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The Base Point Free Theorem and the Fujita Conjecture

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Introduction

The aim of these notes is to present a general framework for proving the absence of base points of a linear system under certain numerical assumptions. For example, if H is an ample divisor on a smooth projective variety X of dimension d, then, by a famous conjecture of T. Fujuta, the linear system $|K_X + mH|$ should be base point free, if $m > d$, where K_X is the canonical divisor on X. For $m > \frac{1}{2}d(d+1)$, it was proved by U. Angehrn and Y.-T. Siu [1], that the above linear system has indeed no base points. In [5] this bound on m was slightly improved by a different method. But both methods are based on techniques which were used in the eighties of the last century, to prove the so-called base point free theorem. Roughly speaking it says, if L is a nef divisor on X such that $M := L - K_X$ is nef and big, then L is semi-ample, i.e. the linear system $|mL|$ is base point free for large integers m. The idea of the proof of this result is, first to use the Riemann-Roch theorem to get a divisor $D \in |nM|$ for some large integer n, which is very singular at a given point $x \in X$, and then applying a generalized form of Kodaira's vanishing theorem to show that the restriction of global sections of the sheaf $\mathcal{O}_X(mL)$ to the 'most singular locus' Z of the divisor D is surjective for $m \gg 0$. Since this locus Z has smaller dimension than X, the result basically follows by induction. But there are a few technical difficulties with this approach. First, the original point x might be not contained in the most singular locus Z of D. Therefore, we cannot prove the existence of a section of $\mathcal{O}_X(mL)$ which is non-vanishing at x, but only non-vanishing somewhere else. In fact, the traditional argument proves only non-vanishing at the first step and then indirectly from this, the base point free theorem. In Section 3 we will follow a modified strategy, to get the base point freeness directly. This will be important in the proof of effective results like Fujita's conjecture. There are mainly two new ingredients in this approach. On the one hand, since we basically want to ignore singular components of the divisor D not containing the point x, we can only show, that the restriction of global sections of the sheaf $\mathcal{O}_X(mL)$ twisted by a certain ideal sheaf, the 'multiplier ideal sheaf' (cf. §1), to the subvariety Z is surjective. This causes no real problem, since the multiplier ideal sheaf will be supported away from x and in a more general context, e.g. separation of points, the use of multiplier ideal sheaf is even very natural. On the other hand, before we can apply the vanishing theorem, we have to resolve the singularities of D by blowing-up X. The existence of such a resolution is guaranteed by Hironaka's theorem, but the blow-up produces certain 'pole-divisors', disturbing the picture slightly. From the vanishing theorem we only get sections of $\mathcal{O}_X(mL)$ with possibly poles along those pole-divisors. But, since those divisors are exceptional with respect to the resolution, there is actually no contribution from them. The problem is, that after restricting to the subvariety Z, the pole-divisors may not be exceptional anymore. In Section 1, we therefore introduce a new concept which handles this problem.

In Section 2, we will study the basic properties of 'adjoint systems'. Roughly speaking, an adjoint system is a complete linear system on a normal variety together with a generalized resolution and a certain semi-positivety property. The advantage of formulating base point freeness in terms of abstract adjoint systems is, that they behave quite stable under basic operations like resolutions of singularities and restricting to certain subvarieties. They form good a framework for both, the proof of the base point free theorem as well as the proof of Fujita's conjecture.

The final section is devoted to effective base point freeness. We will use the results of Section 1 and 2, to prove Angehrn and Siu's theorem. The crucial point is the construction of a filtration of a local ring, with small 'multiplicity' and small 'log-canonical threshold'. At a smooth point, this filtration is just given by powers of the maximal ideal. In general, the filtration is obtained, by degenerating the corresponding filtration at a smooth point along a curve to the singular point. The rest of the argument is basically the same as in the proof of the base point free theorem. The result is far away from the Fujita conjecture, but together with a much more complicated convexity argument discussed in its simplest form at the end of this section, one can actually obtain optimal results.

Preliminaries

Let X be a normal variety over an algebraically closed field of characteristic zero. A *Weil-prime divisor* of X is an irreducible subvariety $P \subset X$ of codimension 1. A $\mathbb{Q}$-*divisor* on X is a finite formal linear combination $\Delta = \sum q_i P_i$ of Weil-prime divisors P_i with rational coefficients q_i. The *round-up*, resp. *round-down* of Δ is defined by

$$\ulcorner \Delta \urcorner = \sum \ulcorner q_i \urcorner P_i \qquad \text{resp.} \qquad \llcorner \Delta \lrcorner = \sum \llcorner q_i \lrcorner P_i,$$

where $\ulcorner q_i \urcorner$ resp. $\llcorner q_i \lrcorner$ denotes the round-up, resp. round-down of the rational number q_i. A $\mathbb{Q}$-divisor Δ is called $\mathbb{Q}$-Cartier, if an integral multiple $m\Delta$ of Δ is a Cartier divisor, i.e. if the reflexive sheaf $\mathcal{O}_X(m\Delta)$ associated to $m\Delta$ is locally free for some positive integer m. If $g : X \longrightarrow S$ is morhism onto another normal variety S, a $\mathbb{Q}$-Cartier divisor M on X is called *g-nef*, if the degree of M restricted to any curve contained in a fiber of g is non-negative. The $\mathbb{Q}$-Cartier divisor M is called *g-big*, if for any open affine subset $U \subset S$, the linear system of sections of $\mathcal{O}_X(nM)$ on $g^{-1}(U)$ defines a birational map, for some positive integer m. The use of $\mathbb{Q}$-divisors is based on the following tricky generalization of Kodaira's vanishing theorem.

(V) **Theorem** (Kawamata [8], Viehweg [14]). *Let $g : X \longrightarrow S$ be a proper morphism and let M be a g-nef and g-big $\mathbb{Q}$-divisor on X. Assume that X is smooth and the fractional part of M has simple normal crossings. Then, the higher direct image sheaves $R^i f_* \mathcal{O}_X(K_X + \ulcorner M \urcorner)$ are all zero for all $i > 0$.*

Because of the assumption on X being smooth and the fractional part of the $\mathbb{Q}$-divisor M having simple normal crossings, one usually has to blow-up X, before one can apply the vanishing theorem. If X is a normal variety and Δ is a $\mathbb{Q}$-divisor on X, then we call a proper birational morphism $f : \tilde{X} \longrightarrow X$ from a smooth variety $\tilde{X}$, a *log-resolution* of the pair (X, Δ), if the union of the exceptional locus of f and the preimage of the support of Δ is a divisor with simple normal crossings. The existence of such a log-resolution is provided by the following famous theorem.

(R) **Theorem** (Hironaka [7]). *If X is a normal variety and Δ is a $\mathbb{Q}$-divisor on X, then the pair (X, Δ) has a log-resolution.*

Actually, H. Hironaka proved that it is possible to resolve the singularities of the pair (X, Δ), by only blowing-up smooth subvarieties of its singular locus. This is sometimes useful to know, if one wants to check the independends of a certain invariant or property on the log-resolution.

1. Multiplier Ideals

In this section we will introduce the concept of quasi-effectivety. The purpose of this concept is to handle the problem of non-effective divisors coming from the resolution of singularities. We will see that the property is stable under many basic operations, most important the restriction to critical components. Moreover, we will define multiplier ideal sheaves for such non-effective divisors.

(1.1) THE ADJOINT TRANSFORM. Let $f : \tilde{X} \longrightarrow X$ be a proper birational morphism from a smooth variety $\tilde{X}$ onto a normal variety X and let Δ be a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then, there is a unique $\mathbb{Q}$-divisor $f^\# \Delta$ on $\tilde{X}$, such that

$$f^*(K_X + \Delta) \equiv K_{\tilde{X}} + f^\# \Delta \tag{1.1.1}$$

and the difference of $f^\# \Delta$ and the strict transform of Δ is f-exceptional. The $\mathbb{Q}$-divisor $f^\# \Delta$ is called the *f-adjoint transform* of Δ. In particular, if the birational morphism f is a log-resolution of the pair (X, Δ), then the support of the adjoint transform $f^\# \Delta$ has simple normal crossings. We will write the round-down of Δ as

$$\llcorner \Delta \lrcorner = \Delta_+ - \Delta_- \tag{1.1.2}$$

with two effective divisors Δ_+ and Δ_- having no common component. The divisor Δ_+ is the *positive part* and Δ_- is the *negative part* of Δ. The $\mathbb{Q}$-divisor Δ is effective if and only if $\Delta_- = 0$. For our purpose, the following generalization will be quite useful.

(1.2) **Definition.** Let $g : X \longrightarrow S$ be a proper morphism with connected fibers between normal varieties and let Δ be a $\mathbb{Q}$-divisor on X. If the natural inclusion map

$$g_*\mathcal{O}_X(-\Delta_+) \hookrightarrow g_*\mathcal{O}_X(\Delta_- - \Delta_+) \tag{1.2.1}$$

is an isomorphism, then we say that Δ is *g-quasi-effective.*

(1.3) EXAMPLE. If $g : X \longrightarrow S$ is a proper birational morphism between normal varieties and E is an effective divisor on X, then $-E$ is g-quasi-effective, if and only if E is g-exceptional. But if the general fiber of g has positive dimension, then the divisor $-E$ can be g-quasi-effective without being g-exceptional. This happens for example, if E is a prime divisor with a one-codimesional center on S and there is another prime divisor P on X, having the same center on S. Then, any rational function on S which has a pole at E must be singular along P as well. Hence, $-E$ is g-quasi-effective.

(1.4) **Proposition.** *Let $g : X \longrightarrow S$ be a proper morphism and let Δ be a g-quasi-effective $\mathbb{Q}$-divisor on X such that Δ_+ and Δ_- are Cartier divisors.*

(1.4.1) *Let $\tilde{\Delta}$ be a $\mathbb{Q}$-divisor on X with $\tilde{\Delta} \geqslant \Delta$. If the two divisors $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ are Cartier divisors, then the $\mathbb{Q}$-divisor $\tilde{\Delta}$ is also g-quasi-effective.*

(1.4.2) *Let $h : \tilde{S} \longrightarrow S$ be a flat morphism and denote by p and q the projections of the fiber product $X \times_S \tilde{S}$ onto $\tilde{S}$ and X. If Δ is $\mathbb{Q}$-Cartier, then $q^*\Delta$ is p-quasi-effective.*

(1.4.3) *Let $f : \tilde{X} \longrightarrow X$ be a proper birational morphism from a smooth variety $\tilde{X}$ and let $\tilde{\Delta}$ be a $\mathbb{Q}$-divisor on $\tilde{X}$, such that $\tilde{\Delta} - f_*^{-1}\Delta$ is f-exceptional. Then,*

$$\tilde{g}_*\mathcal{O}_{\tilde{X}}(-\tilde{\Delta}_+) \subset g_*\mathcal{O}_X(-\Delta_+), \qquad \text{where} \quad \tilde{g} = g \circ f \tag{1.4.4}$$

and $\tilde{\Delta}$ is $\tilde{g}$-quasi-effective. Moreover, if X is smooth, Δ has simple normal crossings and $\tilde{\Delta} = f^{\#}\Delta$, then the above inclusion is in fact an equality.

(1.5) PROOF. First note that $\tilde{\Delta}_+ \geqslant \Delta_+$ and $\tilde{\Delta}_- \leqslant \Delta_-$. In particular the divisor $P = \tilde{\Delta}_+ - \Delta_+$ is effective and the two divisors $\tilde{\Delta}_-$ and P have no common component. Hence, the restriction of $\tilde{\Delta}_-$ to $X' = P$ is also an effective divisor and therefore, the locally free sheaf $\mathcal{O}_{X'}(-\Delta_+)$ is a subsheaf of $\mathcal{O}_{X'}(\tilde{\Delta}_- - \Delta_+)$. From this, we get the following commutative diagram with exact rows

$$\tag{1.5.1}\begin{array}{ccccccc}
0 \longrightarrow & g_*\mathcal{O}_X(-\tilde{\Delta}_+) & \longrightarrow & g_*\mathcal{O}_X(-\Delta_+) & \longrightarrow & g_*\mathcal{O}_{X'}(-\Delta_+) \\
& \cap & & \cap & & \cap \\
0 \longrightarrow & g_*\mathcal{O}_X(\tilde{\Delta}_- - \tilde{\Delta}_+) & \longrightarrow & g_*\mathcal{O}_X(\tilde{\Delta}_- - \Delta_+) & \longrightarrow & g_*\mathcal{O}_{X'}(\tilde{\Delta}_- - \Delta_+) \\
& & & \cap & & \\
& & & g_*\mathcal{O}_X(\Delta_- - \Delta_+). & &
\end{array}$$

Since Δ is g-quasi-effective, the two inclusions in the middle column are in fact equalities and with a diagram chasing we see that the inclusion in the left column is also an equality which proves the first statement.

(1.5.2) To see (1.4.2), note that $p_*(q^*\mathcal{F}) = h^*(f_*\mathcal{F})$ for every coherent sheaf $\mathcal{F}$ on X, because h is flat (see e.g. [4, Prop. III 9.3]). Moreover, q is also flat and hence $(q^*\Delta)_- = q^*(\Delta_-)$ and $(q^*\Delta)_+ = q^*(\Delta_+)$. Therefore, applying the base change formula to $\mathcal{F} = \mathcal{O}_X(-\Delta_+)$ and $\mathcal{O}_X(\Delta_- - \Delta_+)$ we see that $q^*\Delta$ is p-quasi-effective.

(1.5.3) The divisor Δ is g-quasi-effective if and only if $\llcorner\Delta\lrcorner$ is g-quasi-effective. Therefore, because of (1.4.1), in the first statement of (1.4.3) we may assume than Δ is a Cartier divisor and that there exists an effective f-exceptional divisor E on $\tilde{X}$ such that $\tilde{\Delta} = f^*\Delta - E$. Since we have $f_*\mathcal{O}_{\tilde{X}}(E) = \mathcal{O}_X$ the projection formula implies

$$\tilde{g}_*\mathcal{O}_{\tilde{X}}(\tilde{\Delta}_- - \tilde{\Delta}_+) = g_*\mathcal{O}_X(\Delta_- - \Delta_+). \tag{1.5.4}$$

On the other hand $\tilde{\Delta}_+ \leqslant f^*(\Delta_+)$ and hence the difference $f^*(\Delta_+) - \tilde{\Delta}_+$ is effective and f-exceptional too. Applying the projection formula once again, we see that

$$\tilde{g}_*\mathcal{O}_{\tilde{X}}(-\tilde{\Delta}_+) = g_*\mathcal{O}_X(-\Delta_+). \tag{1.5.5}$$

Since Δ is g-quasi-effective, this shows that $\tilde{\Delta}$ is $\tilde{g}$-quasi-effective and under the assumptions made at the beginning of (1.5.3), the inclusion (1.4.4) is an equality.

(1.5.6) Finally, assume that X is smooth, Δ has simple normal crossings and $\tilde{\Delta} = f^\#\Delta$. The morphism f can be dominated by a sequence of blow-ups along smooth subvarieties by Hironaka's Theorem (R), and it is therefore enough to prove equality in (1.4.4) only for such a blow-up. But for such a single blow-up it is quite easy to see that $\tilde{\Delta}_+ \leqslant f^*\Delta_+$ and this implies the desired equality. □

(1.6) **Definition.** Let $g : X \longrightarrow S$ be a proper morphism with connected fibers between normal varieties and let Δ be a g-quasi-effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Choose a log-resolution $f : \tilde{X} \longrightarrow X$ of the pair (X, Δ). The ideal sheaf

$$\mathcal{I}_\Delta := \tilde{g}_*\mathcal{O}_{\tilde{X}}\big(-(f^\#\Delta)_+\big) \subset \mathcal{O}_S, \qquad \text{where} \quad \tilde{g} = g \circ f \tag{1.6.1}$$

is independent of the choice of f by (1.4.3), and will be called the *multiplier ideal sheaf* of (X, Δ). We say that (X, Δ) is *regular* over $\sigma \in S$, if σ is not contained in the support of the multiplier ideal sheaf $\mathcal{I}$. Otherwise (X, Δ) is called *singular* over σ.

(1.7) REMARK. The pair (X, Δ) is called *log-terminal* at a closed point $x \in X$, if there is a log-resolution $f : \tilde{X} \longrightarrow X$ of (X, Δ) such that the adjoint transform $f^\#\Delta$ has no prime component with coefficient larger or equal than 1, which meets the fiber $f^{-1}(x)$. Therefore, the pair (X, Δ) is regular over σ, if and only if Δ is g-quasi-effective and (X, Δ) is log-terminal at all closed points $x \in g^{-1}(\sigma)$.

(1.8) **Definition.** Let X be a normal variety and let Δ be a $\mathbb{Q}$-divisor on X. If P is a prime divisor on X whose coefficient in Δ is equal to 1, then P is called a *critical component* of the $\mathbb{Q}$-divisor Δ.

(1.9) **Reduction Lemma.** *Let $g : X \longrightarrow S$ be a proper morphism from a smooth variety X onto a normal variety S and let Δ be a $\mathbb{Q}$-divisor on X with simple normal crossings. Let P be a critical component of Δ and assume that*

$$M := -(K_X + \Delta) \quad \textit{is } g\textit{-nef and } g\textit{-big,} \tag{1.9.1}$$

$$\tilde{\Delta} := \Delta - P \quad \textit{is } g\textit{-quasi-effective.} \tag{1.9.2}$$

Denote by $X' \xrightarrow{g'} S' \xrightarrow{h} g(X')$ the Stein factorization of the restriction of g to the subvariety $X' = P$. Then, the $\mathbb{Q}$-divisor $\Delta' = \tilde{\Delta}|_{X'}$ is g'-quasi-effective and

$$g'_*\mathcal{O}_{X'}(-\Delta'_+) = g_*\mathcal{O}_X(-\tilde{\Delta}_+) \cdot \mathcal{O}_{S'}. \tag{1.9.3}$$

Moreover, if $(X, \tilde{\Delta})$ is regular over $\sigma \in S$, then h is an isomorphism over σ.

(1.10) PROOF. From the definition of M and $\tilde{\Delta}$ we see that

$$K_X + M \equiv \tilde{\Delta} - P \quad \text{and} \quad K_X + \ulcorner M \urcorner \equiv \tilde{\Delta}_- - \tilde{\Delta}_+ - P. \tag{1.10.1}$$

Therefore, we have an exact sequence of coherent sheaves on S

$$g_*\mathcal{O}_X(\tilde{\Delta}_- - \tilde{\Delta}_+) \longrightarrow g_*\mathcal{O}_{X'}(\Delta'_- - \Delta'_+) \longrightarrow R^1 g_*\mathcal{O}_X(K_X + \ulcorner M \urcorner) \tag{1.10.2}$$

and from by the Vanishing Theorem (V), the last higher direct image sheaf in this sequence is zero. Hence, the right vertical map in the commutative diagram

$$\begin{array}{ccc} g_*\mathcal{O}_X(-\tilde{\Delta}_+) & \hookrightarrow & g_*\mathcal{O}_X(\tilde{\Delta}_- - \tilde{\Delta}_+) \\ \downarrow & & \downarrow \\ g_*\mathcal{O}_{X'}(-\Delta'_+) & \hookrightarrow & g_*\mathcal{O}_{X'}(\Delta'_- - \Delta'_+) \end{array} \tag{1.10.3}$$

is surjective. Since $\tilde{\Delta}$ is g-quasi-effective, the upper horizontal map is an isomorphism and therefore, the lower horizontal map is also an isomorphism and the left vertical map is surjective too. This proves that Δ' is g'-quasi-effective and since $g_*\mathcal{O}_{X'} = \mathcal{O}_{S'}$, we also have the Equality (1.9.3). In particular, if $(X, \tilde{\Delta})$ is regular over $\sigma \in S$, then the map $\mathcal{O}_S \longrightarrow \mathcal{O}_{S'}$ is surjective in a neighborhood of σ and h has to be an isomorphism over σ. □

(1.11) REMARK. Without the assumption on Δ having simple normal crossing in the Reduction Lemma (1.9), the conclusion is not anymore true. But there is still an inclusion of multiplier ideal sheaves as we will see now. In general, let $g : X \longrightarrow S$ be a proper morphism with connected fibers between normal varieties and let Δ be a $\mathbb{Q}$-divisor on X such that $-(K_X + \Delta)$ is g-nef and g-big. Let P be a critical component of Δ and assume that P is normal and Cartier and $\tilde{\Delta} := \Delta - P$ is g-quasi-effective. Let $g' : X' \longrightarrow S'$ be the connected map from the Stein factorization of the restriction of g to the subvariety $X' = P$ and assume that the $\mathbb{Q}$-divisor $\Delta' = \tilde{\Delta}|_{X'}$ is g'-quasi-effective. Denote by $\mathcal{I}_{\tilde{\Delta}} \subset \mathcal{O}_S$ the multiplier ideal sheaf of the pair $(X, \tilde{\Delta})$ and by $\mathcal{I}_{\Delta'} \subset \mathcal{O}_{S'}$ the multiplier ideal sheaf of (X', Δ'). Then, we have the inclusion

$$\mathcal{I}_{\Delta'} \subset \mathcal{I}_{\tilde{\Delta}} \cdot \mathcal{O}_{S'}. \tag{1.11.1}$$

(1.11.2) PROOF. Let $f : \tilde{X} \longrightarrow X$ be a log-resolution of the pair (X, Δ) and denote by $\tilde{X}' \subset \tilde{X}$ the strict transform of X'. Then, the restriction f' of the morphism f to the subvariety $\tilde{X}'$ is a log-resolution of the pair (X', Δ') and the f'-adjoint transform of Δ' is related to the f-adjoint transform of Δ by the formula

$$(f^\#\Delta - \tilde{X}')|_{\tilde{X}'} = f'^\#\Delta'. \tag{1.11.3}$$

To see this, note that both hand sides of (1.11.3) agree on all non-f'-exceptional prime components, because P is a critical component of the $\mathbb{Q}$-divisor Δ. It is therefore enough to check their numerical equivalence. The definition of the f-adjoint transform of Δ on the left hand side implies

$$f^\#\Delta - \tilde{X}' \equiv f^*(K_X + \Delta) - (K_{\tilde{X}} + \tilde{X}') \tag{1.11.4}$$

and applying the adjunction formula twice, we see that

$$(1.11.5)\qquad (K_X + \Delta)\big|_{X'} \equiv K_{X'} + \Delta' \quad\text{and}\quad (K_{\tilde{X}} + \tilde{X}')\big|_{\tilde{X}'} \equiv K_{\tilde{X}'}.$$

Hence, the restriction of the first term on the left hand side in Equation (1.11.4) to the subvariety $\tilde{X}'$ is numerically equivalent to $f'^*(K_{X'} + \Delta')$ and that of the second term is numerically equivalent to $K_{\tilde{X}'}$. But their difference is by definition numerically equivalent to the f'-adjoint transform of Δ', which proves (1.11.3). Now, the Reduction Lemma (1.9) implies that the multiplier ideal sheaf $\mathcal{I}_{\Delta'}$ is the restriction of the multiplier ideal sheaf of the pair $(\tilde{X}, f^\#\Delta - \tilde{X}')$. But since $f^\#\Delta - \tilde{X}' \geqslant f^\#\tilde{\Delta}$, this last multiplier ideal sheaf is contained in $\mathcal{I}_{\tilde{\Delta}}$ and this proves (1.11.1). □

(1.12) **Definition.** Let $g : X \longrightarrow S$ be a proper morphism with connected fibers between normal varieties and let Δ be $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that Δ is *stably* g-quasi-effective if there is a log-resolution $f : \tilde{X} \longrightarrow X$ of the pair (X, Δ) such that $\lambda f^\#\Delta$ is $\tilde{g}$-quasi-effective for all positive rational numbers $\lambda \leqslant 1$ near to 1, where $\tilde{g} = g \circ f$. Then, the ideal

$$(1.12.1)\qquad \mathcal{I}^+_\Delta := \bigcap_{\lambda<1} \tilde{g}_*\mathcal{O}_{\tilde{X}}(-(\lambda f^\#\Delta)_+) \subset \mathcal{O}_S$$

is called the *stable* multiplier ideal sheaf of (X, Δ). We say that the pair (X, Δ) is *critical* over $\sigma \in \Sigma$, if it is not regular over σ and σ is not contained in the support of the stable multiplier ideal sheaf $\mathcal{I}^+_\Delta$.

(1.13) Remark. Let X be a smooth variety and Δ a $\mathbb{Q}$-divisor on X with simple normal crossings. The difference of $\llcorner\Delta\lrcorner$ and $\llcorner\lambda\Delta\lrcorner$ is an effective and reduced divisor if $\lambda < 1$ is a rational number which is near enough to 1. Its components are exactly the prime divisors on X, whose coefficients in Δ are positive integers. In particular, every critical component of Δ appears in the difference of the two round-downs. Moreover, if $f : \tilde{X} \longrightarrow X$ is a proper birational morphism from a smooth variety $\tilde{X}$, then the difference $f^*(\lambda\Delta)_+ - (\lambda f^\#\Delta)_+$ is effective and f-exceptional. This implies, that the stable multiplier ideal sheaf is independent of the choice of a log-resolution.

(1.14) Remark. The pair (X, Δ) is called *log-canonical* at a closed point $x \in X$, if there is a log-resolution $f : \tilde{X} \longrightarrow X$ of (X, Δ) such that the adjoint transform $f^\#\Delta$ has no prime component with coefficient larger than 1, which meets the fiber $f^{-1}(x)$. Therefore, the pair (X, Δ) is critical over σ, if and only if Δ is stably g-quasi-effective and (X, Δ) is log-canonical at all closed points $x \in g^{-1}(\sigma)$, but not log-terminal at some closed point $x \in g^{-1}(\sigma)$.

(1.15) Exercise (Connectedness Lemma). Let $g : X \longrightarrow S$ be a proper morphism with connected fibers between normal varieties and let Δ be a $\mathbb{Q}$-divisor on X. Assume that Δ satisfies the following two conditions

$$(1.15.1)\qquad -(K_X + \Delta) \quad \text{is } g\text{-nef and } g\text{-big},$$

$$(1.15.2)\qquad -(\Delta_-) \quad \text{is } g\text{-quasi-effective}.$$

Let $Z \subset X$ be the closed subset consisting of all closed points $x \in X$ such that the pair (X, Δ) is not log-terminal at x. Prove that the intersection $Z \cap g^{-1}(\sigma)$ is connected for every closed point $\sigma \in S$.

(1.16) EXERCISE (Multiple Reductions). Let R be a normal local Noetherian ring, denote by $S = \operatorname{Spec} R$ its spectrum and let $g : X \longrightarrow S$ be a proper morphism with connected fibers from a smooth variety X. Let Δ be a stably g-quasi-effective $\mathbb{Q}$-divisor with simple normal crossings on X such that

$$(1.16.1) \qquad -(K_X + \Delta) \quad \text{is } g\text{-ample},$$
$$(1.16.2) \qquad (X, \Delta) \quad \text{is critical over } \sigma,$$

where $\sigma \in S$ is the closed point of S. Denote by $P_1, \ldots, P_r$ the critical components of Δ. By our assumption, we know that there is at least one critical component. Show that the intersection $P_1 \cap \ldots \cap P_r$ is non-empty. In particular $r \leqslant \dim X$.

2. Adjoint Systems

The general framework for the proof of effective as well as non-effective base point free theorems is that of adjoint systems. We will see that, if an adjoint system has a specialization which is singular enough, then the freeness of this adjoint system can be reduced to a smaller dimensional adjoint system. In this way, one can prove the freeness by induction on the dimension. The idea of this procedure goes back to Shokurov's proof of the non-vanishing theorem [13]. But the more general concept of specializations introduced in this section, plays a crucial rule in the proof of Fujita's conjecture.

(2.1) **Definition.** Let X be a smooth, S a normal variety, $g : X \longrightarrow S$ a proper morphism with connected fibers, Δ be a stably g-quasi-effective $\mathbb{Q}$-divisor on X and $\mathcal{L}$ a line bundle on S. There is a Cartier divisor L and a $\mathbb{Q}$-divisor M on X such that

$$(2.1.1) \qquad g^*\mathcal{L} \simeq \mathcal{O}_X(L) \quad \text{and} \quad M \equiv L - (K_X + \Delta).$$

The triple $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ is called an *adjoint system*, if the $\mathbb{Q}$-divisor M is semi-ample. In this case, Δ is called the *fixed part* and M the *free part* of $\mathfrak{X}$. The variety X is the *total space* and S is the *base* of the adjoint system $\mathfrak{X}$.

(2.2) EXAMPLE. Let X be a normal variety, $\mathcal{L}$ be an invertible sheaf on X and let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume that for some positive integer n, the invertible sheaf $\mathcal{L}^n(-n(K_X + \Delta))$ is generated by its global sections and let $f : \tilde{X} \longrightarrow X$ be resolution of singularities of X. Then, the triple $(\tilde{X} \longrightarrow X, f^\#\Delta, \mathcal{L})$ is an adjoint system.

(2.3) MODIFICATIONS. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be an adjoint system, $\tilde{X}$ a smooth variety and $f : \tilde{X} \longrightarrow X$ a proper birational morphism. Then, the triple

$$(2.3.1) \qquad \tilde{\mathfrak{X}} := \left(\tilde{X} \xrightarrow{\tilde{g}} S, \tilde{\Delta}, \mathcal{L}\right), \quad \text{where} \quad \tilde{g} = g \circ f \quad \text{and} \quad \tilde{\Delta} = f^\#\Delta,$$

is also an adjoint system. In fact, if M is the free part of $\mathfrak{X}$, the free part of $\tilde{\mathfrak{X}}$ is just f^*M which is semi-ample. The adjoint system $\tilde{\mathfrak{X}}$ is called a *modification* of $\mathfrak{X}$. If the fixed part $\tilde{\Delta}$ of the modification $\tilde{\mathfrak{X}}$ has simple normal crossings, then the modification is a *resolution* of the adjoint system $\mathfrak{X}$.

(2.4) SPECIALIZATIONS. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be an adjoint system. A *specialization* of $\mathfrak{X}$ is a proper birational morphism $f : \tilde{X} \longrightarrow X$ from a smooth variety $\tilde{X}$ onto the total space X, together with an adjoint system

$$(2.4.1) \qquad \tilde{\mathfrak{X}} = \left(\tilde{X} \xrightarrow{\tilde{g}} S, \tilde{\Delta}, \mathcal{L}\right) \quad \text{with} \quad \tilde{\Delta} \geqslant f^{\#}\Delta,$$

where $\tilde{g} = g \circ f$. For short, we will say that $\tilde{\mathfrak{X}}$ is a specialization of $\mathfrak{X}$ without mentioning the morphism f explicitly. The effective $\mathbb{Q}$-divisor $F = \tilde{\Delta} - f^{\#}\Delta$ is called the *increment* of the specialization. If M is the free part of $\mathfrak{X}$, then the free part of $\tilde{\mathfrak{X}}$ is $\tilde{M} := f^*M - F$. Reversely, if $f : \tilde{X} \longrightarrow X$ is a proper birational morphism from a smooth variety $\tilde{X}$ and F is an effective $\mathbb{Q}$-divisor on $\tilde{X}$ such that $f^*M - F$ is semi-ample, then the triple

$$(2.4.2) \qquad \left(\tilde{X} \xrightarrow{\tilde{g}} S,\, f^{\#}\Delta + F,\, \mathcal{L}\right)$$

is a specialization of $\mathfrak{X}$ and every specialization is of this form.

(2.4.3) Specializations of the adjoint system $\mathfrak{X}$ are in natural relation to linear subsystems $\mathfrak{d} \subset |mM|$, where m is a positive integer such that mM is a Cartier divisor. To see this, let $f : \tilde{X} \longrightarrow X$ be a proper birational morphism from a smooth variety $\tilde{X}$ and denote by $f^*\mathfrak{d}$ the linear system on $\tilde{X}$ consisting of all divisors f^*D with $D \in \mathfrak{d}$. The morphism f is called a *resolution* of the linear system $\mathfrak{d}$, if there exists a Cartier divisor $f^*\mathfrak{d}_{\text{fix}}$ on $\tilde{X}$, such that

$$(2.4.4) \qquad \begin{array}{l} D \geqslant f^*\mathfrak{d}_{\text{fix}} \text{ for all effective divisors } D \in f^*\mathfrak{d} \text{ and the} \\ \text{linear system } f^*\mathfrak{d}_{\text{free}} = \{ D - f^*\mathfrak{d}_{\text{fix}} \mid D \in f^*\mathfrak{d} \} \text{ is free.} \end{array}$$

In this case, the divisor $f^*\mathfrak{d}_{\text{fix}}$ is unique and is called the *fixed part* of $f^*\mathfrak{d}$. The linear system $f^*\mathfrak{d}_{\text{free}}$ is called the *free part* of $f^*\mathfrak{d}$. By a theorem of O. Zariski, every finitely generated linear system has a resolution. Now, let $f : \tilde{X} \longrightarrow X$ be a resolution of a linear subsystem $\mathfrak{d} \subset |mM|$ and let $F = \frac{1}{m} f^*\mathfrak{d}_{\text{fix}}$. Then, the adjoint system defined by (2.4.2) is a specialization of $\mathfrak{X}$ with increment F and free part $\tilde{M} = f^*M - F$. Note that $f^*\mathfrak{d}_{\text{free}} = |m\tilde{M}|$ and therefore, $\tilde{M}$ is indeed semi-ample. Reversely, assume that the adjoint system (2.4.1) is a specialization of $\mathfrak{X}$ with increment F and semi-ample free part $\tilde{M}$. Then, there is a positive integer m, such that $|m\tilde{M}|$ is base point free. Therefore, the morphism f is a resolution of the linear system

$$(2.4.5) \qquad \mathfrak{d} := \left\{ D \in |mM| \mid f^*D \geqslant mF \right\}$$

and the fixed part of the linear system $f^*\mathfrak{d}$ is mF. The specialization corresponding to the linear system $\mathfrak{d}$ is unique up to modifications.

(2.5) **Definition.** An adjoint system $\mathfrak{X}$ is called *big*, if its free part is big. $\mathfrak{X}$ is called *stable* if its free part is ample and the support of its fixed part has simple normal crossings. We say that $\mathfrak{X}$ is *complete*, if its total space is complete. A *stable specialization* of $\mathfrak{X}$ is a specialization of $\mathfrak{X}$ which is stable as an adjoint system.

(2.6) **Lemma.** *An adjoint system $\mathfrak{X}$ has a stable specialization if and only if it is big.*

(2.7) PROOF. If $\mathfrak{X}$ has a specialization which is big, then $\mathfrak{X}$ is of course also big. Hence, the condition that $\mathfrak{X}$ is big is necessary for the adjoint system to have a stable specialization. Now assume that $\mathfrak{X}$ is big. By Chow's lemma, $\mathfrak{X}$ has a modification with a quasi-projective total space X. Therefore, we may assume that there is an ample divisor H on X. Since the free part M of $\mathfrak{X}$ is big, the linear system $|mM - H|$ has a non-trivial member D, if $m \gg 1$. By Hironaka's theorem, there is a projective morphism $f : \tilde{X} \longrightarrow X$ from a smooth variety $\tilde{X}$ with an effective $\mathbb{Q}$-divisor E, such that $f^*H - E$ is ample and the union of the supports of $f^*D + E$ and $f^{\#}\Delta$ has simple normal crossings. Let $\varepsilon > 0$ be a rational number with $\varepsilon m \leqslant 1$ and let $F = \varepsilon(f^*D+E)$. Then, the divisor $f^*M - F \equiv (1 - \varepsilon m)f^*M + \varepsilon(f^*H - E)$ is ample and hence, $\mathfrak{X}$ has a stable specialization with increment F. □

(2.8) REMARK. Note that in the Proof (2.7) we only use that the free part M of the adjoint system $\mathfrak{X}$ is nef and big. Nevertheless, because of the correspondence between linear subsystems of $|mM|$ and specializations of $\mathfrak{X}$ explained in (2.4.3), it is more natural to assume that M is semi-ample instead of just being nef.

(2.9) REDUCTIONS. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a stable adjoint system. A *critical component* of $\mathfrak{X}$ is a critical component of the pair (X, Δ). Let P be a such critical component of $\mathfrak{X}$. Then we can define a new adjoint system

$$\mathfrak{X}' = (X' \xrightarrow{g'} S', \Delta', \mathcal{L}') \tag{2.9.1}$$

as follows. Let X' be the support of P and let $X' \xrightarrow{g'} S' \xrightarrow{h} g(X')$ be the Stein factorization of the restriction of g to X'. Since the coefficient of P in the $\mathbb{Q}$-Cartier divisor $\Delta - P$ is zero, we may restrict it to X' and this restriction is a simple normal crossing $\mathbb{Q}$-divisor Δ' on the smooth variety X'. Finally, we define $\mathcal{L}' = h^*\mathcal{L}$. Then, the pull-back of $\mathcal{L}'$ to X' is the restriction of $g^*\mathcal{L}$ to X' and the $\mathbb{Q}$-Cartier divisor $K_{X'} + \Delta'$ is $\mathbb{Q}$-equivalent to the restriction of $K_X + \Delta$ to X', by the adjunction formula. Hence, the free part of $\mathfrak{X}'$ is simply the restriction of the free part of $\mathfrak{X}$ to X' and is therefore ample. Moreover, the Reduction Lemma (1.9) implies that Δ' is stably g'-quasi-effective. In particular, $\mathfrak{X}'$ is a stable adjoint system, which is called the *elementary reduction* of $\mathfrak{X}$ with respect to P.

(2.10) **Definition.** Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be an adjoint system and $\sigma \in S$ be a closed point of the base S of $\mathfrak{X}$. We say that $\mathfrak{X}$ is *regular*, resp. *singular*, resp. *critical* over σ, if the pair (X, Δ) is regular, resp. singular, resp. critical over σ. The *multiplier ideal sheaf* of $\mathfrak{X}$ is the multiplier ideal sheaf of the pair (X, Δ).

(2.11) SIMPLIFICATIONS. Let $\mathfrak{X}$ be an adjoint system which is regular over σ and let $\tilde{\mathfrak{X}}$ be a stable specialization of $\mathfrak{X}$ which is critical over σ. We assume that $\tilde{\mathfrak{X}}$ has only one critical component P which meets the fiber over σ. By wiggling the coefficients of the fixed part of $\tilde{\mathfrak{X}}$, this can always be achieved. Let $\mathfrak{X}'$ be the elementary reduction of $\tilde{\mathfrak{X}}$ with respect to P. Then, the pair $(\mathfrak{X}', \sigma)$ is called an *elementary simplification* of the pair $(\mathfrak{X}, \sigma)$.

(2.12) **Definition.** Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a complete adjoint system which is regular over σ and denote by $\mathcal{I}_\Delta$ the multiplier ideal sheaf of $\mathfrak{X}$. We say that $\mathfrak{X}$ is *free* at σ, if the sheaf $\mathcal{L} \otimes \mathcal{I}_\Delta$ has a global section, which is non-vanishing at the point σ.

(2.13) **Theorem.** *Let $\mathfrak{X}$ be a complete adjoint system which is regular over σ and let $(\mathfrak{X}', \sigma)$ be an elementary simplification of $(\mathfrak{X}, \sigma)$. If $\mathfrak{X}'$ is free at σ, then $\mathfrak{X}$ is also free at the point σ.*

(2.14) PROOF. There is a stable specialization $\tilde{\mathfrak{X}} = (\tilde{X} \longrightarrow S, \tilde{\Delta}, \mathcal{L})$ of the adjoint system $\mathfrak{X}$ such that $\mathfrak{X}' = (X' \longrightarrow S', \Delta', \mathcal{L}')$ is an elementary reduction of $\tilde{\mathfrak{X}}$ with respect to some critical component P of the adjoint system $\tilde{\mathfrak{X}}$. For a small rational number $\varepsilon > 0$, let $D = \tilde{\Delta} - \varepsilon P$. Then, the triple $\mathfrak{X}_\varepsilon = (\tilde{X} \longrightarrow S, D, \mathcal{L})$ is another stable specialization of $\mathfrak{X}$. Let $\mathcal{L}_{\tilde{X}}$ be the pull-back of $\mathcal{L}$ to $\tilde{X}$ and denote by M_ε the free part the adjoint system $\mathfrak{X}_\varepsilon$. Then,

$$\mathcal{L}_{\tilde{X}}(D_- - D_+ - P) \simeq \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + \ulcorner M_\varepsilon \urcorner). \tag{2.14.1}$$

and since M_ε is ample, the Vanishing Theorem (V) implies that

$$H^1\big(\tilde{X}, \mathcal{L}_{\tilde{X}}(D_- - D_+ - P)\big) = 0. \tag{2.14.2}$$

Denote by $\mathcal{L}_{X'}$ the restriction of $\mathcal{L}_{\tilde{X}}$ to X' and note that, by the construction of the reduction $\mathfrak{X}'$ we have $\Delta'_+ = D_+|_{X'}$ and $\Delta'_- = D_-|_{X'}$. Therefore, the restriction map

$$\Gamma\big(\tilde{X}, \mathcal{L}_{\tilde{X}}(D_- - D_+)\big) \twoheadrightarrow \Gamma\big(X', \mathcal{L}_{X'}(\Delta'_- - \Delta'_+)\big) \tag{2.14.3}$$

is surjective. Let $\mathcal{I} \subset \mathcal{O}_S$ be the multiplier ideal sheaf of the adjoint system $\mathfrak{X}$ and denote by $\mathcal{J} \subset \mathcal{O}_S$ the multiplier ideal sheaf of the specialization $\mathfrak{X}_\varepsilon$. Since the multiplier ideal sheaf $\mathcal{I}' \subset \mathcal{O}_{S'}$ of the reduction $\mathfrak{X}'$ is the pull-back of $\mathcal{J}$ by the Reduction Lemma (1.9), the push-forward of the map (2.14.3) is

$$\Gamma(S, \mathcal{L} \otimes \mathcal{J}) \twoheadrightarrow \Gamma(S', \mathcal{L}' \otimes \mathcal{I}'). \tag{2.14.4}$$

Since $\mathcal{J} \subset \mathcal{I}$, this proves that every global section of the sheaf $\mathcal{L}' \otimes \mathcal{I}'$ extends to a global section of $\mathcal{L} \otimes \mathcal{I}$ and this extension is non-vanishing at σ if and only if its restriction to S' is non-vanishing at σ. □

(2.15) LOCAL ADJOINT SYSTEMS. An adjoint system $(X \xrightarrow{g} S, \Delta, \mathcal{L})$ is called *local*, if $S = \operatorname{Spec} R$ and R is a local Noetherian ring. In this case, $R = \Gamma(X, \mathcal{O}_X)$, the morphism g is the universal morphism onto an affine variety and the invertible sheaf $\mathcal{L}$ is trivial. Hence, a local adjoint system is completely determinate by the pair $\mathfrak{X} = (X, \Delta)$ and we will also call the pair $\mathfrak{X}$ a local adjoint system. In other words, a local adjoint system is a pair (X, Δ), where X is a smooth variety such that $R = \Gamma(X, \mathcal{O}_X)$ is a local Noetherian ring, X is proper over $S = \operatorname{Spec} R$ and Δ is a stably g-quasi effective $\mathbb{Q}$-divisor on X such that its free part $M = -(K_X + \Delta)$ is semi-ample. A local adjoint system is called regular, if it is regular over the only closed point $\sigma \in S$ and we say that another local adjoint system $\mathfrak{X}'$ is a simplification of $\mathfrak{X}$ if the pair $(\mathfrak{X}', \sigma)$ is a simplification of $(\mathfrak{X}, \sigma)$.

(2.16) EXAMPLE (Localizations). Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be an adjoint system and let $\sigma \in S$ be a closed point. Denote by X_σ the fiber product $X \times_S \operatorname{Spec} \mathcal{O}_{S,\sigma}$ and let Δ_σ be the restriction of Δ to X_σ. The pair $\mathfrak{X}_\sigma = (X_\sigma, \Delta_\sigma)$ is then a local adjoint system, the *localization* of $\mathfrak{X}$ at the point σ. The adjoint system $\mathfrak{X}$ is regular over σ if and only if $\mathfrak{X}_\sigma$ is regular.

(2.17) EXERCISE (Point Separation). Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a complete adjoint system and $\Sigma \subset S$ be a finite subset of closed points, such that $\mathfrak{X}$ is regular over every point of Σ. For any subset $\Sigma' \subset \Sigma$ let $\mathcal{I}_{\Sigma'}$ be the intersection of the multiplier ideal sheaf $\mathcal{I}_\Delta$ of the adjoint system $\mathfrak{X}$ with the ideal sheaf of Σ'. Then, for any such subset we have an exact sequence

$$0 \longrightarrow \mathcal{I}_\Sigma \longrightarrow \mathcal{I}_{\Sigma\setminus\Sigma'} \longrightarrow \mathcal{O}_{\Sigma'} \longrightarrow 0. \tag{2.17.1}$$

We say that $\mathfrak{X}$ *separates* the pair (Σ, Σ') if the natural restriction map

$$\Gamma(S, \mathcal{L} \otimes \mathcal{I}_{\Sigma\setminus\Sigma'}) \longrightarrow \Gamma(S, \mathcal{L} \otimes \mathcal{O}_{\Sigma'}) \tag{2.17.2}$$

is surjective. Generalize the concept of a simplification for pairs $(\mathfrak{X}, \Sigma)$ and proof a version of Theorem (2.13), which applies to this situation.

3. The Base Point Free Theorem

The base point free theorem was proved in [13] by using an idea of [12]. Instead of this, we will follow and refine the ideas of the proof of the non-vanishing theorem from [13] to get base point freeness directly. There are basically two cases to consider. If line bundle $\mathcal{L}$ is numerically trivial, we will see in Lemma (3.3) that it is indeed trivial and hence, without base points. Otherwise, we will show in Proposition (3.6), that the corresponding adjoint system has a simplification, which concludes the proof by induction. In this non-effective version of base point freeness, we have to take special care on curves C with the property $\deg \mathcal{L}|_C = 0$, since any global section of $\mathcal{L}$ will be constant along C. Hence, if such a curve meets the support of the multiplier ideal sheaf, all the points of C are base points of the adjoint system.

(3.1) **Definition.** Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be an adjoint system. $\mathfrak{X}$ is called *nef*, resp. *numerically trivial*, if the degree of the line bundle $\mathcal{L}|_C$ is non-negative, resp. zero, for all curves $C \subset S$. It is called *trivial* if the line bundle $\mathcal{L}$ is trivial.

(3.2) NUMERICALLY FREE ADJOINT SYSTEMS. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a nef adjoint system and $\sigma \in S$ be a closed point of the base of $\mathfrak{X}$. Then, we define a subset of S as follows

$$\langle \mathfrak{X}, \sigma \rangle := \left\{ \sigma' \in S \;\middle|\; \begin{array}{l} \text{there is a connected curve } C \subset S \\ \text{with } \sigma, \sigma' \in C \text{ and } \deg \mathcal{L}|_C = 0 \end{array} \right\}. \tag{3.2.1}$$

In particular, $\mathfrak{X}$ is numerically trivial if and only if $\langle \mathfrak{X}, \sigma \rangle = S$. Denote by $\mathcal{I}_\Delta$ the multiplier ideal sheaf of $\mathfrak{X}$. The adjoint system $\mathfrak{X}$ is called *numerically free* at σ, if the support of $\mathcal{I}_\Delta$ and the set $\langle \mathfrak{X}, \sigma \rangle$ are disjoint. If $\mathfrak{X}$ is numerically trivial, then it is numerically free at σ if and only if its multiplier ideal sheaf $\mathcal{I}_\Delta$ is trivial.

(3.3) **Lemma.** *Let $\mathfrak{X}$ be a complete, big and numerically trivial adjoint system. If $\mathfrak{X}$ is numerically free at some point σ, then it is trivial.*

(3.4) PROOF. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ and denote by M the free part of $\mathfrak{X}$. By Hironaka's Theorem (R), the adjoint system $\mathfrak{X}$ has a resolution and hence, we may assume that the fixed part Δ has simple normal crossings. Since $\mathfrak{X}$ is numerically trivial and numerically free at some closed point $\sigma \in S$, the multiplier ideal sheaf of $\mathfrak{X}$ is trivial. Hence, the projection formula implies that

$$(3.4.1) \qquad g_*\mathcal{O}_X(K_X + \ulcorner M \urcorner) \;=\; g_*\big(g^*\mathcal{L} \otimes \mathcal{O}_X(\Delta_- - \Delta_+)\big) \;=\; \mathcal{L}.$$

By the Vanishing Theorem (V), the higher cohomology groups of the locally free sheaf $\mathcal{O}_X(K_X + \ulcorner M \urcorner)$ are zero, because the $\mathbb{Q}$-divisor M is nef and big and its fractional part has simple normal crossings. Therefore,

$$(3.4.2) \qquad h^0(S, \mathcal{L}) \;=\; h^0(X, \mathcal{O}_X(K_X + \ulcorner M \urcorner)) \;=\; \chi\big(X, \mathcal{O}_X(K_X + \ulcorner M \urcorner)\big).$$

Let L be a Cartier divisor on X with $g^*\mathcal{L} \simeq \mathcal{O}_X(L)$. Since $\mathcal{L}$ is numerically trivial, the divisor L is also numerically trivial. Therefore, tensoring the Equation (3.4.1) with the locally free sheaf $\mathcal{L}^* = g_*\mathcal{O}_X(-L)$ and applying the Vanishing Theorem (V) once more, we also get

$$(3.4.3) \qquad h^0(S, \mathcal{O}_S) \;=\; \chi\big(X, \mathcal{O}_X(K_X + \ulcorner M \urcorner - L)\big).$$

Of course, $h^0(S, \mathcal{O}_S) = 1$. On the other hand, the Euler characteristic of a coherent sheaf is a deformation invariant (cf. e.g. [4, proof of Theo. III 9.9]) and since L is numerically trivial, i.e. deformation equivalent to 0, the two Euler characteristics (3.4.2) and (3.4.3) coincide. (The numerical invariance of the Euler characteristic is also a consequence of the Riemann-Roch theorem.) In particular we find $h^0(S, \mathcal{L}) = 1$ and since $\mathcal{L}$ is numerically trivial, this is only possible if $\mathcal{L} \simeq \mathcal{O}_S$. □

(3.5) EVENTUALLY FREE ADJOINT SYSTEMS. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a nef adjoint system. A nef line bundle $\mathcal{N}$ on S is called *numerically $\mathcal{L}$-adjacent*, if $\deg \mathcal{N}|_C = 0$ for every connected curve $C \subset S$ with $\deg \mathcal{L}|_C = 0$. Assume that the free part M of the adjoint system $\mathfrak{X}$ is ample. Then, for every numerically $\mathcal{L}$-adjacent line bundle $\mathcal{N}$,

$$(3.5.1) \qquad \mathfrak{X}(\mathcal{N}) \;:=\; \big(X \xrightarrow{g} S,\, \Delta,\, \mathcal{L} \otimes \mathcal{N}\big)$$

is an adjoint system with the property $\langle \mathfrak{X}(\mathcal{N}), \sigma \rangle = \langle \mathfrak{X}, \sigma \rangle$ for all closed points $\sigma \in S$. Actually, the free part of $\mathfrak{X}(\mathcal{N})$ is the $\mathbb{Q}$-Cartier divisor

$$(3.5.2) \qquad M(\mathcal{N}) \;:=\; M + N, \quad \text{where} \quad \mathcal{O}_X(N) \;\simeq\; g^*\mathcal{N}.$$

Since M is ample and N is nef by assumption, the $\mathbb{Q}$-Cartier divisor $M(\mathcal{N})$ is also ample. Let $\sigma \in S$ be a closed point. We say that $\mathfrak{X}$ is *eventually free* at σ, if there is a positive integer ℓ such that the adjoint system $\mathfrak{X}(\mathcal{L}^\ell \otimes \mathcal{N})$ is free at σ for every numerically $\mathcal{L}$-adjacent line bundle $\mathcal{N}$. Our aim is to prove that numerically free and eventually free are equivalent.

(3.6) **Proposition.** *Let $\mathfrak{X}$ be a complete, stable and nef adjoint system. Assume that $\mathfrak{X}$ is not numerically trivial and numerically free at some point σ. Then, there is a positive integer n and a closed point $\sigma' \in \langle \mathfrak{X}, \sigma \rangle$, such that $\big(\mathfrak{X}(\mathcal{L}^n), \sigma'\big)$ has an elementary simplification which is numerically free at σ'.*

(3.7) PROOF. Denote by $M(\mathcal{L}^n)$ the free part of $\mathfrak{X}(\mathcal{L}^n)$ and let d be the dimension of the total space of $\mathfrak{X}$. The construction of a critical specialization is based on:

(3.7.1) **Claim.** *For every $t > 0$ there is a positive integer n, such that $M(\mathcal{L}^n)^d > t^d$.*

(3.7.2) PROOF. Let $C \subset S$ be a curve with $\deg \mathcal{L}|_C > 0$ and choose a curve $\tilde{C} \subset X$ which maps onto C. Since M is ample, for large integers m, the linear system $|mM|$ contains elements $H_1, \ldots, H_{d-1}$ such that the intersection $H_1 \cap \cdots \cap H_{d-1}$ is a curve which contains $\tilde{C}$ as a component. Let L be a Cartier divisor on X such that $\mathcal{O}_X(L) \equiv g^*\mathcal{L}$. Then, L is nef and from Kleiman's criterion [9] we get

$$(3.7.3) \qquad (M + nL)^d \geqslant n \cdot L.M^{d-1} \geqslant n \cdot L.\tilde{C}/m^{d-1}.$$

Our choice of $\tilde{C}$ implies $L.\tilde{C} > 0$ and this proves the claim. □

Now, choose a closed point $x \in X$ with $g(x) = \sigma$ and $t := d - s$, where s is the multiplicity of the fixed part Δ at the point x. By Claim (3.7.1) there is a positive integer n, such that $M(\mathcal{L}^n)^d > t^d$. Since $M(\mathcal{L}^n)$ is ample, the Riemann-Roch theorem implies

$$(3.7.4) \qquad h^0\Big(X, \mathcal{O}_X\big(m \cdot M(\mathcal{L}^n)\big)\Big) > \frac{(m \cdot t)^d}{d!} \qquad \text{for} \quad m \gg 0.$$

Hence, for large integers m there exists elements $D \in \big|m \cdot M(\mathcal{L}^n)\big|$ with multiplicity larger than $m \cdot t$ at x. Note that (X, Δ) is regular over all closed points $\sigma' \in \langle \mathfrak{X}, \sigma \rangle$ and that $(X, \Delta + qD)$ is singular over σ, if the multiplicity of $\Delta + qD$ at x is larger than the dimension d of the total space X. Hence,

$$(3.7.5) \qquad c := \sup \left\{ q \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X, \Delta + qD) \text{ is regular over} \\ \text{all closed points } \sigma' \in \langle \mathfrak{X}, \sigma \rangle \end{array} \right\}$$

is a positive rational number with $mc < 1$ by our choice of D. In particular, $F = cD$ is an effective $\mathbb{Q}$-divisor on X, such that $M(\mathcal{L}^n) - F \equiv (1-mc)\, M(\mathcal{L}^n)$ is ample and the pair $(X, \Delta + F)$ is critical over some point $\sigma' \in \langle \mathfrak{X}, \sigma \rangle$. Therefore, F is the increment of a big specialization $\mathfrak{X}_c$ of the adjoint system $\mathfrak{X}(\mathcal{L}^n)$. By Lemma (2.6) there is a stable specialization $\tilde{\mathfrak{X}}$ of the adjoint system $\mathfrak{X}_c$. Moreover, the increment of this last specialization can be chosen arbitrary small and hence, by decreasing c slightly and wiggling the coefficients of the fixed part of $\tilde{\mathfrak{X}}$, we can achieve that $\tilde{\mathfrak{X}}$ is regular or critical over all closed points of the set $\langle \mathfrak{X}, \sigma \rangle$ and there is unique critical component P of $\tilde{\mathfrak{X}}$ which meets $g^{-1}\langle \mathfrak{X}, \sigma \rangle$. Let $\mathfrak{X}'$ be the elementary reduction of $\tilde{\mathfrak{X}}$ with respect to the critical component P and choose a closed point $\sigma' \in \langle \mathfrak{X}, \sigma \rangle \cap g(P)$. Then, $(\mathfrak{X}', \sigma')$ is an elementary simplification of $\big(\mathfrak{X}(\mathcal{L}^n), \sigma'\big)$ which is numerically free at σ'. □

(3.8) **Theorem.** *Let $\mathfrak{X}$ be a complete, stable and nef adjoint system and σ a closed point of its base. If $\mathfrak{X}$ is numerically free at σ, then it is eventually free at σ.*

(3.9) PROOF. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$. If $\mathfrak{X}$ is numerically trivial, then its multiplier ideal sheaf is trivial and if $\mathcal{N}$ is a numerically $\mathcal{L}$-adjacent line bundle on S, then, $\mathcal{L}$ and $\mathcal{N}$ are trivial by Lemma (3.3). Therefore, $\mathfrak{X}(\mathcal{N})$ is obviously free at σ. Hence, we may assume that $\mathfrak{X}$ is not numerically trivial. Then, by Proposition (3.6) there is a positive integer n and a closed point $\sigma' \in \langle \mathfrak{X}, \sigma \rangle$, such that $\big(\mathfrak{X}(\mathcal{L}^n), \sigma'\big)$ has an elementary simplification $(\mathfrak{X}', \sigma')$ which is numerically free at σ'. Denote by S' the base of $\mathfrak{X}'$, let $h : S' \longrightarrow S$ be the natural morphism and define $\mathcal{L}' = h^*\mathcal{L}^n$. By induction on the dimension of the total space of the adjoint system, we may assume that $\mathfrak{X}'$ is eventually free at σ'. In other words, there is a positive integer ℓ' such that for every numerically $\mathcal{L}'$-adjacent line bundle $\mathcal{N}'$ on S', the adjoint system $\mathfrak{X}'(\mathcal{L}'^{\ell'} \otimes \mathcal{N}')$ is free at σ'. Let $\ell = \ell' \cdot (n+1) - 1$ and $\mathcal{N}$ a numerically $\mathcal{L}$-adjacent line bundle on S. Then, the line bundle $\mathcal{N}' = h^*\mathcal{N}$ is numerically $\mathcal{L}'$-adjacent and $\big(\mathfrak{X}'(\mathcal{L}'^{\ell'} \otimes \mathcal{N}'), \sigma'\big)$ is a simplification of $\big(\mathfrak{X}(\mathcal{L}^{\ell} \otimes \mathcal{N}), \sigma'\big)$. Hence, $\mathfrak{X}(\mathcal{L}^{\ell} \otimes \mathcal{N})$ is also free at σ' by Theorem (2.13), i.e. there is a global section of the sheaf $\mathcal{L}^{\ell+1} \otimes \mathcal{N} \otimes \mathcal{I}_{\Delta}$ which is non-vanishing at σ'. But there is a connected curve C containing σ and σ' such that the degree of the line bundle $\mathcal{L}^{\ell+1} \otimes \mathcal{N}$ restricted to C is zero. Therefore, the section is constant on C and in particular non-vanishing at σ. □

(3.10) **Corollary** (Base Point Free Theorem). *Let X be a normal variety, Δ an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is log-terminal and L a nef Cartier divisor on X, such that $mL - (K_X + \Delta)$ is nef and big for some $m > 0$. Then, there is a positive integer n_0 such that the linear system $|nL|$ is base point free for all $n \geqslant n_0$.*

(3.11) Proof. Let $\mathcal{L} = \mathcal{O}_X(mL)$. By Lemma (2.6) and Remark (2.8) there is a log-resolution $f : \tilde{X} \longrightarrow X$ of the pair (X, Δ) and an effective $\mathbb{Q}$-divisor F on the smooth projective variety $\tilde{X}$, such that the triple

$$\mathfrak{X} := \big(\tilde{X} \xrightarrow{f} X,\ f^{\#}\Delta + F,\ \mathcal{L}\big) \tag{3.11.1}$$

is a stable adjoint system. Moreover, since we may replace F by εF for any positive rational number $\varepsilon \leqslant 1$, we can achieve that $\mathfrak{X}$ is regular over every closed point $x \in X$. Hence, there is an integer ℓ, such that $\mathfrak{X}(\mathcal{L}^{\ell} \otimes \mathcal{N})$ is free for every numerically $\mathcal{L}$-adjacent line bundle $\mathcal{N}$. Let $n_0 = \ell \cdot m$. Since the line bundle $\mathcal{O}_X(nL)$ is numerically $\mathcal{L}$-adjacent for all $n \geqslant 0$, it follows that $|nL|$ is base point free for all $n \geqslant n_0$. □

(3.12) Exercise. Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a complete, stable and nef adjoint system and $\Sigma \subset S$ be a finite subset of its base. We say that Σ is *numerically separated* by $\mathfrak{X}$, if the adjoint system is numerically free at all points $\sigma \in \Sigma$ and

$$\langle \mathfrak{X}, \sigma \rangle \cap \langle \mathfrak{X}, \sigma' \rangle = \emptyset \quad \text{for all } \sigma, \sigma' \in \Sigma \text{ with } \sigma \neq \sigma'. \tag{3.12.1}$$

Show that there is a positive integer ℓ, such that $\mathfrak{X}(\mathcal{L}^{\ell} \otimes \mathcal{N})$ separates Σ for every numerically $\mathcal{L}$-adjacent line bundle $\mathcal{N}$, if the set Σ is numerically separated by $\mathfrak{X}$.

4. Some Effective Methods

In this section, we will reformulate results of [1] in terms of filtrations of a local ring. This makes the proof of those results particularly coherent. We have to calculate two invariants of a filtration, which comes from a degeneration of the filtration of a regular local ring by the powers of its maximal ideal, the multiplicity and the log-canonical threshold. The first calculation is elementary commutative algebra and the second is based on the Reduction Lemma (1.9). The original argument used the Ohsawa-Takegoshi Extension Theorem [11] instead. Once the filtration is constructed, the existence of simplifications is easy to prove, and almost the same as in the proof of the base point free theorem. But it is still difficult to prove the Fujita conjecture, since one has to apply the same argument to the simplification again, which is only possible under much stronger assumptions on the linear system. At the end of this section, we will discuss some prospects how to overcome those difficulties.

(4.1) **Definition.** Let R be a ring. A family $\mathfrak{q} = (\mathfrak{q}_t)_t$ of ideals $\mathfrak{q}_t$ of the ring R for all positive real numbers t with trivial intersection is called a *filtration* of R, if

(4.1.1) $$\mathfrak{q}_s \subset \mathfrak{q}_t \quad \text{for all } s \geqslant t > 0,$$

(4.1.2) $$\mathfrak{q}_s \cdot \mathfrak{q}_t \subset \mathfrak{q}_{s+t} \quad \text{for all } s, t > 0.$$

If R be a local ring with maximal ideal $\mathfrak{m}$, the filtration $\mathfrak{q}$ is called *primary*, if all the ideals $\mathfrak{q}_t$ are $\mathfrak{m}$-primary ideals of R, for all $t > 0$.

(4.2) THE MULTIPLICITY OF A FILTRATION. Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{q} = (\mathfrak{q}_t)_t$ be a primary filtration of the ring R. We define the *Hilbert function* of the filtration $\mathfrak{q}$ by

(4.2.1) $$\varphi_{\mathfrak{q}}(t) = \operatorname{length}_R R/\mathfrak{q}_t.$$

Denote by $d = \dim R$ the dimension of the local ring R. The limit

(4.2.2) $$\operatorname{mult} \mathfrak{q} = d! \cdot \limsup_{t \to \infty} \varphi_{\mathfrak{q}}(t)/t^d$$

is called the *multiplicity* of the filtration $\mathfrak{q}$. Note that the multiplicity of a filtration is always finite. To see this, let ℓ be a positive integer such that $\mathfrak{m}^\ell \subset \mathfrak{q}_1$ and let e be the multiplicity of the local ring R. Then, it follows from (4.1.1) and (4.1.2) that $\mathfrak{m}^{n\ell} \subset \mathfrak{q}_n$ for all positive integers n and hence, $\operatorname{mult} \mathfrak{q} \leqslant e \cdot \ell^d$.

(4.3) **Definition.** Let $\mathfrak{X} = (X, \Delta)$ be a regular local adjoint system and denote by R the local Noetherian ring of regular functions on X. Let $\mathfrak{q}$ be a filtration of the local ring R and $g : X \longrightarrow \operatorname{Spec} R$ the natural morphism. Then, the supremum

(4.3.1) $$c(\mathfrak{X}, \mathfrak{q}) := \sup \left\{ c \cdot t \;\middle|\; \begin{array}{l} c, t \in \mathbb{Q} \text{ and } a \in \mathfrak{q}_t \text{ such that the} \\ \text{pair } (X, \Delta + c \cdot g^*(a)) \text{ is regular} \end{array} \right\}$$

is called the *log-canonical threshold* of the filtration $\mathfrak{q}$.

(4.4) EXAMPLE. Let R be the localization of the polynomial ring $k[x_1, \ldots x_d]$ at the maximal ideal $(x_1, \ldots x_d)$. Then, the pair $\mathfrak{X} = (X, \Delta)$, where $X = \operatorname{Spec} R$ and $\Delta = 0$ is a local adjoint system. Let $w_1, \ldots, w_d$ be positive real numbers. On $R \setminus \{0\}$, we define a function

(4.4.1) $$v\left(\sum a_{i_1 \ldots i_d} x_1^{i_1} \cdots x_d^{i_d}\right) = \min\{ w_1 i_1 + \ldots + w_d i_d \mid a_{i_1 \ldots i_d} \neq 0 \}.$$

The function v is actually the restriction of a valuation of the field of rational functions on X. Therefore, for any non-negative real number t, the subset

(4.4.2) $$\mathfrak{q}_t = \{ a \in R \setminus \{0\} \mid v(a) \geqslant t \} \cup \{0\}$$

is an ideal of R and the family $(\mathfrak{q}_t)_t$ is a filtration of the local ring R. This filtration turns out to have the following log-canonical threshold and multiplicity

(4.4.3) $$c(\mathfrak{X}, \mathfrak{q}) = w_1 + \cdots + w_d \quad \text{and} \quad \operatorname{mult} \mathfrak{q} = \frac{1}{w_1 \cdots w_d}.$$

The calculation of the log-canonical threshold can be found in [10] for the case, where all the weights w_i are rational numbers. The general case follows from this by an approximation argument. The calculation of the multiplicity is easy. Note that, if we normalize the multiplicity to 1 (i.e. $w_1 \cdots w_d = 1$), then the filtration corresponding to the case with all weights equal to 1 has the minimal log-canonical threshold d. Actually, there is no filtration of R with multiplicity 1 and a smaller log-canonical threshold than d, but this is not so easy to see.

(4.5) **Lemma** (Angehrn and Siu [1]). *Let $\mathfrak{X} = (X, \Delta)$ be a regular and big local adjoint system. Then there is a filtration $\mathfrak{q}$ of the ring of regular functions R on X, such that*

$$c(\mathfrak{X}, \mathfrak{q}) \leqslant \dim R \qquad \text{and} \qquad \operatorname{mult} \mathfrak{q} = 1. \tag{4.5.1}$$

(4.6) PROOF. Let $S = \operatorname{Spec} R$ and denote by σ the unique closed point of S. We choose a curve $C \subset S$ through the closed point σ of S, such that the general point ξ of C is a smooth point of S and a regular value of the natural morphism $g : X \longrightarrow S$, and the smooth fiber $g^{-1}(\xi)$ is not contained in the support of Δ. Let $h : \tilde{C} \longrightarrow C$ be the normalization of C and denote by R_C the ring of regular functions of $S_C := S \times \tilde{C}$. Define $h_C : \tilde{C} \longrightarrow S_C$, $\zeta \longmapsto (h(\zeta), \zeta)$ and for any real number t let

$$\mathfrak{q}_{C,t} := \{ \bar{a} \in R_C \mid \operatorname{ord}_{h_C(\xi)} \bar{a} \geqslant t \}, \tag{4.6.1}$$

where ξ is the general point of $\tilde{C}$. The ideal $\mathfrak{q}_{C,t}$ of R_C is the kth symbolic power of the vanishing ideal of the curve $h_C(\tilde{C}) \subset S_C$, where $k = \ulcorner t \urcorner$. Choose a closed point $\gamma \in \tilde{C}$ with $h(\gamma) = \sigma$. Then, $\iota : S \hookrightarrow S_C$, $\zeta \longmapsto (\zeta, \gamma)$ is a closed embedding and for every real number t the restriction $\mathfrak{q}_t := \iota^*(\mathfrak{q}_{C,t})$ is an ideal of R. Moreover, the family $\mathfrak{q} = (\mathfrak{q}_t)_t$ is a primary filtration of the local ring R.

(4.6.2) THE MULTIPLICITY OF $\mathfrak{q}$. First note that the order of rational functions $\bar{a}$ on S_C at the general point of the curve $h_C(\tilde{C})$ is actually a valuation of the function field of the variety S_C. Therefore, the graded algebra

$$\operatorname{Gr} \mathfrak{q}_C := \bigoplus_{k=0}^{\infty} (\mathfrak{q}_{C,k}/\mathfrak{q}_{C,k+1}) \tag{4.6.3}$$

is an integral domain. In particular, the quotients $\mathfrak{q}_{C,k}/\mathfrak{q}_{C,k+1}$ are torsion free $\mathcal{O}_{\tilde{C}}$-modules and since $\tilde{C}$ is a smooth curve, those modules are locally free. Moreover, since the general point of C is a smooth point of S, their rank is

$$\operatorname{rank}_{\mathcal{O}_{\tilde{C}}}(\mathfrak{q}_{C,k}/\mathfrak{q}_{C,k+1}) = \binom{d+k-1}{d-1}, \tag{4.6.4}$$

where $d = \dim R$. The vector space $\mathfrak{q}_k/\mathfrak{q}_{k+1}$ is just the stalk of the vector bundle $\mathfrak{q}_{C,k}/\mathfrak{q}_{C,k+1}$ at the point $\gamma \in \tilde{C}$. Hence, the length of $R/\mathfrak{q}_t$ as an R-module is

$$\varphi_{\mathfrak{q}}(t) = \binom{d + \ulcorner t \urcorner - 1}{d} \sim \frac{t^d}{d!} \tag{4.6.5}$$

and the multiplicity of the filtration $\mathfrak{q}$ is equal to 1.

(4.6.6) THE LOG-CANONICAL THRESHOLD OF $\mathfrak{q}$. Let t be a positive rational number and let a be a regular function on S with $a \in \mathfrak{q}_t$. We want to show

(4.6.7) **Claim.** *The pair $(X, \Delta + c \cdot g^*(a))$ is not regular if $c \cdot t \geqslant \dim R$.*

(4.6.8) To see this, let $X_C := X \times_S S_C = X \times \tilde{C}$ and denote by g_C and q the two projections from X_C onto S_C and X respectively. Choose an element $\bar{a} \in \mathfrak{q}_{C,t}$ with $\iota^*\bar{a} = a$ and denote by P the prime Cartier divisor $X \times \{\gamma\}$ of X_C. Then,

$$\Delta_C := q^*\Delta + P + c \cdot g_C^*(\bar{a}), \tag{4.6.9}$$

is stably g_C-quasi-effective by (1.4.1) and (1.4.2). Let $Z \subset X_C$ be the closure of the preimage $g_C^{-1}(h_C(\xi))$. Note that the codimension of the subvariety Z in X_C is equal to $d = \dim R$ and that the order of $\Delta_C - P$ along Z is at least d. Therefore, the pair $(X_C, \Delta_C - P)$ is singular over every closed point of the image of h_C. In other words, the support of the multiplier ideal sheaf of $\Delta_C - P$ contains the curve $\tilde{C} \subset S_C$. To compare this multiplier ideal sheaf with that of the restriction

$$(\Delta_C - P)\big|_X = \Delta + c \cdot g^*(a), \tag{4.6.10}$$

we use (1.11.1). The $\mathbb{Q}$-divisor $K_{X_C} + \Delta_C$ is numerically equivalent to $q^*(K_X + \Delta)$. Since q is an affine morphism and the pair (X, Δ) is a big local adjoint system, this implies that the $\mathbb{Q}$-divisor $-(K_{X_C} + \Delta_C)$ is nef and big. Hence, the multiplier ideal sheaf of the pair $(X, \Delta + c \cdot g^*(a))$ is contained in the restriction of the multiplier ideal sheaf of $(X_C, \Delta_C - P)$ to the local ring $\iota^*(R_C) \simeq R$. The support of this last ideal sheaf contains the point (σ, γ) as we have seen above and this proves the claim and the inequality $c(\mathfrak{X}, \mathfrak{q}) \leqslant d$. □

(4.7) **Definition.** Let $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ be a complete adjoint system with free part M and $d = \dim S$. We say that $\mathfrak{X}$ is *large*, resp. *very large*, if there is an ample Cartier divisor H on S such that

(4.7.1) $M - d \cdot g^*H$, resp. $M - \frac{1}{2}d(d+1) \cdot g^*H$ is ample.

(4.8) **Theorem.** *Let $\mathfrak{X}$ be a complete adjoint system. If $\mathfrak{X}$ is large, and σ is a closed point of the base of $\mathfrak{X}$, then $(\mathfrak{X}, \sigma)$ has an elementary simplification. If $\mathfrak{X}$ is very large, then $(\mathfrak{X}, \sigma)$ has an elementary simplification which is also very large.*

(4.9) PROOF. Let $\mathfrak{X}_\sigma = (X_\sigma, \Delta_\sigma)$ be the localization of the complete adjoint system $\mathfrak{X} = (X \xrightarrow{g} S, \Delta, \mathcal{L})$ at σ and choose a filtration $\mathfrak{q}$ of the local ring $\mathcal{O}_{S,\sigma}$ with multiplicity 1 and log-canonical threshold less or equal that $d = \dim S$, which exists by Lemma (4.5). Let M be the free part of $\mathfrak{X}$ and H an ample Cartier divisor on S, such that $M - d \cdot g^*H$ is ample. Then, $\lambda \cdot M - d \cdot g^*H$ is still ample, if $\lambda < 1$ is a positive rational number near to one. Since the degree of H on S is at least 1, the Riemann-Roch theorem implies that

$$h^0(X, \mathcal{O}_S(mH)) > \frac{(m \cdot \lambda)^d}{d!} \quad \text{for} \quad m \gg 0. \tag{4.9.1}$$

Hence, there exists an element $D \in |mH|$, such that the local equation of D at σ lies in the ideal $\mathfrak{q}_{m\lambda}$. Then, since the log-canonical threshold of $\mathfrak{q}$ is at most d, the supremum

$$c := \sup\{\, q \in \mathbb{Q} \mid (X, \Delta + q \cdot g^*D) \text{ is regular over } \sigma \,\} \tag{4.9.2}$$

is a rational number with $m\lambda c \leqslant d$. Define $F = c \cdot g^*D$. Since $M - F \equiv M - mcH$ is ample, the effective $\mathbb{Q}$-divisor F is the increment of a specialization $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ which is critical over the closed point σ. By using Lemma (2.6) and arguing as in (3.7), we may assume that $\tilde{\mathfrak{X}}$ is stable, critical over σ and $\tilde{\mathfrak{X}}$ has exactly one critical component P over σ. Then, the elementary reduction of $\tilde{\mathfrak{X}}$ with respect to P is an elementary simplification of the pair $(\mathfrak{X}, \sigma)$.

(4.9.3) Now, assume that $\lambda \cdot M - \frac{1}{2}d(d+1) \cdot g^*H$ is ample and let H' be the pull-back of H to the base S' of the simplification of $\mathfrak{X}$ constructed in the previous step. Since the natural morphism $S' \longrightarrow S$ is finite, H' is ample. Moreover, the free part M' of this simplification is the restriction of the free part $\tilde{M} = M - F$ of the specialization $\tilde{\mathfrak{X}}$ and $\tilde{M} - \frac{1}{2}d(d-1) \cdot g^*H$ is still ample. But $d' = \dim S' \leqslant d - 1$ and therefore, the $\mathbb{Q}$-divisor $M' - \frac{1}{2}d'(d'+1) \cdot g'^*H'$ is also ample. □

(4.10) **Corollary.** *Let $\mathfrak{X}$ be a complete and very large adjoint system. If $\mathfrak{X}$ is regular over a closed point σ of the base of $\mathfrak{X}$, then $\mathfrak{X}$ is free at σ.*

(4.11) PROOF. By Theorem (4.8), the pair $(\mathfrak{X}, \sigma)$ has a very large simplification $(\mathfrak{X}', \sigma)$ and by induction on the dimension of the total space of the adjoint system, we may assume that $\mathfrak{X}'$ is free at the closed point σ. But then, Theorem (2.13) implies that $\mathfrak{X}$ is also free at σ. □

(4.12) **Corollary** (Angehrn and Siu [1]). *Let X be a smooth projective variety with canonical divisor K_X of dimension d and H an ample Cartier divisor on X. Then, the linear system $|K_X + mH|$ is base point free, if $m > \frac{1}{2}d(d+1)$.*

(4.13) PROOF. The triple $(X \xrightarrow{\text{id}} X,\ 0,\ K_X + mH)$ is a very large adjoint system which is regular over every closed point of X. Hence, the linear system $|K_X + mH|$ has no base points by the previous corollary. □

(4.14) REMARK. There are basically two reasons, why the simple arguments of (4.9) are not sufficient for a proof of the Fujita conjecture. First of all, the estimate of Lemma (4.5) can be considerably improved, if one knows more about the fixed part Δ. Actually, from the construction of the simplification in (4.9), we see that the fixed part Δ is already quite singular over σ, at least after the first induction step. It was shown in [5], that this idea leads already to an essential improvement of Corollary (4.12). The other important observation is, that if the simplification at the first step has to have a positive dimensional base, then there must be an even more singular specialization. We will conclude these notes with an explanation of this phenomenon.

(4.15) CONVEXITY. Let S be a normal projective variety and $\sigma \in S$ a closed point of S. For simplicity we assume that σ is a smooth point. Choose a local coordinate system $(x_1, \ldots, x_d)$ at the point σ and some monomial order for those coordinates (cf. e.g. [2, Section 15.2]). For any formal power series in the x_i, we define

$$\mu\left(\sum a_{i_1 \ldots i_d} x_1^{i_1} \cdots x_d^{i_d}\right) = \min\{(i_1, \ldots, i_d) \mid a_{i_1 \ldots i_d} \neq 0\}, \tag{4.15.1}$$

where the minimum is taken with respect to the chosen monomial order. Again, μ is the restriction of a valuation. If D is a Cartier divisor on S and p is the Tailor Expansion of a local equation for D at σ, then we define $\mu(D) = \mu(p)$. Moreover, let

$$\bar{\mu}(D) := (i_1 \cdots i_d)^{1/d}, \quad \text{where} \quad \mu(D) = (i_1, \ldots, i_d). \tag{4.15.2}$$

(4.15.3) **Claim.** *Let H be an ample divisor on S and λ a positive real number with $\lambda^d < H^d$. Then, for large m there are divisors $D \in |mH|$ with $\bar{\mu}(D) > m \cdot \lambda/d$.*

(4.15.4) PROOF. For any positive integer m we are defining subsets $\Phi(m) \subset \mathbb{R}^d$ by

$$\Phi(m) := \{ \mu(D)/m \mid D \in |mH| \} \tag{4.15.5}$$

and denote by Φ the closure of their union. Note that representatives of the set $\Phi(m)$ form a basis of the vector space of global sections of $\mathcal{O}_S(mH)$ and therefore,

$$h^0(S, \mathcal{O}_S(mH)) = \# \Phi(m) \sim \operatorname{Vol}\Phi \cdot m^d. \tag{4.15.6}$$

Comparing this with the Riemann-Roch theorem, we see that $d! \cdot \operatorname{Vol}\Phi = H^d$. On the other hand, since μ is a valuation, the set Φ is actually a convex subset of $\mathbb{R}^d$. If the conclusion of the claim would be wrong, the set Φ would be disjoint from the set

$$\Psi := \{ (w_1, \ldots, w_d) \mid w_1 \cdots w_d > (\lambda/d)^d \} \tag{4.15.7}$$

and since this set is also convex, there is a tangent plane to the boundary of Ψ, which separates Ψ from Φ (cf. Fig. 1). Now, the subset of $\mathbb{R}^d_{\geqslant 0}$ consisting of all points below a tangent plane to the boundary Ψ is a simplex with volume $\lambda^d/d!$, independently of the tangent plane. Since Φ is contained in this simplex, we find $d! \cdot \operatorname{Vol}\Phi \leqslant \lambda^d$, in contradiction to the assumption on the degree of H. □

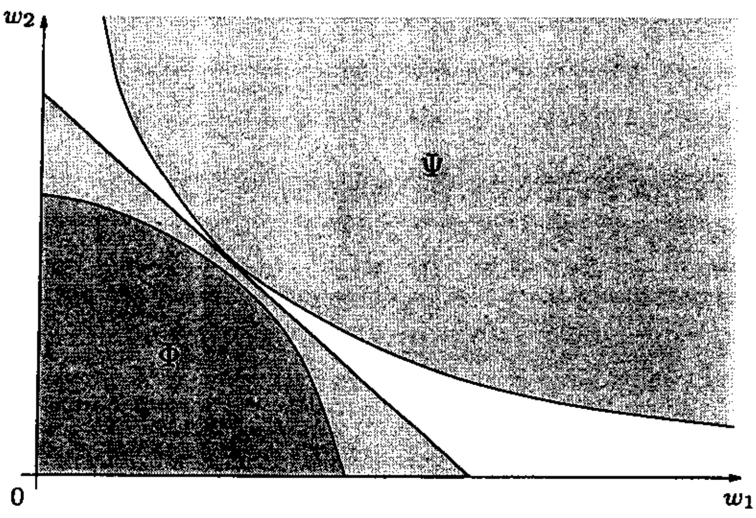

Fig. 1. The two convex subsets of the Proof (4.15.4).

If we choose a monomial order which starts with the vanishing order $i_1 + \cdots + i_d$ then, the divisor D with $\bar{\mu}(D) > m \cdot \lambda/d$ has vanishing order greater than $m \cdot \lambda$, since the geometric mean of positive real numbers is bounded by their arithmetic mean. Therefore, the divisor D has at least the same properties as the divisor from (4.9). But the invariant $\bar{\mu}(D)$ controls also the order of D in a small neighborhood of σ. This idea was first used by T. Fujita [3] to get some effective results in 3 dimensions and further developed in [5] and [6]. It is in general not easy to make use of Claim (4.15.3) but at least its idea is the key observation to the proof of Fujita's conjecture.

(4.16) EXERCISE. The estimate of Lemma (4.5) is in general far from being optimal, even if $\Delta = 0$. Show by example, that for every $\varepsilon > 0$ there exists a regular and big local adjoint system $\mathfrak{X} = (X, 0)$ and a filtration $\mathfrak{q}$ of the local ring of regular functions R on X such that

(4.16.1) $$c(\mathfrak{X}, \mathfrak{q}) < \varepsilon \quad \text{and} \quad \operatorname{mult} \mathfrak{q} = 1.$$

Note, that by the remark at the end of Example (4.4), this is impossible if X is smooth.

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Index of Definitions

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