = (1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)d''\sigma_\varepsilon\wedge F_\infty + \theta(\varepsilon^{-2}|s|^2)D''F_\infty$$

[as is easily seen from the equality $1+\varepsilon^{-2}|s|^2 = \varepsilon^{-2}e^{\sigma_\varepsilon}$]. We observe that g_ε has its support contained in the tubular neighborhood $|s| < \varepsilon$; moreover, as $\varepsilon \to 0$, the second term in the right hand side converges uniformly to 0 on every compact set; it will therefore produce no contribution in the limit. On the other hand, the first term has the same order of magnitude as $d''\sigma_\varepsilon$ and $d''\eta_\varepsilon$, and can be controlled in terms of B_ε. In fact, for any (n,q)-form u and any $(n,q+1)$-form v we have

$$\begin{aligned}
|\langle d''\eta_\varepsilon\wedge u, v\rangle|^2 &= |\langle u, (d''\eta_\varepsilon)^*v\rangle|^2 \le |u|^2|(d''\eta_\varepsilon)^*v|^2 = |u|^2\langle (d''\eta_\varepsilon)(d''\eta_\varepsilon)^*v, v\rangle\\
&\le \frac{2|s|^2}{\varepsilon^2}|u|^2\langle B_\varepsilon v, v\rangle.
\end{aligned}$$

This implies

$$\langle B_\varepsilon^{-1}(d''\eta_\varepsilon\wedge u), (d''\eta_\varepsilon\wedge u)\rangle \le \frac{2|s|^2}{\varepsilon^2}|u|^2.$$

The main term in g_ε can be written

$$g_\varepsilon^{(1)} := (1+\varepsilon^{-2}|s|^2)\theta'(\varepsilon^{-2}|s|^2)\chi_0'(\sigma_\varepsilon)^{-1}d''\eta_\varepsilon\wedge F_\infty.$$

As $\frac{2|s|^2}{\varepsilon^2} \le 2$ and $(1+\varepsilon^{-2}|s|^2)\chi_0'(\sigma_\varepsilon)^{-1} \le 2$ on $\mathrm{Supp}\, g_\varepsilon^{(1)} \subset \{|s| < \varepsilon\}$, we find

$$\langle B_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\rangle \le 8\,\theta'(\varepsilon^{-2}|s|^2)^2|F_\infty|^2.$$

Instead of working on X itself, we will work rather on the relatively compact subset $X_c \setminus Y_c$, where $Y_c = Y \cap X_c = Y \cap \{\psi < c\}$. We know that $X_c \setminus Y_c$ is again complete Kähler by Lemma 7.9. In this way, we avoid the singularity of the weight $|s|^{-2r}$ along Y. We find

$$\int_{X_c\setminus Y_c}\langle B_\varepsilon^{-1}g_\varepsilon^{(1)}, g_\varepsilon^{(1)}\rangle\,|s|^{-2r}dV_\omega \le 8\int_{X_c\setminus Y_c}|F_\infty|^2\theta'(\varepsilon^{-2}|s|^2)^2|s|^{-2r}dV_\omega.$$

As F_∞ coincides with f on Y, it is not hard to see that the right hand side converges to $8\,c_r\int_{Y_c}|f|^2|\Lambda^r(ds)|^{-2}dV_{Y,\omega}$ where c_r is the "universal" constant

$$c_r = \int_{z\in\mathbb{C}^r,\,|z|\le 1}\theta'(|z|^2)^2\frac{\mathrm{i}^{r^2}\Lambda^r(dz)\wedge\Lambda^r(d\bar z)}{|z|^{2r}} < +\infty$$

depending only on r. The second term

$$g_\varepsilon^{(2)} = \theta(\varepsilon^{-2}|s|^2)d''F_\infty$$

in g_ε satisfies $\mathrm{Supp}(g_\varepsilon^{(2)}) \subset \{|s| < \varepsilon\}$ and $|g_\varepsilon^{(2)}| = O(|s|)$ (just look at the Taylor expansion of $d''F_\infty$ near Y). From this we easily conclude that

$$\int_{X_c\setminus Y_c}\langle B_\varepsilon^{-1}g_\varepsilon^{(2)}, g_\varepsilon^{(2)}\rangle\,|s|^{-2r}dV_{X,\omega} = O(\varepsilon^2),$$

provided that B_ε remains locally uniformly bounded below near Y (this is the case for instance if we have strict inequalities in the curvature assumption a)). If this holds true, we apply Proposition 12.4 on $X_c \setminus Y_c$ with the additional weight factor $|s|^{-2r}$. Otherwise, we use the modified estimate stated in Remark 12.5 in order to solve the approximate equation $D''u + \delta^{1/2}h = g_\varepsilon$ with $\delta > 0$ small. This yields sections $u = u_{c,\varepsilon,\delta}$, $h = h_{c,\varepsilon,\delta}$ such that

$$\int_{X_c\setminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1}|u_{c,\varepsilon,\delta}|^2|s|^{-2r}\,dV_\omega + \int_{X_c\setminus Y_c} |h_{c,\varepsilon,\delta}|^2|s|^{-2r}\,dV_\omega \le 2\int_{X_c\setminus Y_c} \langle (B_\varepsilon+\delta I)^{-1}g_\varepsilon, g_\varepsilon\rangle |s|^{-2r}\,dV_\omega,$$

and the right hand side is under control in all cases. The extra error term $\delta^{1/2}h$ can be removed at the end by letting δ tend to 0. Since there is essentially no additional difficulty involved in this process, we will assume for simplicity of exposition that we do have the required lower bound for B_ε and the estimates of $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ as above. For $\delta = 0$, the above estimate provides a solution $u_{c,\varepsilon}$ of the equation $D''u_{c,\varepsilon} = g_\varepsilon = D''F_\varepsilon$ on $X_c \setminus Y_c$, such that

$$\int_{X_c\setminus Y_c} (\eta_\varepsilon + \lambda_\varepsilon)^{-1}|u_{c,\varepsilon}|^2|s|^{-2r}dV_{X,\omega} \le 2\int_{X_c\setminus Y_c} \langle B_\varepsilon^{-1}g_\varepsilon, g_\varepsilon\rangle\,|s|^{-2r}dV_{X,\omega} \le 16\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega} + O(\varepsilon).$$

Here we have

$$\sigma_\varepsilon = \log(|s|^2+\varepsilon^2) \le \log(e^{-2\alpha}+\varepsilon^2) \le -2\alpha + O(\varepsilon^2) \le -2 + O(\varepsilon^2),$$
$$\eta_\varepsilon = \varepsilon - \chi_0(\sigma_\varepsilon) \le (1+O(\varepsilon))\sigma_\varepsilon^2,$$
$$\lambda_\varepsilon = \frac{\chi_0'(\sigma_\varepsilon)^2}{\chi_0''(\sigma_\varepsilon)} = (1-\sigma_\varepsilon)^2 + (1-\sigma_\varepsilon) \le (3+O(\varepsilon))\sigma_\varepsilon^2,$$
$$\eta_\varepsilon + \lambda_\varepsilon \le (4+O(\varepsilon))\sigma_\varepsilon^2 \le (4+O(\varepsilon))\big(-\log(|s|^2+\varepsilon^2)\big)^2.$$

As F_ε is uniformly bounded with support in $\{|s| < \varepsilon\}$, we conclude from an obvious volume estimate that

$$\int_{X_c}\frac{|F_\varepsilon|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega} \le \frac{\text{Const}}{(\log\varepsilon)^2},$$

hence $F_{c,\varepsilon} = F_\varepsilon - u_{c,\varepsilon}$ satisfies

$$\int_{X_c\setminus Y_c}\frac{|F_{c,\varepsilon}|^2}{(|s|^2+\varepsilon^2)^r(-\log(|s|^2+\varepsilon^2))^2}dV_{X,\omega} \le 64\,c_r\int_{Y_c}\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega} + \frac{\text{Const}}{(\log\varepsilon)^2}.$$

In addition to this, we have $d''F_{c,\varepsilon} = 0$ by construction, and this equation extends from $X_c \setminus Y_c$ to X_c by Lemma 7.10.

If $q = 0$, then $u_{c,\varepsilon}$ must be smooth also, and the non integrability of the weight $|s|^{-2r}$ along Y shows that $u_{c,\varepsilon}$ vanishes on Y, therefore

$$F_{c,\varepsilon|Y} = F_{\varepsilon|Y} = F_{\infty|Y} = f.$$

The theorem and its final estimate are thus obtained by extracting weak limits, first as $\varepsilon \to 0$, and then as $c \to +\infty$. The initial assumption that $\Sigma = \{s = \Lambda^r(ds) = 0\}$ is empty can be easily removed in two steps: i) the result is true if X is Stein, since we can always find a complex hypersurface Z in X such that $\Sigma \subset \overline{Y} \cap Z \subsetneq \overline{Y}$, and then apply the extension theorem on the Stein manifold $X \setminus Z$, in combination with Lemma 7.10; ii) the whole procedure still works when Σ is nowhere dense in $\overline{Y}$ (and possibly nonempty). Indeed local L^2 extensions $\widetilde{f}_j$ still exist by step i) applied on small coordinate balls U_j; we then set $F_\infty = \sum\theta_j\widetilde{f}_j$ and observe that $|D''F_\infty|^2|s|^{-2r}$ is locally integrable, thanks to the estimate $\int_{U_j}|\widetilde{f}_j|^2|s|^{-2r}(\log|s|)^{-2}dV < +\infty$ and the fact that $|\sum d''\theta_j\wedge\widetilde{f}_j| = O(|s|^\delta)$ for suitable $\delta > 0$ [as follows from Hilbert's Nullstensatz applied to $\widetilde{f}_j - \widetilde{f}_k$ at singular points of $\overline{Y}$].

When $q \ge 1$, the arguments needed to get a smooth solution involve more delicate considerations, and we will skip the details, which are extremely technical and not very enlightening.

(12.7) Remarks.

a) When $q = 0$, the estimates provided by Theorem 12.6 are independent of the Kähler metric ω. In fact, if f and F are holomorphic sections of $\Lambda^n T_X^*\otimes L$ over Y (resp. X), viewed as $(n,0)$-forms with values in L, we can "divide" f by $\Lambda^r(ds) \in \Lambda^r(TX/TY)^*\otimes\det E$ to get a section $f/\Lambda^r(ds)$ of $\Lambda^p T_Y^*\otimes L\otimes(\det E)^{-1}$ over Y. We then find

$$|F|^2dV_{X,\omega} = i^{n^2}\{F,F\},$$
$$\frac{|f|^2}{|\Lambda^r(ds)|^2}dV_{Y,\omega} = i^{p^2}\{f/\Lambda^r(ds), f/\Lambda^r(ds)\},$$

where $\{\bullet,\bullet\}$ is the canonical bilinear pairing described in (6.3).

b) The hermitian structure on E is not really used in depth. In fact, one only needs E to be equipped with a *Finsler metric*, that is, a smooth complex homogeneous function of degree 2 on E [or equivalently, a smooth hermitian metric on the tautological bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $P(E)$]. The section s of E induces a section $[s]$ of $P(E)$ over $X \setminus s^{-1}(0)$ and a corresponding section $\widetilde{s}$ of the pull-back line bundle $[s]^*\mathcal{O}_{P(E)}(-1)$. A trivial check shows that Theorem 12.6 as well as its proof extend to the case of a Finsler metric on E, if we replace everywhere $\{i\Theta(E)s, s\}$ by $\{i\Theta([s]^*\mathcal{O}_{P(E)}(-1))\widetilde{s}, \widetilde{s}\}$ (especially in hypothesis 12.6 b)). A minor issue is that $|\Lambda^r(ds)|$ is (a priori) no longer defined, since no obvious hermitian norm exists on $\det E$. A posteriori, we have the following ad hoc definition of a metric on $(\det E)^*$ which makes the L^2 estimates work as before: for $x \in X$ and $\xi \in \Lambda^r E_x^*$, we set

$$|\xi|_x^2 = \frac{1}{c_r}\int_{z\in E_x}\theta'(|z|^2)^2\,\frac{i^{r^2}\xi\wedge\overline{\xi}}{|z|^{2r}}$$

where $|z|$ is the Finsler norm on E_x [the constant c_r is there to make the result agree with the hermitian case; it is not hard to see that this metric does not depend on the choice of θ]. □

We now present a few interesting corollaries. The first one is a surjectivity theorem for restriction morphisms in Dolbeault cohomology.

(12.8) Corollary. *Let X be a projective algebraic manifold and E a holomorphic vector bundle of rank r over X, s a holomorphic section of E which is everywhere transverse to the zero section, $Y = s^{-1}(0)$, and let L be a holomorphic line bundle such that $F = L^{1/r} \otimes E^\star$ is Griffiths positive (we just mean formally that $\frac{1}{r}\mathrm{i}\Theta(L)\otimes \mathrm{Id}_E - \mathrm{i}\Theta(E) >_{\mathrm{Grif}} 0$). Then the restriction morphism*

$$H^{0,q}(X, \Lambda^n T^\star_X \otimes L) \to H^{0,q}(Y, \Lambda^n T^\star_X \otimes L)$$

is surjective for every $q \geq 0$.

Proof. A short computation gives

$$\begin{aligned}\mathrm{i}\,d'd''\log|s|^2 &= \mathrm{i}\,d'\Big(\frac{\{s, D's\}}{|s|^2}\Big)\\ &= \mathrm{i}\Big(\frac{\{D's, D's\}}{|s|^2} - \frac{\{D's, s\}\wedge\{s, D's\}}{|s|^4} + \frac{\{s, \Theta(E)s\}}{|s|^2}\Big) \geq -\frac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2}\end{aligned}$$

thanks to Lagrange s inequality and the fact that $\Theta(E)$ is antisymmetric. Hence, if δ is a small positive constant such that

$$-\mathrm{i}\Theta(E) + \frac{1}{r}\mathrm{i}\Theta(L)\otimes\mathrm{Id}_E \geq_{\mathrm{Grif}} \delta\,\omega\otimes\mathrm{Id}_E > 0,$$

we find

$$\mathrm{i}\Theta(L) + r\,\mathrm{i}\,d'd''\log|s|^2 \geq r\delta\,\omega.$$

The compactness of X implies $\mathrm{i}\Theta(E) \leq C\omega\otimes\mathrm{Id}_E$ for some $C > 0$. Theorem 12.6 can thus be applied with $\alpha = r\delta/C$ and Corollary 12.8 follows. By remark 12.7 b), the above surjectivity property even holds if $L^{1/r}\otimes E^\star$ is just assumed to be ample (in the sense that the associated line bundle $\pi^\star L^{1/r}\otimes\mathcal{O}_{P(E)}(1)$ is positive on the projectivized bundle $\pi : P(E) \to X$ of lines of E). □

Another interesting corollary is the following special case, dealing with bounded pseudoconvex domains $\Omega\subset\subset\mathbb{C}^n$. Even this simple version retains highly interesting information on the behavior of holomorphic and plurisubharmonic functions.

(12.9) Corollary. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $Y \subset X$ be a nonsingular complex submanifold defined by a section s of some hermitian vector bundle E with* bounded *curvature tensor on Ω. Assume that s is everywhere transverse to the zero section and that $|s| \leq e^{-1}$ on Ω. Then there is a constant $C > 0$ (depending only on E), with the following property: for every psh function φ on Ω, every holomorphic function f on Y with $\int_Y |f|^2|\Lambda^r(ds)|^{-2}e^{-\varphi}dV_Y < +\infty$, there exists an extension F of f to Ω such that*

$$\int_\Omega \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}e^{-\varphi}dV_\Omega \leq C\int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2}e^{-\varphi}dV_Y.$$

Proof. We apply essentially the same idea as for the previous corollary, in the special case when $L = \Omega\times\mathbb{C}$ is the trivial bundle equipped with a weight function $e^{-\varphi - A|z|^2}$. The choice of a sufficiently large constant $A > 0$ guarantees that the curvature assumption 12.6 a) is satisfied (A just depends on the presupposed bound for the curvature tensor of E). □

(12.10) Remark. The special case when $Y = \{z_0\}$ is a point is especially interesting. In that case, we just take $s(z) = (e\,\mathrm{diam}\,\Omega)^{-1}(z - z_0)$, viewed as a section of the rank $r = n$ trivial vector bundle $\Omega\times\mathbb{C}^n$ with $|s| \leq e^{-1}$. We take $\alpha = 1$ and replace $|s|^{2n}(-\log|s|)^2$ in the denominator by $|s|^{2(n-\varepsilon)}$, using the inequality

$$-\log|s| = \frac{1}{\varepsilon}\log|s|^{-\varepsilon} \leq \frac{1}{\varepsilon}|s|^{-\varepsilon}, \qquad \forall\varepsilon > 0.$$

For any given value f_0, we then find a holomorphic function f such that $f(z_0) = f_0$ and

$$\int_\Omega \frac{|f(z)|^2}{|z - z_0|^{2(n-\varepsilon)}}e^{-\varphi(z)}dV_\Omega \leq \frac{C_n}{\varepsilon^2(\mathrm{diam}\,\Omega)^{2(n-\varepsilon)}}|f_0|^2e^{-\varphi(z_0)}.$$

We prove here, as an application of the Ohsawa-Takegoshi extension theorem, that every psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ can be approximated very accurately by functions of the form $c\log|f|$, where $c > 0$ and f is a holomorphic function. The main idea is taken from [Dem92]. For other applications to algebraic geometry, see [Dem93b] and Demailly-Kollár [DK96]. Recall that the Lelong number of a function $\varphi \in \mathrm{Psh}(\Omega)$ at a point x_0 is defined to be

$$\nu(\varphi, x_0) = \liminf_{z\to x_0}\frac{\log\varphi(z)}{\log|z - x_0|} = \lim_{r\to 0.}\frac{\sup_{B(x_0,r)}\varphi}{\log r}.$$

In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\mathrm{ord}_{x_0}(f) = \sup\{k \in \mathbb{N};\, D^\alpha f(x_0) = 0,\ \forall|\alpha| < k\}$.

(12.11) Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2e^{-2m\varphi}d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m}\log\sum|\sigma_\ell|^2$ where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

a) $\varphi(z) - \dfrac{C_1}{m} \leq \varphi_m(z) \leq \sup\limits_{|\zeta - z|<r}\varphi(\zeta) + \dfrac{1}{m}\log\dfrac{C_2}{r^n}$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

b) $\nu(\varphi, z) - \dfrac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof. Note that $\sum|\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum|\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f\in B(1)} \frac{1}{m}\log|f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z,\partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned}|f(z)|^2 &\le \frac{1}{\pi^n r^{2n}/n!}\int_{|\zeta-z|<r}|f(\zeta)|^2 d\lambda(\zeta)\\ &\le \frac{1}{\pi^n r^{2n}/n!}\exp\Big(2m\sup_{|\zeta-z|<r}\varphi(\zeta)\Big)\int_\Omega |f|^2 e^{-2m\varphi}d\lambda.\end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \le \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{2m}\log\frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in a) is proved. Conversely, the Ohsawa-Takegoshi extension theorem (estimate 4.10) applied to the 0-dimensional subvariety $\{z\}\subset\Omega$ shows that for any $a\in\mathbb{C}$ there is a holomorphic function f on Ω such that $f(z)=a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi}d\lambda \le C_3|a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \ge \frac{1}{m}\log|a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m,z)\le\nu(\varphi,z)$. In the opposite direction, we find

$$\sup_{|x-z|<r}\varphi_m(x) \le \sup_{|\zeta-z|<2r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x,r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m,x) \ge \nu(\varphi,x) - \frac{n}{m}. \qquad \square$$

Theorem 12.11 implies in a straightforward manner the deep result of [Siu74] on the analyticity of the Lelong number sublevel sets.

(12.12) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c>0$, the Lelong number sublevel set*

$$E_c(\varphi) = \big\{z\in X\,;\ \nu(\varphi,z)\ge c\big\}$$

is an analytic subset of X.

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega\subset\mathbb{C}^n$. The inequalities obtained in 12.11 b) imply that

$$E_c(\varphi) = \bigcap_{m\ge m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z)=0$ for all multi-indices α such that $|\alpha|<mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. $\square$

We now translate Theorem 12.11 into a more geometric setting. Let X be a projective manifold and L a line bundle over X. A singular hermitian metric h on L is a metric such that the weight function φ of h is $L^1_{\rm loc}$ in any local trivialization (such that $L_{|U}\simeq U\times\mathbb{C}$ and $\|\xi\|_h = |\xi|e^{-\varphi(x)}$, $\xi\in L_x\simeq\mathbb{C}$). The curvature of L can then be computed in the sense of distributions

$$T = \frac{\mathrm{i}}{2\pi}\Theta_h(L) = \frac{\mathrm{i}}{\pi}\partial\overline{\partial}\varphi,$$

and L is said to be pseudoeffective if L admits a singular hermitian metric h such that the curvature current $T=\frac{\mathrm{i}}{2\pi}\Theta_h(L)$ is semipositive [The weight functions φ of L are thus plurisubharmonic]. Our goal is to approximate T in the weak topology by divisors which have roughly the same Lelong numbers as T. The existence of weak approximations by divisors has already been proved in [Lel72] for currents defined on a pseudoconvex open set $\Omega\subset\mathbb{C}^n$ with $H^2(\Omega,\mathbb{R})=0$, and in [Dem92, 93b] in the situation considered here (cf. also [Dem82b], although the argument given there is somewhat incorrect). We take the opportunity to present here a slightly simpler derivation. In what follows, we use an additive notation for $\operatorname{Pic}(X)$, i.e. kL is meant for the line bundle $L^{\otimes k}$.

(12.13) Proposition. *For any $T=\frac{\mathrm{i}}{2\pi}\Theta_h(L)\ge 0$ and any ample line bundle F, there is a sequence of non zero sections $h_s\in H^0(X,p_sF+q_sL)$ with $p_s,q_s>0$, $\lim q_s=+\infty$ and $\lim p_s/q_s=0$, such that the divisors $D_s=\frac{1}{q_s}\operatorname{div}(h_s)$ satisfy $T=\lim D_s$ in the weak topology and $\sup_{x\in X}|\nu(D_s,x)-\nu(T,x)|\to 0$ as $s\to+\infty$.*

(12.14) Remark. The proof will actually show, with very slight modifications, that Proposition 12.13 also holds when X is a Stein manifold and L is an arbitrary holomorphic line bundle.

Proof. We first use Hörmander's L^2 estimates to construct a suitable family of holomorphic sections and combine this with some ideas of [Lel72] in a second step. Select a smooth metric with positive curvature on F, choose $\omega=\frac{\mathrm{i}}{2\pi}\Theta(F)>0$ as a Kähler metric on X and fix some large integer k (how large k must be will be specified later). For all $m\ge 1$ we define

$$w_m(z) = \sup_{1\le j\le N}\frac{1}{m}\log\|f_j(z)\|,$$

where $(f_1,\dots,f_N)$ is an orthonormal basis of the space of sections of $\mathcal{O}(kF+mL)$ with finite global L^2 norm $\int_X\|f\|^2dV_\omega$. Let e_F and e_L be non vanishing holomorphic sections of F and L on a trivializing open set Ω, and let $e^{-\psi}=\|e_F\|$, $e^{-\varphi}=\|e_L\|$ be the corresponding weights. If f is a section of $\mathcal{O}(kF+mL)$ and if we still denote

by f the associated complex valued function on Ω with respect to the holomorphic frame $e_F^k \otimes e_L^m$, we have $\|f(z)\| = |f(z)|e^{-k\psi(z)-m\varphi(z)}$; here φ is plurisubharmonic, ψ is smooth and strictly plurisubharmonic, and $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, $\omega = \frac{i}{\pi}\partial\overline{\partial}\psi$. In Ω, we can write

$$w_m(z) = \sup_{1\le j\le N} \frac{1}{m}\log|f_j(z)| - \varphi(z) - \frac{k}{m}\psi(z).$$

In particular $T_m := \frac{i}{\pi}\partial\overline{\partial}w_m + T + \frac{k}{m}\omega$ is a closed positive current belonging to the cohomology class $c_1(L) + \frac{k}{m}c_1(F)$.

Step 1. We claim that T_m converges to T as m tends to $+\infty$ and that T_m satisfies the inequalities

$$\nu(T,x) - \frac{n}{m} \le \nu(T_m,x) \le \nu(T,x) \tag{12.14}$$

at every point $x \in X$. Note that T_m is defined on Ω by $T_m = \frac{i}{\pi}\partial\overline{\partial}v_{m,\Omega}$ with

$$v_{m,\Omega}(z) = \sup_{1\le j\le N} \frac{1}{m}\log|f_j(z)|, \qquad \int_\Omega |f_j|^2 e^{-2k\psi-2m\varphi}dV_\omega \le 1.$$

We proceed in the same way as for the proof of Theorem 12.11. We suppose here that Ω is a coordinate open set with analytic coordinates $(z_1,\dots,z_n)$. Take $z \in \Omega' \subset\subset \Omega$ and $r \le r_0 = \frac{1}{2}d(\Omega',\partial\Omega)$. By the L^2 estimate and the mean value inequality for subharmonic functions, we obtain

$$|f_j(z)|^2 \le \frac{C_1}{r^{2n}}\int_{|\zeta-z|<r} |f_j(\zeta)|^2 d\lambda(\zeta) \le \frac{C_2}{r^{2n}} \sup_{|\zeta-z|<r} e^{2m\varphi(\zeta)}$$

with constants C_1, C_2 independent of m and r (the smooth function ψ is bounded on any compact subset of Ω). Hence we infer

$$v_{m,\Omega}(z) \le \sup_{|\zeta-z|<r} \varphi(\zeta) + \frac{1}{2s}\log\frac{C_2}{r^{2n}}. \tag{12.15}$$

If we choose for example $r = 1/m$ and use the upper semi-continuity of φ, we infer $\limsup_{s\to+\infty} v_{m,\Omega} \le \varphi$. Moreover, if $\gamma = \nu(\varphi,x) = \nu(T,x)$, then $\varphi(\zeta) \le \gamma\log|\zeta-x| + O(1)$ near x. By taking $r = |z-x|$ in (12.15), we find

$$v_{m,\Omega}(z) \le \sup_{|\zeta-x|<2r} \varphi(\zeta) - \frac{n}{m}\log r + O(1) \le \Big(\gamma - \frac{n}{m}\Big)\log|z-x| + O(1),$$
$$\nu(T_m,x) = \nu(v_{m,\Omega},x) \ge \Big(\gamma - \frac{n}{m}\Big)_+ \ge \nu(T,x) - \frac{n}{m}.$$

In the opposite direction, the inequalities require deeper arguments since we actually have to construct sections in $H^0(X,kF+mL)$. Assume that Ω is chosen isomorphic to a bounded pseudoconvex open set in $\mathbb{C}^n$. By the Ohsawa-Takegoshi L^2 extension theorem (remark (4.11)), for every point $x \in \Omega$, there is a holomorphic function g on Ω such that $g(x) = e^{m\varphi(x)}$ and

$$\int_\Omega |g(z)|^2 e^{-2m\varphi(z)}d\lambda(z) \le C_3,$$

where C_3 depends only on n and $\operatorname{diam}(\Omega)$. For $x \in \Omega'$, set

$$\sigma(z) = \theta\big(|z-x|/r\big)\, g(z)\, e_F(z)^k \otimes e_L(z)^m, \qquad r = \min\big(1, 2^{-1}d(\Omega',\partial\Omega)\big),$$

where $\theta : \mathbb{R} \to [0,1]$ is a cut-off function such that $\theta(t) = 1$ for $t < 1/2$ and $\theta(t) = 0$ for $t \ge 1$. We solve the global equation $\overline{\partial}u = v$ on X with $v = \overline{\partial}\sigma$, after multiplication of the metric of $kF+mL$ with the weight

$$e^{-2n\rho_x(z)}, \qquad \rho_x(z) = \theta\big(|z-x|/r\big)\,\log|z-x| \le 0.$$

The $(0,1)$-form v can be considered as a $(n,1)$-form with values in the line bundle $\mathcal{O}(-K_X+kF+mL)$ and the resulting curvature form of this bundle is

$$\operatorname{Ricci}(\omega) + k\omega + mT + n\frac{i}{\pi}\partial\overline{\partial}\rho_x.$$

Here the first two summands are smooth, $i\partial\overline{\partial}\rho_x$ is smooth on $X\setminus\{x\}$ and ≥ 0 on $B(x,r/2)$, and T is a positive current. Hence by choosing k large enough, we can suppose that this curvature form is $\ge \omega$, uniformly for $x \in \Omega'$. By Hörmander's standard L^2 estimates [AV65, Hör65, 66], we get a solution u on X such that

$$\int_X \|u\|^2 e^{-2n\rho_x}dV_\omega \le C_4\int_{r/2<|z-x|<r} |g|^2 e^{-2k\psi-2m\varphi-2n\rho_x}dV_\omega \le C_5;$$

to get the estimate, we observe that v has support in the corona $r/2 < |z-x| < r$ and that ρ_x is bounded there. Thanks to the logarithmic pole of ρ_x, we infer that $u(x) = 0$. Moreover

$$\int_\Omega \|\sigma\|^2 dV_\omega \le \int_{\Omega'+B(0,r/2)} |g|^2 e^{-2k\psi-2m\varphi}dV_\omega \le C_6,$$

hence $f = \sigma - u \in H^0(X,kF+mL)$ satisfies $\int_X \|f\|^2 dV_\omega \le C_7$ and

$$\|f(x)\| = \|\sigma(x)\| = \|g(x)\|\,\|e_F(x)\|^m\|e_L(x)\|^m = \|e_F(x)\|^k = e^{-k\psi(x)}.$$

In our orthonormal basis (f_j), we can write $f = \sum\lambda_j f_j$ with $\sum|\lambda_j|^2 \le C_7$. Therefore

$$e^{-k\psi(x)} = \|f(x)\| \le \sum|\lambda_j|\sup\|f_j(x)\| \le \sqrt{C_7 N}\, e^{mw_m(x)},$$
$$w_m(x) \ge \frac{1}{m}\log(C_7N)^{-1/2}\|f(x)\| \ge -\frac{1}{m}\Big(\log(C_7N)^{1/2} + k\,\psi(x)\Big)$$

where $N = \dim H^0(X,kF+mL) = O(m^n)$. By adding $\varphi + \frac{k}{m}\psi$, we get $v_{m,\Omega} \ge \varphi - C_8m^{-1}\log m$. Thus $\lim_{m\to+\infty} v_{m,\Omega} = \varphi$ everywhere, $T_m = \frac{i}{\pi}\partial\overline{\partial}v_{m,\Omega}$ converges weakly to $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, and

$$\nu(T_m,x) = \nu(v_{m,\Omega},x) \le \nu(\varphi,x) = \nu(T,x).$$

Note that $\nu(v_{m,\Omega},x) = \frac{1}{m}\min\operatorname{ord}_x(f_j)$ where $\operatorname{ord}_x(f_j)$ is the vanishing order of f_j at x, so our initial lower bound for $\nu(T_m,x)$ combined with the last inequality gives

$$\nu(T,x) - \frac{n}{m} \le \frac{1}{m}\min\operatorname{ord}_x(f_j) \le \nu(T,x). \tag{12.16}$$

Step 2: *Construction of the divisors D_s.*

Select sections $(g_1,\dots,g_N)\in H^0(X,k_0F)$ with k_0 so large that k_0F is very ample, and set

$$h_{\ell,m}=f_1^\ell g_1+\ldots+f_N^\ell g_N\in H^0\big(X,(k_0+\ell k)F+\ell mL\big).$$

For almost every N-tuple $(g_1,\dots,g_N)$, Lemma 12.17 below and the weak continuity of $\partial\overline{\partial}$ show that

$$\Delta_{\ell,m}=\frac{1}{\ell m}\frac{i}{\pi}\partial\overline{\partial}\log|h_{\ell,m}|=\frac{1}{\ell m}\operatorname{div}(h_{\ell,m})$$

converges weakly to $T_m=\frac{i}{\pi}\partial\overline{\partial}v_{m,\Omega}$ as ℓ tends to $+\infty$, and that

$$\nu(T_m,x)\le\nu\Big(\frac{1}{\ell m}\Delta_{\ell,m},x\Big)\le\nu(T,x)+\frac{1}{\ell m}.$$

This, together with the first step, implies the proposition for some subsequence $D_s=\Delta_{\ell(s),s}$, $\ell(s)\gg s\gg 1$. We even obtain the more explicit inequality

$$\nu(T,x)-\frac{n}{m}\le\nu\Big(\frac{1}{\ell m}\Delta_{\ell,m},x\Big)\le\nu(T,x)+\frac{1}{\ell m}.\qquad\square$$

(12.17) Lemma. *Let Ω be an open subset in $\mathbb{C}^n$ and let $f_1,\dots,f_N\in H^0(\Omega,\mathcal{O}_\Omega)$ be non zero functions. Let $G\subset H^0(\Omega,\mathcal{O}_\Omega)$ be a finite dimensional subspace whose elements generate all 1-jets at any point of Ω. Finally, set $v=\sup\log|f_j|$ and*

$$h_\ell=f_1^\ell g_1+\ldots+f_N^\ell g_N,\qquad g_j\in G\smallsetminus\{0\}.$$

Then for all $(g_1,\dots,g_N)$ in $(G\smallsetminus\{0\})^N$ except a set of measure 0, the sequence $\frac{1}{\ell}\log|h_\ell|$ converges to v in $L^1_{\rm loc}(\Omega)$ and

$$\nu(v,x)\le\nu\Big(\frac{1}{\ell}\log|h_\ell|\Big)\le\nu(v,x)+\frac{1}{\ell},\qquad\forall x\in X,\quad\forall\ell\ge1.$$

Proof. The sequence $\frac{1}{\ell}\log|h_\ell|$ is locally uniformly bounded above and we have

$$\lim_{\ell\to+\infty}\frac{1}{\ell}\log|h_\ell(z)|=v(z)$$

at every point z where all absolute values $|f_j(z)|$ are distinct and all $g_j(z)$ are nonzero. This is a set of full measure in Ω because the sets $\{|f_j|^2=|f_l|^2,\ j\ne l\}$ and $\{g_j=0\}$ are real analytic and thus of zero measure (without loss of generality, we may assume that Ω is connected and that the f_j's are not pairwise proportional). The well-known uniform integrability properties of plurisubharmonic functions then show that $\frac{1}{\ell}\log|h_\ell|$ converges to v in $L^1_{\rm loc}(\Omega)$. It is easy to see that $\nu(v,x)$ is the minimum of the vanishing orders $\operatorname{ord}_x(f_j)$, hence

$$\nu(\log|h_\ell|,x)=\operatorname{ord}_x(h_\ell)\ge\ell\,\nu(v,x).$$

In the opposite direction, consider the set $\mathcal{E}_\ell$ of all $(N+1)$-tuples

$$(x,g_1,\dots,g_N)\in\Omega\times G^N$$

for which $\nu(\log|h_\ell|,x)\ge\ell\,\nu(v,x)+2$. Then $\mathcal{E}_\ell$ is a constructible set in $\Omega\times G^N$: it has a locally finite stratification by analytic sets, since

$$\mathcal{E}_\ell=\bigcup_{s\ge0}\Big(\bigcup_{j,\,|\alpha|=s}\{x\,;\,D^\alpha f_j(x)\ne0\}\times G^N\Big)\cap\bigcap_{|\beta|\le\ell s+1}\{(x,(g_j))\,;\,D^\beta h_\ell(x)=0\}.$$

The fiber $\mathcal{E}_\ell\cap(\{x\}\times G^N)$ over a point $x\in\Omega$ where $\nu(v,x)=\min\operatorname{ord}_x(f_j)=s$ is the vector space of N-tuples $(g_j)\in G^N$ satisfying the equations $D^\beta\big(\sum f_j^\ell g_j(x)\big)=0$, $|\beta|\le\ell s+1$. However, if $\operatorname{ord}_x(f_j)=s$, the linear map

$$(0,\dots,0,g_j,0,\dots,0)\longmapsto\big(D^\beta(f_j^\ell g_j(x))\big)_{|\beta|\le\ell s+1}$$

has rank $n+1$, because it factorizes into an injective map $J^1_x g_j\mapsto J^{\ell s+1}_x(f_j^\ell g_j)$. It follows that the fiber $\mathcal{E}_\ell\cap(\{x\}\times G^N)$ has codimension at least $n+1$. Therefore

$$\dim\mathcal{E}_\ell\le\dim(\Omega\times G^N)-(n+1)=\dim G^N-1$$

and the projection of $\mathcal{E}_\ell$ on G^N has measure zero by Sard's theorem. By definition of $\mathcal{E}_\ell$, any choice of $(g_1,\dots,g_N)\in G^N\smallsetminus\bigcup_{\ell\ge1}\operatorname{pr}(\mathcal{E}_\ell)$ produces functions h_ℓ such that $\nu(\log|h_\ell|,x)\le\ell\,\nu(v,x)+1$ on Ω. $\square$

13. Invariance of Plurigenera of Varieties of General Type

The goal of this section is to give a proof of the following fundamental result on the invariance of the plurigenera, proved by Y.T. Siu [Siu97].

(13.0) Theorem (Siu). *Let $X\to S$ be a smooth projective family of varieties of general type on a connected base S. Then the plurigenus $p_m(X_t)=h^0(X_t,mK_{X_t})$ of fibers is independent of t for all $m\ge0$.*

Given a family $\gamma:X\to S$ of projective varieties, the proof of the invariance of the plurigenera of fibers $X_t=\gamma^{-1}(t)$ is easily reduced to the case when the base is the unit disk $\Delta\subset\mathbb{C}$ (in general, one can always connect two arbitrary points of S by a chain of small analytic disks, and one can take the pull-back of the family in restriction to each of those disks). We can therefore suppose that $\gamma:X\to\Delta$. In that case, we have a canonical isomorphism $K_{X_t}\simeq K_{X|X_t}$ on each fiber, given by $u\mapsto dt\wedge u$ [in this way, we will allow ourselves to identify K_{X_t} and $K_{X|X_t}$ in the sequel]. we know from Grauert's direct image theorem [Gra60] that the direct image sheaves $\gamma_*\mathcal{O}(mK_X)$ are coherent, and moreover, the plurigenera $p_m(X_t)=h^0(X_t,(mK_X)_{|X_t})$ are upper semi-continuous functions of t. The jumps occur precisely if a section on some fiber X_{t_0} does not extend to nearby fibers. Proving the invariance of plurigenera is thus equivalent to proving that a section of $mK_{X|X_{t_0}}$ on a fiber X_{t_0} can be extended to a neighborhood of X_{t_0} in X. We can assume without loss of generality that $t_0=0$. The strategy of Siu's proof consists more or less in the use of suitable singular hermitian metrics on K_X and K_{X_0} (with "minimal singularities"), combined with L^2 extension theorems with respect to these metrics.

13.1. Metrics with minimal singularities

One of the main ideas in the proof of the invariance of plurigenera – although it is not completely explicit in [Siu97] – rests on the fact that the singularities of hermitian metrics on a line bundle L can reflect very accurately the base loci of the sequence of linear systems $|mL|$, precisely when metrics with minimal singularities are used. We follow here the approach of [DPS00].

(13.1.1) Definition. *Let L be a pseudo-effective line bundle on a compact complex manifold X. Consider two hermitian metrics h_1, h_2 on L with curvature $i\Theta_{h_j}(L) \geq 0$ in the sense of currents.*

(i) *We will write $h_1 \preccurlyeq h_2$, and say that h_1 is less singular than h_2, if there exists a constant $C > 0$ such that $h_1 \leq Ch_2$.*

(ii) *We will write $h_1 \sim h_2$, and say that h_1, h_2 are equivalent with respect to singularities, if there exists a constant $C > 0$ such that $C^{-1}h_2 \leq h_1 \leq Ch_2$.*

Of course $h_1 \preccurlyeq h_2$ if and only if the associated weights in suitable trivializations locally satisfy $\varphi_2 \leq \varphi_1 + C$. This implies in particular $\nu(\varphi_1, x) \leq \nu(\varphi_2, x)$ at each point. The above definition is motivated by the following observation.

(13.1.2) Theorem. *For every pseudo-effective line bundle L over a compact complex manifold X, there exists up to equivalence of singularities a unique class of hermitian metrics h with minimal singularities such that $i\Theta_h(L) \geq 0$.*

Proof. The proof is almost trivial. We fix once for all a smooth metric h_∞ (whose curvature is of random sign and signature), and we write singular metrics of L under the form $h = h_\infty e^{-2\psi}$. The condition $i\Theta_h(L) \geq 0$ is equivalent to $\frac{i}{\pi}\partial\overline{\partial}\psi \geq -u$ where $u = i\Theta_{h_\infty}(L)$. This condition implies that ψ is plurisubharmonic up to the addition of the weight φ_∞ of h_∞, and therefore locally bounded from above. Since we are concerned with metrics only up to equivalence of singularities, it is always possible to adjust ψ by a constant in such a way that $\sup_X \psi = 0$. We now set

$$h_{\min} = h_\infty e^{-2\psi_{\min}}, \qquad \psi_{\min}(x) = \sup_\psi \psi(x)$$

where the supremum is extended to all functions ψ such that $\sup_X \psi = 0$ and $\frac{i}{\pi}\partial\overline{\partial}\psi \geq -u$. By standard results on plurisubharmonic functions (see Lelong [Lel69]), $\psi_{\min}$ still satisfies $\frac{i}{\pi}\partial\overline{\partial}\psi_{\min} \geq -u$ (i.e. the weight $\varphi_\infty + \psi_{\min}$ de $h_{\min}$ is plurisubharmonic), and $h_{\min}$ is obviously the metric with minimal singularities that we were looking for. □

Now, given a section $\sigma \in H^0(X, mL)$, the expression $h(\xi) = |\xi^m/\sigma(x)|^{2/m}$ defines a singular metric on L, which therefore necessarily has at least as much singularity as $h_{\min}$ as, i.e. $\frac{1}{m}\log|\sigma|^2 \leq \varphi_{\min} + C$ locally. In particular, $|\sigma|^2 e^{-m\varphi_{\min}}$ is locally bounded, hence $\sigma \in H^0(X, mL \otimes \mathcal{I}(h_{\min}^{\otimes m}))$. For all $m > 0$ we therefore have an isomorphism

$$H^0(X, mL \otimes \mathcal{I}(h_{\min}^{\otimes m})) \xrightarrow{\simeq} H^0(X, mL).$$

By the well-known properties of Lelong numbers (see Skoda [Sk72]), the union of all zero varieties of the ideals $\mathcal{I}(h_{\min}^{\otimes m})$ is equal to the Lelong sublevel set

$$E_+(h_{\min}) = \big\{x \in X\,;\ \nu(\varphi_{\min}, x) > 0\big\}. \tag{13.1.3}$$

We will call this set the *virtual base locus* of L. It is always contained in the "algebraic" base locus

$$B_{\|L\|} = \bigcap_{m>0} B_{|mL|}, \qquad B_{|mL|} = \bigcap_{\sigma \in H^0(X,mL)} \sigma^{-1}(0),$$

but there may be a strict inclusion. This is the case for instance if $L \in \mathrm{Pic}^0(X)$ is such that all positive multiples mL have no nonzero sections; in that case $E_+(h_{\min}) = \emptyset$ but $\bigcap_{m>0} B_{|mL|} = X$. Another general situation where $E_+(h_{\min})$ and $B_{\|L\|}$ can differ is given by the following result.

(13.1.4) Proposition. *Let L be a big nef line bundle. Then $h_{\min}$ has zero Lelong numbers everywhere, i.e. $E_+(h_{\min}) = \emptyset$.*

Proof. Recall that L is big if its Kodaira-Iitaka dimension $\kappa(L)$ is equal to $n = \dim X$. In that case, it is well known that one can write $m_0 L = A + E$ with A ample and E effective, for m_0 sufficiently large. Then $mL = ((m - m_0)L + A) + E$ is the sum of an ample divisor $A_m = (m - m_0)L + A$ plus a (fixed) effective divisor, so that there is a hermitian metric h_m on L for which $i\Theta_{h_m}(L) = \frac{1}{m} i\Theta(A_m) + \frac{1}{m}[E]$, with a suitable smooth positive form $i\Theta(A_m)$. This shows that the Lelong numbers of the weight of h_m are $O(1/m)$, hence in the limit those of $h_{\min}$ are zero.

If h is a singular hermitian metric such that $i\Theta_h(L) \geq 0$ and

$$H^0(X, mL \otimes \mathcal{I}(h^{\otimes m})) \simeq H^0(X, mL) \qquad \text{for all } m \geq 0, \tag{13.1.5}$$

we say that h is an *analytic Zariski decomposition* of L. We have just seen that such a decomposition always exists and that $h = h_{\min}$ is a solution. The concept of analytic Zariski decomposition is motivated by its algebraic counterpart (the existence of which generally fails) : one says that L admits an *algebraic Zariski decomposition* if there exists an integer m_0 such that $m_0 L \simeq \mathcal{O}(E + D)$ where E is an effective divisor and D a nef divisor, in such a way that $H^0(X, kD) \simeq H^0(X, km_0 L)$ for all $k \geq 0$. If $\mathcal{O}(*D)$ is generated by sections, there is a smooth metric with semipositive curvature on $\mathcal{O}(D)$, and this metric induces a singular hermitian metric h on L of curvature current $\frac{1}{m_0}(i\Theta(\mathcal{O}(D)) + [E])$. Its poles are defined by the effective $\mathbb{Q}$-divisor $\frac{1}{m_0}E$. For this metric, we of course have $\mathcal{I}(h^{\otimes km_0}) = \mathcal{O}(-kE)$, hence (13.1.5) holds true at least when m is a multiple of m_0.

13.2. A uniform global generation property

The "uniform global generation property" shows in some sense that the curvature of the tensor product sheaf $L \otimes \mathcal{I}(h)$ is uniformly bounded below, for any singular hermitian metric h with nonnegative curvature on L.

(13.2.1) Lemma. *There exists an ample line bundle G on X such that for every pseudoeffective line bundle (L,h), the sheaf $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$ is generated by its global sections. In fact, G can be chosen as follows: pick any very ample line bundle A, and take G such that $G-(K_X+nA)$ is ample, e.g. $G=K_X+(n+1)A$.*

Proof. Let φ be the weight of the metric h on a small neighborhood of a point $z_0\in X$. Assume that we have a local section u of $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$ on a coordinate open ball $B=B(z_0,\delta)$, such that

$$\int_B |u(z)|^2 e^{-2\varphi(z)}|z-z_0|^{-2(n+\varepsilon)}dV(z)<+\infty.$$

Then Skoda's division theorem [Sko72b] implies $u(z)=\sum(z_j-z_{j,0})v_j(z)$ with

$$\int_B |v_j(z)|^2 e^{-2\varphi(z)}|z-z_0|^{-2(n-1+\varepsilon)}dV(z)<+\infty,$$

in particular $u_{z_0}\in\mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$. Select a very ample line bundle A on X. We take a basis $\sigma=(\sigma_j)$ of sections of $H^0(X,G\otimes\mathfrak{m}_{X,z_0})$ and multiply the metric h of G by the factor $|\sigma|^{-2(n+\varepsilon)}$. The weight of the above metric has singularity $(n+\varepsilon)\log|z-z_0|^2$ at z_0, and its curvature is

$$i\Theta(G)+(n+\varepsilon)i\partial\overline{\partial}\log|\sigma|^2\geq i\Theta(G)-(n+\varepsilon)\Theta(A). \tag{13.2.2}$$

Now, let f be a local section in $H^0(B,\mathcal{O}(G+L)\otimes\mathcal{I}(h))$ on $B=B(z_0,\delta)$, δ small. We solve the global $\overline{\partial}$ equation

$$\overline{\partial}u=\overline{\partial}(\theta f)\qquad\text{on } X$$

with a cut-off function θ supported near z_0 and with the weight associated with our above choice of metric on $G+L$. Thanks to Nadel's theorem 5.11, the solution exists if the metric of $G+L-K_X$ has positive curvature. As $i\Theta_h(L)\geq 0$ in the sense of currents, (13.2.2) shows that a sufficient condition is $G-K_X-nA>0$ (provided that ε is small enough). We then find a smooth solution u such that $u_{z_0}\in\mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$, hence

$$F:=\theta f-u\in H^0(X,\mathcal{O}(G+L)\otimes\mathcal{I}(h))$$

is a global section differing from f by a germ in $\mathcal{O}(G+L)\otimes\mathcal{I}(h)\otimes\mathfrak{m}_{X,z_0}$. Nakayama's lemma implies that $H^0(X,\mathcal{O}(G+L)\otimes\mathcal{I}(h))$ generates the stalks of $\mathcal{O}(G+L)\otimes\mathcal{I}(h)$.

13.3. A special case of the Ohsawa-Takegoshi-Manivel L^2 extension theorem

We will need the special case below of the Ohsawa-Takegoshi L^2 extension theorem. (Notice that, in this way, the proof of the theorem on invariance of plurigenera requires 3 essentially different types of L^2 existence theorems !).

(13.3.1) Theorem. *Let $\gamma: X\to\Delta$ be a projective family of projective manifolds parametrized by the open unit disk $\Delta\subset\mathbb{C}$. Let $X_0=\gamma^{-1}(0)$, $n=\dim_{\mathbb{C}}X_0$, and let*

$L\to X$ be a line bundle equipped with a hermitian metric written locally as $e^{-\chi}$, such that $i\partial\overline{\partial}\chi\geq\omega$ in the sense of currents, with a suitable positive $(1,1)$-form ω on X. Let $0<r<1$ and $\Delta_r=\{t\in\Delta;\ |t|<r\}$. Then there exists a positive constant A_r such that the following statement holds. For every holomorphic n-form f on X_0 with values in L such that

$$\int_{X_0}|f|^2e^{-\chi}dV<\infty,$$

there exists a holomorphic $(n+1)$-form $\tilde{f}$ on $\gamma^{-1}(\Delta_r)$ with values in L, such that $\tilde{f}|_{X_0}=f\wedge\gamma^*(dt)$ on X_0 and

$$\int_X|\tilde{f}|^2e^{-\chi}dV\leq A_r\int_{X_0}|f|^2e^{-\chi}dV.$$

(13.3.2) Remark. It should be noticed that no hermitian metric on the tangent bundle of X_0 or X is required in order to define the integral of the square of the norm of holomorphic forms f and $\tilde{f}$ on X_0 and X; in fact, it suffices to integrate the volume forms $i^{n^2}f\wedge\overline{f}$ and $i^{(n+1)^2}\tilde{f}\wedge\overline{\tilde{f}}$.

13.4. Construction of a hermitian metric with positive curvature on K_{X_0}

From now on, we suppose that $X\to\Delta$ is a family of projective manifolds of general type (i.e. that all fibers X_t are of general type). A technical point to be settled first is that the hermitian metric on K_X must be chosen in such a way that its curvature current dominates a smooth positive definite $(1,1)$-form, so that the Skoda division theorem and the Ohsawa-Takegoshi-Manivel L^2 extension theorem can be applied. This point is easily settled by means of a variation of Kodaira's technique which consists in expressing a sufficiently large multiple of a big line bundle as the sum of an effective divisor and of an ample line bundle.

(13.4.1) Lemma. *There exists a positive integer a such that $aK_X=D+F$, where D is an effective divisor on X which does not contain X_0, and where F is a positive line bundle on X.*

Proof. Let F be a positive line bundle on X, and let r_a be the generic rank of $H^0(X_t,aK_{X_t}-F)$, which is achieved for $t\in\Delta\setminus S_a$ in the complement of a suitable locally finite subset $S_a\subset\Delta$. Fix $t_1\in\Delta\setminus\bigcup S_a$. Since X_{t_1} is of general type, we know that $h^0(X_{t_1},aK_{X_{t_1}})\geq ca^n$, hence $h^0(X_{t_1},aK_{X_{t_1}}-F)\geq c'a^n$ for suitable $c,c'>0$ and a large enough. By the choice of t_1, every non zero section of $H^0(X_{t_1},aK_{X_{t_1}}-F)$ can be extended into a section s of $H^0(X,aK_X-F)$, hence $aK_X=D+F$ where D is the zero divisor of s. If necessary, we can eliminate any unwanted component X_0 in the decomposition of D by dividing s with a suitable power of t. □

The next step is to construct hermitian metrics on K_{X_0} and K_X, respectively, and to compare their multiplier ideal sheaves. By the results of section 13.1, there

exists a metric with minimal singularities on X_0 (unique up to equivalence of singularities). We will denoter by φ_0 the weight of this metric. Similarly, there exists a metric with minimal singularities on every relatively compact neighborhood of X_0 in X, and by shrinking the base Δ, we can suppose that this metric exists on the whole space X. We will denote by φ the *restriction to* X_0 of the weight of this metric. By definition, φ is at least as singular as φ_0 on X_0, hence by adding a constant, we may eventually assume that $\varphi \le \varphi_0$.

On the other hand, Lemma 13.4.1 shows that we can choose an integer $a \ge 2$ such that $aK_X = D+F$, where D is an effective divisor on X which does not contain X_0 as a component, and F is an ample line bundle on X. After a has been replaced with a sufficiently large multiple, we can make the following additional assumptions:

(13.4.2) *$F - K_X > K_X + nA$ for some very ample line bundle A on X.*

(13.4.3) *there is a basis of sections of $H^0(X,F)_{|X_0}$ providing an embedding of X_0 onto a subvariety of projective space.*

Let s_D be the canonical section of the line bundle $\mathcal{O}(D)$, so that the divisor of s_D is D. Let $u_1,\dots,u_N \in H^0(X,F)$ be sections such that

$$u_{1|X_0},\dots,u_{N|X_0}$$

form a basis of $H^0(X,F)_{|X_0}$. Since $s_D u_j \in H^0(X,aK_X)$ $(1 \le j \le N)$, we get a hermitian metric

$$e^{-\psi} = \left(\frac{1}{|s_D|^2 \sum |u_j|^2}\right)^{\frac{1}{a}}$$

on the line bundle K_X. Moreover

$$\psi = \frac{1}{a}\left(\log|s_D|^2 + \chi\right)$$

where $\chi = \log(\sum |u_j|^2)$ defines a smooth hermitian metric with positive curvature on F. Therefore, $i\partial\bar\partial\psi$ is a positive definite current. Furthermore, the singularities of ψ on X are at least as large as those of the weight φ which is by definition the weight with minimal singularities on X. By adjusting again the weights with constants, we can state:

(13.4.4) Lemma. *The hermitian metric $e^{-\psi}$ has positive definite curvature and $\psi \le \varphi \le \varphi_0$ on X_0.*

The crucial argument is a comparison result for the multiplier ideal sheaves on X_0 defined by $\ell\varphi_0$ and $\ell\varphi$, respectively, when ℓ is large. In the sequel, the notation $\mathcal{I}((\ell+a-\varepsilon)\varphi+\varepsilon\psi)$ will stand for an ideal sheaf on X_0 (and not for an ideal sheaf on X, even when the weight is possibly defined on the whole space X).

(13.4.5) Proposition. *Select $0<\varepsilon<1$ so small that $e^{-\varepsilon\psi}$ is locally integrable on X (possibly after shrinking the base Δ) and $e^{-\varepsilon\psi}_{|X_0}$ is locally integrable on X_0. Then*

$$\mathcal{I}((\ell-\varepsilon)\varphi_0+(a+\varepsilon)\psi) \subset \mathcal{I}((\ell-1+a-\varepsilon)\varphi+\varepsilon\psi)$$

for every integer $\ell \ge 1$.

Proof. We argue by induction on ℓ. For $\ell = 1$, Lemma 13.4.4 implies

$$(1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi \le (1-\varepsilon)\varphi_0 + a\varphi + \varepsilon\psi \le C + (a-\varepsilon)\varphi + \varepsilon\psi$$

since φ_0 is locally bounded above by a constant C. Thus we get

$$\mathcal{I}((1-\varepsilon)\varphi_0+(a+\varepsilon)\psi) \subset \mathcal{I}((a-\varepsilon)\varphi+\varepsilon\psi),$$

as desired.

Now, suppose that the induction step ℓ has been settled. Take an arbitrary germ of function f in the ideal sheaf $\mathcal{I}((\ell+1-\varepsilon)\varphi_0+(a+\varepsilon)\psi)$, defined on a small neighborhood U of a point $P \in X_0$. Fix a local non vanishing holomorphic section e of $(\ell+a)K_{X_0}$ on U. Then $s = fe$ is a section of

$$\mathcal{O}((\ell+a)K_{X_0}) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0+(a+\varepsilon)\psi)$$

on U. Observe that, given an arbitrary plurisubharmonic function ξ, we have

$$\mathcal{I}(\xi+\log|s_D|^2) = \mathcal{I}(\xi)\otimes\mathcal{O}(-D).$$

Writing $aK_{X_0} = (D+F)_{|X_0}$ thanks to Lemma 13.4.1 and $a\psi = \log|s_D|^2 + \chi$ by definition of ψ, we can reinterpret s as a section of

$$\begin{aligned}\mathcal{O}(\ell K_{X_0} + (D+F)_{|X_0}) &\otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0+\varepsilon\psi+\log|s_D|^2+\chi)\\ &= \mathcal{O}((\ell+2)K_{X_0}+E_{|X_0})\otimes\mathcal{I}((\ell+1-\varepsilon)\varphi_0+\varepsilon\psi),\end{aligned}$$

where $E = F - 2K_X$ (since χ is smooth, adding χ does not change the multiplier idela sheaf). Let us observe that $(\ell+1-\varepsilon)\varphi_0+\varepsilon\psi$ defines a hermitian metric with positive curvature on $(\ell+1)K_{X_0}$. By the above hypothesis (13.4.2) and Proposition 13.2.1 applied with $Y = X_0$ and $L = (\ell+1)K_{X_0}$, we conclude that

$$\mathcal{O}((\ell+2)K_{X_0}+E_{|X_0})\otimes\mathcal{I}((\ell+1-\varepsilon)\varphi_0+\varepsilon\psi)$$

is generated by global sections on X_0. We can therefore without loss of generality restrict ourselves to the case of germs f such that fe coincides on U with a global section

$$s \in H^0\left(X_0, \mathcal{O}((\ell+2)K_{X_0}+E_{|X_0})\otimes\mathcal{I}((\ell+1-\varepsilon)\varphi_0+\varepsilon\psi)\right).$$

By reversing the order of calculations, we find

$$\begin{aligned}H^0(X_0,\mathcal{O}((\ell+2)K_{X_0}&+E_{|X_0})\otimes\mathcal{I}((\ell+1-\varepsilon)\varphi_0+\varepsilon\psi))\\ &= H^0(X_0,\mathcal{O}((\ell+a)K_{X_0})\otimes\mathcal{I}((\ell+1-\varepsilon)\varphi_0+(a+\varepsilon)\psi))\\ &\subset H^0(X_0,\mathcal{O}((\ell+a)K_{X_0})\otimes\mathcal{I}((\ell-\varepsilon)\varphi_0+(a+\varepsilon)\psi))\\ &\subset H^0(X_0,\mathcal{O}((\ell+a)K_{X_0})\otimes\mathcal{I}((\ell+a-1-\varepsilon)\varphi+\varepsilon\psi)).\end{aligned}$$

[the first inclusion is obtained by neglecting the term φ_0 in the weight, et the second one is a consequence of the induction hypothesis for step ℓ]. Now, the weight

$$(\ell+a-1-\varepsilon)\varphi+\varepsilon\psi$$

defines a hermitian metric with positive definite curvature on $L = (\ell + a - 1)K_X$, and Proposition 13.3.1 implies that s can be extended into a global section $\tilde{s} \in H^0(X, (\ell + a)K_X)$ (maybe after shrinking Δ). The definition of φ implies $|\tilde{s}|^2 \leq C e^{(\ell+a)\varphi}$, hence

$$|\tilde{s}|^2 e^{-(\ell+a)\varphi - \varepsilon\psi} \leq C e^{-\varepsilon\psi}$$

is integrable on X. from there, we conclude that

$$fe = s = \tilde{s}_{|X_0} \in \mathcal{O}((\ell + a)K_{X_0}) \otimes \mathcal{I}((\ell + a)\varphi + \varepsilon\psi),$$

hence $f \in \mathcal{I}((\ell + a)\varphi + \varepsilon\psi)$. The step $\ell + 1$ of the induction is proved.

13.5. Proof of the invariance of plurigenera.

Fix an integer $m > 0$ and an arbitrary section $s \in H^0(X_0, mK_{X_0})$. By definition of φ_0, we have $|s|^2 \leq Ce^{m\varphi_0}$ on X_0. If s_D is the canonical section of $\mathcal{O}(-D)$ of divisor D, we conclude that $s^\ell s_D$ is locally L^2 with respect to the weight $e^{-\ell m\varphi_0 - (a+\varepsilon)\psi}$, for $a\psi$ has the same singularities as $\log|s_D|^2$ and $e^{-\varepsilon\psi}$ is supposed to be locally integrable on X_0. The functions $s^\ell s_D$ are locally in the ideal sheaf $\mathcal{I}((\ell m - \varepsilon)\varphi_0 + (a + \varepsilon)\psi)$. By Proposition 13.4.5, they belong to $\mathcal{I}((\ell m - 1 - \varepsilon)\varphi + \varepsilon\psi)$, i.e.

$$\int_U |s^\ell s_D|^2 e^{-(\ell m - 1 - \varepsilon)\varphi - \varepsilon\psi} < +\infty$$

on each sufficiently small open set U such that $K_{X_0}|U$ and $\mathcal{O}(-D)|U$ are trivial. In an equivalent way, we can write

$$\int_U |s|^{2\ell} e^{-(\ell m - 1 - \varepsilon)\varphi + (a-\varepsilon)\psi} < +\infty.$$

Take ℓ large enough so that $a/(\ell - 1) \leq \varepsilon$. Then $e^{\varphi - \frac{a}{\ell-1}\psi} \leq Ce^{-\varepsilon\psi}$ soit integrable sur U. By Hölder inequality with conjugate exponents ℓ, $\ell' = \ell/(\ell - 1)$, we find

$$\begin{aligned} +\infty > &\left(\int_U |s|^{2\ell} e^{-(\ell m - 1 - \epsilon)\varphi + (a-\epsilon)\psi}\right)^{1/\ell} \left(\int_U e^{\varphi - \frac{a}{\ell-1}\psi}\right)^{(\ell-1)/\ell} \\ &\geq \int_U |s|^2 e^{-(m - \frac{1}{\ell} - \frac{\delta}{\ell})\varphi + (\frac{a}{\ell} - \frac{\delta}{\ell})\psi} e^{(1-\frac{1}{\ell})\varphi - \frac{a}{\ell}\psi} = \int_U |s|^2 e^{-(m-1-\delta)\varphi - \delta\psi} \end{aligned}$$

with $\delta = \varepsilon/\ell$. We can consider s as a section of $K_{X_0} + L_{|X_0}$ with $L = (m - 1)K_X$. The weight $(m - 1 - \delta)\varphi - \delta\psi$ defines a hermitian metric on L with positive definite curvature, and s is globally L^2 with respect to this metric. By the Ohsawa-Takegoshi-Manivel extension theorem, we can extend s into a section $\tilde{s} \in H^0(X, K_X + L) = H^0(X, mK_X)$, possibly after shrinking Δ again. This achieves the proof of the main theorem. □

14. Subadditivity of Multiplier Ideals and Fujita's Approximate Zariski Decompostion

We first notice the following basic restriction formula, which is just a rephrasing of the Ohsawa-Takegoshi extension theorem.

(14.1) Restriction formula. *Let φ be a plurisubharmonic function on a complex manifold X, and let $Y \subset X$ be a submanifold. Then*

$$\mathcal{I}(\varphi_{|Y}) \subset \mathcal{I}(\varphi)_{|Y}.$$

Thus, in some sense, the singularities of φ can only get worse if we restrict to a submanifold (if the restriction of φ to some connected component of Y is identically $-\infty$, we agree that the corresponding multiplier ideal sheaf is zero). The proof is straightforward and just amounts to extend locally a germ of function f on Y near a point $y_0 \in Y$ to a function $\tilde{f}$ on a small Stein neighborhood of y_0 in X, which is possible by the Ohsawa-Takegoshi extension theorem. As a direct consequence, we get:

(14.2) Subadditivity Theorem.

(i) *Let X_1, X_2 be complex manifolds and let φ_i be a plurisubharmonic function on X_i. Then*

$$\mathcal{I}(\varphi_1 \circ p_1 + \varphi_2 \circ p_2) = p_1^{-1}\mathcal{I}(\varphi_1) \cdot p_2^{-1}\mathcal{I}(\varphi_2).$$

(ii) *Let X be a complex manifold and let φ, ψ be plurisubharmonic functions on X. Then*

$$\mathcal{I}(\varphi + \psi) \subset \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi)$$

Proof. (i) Let us fix two relatively compact Stein open subsets $U_1 \subset X_1$, $U_2 \subset X_2$. Then $\mathcal{H}^2(U_1 \times U_2, \varphi_1 \circ p_1 + \varphi_2 \circ p_2, p_1^* dV_1 \otimes p_2^* dV_2)$ is the Hilbert tensor product of $\mathcal{H}^2(U_1, \varphi_1, dV_1)$ and $\mathcal{H}^2(U_2, \varphi_2, dV_2)$, and admits $(f'_k \boxtimes f''_l)$ as a Hilbert basis, where (f'_k) and (f''_l) are respective Hilbert bases. Since $\mathcal{I}(\varphi_1 \circ p_1 + \varphi_2 \circ p_2)_{|U_1 \times U_2}$ is generated as an $\mathcal{O}_{U_1 \times U_2}$ module by the $(f'_k \boxtimes f''_l)$ (Theorem 5.7), we conclude that (i) holds true.

(ii) We apply (i) to $X_1 = X_2 = X$ and the restriction formula to $Y =$ diagonal of $X \times X$. Then

$$\begin{aligned} \mathcal{I}(\varphi + \psi) &= \mathcal{I}\big((\varphi \circ p_1 + \psi \circ p_2)_{|Y}\big) \subset \mathcal{I}\big(\varphi \circ p_1 + \psi \circ p_2\big)_{|Y} \\ &= \big(p_1^* \mathcal{I}(\varphi) \otimes p_2^* \mathcal{I}(\psi)\big)_{|Y} = \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi). \end{aligned}$$

As an application of subadditivity, we now reprove a result of Fujita [Fuj93], relating the growth of sections of multiples of a line bundle to the Chern numbers of its "largest nef part". Fujita's original proof is by contradiction, using the Hodge

index theorem and intersection inequalities. The present methode arose in the course of joint work with R. Lazarsfeld [Laz99].

Let X be a projective n-dimensional algebraic variety and L a line bundle over X. We define the *volume* of L to be

$$v(L) = \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL) \in [0, +\infty[.$$

The line bundle is said to be *big* if $v(L) > 0$. If L is ample, we have $h^q(X, kL) = 0$ for $q \geq 1$ and $k \gg 1$ by the Kodaira-Serre vanishing theorem, hence

$$h^0(X, kL) \sim \chi(X, kL) \sim \frac{L^n}{n!} k^n$$

by the Riemann-Roch formula. Thus $v(L) = L^n\ (= c_1(L)^n)$ if L is ample. This is still true if L is nef (numerically effective), i.e. if $L \cdot C \geq 0$ for every effective curve C. In fact, one can show that $h^q(X, kL) = O(k^{n-q})$ in that case. The following well-known proposition characterizes big line bundles.

(14.3) Proposition. *The line bundle L is big if and only if there a multiple $m_0 L$ such that $m_0 L = E + A$, where E is an effective divisor and A an ample divisor.*

Proof. If the condition is satisfied, the decomposition $km_0 L = kE + kA$ gives rise to an injection $H^0(X, kA) \hookrightarrow H^0(X, km_0 L)$, thus $v(L) \geq m_0^{-n} v(A) > 0$. Conversely, assume that L is big, and take A to be a very ample nonsingular divisor in X. The exact sequence

$$0 \longrightarrow \mathcal{O}_X(kL - A) \longrightarrow \mathcal{O}_X(kL) \longrightarrow \mathcal{O}_A(kL_{|A}) \longrightarrow 0$$

gives rise to a cohomology exact sequence

$$0 \to H^0(X, kL - A) \longrightarrow H^0(X, kL) \longrightarrow H^0(A, kL_{|A}),$$

and $h^0(A, kL_{|A}) = O(k^{n-1})$ since $\dim A = n - 1$. Now, the assumption that L is big implies that $h^0(X, kL) > ck^n$ for infinitely many k, hence $H^0(X, m_0 L - A) \neq 0$ for some large integer m_0. If E is the divisor of a section in $H^0(X, m_0 L - A)$, we find $m_0 L - A = E$, as required. □

(14.4) Lemma. *Let G be an arbitrary line bundle. For every $\varepsilon > 0$, there exists a positive integer m and a sequence $\ell_\nu \uparrow +\infty$ such that*

$$h^0\big(X, \ell_\nu(mL - G)\big) \geq \frac{\ell_\nu^m m^n}{n!} \big(v(L) - \varepsilon\big),$$

in other words, $v(mL - G) \geq m^n(v(L) - \varepsilon)$ for m large enough.

Proof. Clearly, $v(mL - G) \geq v(mL - (G + E))$ for every effective divisor E. We can take E so large that $G + E$ is very ample, and we are thus reduced to the case where G is very ample by replacing G with $G + E$. By definition of $v(L)$, there exists a sequence $k_\nu \uparrow +\infty$ such that

$$h^0(X, k_\nu L) \geq \frac{k_\nu^n}{n!} \Big(v(L) - \frac{\varepsilon}{2}\Big).$$

We take $m \gg 1$ (to be precisely chosen later), and $\ell_\nu = \big[\frac{k_\nu}{m}\big]$, so that $k_\nu = \ell_\nu m + r_\nu$, $0 \leq r_\nu < m$. Then

$$\ell_\nu(mL - G) = k_\nu L - (r_\nu L + \ell_\nu G).$$

Fix a constant $a \in \mathbb{N}$ such that $aG - L$ is an effective divisor. Then $r_\nu L \leq maG$ (with respect to the cone of effective divisors), hence

$$h^0\big(X, \ell_\nu(mL - G)\big) \geq h^0\big(X, k_\nu L - (\ell_\nu + am)G\big).$$

We select a smooth divisor D in the very ample linear system $|G|$. By looking at global sections associated with the exact sequences of sheaves

$$0 \to \mathcal{O}(-(j+1)D) \otimes \mathcal{O}(k_\nu L) \to \mathcal{O}(-jD) \otimes \mathcal{O}(k_\nu L) \to \mathcal{O}_D(k_\nu L - jD) \to 0,$$

$0 \leq j < s$, we infer inductively that

$$\begin{aligned} h^0(X, k_\nu L - sD) &\geq h^0(X, k_\nu L) - \sum_{0\leq j<s} h^0\big(D, \mathcal{O}_D(k_\nu L - jD)\big) \\ &\geq h^0(X, k_\nu L) - s\, h^0\big(D, k_\nu L_{|D}\big) \\ &\geq \frac{k_\nu^n}{n!} \Big(v(L) - \frac{\varepsilon}{2}\Big) - s\, C k_\nu^{n-1} \end{aligned}$$

where C depends only on L and G. Hence, by putting $s = \ell_\nu + am$, we get

$$\begin{aligned} h^0\big(X, \ell_\nu(mL - G)\big) &\geq \frac{k_\nu^n}{n!} \Big(v(L) - \frac{\varepsilon}{2}\Big) - C(\ell_\nu + am) k_\nu^{n-1} \\ &\geq \frac{\ell_\nu^n m^n}{n!} \Big(v(L) - \frac{\varepsilon}{2}\Big) - C(\ell_\nu + am)(\ell_\nu + 1)^{n-1} m^{n-1} \end{aligned}$$

and the desired conclusion follows by taking $\ell_\nu \gg m \gg 1$. □

We are now ready to prove Fujita's decomposition theorem, as reproved in [DEL00].

(14.5) Theorem (Fujita). *Let L be a big line bundle. Then for every $\varepsilon > 0$, there exists a modification $\mu : \widetilde{X} \to X$ and a decomposition $\mu^* L = E + A$, where E is an effective $\mathbb{Q}$-divisor and A an ample $\mathbb{Q}$-divisor, such that $A^n > v(L) - \varepsilon$.*

(14.6) Remark. Of course, if $\mu^* L = E + A$ with E effective and A nef, we get an injection

$$H^0(\widetilde{X}, kA) \hookrightarrow H^0(\widetilde{X}, kE + kA) = H^0(\widetilde{X}, k\mu^* L) = H^0(X, kL)$$

for every integer k which is a multiple of the denominator of E, hence $A^n \leq v(L)$.

(14.7) Remark. Once Theorem 14.5 is proved, the same kind of argument easily shows that

$$v(L) = \lim_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL).$$

because the formula is true for every ample line bundle A.

Proof of Theorem 14.5. It is enough to prove the theorem with A being a big and nef divisor. In fact, Proposition 14.3 then shows that we can write $A = E' + A'$ where E' is an effective $\mathbb{Q}$-divisor and A' an ample $\mathbb{Q}$-divisor, hence

$$E + A = E + \varepsilon E' + (1-\varepsilon)A + \varepsilon A'$$

where $A'' = (1-\varepsilon)A + \varepsilon A'$ is ample and the intersection number A''^n approaches A^n as closely as we want. Let G be as in Lemma 13.2.1 (uniform global generation). Lemma 14.4 implies that $v(mL-G) > m^n(v(L)-\varepsilon)$ for m large. By Theorem 13.1.2, there exists a hermitian metric h_m of weight φ_m on $mL-G$ such that

$$H^0\big(X, \ell(mL-G)\big) = H^0\big(X, \ell(mL-G)\otimes \mathcal{I}(\ell\varphi_m)\big)$$

for every $\ell \geq 0$. We take a smooth modification $\mu : \widetilde{X} \to X$ such that

$$\mu^*\mathcal{I}(\varphi_m) = \mathcal{O}_{\widetilde{X}}(-E)$$

is an invertible ideal sheaf in $\mathcal{O}_{\widetilde{X}}$. This is possible by taking the blow-up of X with respect to the ideal $\mathcal{I}(\varphi_m)$ and by resolving singularities (Hironaka [Hir64]). Lemma 13.2.1 applied to $L' = mL - G$ implies that $\mathcal{O}(mL)\otimes\mathcal{I}(\varphi_m)$ is generated by its global sections, hence its pull-back $\mathcal{O}(m\,\mu^*L - E)$ is also generated. This implies

$$m\,\mu^*L = E + A$$

where E is an effective divisor and A is a nef (semi-ample) divisor in $\widetilde{X}$. We find

$$\begin{aligned} H^0(\widetilde{X}, \ell A) &= H^0\big(\widetilde{X}, \ell(m\,\mu^*L - E)\big) \\ &\supset H^0\big(\widetilde{X}, \mu^*\big(\mathcal{O}(\ell mL)\otimes\mathcal{I}(\varphi_m)^\ell\big)\big) \\ &\supset H^0\big(\widetilde{X}, \mu^*\big(\mathcal{O}(\ell mL)\otimes\mathcal{I}(\ell\varphi_m)\big)\big), \end{aligned}$$

thanks to the subadditivity property of multiplier ideals. Moreover, the direct image $\mu_*\mu^*\mathcal{I}(\ell\varphi_m)$ coincides with the integral closure of $\mathcal{I}(\ell\varphi_m)$, hence with $\mathcal{I}(\ell\varphi_m)$, because a multiplier ideal sheaf is always integrally closed. From this we infer

$$\begin{aligned} H^0(\widetilde{X}, \ell A) &\supset H^0\big(X, \mathcal{O}(\ell mL)\otimes\mathcal{I}(\ell\varphi_m)\big) \\ &\supset H^0\big(X, \mathcal{O}(\ell(mL-G))\otimes\mathcal{I}(\ell\varphi_m)\big) \\ &= H^0\big(X, \mathcal{O}(\ell(mL-G))\big). \end{aligned}$$

By Lemma 14.4, we find

$$h^0(\widetilde{X}, \ell A) \geq \frac{\ell^n}{n!} m^n\big(v(L)-\varepsilon\big)$$

for infinitely many ℓ, therefore $v(A) = A^n \geq m^n(v(L)-\varepsilon)$. Theorem 14.5 is proved, up to a minor change of notation $E \mapsto \frac{1}{m}E$, $A \mapsto \frac{1}{m}A$. □

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