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Semipositivity, vanishing and applications

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1 Semipositivity theorem

The semipositivity theorem proved in [6] was used for the study of algebraic fiber spaces whose fibers have nonnegative Kodaira dimension. Now its logarithmic generalization will be applied for those with negative Kodaira dimension as well.

We start with recalling the semipositivity theorem ([6] Theorem 5) with slightly different expression:

Theorem 1.1. *Let X and S be smooth projective varieties and let $f : X \to S$ be a surjective morphism. Let $n = \dim X - \dim Y$. Assume that there exists a normal crossing divisor Γ on S such that f is smooth over $S_0 = S \setminus \Gamma$. Then the following hold:*

(1) $\mathcal{F} = f_*\mathcal{O}_X(K_{X/S})$ is a locally free sheaf, where $K_{X/S} = K_X - f^*K_S$.

(2) Let $\pi : P = \mathbb{P}(\mathcal{F}) \to S$ be the associated projective space bundle, and let $P_0 = \pi^{-1}(S_0)$. Then the tautological invertible sheaf $\mathcal{O}_P(1)$ on P has a singular hermitian metric h which is smooth over P_0 and such that the curvature current Θ is semipositive and that the corresponding multiplier ideal sheaf coincides with $\mathcal{O}_P$.

(3) Let $X_0 = f^{-1}(S_0)$ and $f_0 = f|_{X_0}$. If the local monodromies of $R^n f_{0*}\mathbb{Q}_{X_0}$ around the branches of Γ are unipotent, then the Lelong number of Θ vanishes at any point of P. In particular, $\mathcal{F}$ is numerically semipositive. If Θ is strictly positive at a point on P_0, then $\mathcal{O}_P(1)$ is also big.

Proof. The hermitian metric $\tilde{h}$ on $\mathcal{F}|_{S_0}$ is defined by the integration along the fiber: for $s \in S_0$ and $u, v \in \mathcal{F}_s$,

$$\tilde{h}_s(u,v) = \text{const.} \int_{f^{-1}(x)} u \wedge \bar{v}.$$

By [3], the curvature form of $\tilde{h}$ is Griffiths semipositive. Hence the smooth metric h on $\mathcal{O}_P(1)|_{P_0}$ induced from $\tilde{h}$ has semipositive curvature form as well. Moreover, it is extended to a singular hermitian metric h over P. The multiplier ideal sheaf is trivial because the sections of $\mathcal{F}$ are L^2. In the case (3), the growth of the metric is logarithmic. So the Lelong number vanishes, and the last statements follow from the regularization of positive currents ([2]). □

Let X be a smooth projective variety, and B a normal crossing divisor. Let $H_{\mathbb{Z}}$ be a local system on $X_0 = X \setminus B$. A *variation of Hodge structures* on X_0 is defined by a decreasing filtration $\{F_0^p\}$ of $\mathcal{H}_0 = H_{\mathbb{Z}} \otimes \mathcal{O}_{X_0}$ by locally free subsheaves which satisfy certain axioms ([3]). Assume in addition that the local monodromies of $H_{\mathbb{Z}}$ around the branches of B are unipotent. Then the *canonical extension* $\mathcal{H}$ of $\mathcal{H}_0$ is defined as a locally free sheaf on X as follows. Let $\{z_1, \cdots, z_n\}$ be a local coordinates at a given point $x \in X$ such that B is defined by an equation $z_1 \cdots z_r$ near x, and T_i the monodromies of $H_{\mathbb{Z}}$ around the brances of B defined by $z_i = 0$. Then for multi-valued flat sections v of $H_{\mathbb{Z}}$, the expression

$$s = \exp(-\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{r} \log T_i \log z_i)v$$

is single-valued and give a holomorphic section of $\mathcal{H}_0$. The canonical extension $\mathcal{H}$ is defined as a locally free sheaf on X which has a basis near x consisting of the s for v varying a basis of $H_{\mathbb{Z}}$. The filtration $\{F_0^p\}$ extends to a filtration $\{F^p\}$ of $\mathcal{H}$ by locally free subsheaves such that $\mathcal{H}/F^p$ are also locally free for all p ([32]).

Let $f : Y \to X$ be a surjective morphism from another smooth projective variety whose geometric fibers are not necessarily connected. Assume that f is smooth over X_0. Let $d = \dim Y - \dim X$, $Y_0 = f^{-1}(X_0)$ and $f_0 = f|_{Y_0}$. Then $H_{\mathbb{Z}} = R^{d+q} f_{0*}\mathbb{Z}_{Y_0}$ gives a variation of Hodge structures on X_0 for any $q \geq 0$, and $F^d(\mathcal{H})$ coincides with $R^q f_*\omega_{Y/X}$ ([6] and [19]).

The following lemma is obvious:

Lemma 1.2. *Let X be a smooth projective variety, and B a normal crossing divisor. Let $H_{\mathbb{Z}}$ be a variation of Hodge structures on $X_0 = X \setminus B$ whose local monodromies around the branches of B are unipotent, and $\mathcal{H}$ the canonical extension of $\mathcal{H}_0 = H_{\mathbb{Z}} \otimes \mathcal{O}_{X_0}$ on X. Let $\pi : X' \to X$ be a generically finite and surjective morphism from a smoth projective variety such that $B' = (\pi^* B)_{\text{red}}$*

is a normal crossing divisor. Let $H'_{\mathbb{Z}} = \pi^* H_{\mathbb{Z}}$ be the induced variation of Hodge structures on $X'_0 = X' \setminus B'$, and $\mathcal{H}'$ the canonical extension of $\mathcal{H}'_0 = H'_{\mathbb{Z}} \otimes \mathcal{O}_{X'_0}$ on X'. Then $\mathcal{H}' = \pi^*\mathcal{H}$.

The semipositivity theorem is generalized to the logarithmic case by the covering method:

Theorem 1.3. *Let X and S be smooth projective varieties, let $f : X \to S$ be a surjective morphism, and let B be an effective $\mathbb{Q}$-divisor on X whose support is a normal crossing divisor and whose coefficients are strictly less than 1. Assume that there exists a normal crossing divisor Γ on S such that f is smooth and* Supp(B) *is relative normal crossing over $S_0 = S \setminus \Gamma$. Let D be a Cartier divisor on X. Assume that $D \sim_{\mathbb{Q}} K_{X/S} + B$. Then the following hold:*

(1) $\mathcal{F} = f_*\mathcal{O}_X(D)$ is a locally free sheaf.

(2) Let $\pi : P = \mathbb{P}(\mathcal{F}) \to S$ be the associated projective space bundle, and let $P_0 = \pi^{-1}(S_0)$. Then the tautological invertible sheaf $\mathcal{O}_P(1)$ on P has a singular hermitian metric h which is smooth over P_0 and such that the curvature current Θ is semipositive and that the corresponding multiplier ideal sheaf coincides with $\mathcal{O}_P$.

(3) There exists a finite surjective morphism $\sigma : S' \to S$ from a smooth projective variety S' such that $\Gamma' = \sigma^{-1}(\Gamma)$ is a normal crossing divisor and satisfies the following conditions: Let $X' \to X \times_S S'$ be a birational morphism from a smooth projective variety which is isomorphic over S_0 and such that the union of the pull-back of the support of B, the pull-back of the support of Γ and the exceptional locus is a normal crossing divisor. Let $f' : X' \to S'$ and $\tau : X' \to X$ be the induced morphisms. An effective $\mathbb{Q}$-divisor B' on X' is defined such that its coefficients are strictly less than 1 and that $R = \tau^*(K_{X/S} + B) - (K_{X'/S'} + B')$ is a divisor. Let $D' = \tau^* D - R$. Then R is effective, and the assumptions of the theorem are satisfied by $f' : X' \to S'$, B' and D'. The locally free sheaf $\mathcal{F}' = f'_*\mathcal{O}_{X'}(D')$ on S' satisfies that $\sigma^*\mathcal{F} \supset \mathcal{F}'$. The singular hermitian metric h induces a singular hermitian metric h' on the tautological invertible sheaf $\mathcal{O}_{P'}(1)$ on $P' = \mathbb{P}(\mathcal{F}')$, and the Lelong number of the curvature current Θ' vanishes at any point of P'. In particular, $\mathcal{F}'$ is numerically semipositive. If Θ' is strictly positive at a point on P'_0, then $\mathcal{O}_{P'}(1)$ is also big.

Proof. Let m be the minimal positive number such that $mD \sim m(K_{X/S}+B)$. We take a rational function h on X such that $\text{div}(h) = -mD+m(K_{X/S}+B)$.

Let $\pi : Y \to X$ be the normalization of X in the field $\mathbb{C}(X)(h^{1/m})$, and let $\mu : Y' \to Y$ be a desingularization such that the composite morphism $g : Y' \to S$ is smooth over S_0. We have

$$\pi_*\mathcal{O}_Y \cong \bigoplus_{k=0}^{m-1} \mathcal{O}_X(-kD + kK_{X/S} + \llcorner kB \lrcorner).$$

The Galois group $G \cong \mathbb{Z}/m\mathbb{Z}$ acts on Y such that the above direct summands of $\pi_*\mathcal{O}_Y$ are eigenspaces with eigenvalues $\exp(2\pi\sqrt{-1}k/m)$.

Since π is etale outside the support of B, Y has only rational singularities, hence $\mu_*\mathcal{O}_{Y'}(K_{Y'}) = \mathcal{O}_Y(K_Y)$. We apply Theorem 1.1 to the sheaf $g_*\mathcal{O}_{Y'}(K_{Y'/S}) = f_*\pi_*\mathcal{O}_Y(K_{Y/S})$. By duality, we have

$$\pi_*\mathcal{O}_Y(K_Y) \cong \bigoplus_{k=0}^{m-1} \mathcal{O}_X(K_X + kD - kK_{X/S} - \llcorner kB \lrcorner).$$

By taking $k = 1$ (we may assume that $m \geq 2$), we obtain our assertions (1) and (2) since $f_*\mathcal{O}_X(D - \llcorner B \lrcorner) = f_*\mathcal{O}_X(D)$.

For (3), we use the unipotent reduction theorem for the local monodromies of g ([6]). □

If the base space is 1-dimensional, we have a simpler expression:

Corollary 1.4. *Let X be a complete normal variety, and B an effective $\mathbb{Q}$-divisor on X such that the pair (X, B) is KLT. Let $f : X \to C$ be a surjective morphism to a smooth curve. Let D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_{X/C} + B$. Then $f_*\mathcal{O}_X(D)$ is a numerically semipositive locally free sheaf on C.*

Proof. Let $\mu : X' \to X$ be a log resolution for the pair (X, B), and set $\mu^*(K_X + B) = K_{X'} + B'$. The coefficients of B' are less than 1 and negative coefficients appear only for exceptional divisors of μ. We set $B' = -B'_I + B'_F$ where B'_I is an effective integral divisor and B'_F is a $\mathbb{Q}$-divisor whose coefficients belong to the interval $(0, 1)$. Since the support of B'_I is exceptional for μ, we have $\mu_*\mathcal{O}_{X'}(B'_I) = \mathcal{O}_X$. By applying the theorem to the pair (X', B'_F), we deduce that the sheaf $f_*\mathcal{O}_X(D) = f_*\mu_*\mathcal{O}_{X'}(\mu^*D + B'_I)$ is numerically semipositive. □

Corollary 1.5. *Let X, B and $f : X \to C$ be as in Corollary 1.4. Let D be a Cartier divisor on X such that $H = D - (K_{X/C} + B)$ is nef and big. Then $f_*\mathcal{O}_X(D)$ is a numerically semipositive locally free sheaf on C.*

Proof. There exists an effective $\mathbb{Q}$-divisor B' such that $(X, B + B')$ is KLT and $D \sim_{\mathbb{Q}} K_{X/C} + B + B'$. □

In the case of rank one sheaf, we have a more precise result which is not used later:

Corollary 1.6. *In Theorem 1.3, assume that $\mathcal{F} = \mathcal{O}_S(F)$ is an invertible sheaf. Let E be an effective divisor such that $E \sim D - f^*F$. Let Δ be the smallest $\mathbb{Q}$-divisor supported on Γ such that $(X, B - E + f^*(\Gamma - \Delta))$ is subLC over the generic points of Γ. Then $F - \Delta$ is nef.*

Proof. Let $\mu : X' \to X$ be the log resolution of the pair $(X, B + E + f^*\Gamma)$. We have $\mu^* f^* F \sim \mu^*(D - E) \sim_{\mathbb{Q}} \mu^*(K_{X/S} + B - E) \sim_{\mathbb{Q}} K_{X'/S} + B'$ for some $\mathbb{Q}$-divisor B'. Then our assertion is proved in [15] Theorem 2. □

2 Adjunction

Theorem 2.1. *Let X be a normal projective variety. Let D^o and D be effective $\mathbb{Q}$-divisors on X such that $D^o < D$, (X, D^o) is log terminal, and (X, D) is log canonical. Let W be a minimal center of log canonical singularities for (X, D). Let H be an ample Cartier divisor on X, and ϵ a positive rational number. Then there exists an effective $\mathbb{Q}$-divisor D_W on W such that*

$$(K_X + D + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + D_W$$

and that the pair (W, D_W) is log terminal. In particular, W has only rational singularities.

We recall the terminology. A pair (X, D) of a normal variety and an effective $\mathbb{Q}$-divisor is said to be *log terminal (KLT)* (resp. *log canonical (LC)*) if the following conditions are satisfied:

(1) $K_X + D$ is a $\mathbb{Q}$-Cartier divisor.

(2) There exists a projective birational morphism $\mu : Y \to X$ from a smooth variety Y with a normal crossing divisor $\sum_j E_j$ such that a formula

$$K_Y + \sum_j e_j E_j \sim_{\mathbb{Q}} \mu^*(K_X + D)$$

holds with $e_j < 1$ (resp. ≤ 1) for all j, where $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence.

If $e_j = 1$, then the subvariety $\mu(E_j)$ of X is called a *center of log canonical singularities*, and the discrete valuation of $\mathbb{C}(X)$ corresponding to the prime divisor E_j is called a *place of log canonical singularities*.

If D^o and D are effective $\mathbb{Q}$-divisors on a normal variety X such that $D^o < D$, (X, D^o) is KLT, and that (X, D) is LC, then there exists a minimal element among the centers of log canonical singularities for (X, D) with respect to the inclusions (cf. [13]). Though the argument in [13] on the *minimal center of log canonical singularities* treats only the case where $D^o = 0$, it can be easily extended to our case. In particular, a minimal center of log canonical singularities is always normal.

Proof. There exists an effective $\mathbb{Q}$-divisor D' which passes through W and satisfies the following conditions: $(X, (1 - \alpha)D + D')$ is LC for a rational number α such that $0 < \alpha \ll 1$, W is a minimal center of log canonical singularities for $(X, (1 - \alpha)D + D')$, and there exists only one place of log canonical singularities for $(X, (1 - \alpha)D + D')$ above W.

Let $D_t = (1 - \alpha t)D + tD'$ for $0 \leq t \leq 1$, and let $\mu : Y \to X$ be an embedded resolution for the pairs (X, D_0) and (X, D_1) simultaneously. We write

$$K_Y + E + F_t \sim_{\mathbb{Q}} \mu^*(K_X + D_t)$$

where E is the only place of log canonical singularities for (X, D_t) above W if $t \neq 0$. By construction, the coefficients of $F_t|_E$ are less than 1 if $t \neq 0$. Moreover, even if $t = 0$, the coefficients are less than 1 for vertical components of $F_t|_E$ with respect to μ, because W is a minimal center.

We have $R^1\mu_*\mathcal{O}_Y(-E + \ulcorner -F_t \urcorner) = 0$ by Kawamata-Viehweg vanishing theorem, and the natural homomorphism $\mu_*\mathcal{O}_Y(\ulcorner -F_t \urcorner) \to \mu_*\mathcal{O}_E(\ulcorner -F_t|_E \urcorner)$ is surjective. Since the negative part of F_t is exceptional for μ, it follows that there is a natural isomorphism $\mathcal{O}_W \to \mu_*\mathcal{O}_E(\ulcorner -F_t|_E \urcorner)$ if $t \neq 0$.

We may assume that there is a resolution of singularities $\sigma : V \to W$ which factors $\mu : E \to W$. Let $f : E \to V$ be the induced morphism:

$$\begin{array}{ccc} E & \longrightarrow & Y \\ f\downarrow & & \\ V & & \\ \sigma\downarrow & & \downarrow\mu \\ W & \longrightarrow & X. \end{array}$$

We may also assume that there exist normal crossing divisors P and Q on E and V, respectively, such that the conditions of Corollary 1.6 are satisfied for $F_t|_E$ if $t \neq 0$, since we have

$$\mu^*(K_X + D_t)|_E \sim_{\mathbf{Q}} (K_Y + E + F_t)|_E = K_E + F_t|_E.$$

We define $\mathbf{Q}$-divisors M_t and Δ_t on V for $0 \le t \le 1$ such that $K_E + F_t|_E \sim_{\mathbf{Q}} f^*(K_V + M_t + \Delta_t)$ as in the semipositivitry theorem. By construction, the coefficients of Δ_t are less than 1 for any t. By the semipositivity theorem, M_t is nef for $t \neq 0$, hence for any t.

The surjectivity of the homomorphism $\mathcal{O}_W \to \mu_*\mathcal{O}_E(\ulcorner -F_t|_E \urcorner)$ implies that, if $\sigma_* Q_\ell \neq 0$, then there exists a j such that $f(P_j) = Q_\ell$ and $d_j \ge 1 - w_{\ell j}$. Thus $0 \le \delta_\ell$ and $\sigma_*\Delta_t$ is effective.

We let $t = 0$, and set $M = M_0$ and $\Delta = \Delta_0$. Since M is nef, we may assume that there exists rational numbers q_ℓ such that $q_\ell > 0$ (resp. $= 0$) if $\sigma_* Q_\ell = 0$ (resp. $\neq 0$) and that $M + \epsilon\sigma^*H - \epsilon' \sum_\ell q_\ell Q_\ell$ is ample for $0 < \epsilon' \ll \epsilon$. We take a general effective $\mathbf{Q}$-divisor $M' \sim_{\mathbf{Q}} M + \epsilon\sigma^*H - \epsilon' \sum_\ell q_\ell Q_\ell$ with very small coefficients and a very ample divisor as a support. Let $D_W = \sigma_*(M' + \Delta)$. Then we have $(K_X + D + \epsilon H)|_W \sim_{\mathbf{Q}} K_W + D_W$, and

$$\sigma^*(K_W + D_W) \sim_{\mathbf{Q}} K_V + M' + \Delta + \epsilon' \sum_\ell q_\ell Q_\ell.$$

If ϵ' is chosen small enough, then the coefficients on the right hand side are less than 1, and (W, D_W) is KLT. □

Remark 2.2. Since the choice of M' is generic in the proof of Theorem 2.1, we can take D_W such that the following holds. Let D' be an effective $\mathbf{Q}$-Cartier divisor on X whose support does not contain W. Assume that $(W, D_W + D'|_W)$ is not KLT. Then W is not a minimal center of log canonical singularities for $(X, D + D')$. Indeed, the variation of Hodge structures considered in the proof does not change if we replace D by $D + D'$. Then Δ is replaced by $\Delta + \sigma^*(D'|_W)$, and D_W by $D_W + D'|_W$.

3 Effective non-vanishing and base-point-freeness

The celebrated Kodaira vanishing theorem implies that the cohomology groups $H^p(X, K_X + H)$ vanish for $p > 0$ if X is a smooth projective variety and H is an ample divisor. It is natural to ask when $H^0(X, K_X + H)$ does not vanish.

More generally, we consider the following problem in this article. Let X be a complete normal variety, B an effective $\mathbf{R}$-divisor on X, and D a Cartier divisor on X. Assume that the pair (X, B) is KLT (log terminal), D is nef, and that $H = D - (K_X + B)$ is nef and big (cf. [18] for the terminology). By a generalization of the Kodaira vanishing theorem ([18] Theorem 1.2.5), we have $H^p(X, mD) = 0$ for any positive integer m. The problem is to find a condition on the integer m for which the non-vanishing $H^0(X, mD) \neq 0$ holds or moreover that the linear system $|mD|$ is free. By the base point free theorem ([18] Theorem 3.1.1), it is known that $|mD|$ is free for sufficiently large integer m. Fujita's freeness conjecture implies that it should be free if $m \ge \dim X + 1$. Our prediction is that $H^0(X, D) \neq 0$ always holds (Conjecture 3.1).

3.1 Reduction

We consider the following problem:

Conjecture 3.1. *Let X be a complete normal variety, B an effective $\mathbf{R}$-divisor on X such that the pair (X, B) is KLT, and D a Cartier divisor on X. Assume that D is nef, and that $H = D - (K_X + B)$ is nef and big. Then $H^0(X, D) \neq 0$.*

This problem was considered in [1] in order to construct ladders on log Fano varieties. By the generalization of the Kodaira Vanishing Theorem ([18] Theorem 1.2.5), we have $H^p(X, D) = 0$ for any positive integer p. Thus the condition $H^0(X, D) \neq 0$ is equivalent to saying that $\chi(X, D) \neq 0$. Our problem is a topological question, unlike the case of the Abundance Conjectures.

The base point free theorem says that there exists a positive integer m_1 such that the linear system $|mD|$ is free for $m \ge m_1$. The following reduction theorem is obtained as an application of the base point free theorem and the semipositivity theorem with the help of the perturbation technique.

Theorem 3.2. *In Conjecture 3.1, one may assume that B is a $\mathbf{Q}$-divisor and that H is ample. Moreover, one may assume that D is also ample if one replaces X suitably.*

Proof. By the Kodaira lemma, there exists an effecive $\mathbf{R}$-divisor E such that $B + E$ is a $\mathbf{Q}$-divisor, the pair $(X, B + E)$ is KLT, and that $H - E$ is ample. Therefore, we may assume that B is a $\mathbf{Q}$-divisor and that H is ample.

By the Base Point Free Theorem, there exists a proper surjective morphism $\phi : X \to X'$ with connected fibers to a normal projective variety such that $D \sim \phi^* D'$ for an ample Cartier divisor D' on X'. We have $H^0(X, D) \neq 0$ if and only if $H^0(X', D') \neq 0$. We shall show that there exists an effective $\mathbb{Q}$-divisor B' on X' such that (X', B') is KLT and $D' - (K_{X'} + B')$ is ample.

Since H is already assumed to be ample, we can write $H = H_0 + 2\phi^* H'$ with H_0 and H' being ample $\mathbb{Q}$-divisors. Since H_0 is ample, there exists an effective $\mathbb{Q}$-divisor B_0 such that $B + H_0 \sim_{\mathbb{Q}} B_0$ and that (X, B_0) is KLT. We set $D_0 = K_X + B_0$. Then $D_0 \sim_{\mathbb{Q}} \phi^* D'_0$ for $D'_0 = D' - 2H'$.

We construct birational morphisms $\mu : Y \to X$ and $\mu' : Y' \to X'$ from smooth projective varieties such that $\phi \circ \mu = \mu' \circ \psi$ for a morphism $\psi : Y \to Y'$. We write $\mu^*(K_X + B_0) \sim K_Y + E_0$. If μ and μ' are chosen suitably, then we may assume that the conditions of Corollary 1.6 are satisfied for ψ and E_0. Then there exist $\mathbb{Q}$-divisors E'_0 and M on Y' such that $K_Y + E_0 \sim_{\mathbb{Q}} K_{Y'} + E'_0 + M$, $\mu'_* E'_0$ is effective, $\llcorner E'_0 \lrcorner \leq 0$ and M is nef. Since H' is ample, there exists a $\mathbb{Q}$-divisor $E' \sim_{\mathbb{Q}} E'_0 + M + \mu'^* H'$ on Y' such that $B' = \mu'_* E'$ is effective and $\llcorner E' \lrcorner \leq 0$. Then we have $D'_0 + H' \sim_{\mathbb{Q}} K_{X'} + B'$ and (X', B') is KLT. Since $D' \sim_{\mathbb{Q}} K_{X'} + B' + H'$, we obtain our assertion. □

3.2 Surface case

We have a complete answer in dimension 2.

Theorem 3.3. *Let X, B and D be as in Conjecture 3.1. Assume that the numerical Kodaira dimension $\nu(X, D)$ is at most 2; namely, assume that $D^3 \equiv 0$. Then the following hold.*

(1) $H^0(X, D) \neq 0$.

(2) The linear system $|mD|$ is free for any integer m such that $m \geq 2$.

Proof. We may assume that $\dim X = \nu(X, D) \leq 2$ by Theorem 3.2. Let $\mu : X' \to X$ be the minimal resolution of singularities. Since $\mu^* K_X - K_{X'}$ is effective, we can write $\mu^*(K_X + B) = K_{X'} + B'$ with (X', B') being KLT. Therefore, we may assume that X is smooth. By Theorem 3.2 again, we may also assume that H is ample and that D is big.

Assume first that $\dim X = 1$. Then the assertions follow immediately from the Riemann-Roch theorem.

We assume that $\dim X = 2$ in the following. We prove (1). By the Riemann-Roch theorem, $\chi(X, D) = \frac{1}{2} D(B + H) + \chi(X, \mathcal{O}_X)$. Thus, if $\chi(X, \mathcal{O}_X) \geq 0$, then $\chi(X, D) > 0$. Let us assume that $\chi(X, \mathcal{O}_X) = 1 - g < 0$.

Then there exists a surjective morphism $f : X \to C$ to a curve of genus g whose generic fiber is isomorphism to $\mathbb{P}^1$. By Corollary 1.5, the vector bundle $f_* \mathcal{O}_X(D - f^* K_C)$ is numerically semipositive. Since $\mathcal{O}_X(D)$ is f-nef, it is f-generated, hence we have a surjective homomorphism $f^* f_* \mathcal{O}_X(D - f^* K_C) \to \mathcal{O}_X(D - f^* K_C)$, and the latter sheaf is nef. Thus $(D - f^* K_C)(B + H) \geq 0$. Since $f^* K_C(B + H) \geq -f^* K_C \cdot K_X = 4g - 4$, we have $\chi(X, D) \geq g - 1 > 0$.

In order to prove (2), we take a general member $Y \in |D|$ as a subscheme of X. We have an exact sequence $0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_Y(mD) \to 0$ and $H^1(X, (m-1)D) = 0$, hence it is sufficient to prove the freeness of $|\mathcal{O}_Y(mD)|$. Let $\mathfrak{m}$ be any ideal sheaf of $\mathcal{O}_Y$ of colength 1. We shall prove that $H^1(Y, \mathfrak{m}(mD)) = 0$. By duality, it is equivalent to $\mathrm{Hom}(\mathfrak{m}, \omega_Y(-mD)) = 0$, where $\omega_Y \cong \mathcal{O}_Y(K_X + Y)$. Since $\deg \omega_Y = \chi(Y, \omega_Y) - \chi(Y, \mathcal{O}_Y) = Y(K_X + Y)$ is even and $\deg \omega_Y(-2D) = Y(K_X + Y - 2D) = -Y(B + H) < 0$, we have $\deg \omega_Y(-mD) \leq -2$. Since $\chi(Y, \mathfrak{m}) = \chi(Y, \mathcal{O}_Y) - 1$, we have the desired vanishing. □

Our bound for the freeness in Theorem 3.3 is better than the one given by the Fujita conjecture. But we cannot expect similar thing in higher dimensions:

Example 3.4. (1) (Oguiso) Let X be a general weighted hypersurface of degree 10 in a weighted projective space $\mathbb{P}(1, 1, 1, 2, 5)$. Then X is smooth, $\dim X = 3$, and $K_X \sim 0$. Let $D = H = \mathcal{O}_X(1)$. We have $H^0(X, D) \neq 0$, and $|2D|$ is free. But $|3D|$ is not free, and $|4D|$ is not very ample.

(2) Let d be an odd integer such that $d \geq 3$, and let X be a general weighted hypersurface of degree $2d$ in $\mathbb{P}(1, \cdots, 1, 2, d)$, where the number of 1's is equal to $n = \dim X$. Then X is smooth. Let $D = \mathcal{O}_X(1)$. We have $K_X \sim (d - n - 2)D$, $D^n = 1$ and $|mD|$ is not free if m is odd and $m < d$. For example, if $n = d - 2$, then $K_X \sim 0$ and $|nD|$ is not free.

(3) Let d be an integer such that $d \not\equiv 0 \pmod 3$ and $d \geq 4$. Let X be a general weighted hypersurface of degree $3d$ in $\mathbb{P}(1, \cdots, 1, 3, d)$ as in (2). Then X is smooth. Let $D = \mathcal{O}_X(1)$. We have $K_X \sim (2d - n - 3)D$, $D^n = 1$, and $|mD|$ is not free if $m \not\equiv 0 \pmod 3$ and $m < d$. For example, $|2D|$ is not free, and $|(d-1)D|$ is not free if $d \equiv 2 \pmod 3$.

3.3 Minimal 3-fold

We have so far an affirmative answer only for minimal varieties in the case of dimension 3.

Proposition 3.5. *Let X be a 3-dimensional projective variety with at most canonical singularities, and D a Cartier divisor. Assume that K_X is nef, and $D - K_X$ is nef and big. Then $H^0(X, D) \neq 0$.*

Proof. By a crepant blowings-up, we may assume that X has only terminal singularities. Then we have $\chi(\mathcal{O}_X) \geq -\frac{1}{24}K_X c_2$ by [10], and $3c_2 - K_X^2$ is pseudo-effective by Miyaoka [23] (see also [33]). By the Riemann-Roch theorem, we calculate

$$\begin{aligned} h^0(X, D) &= \frac{1}{6}D^3 - \frac{1}{4}D^2K_X + \frac{1}{12}DK_X^2 + \frac{1}{12}Dc_2 + \chi(\mathcal{O}_X) \\ &= \frac{1}{12}(2D - K_X)\{\frac{1}{6}D^2 + \frac{2}{3}D(D - K_X) + \frac{1}{6}(D - K_X)^2\} \\ &+ \frac{1}{72}(2D - K_X)(3c_2 - K_X^2) + \frac{1}{24}K_X c_2 + \chi(\mathcal{O}_X) \\ &> 0. \end{aligned}$$

□

Proposition 3.6. *Let X be a complete variety of dimension 3 with at most Gorenstein canonical singularities, and D a Cartier divisor. Assume that $K_X \sim 0$ and D is ample. Let $Y \in |D|$ be a general member whose existence is guaranteed by Proposition 3.5. Then the pair (X, Y) is LC. In particular, Y is SLC.*

Proof. Assume that (X, Y) is not LC. Let c be the LC threshold for $(X, 0)$ so that $c < 1$ and (X, cY) is properly LC. Let W be a minimal center. By [15] Theorem 1, for any positive rational number ϵ, there exists an effective $\mathbb{Q}$-divisor B' on W such that $(K_X + cY + \epsilon D)|_W \sim_{\mathbb{Q}} K_W + B'$ and (W, B') is KLT. By the perturbation technique, we may assume that W is the only LC center for $(X, cY + \epsilon D)$ and there exists only one LC place E above W if we replace c and ϵ suitably (cf. [13] Proposition 2.3).

Therefore, there exists a birational morphism $\mu : Y \to X$ from a smooth projective variety such that we can write $\mu^*(K_X + cY + \epsilon D) = K_Y + E + F$, where the support of $E + F$ is a normal crossing divisor and the coefficients of F are strictly less than 1.

We consider an exact sequence

$$0 \to \mathcal{I}_W(D) \to \mathcal{O}_X(D) \to \mathcal{O}_W(D|_W) \to 0,$$

where $\mathcal{I}_W = \mu_*\mathcal{O}_Y(-E)$ is the ideal sheaf for W. Since $D - (K_X + cY + \epsilon D)$ is ample, we have $H^p(Y, \mu^*D - E + \llcorner -F \lrcorner) = 0$ and $R^p\mu_*\mathcal{O}_Y(\mu^*D - E + \llcorner -F \lrcorner) = 0$ for $p > 0$ by the generalization of the Kodaira vanishing theorem. Since $\mu_*\mathcal{O}_Y(\llcorner -F \lrcorner) = \mathcal{O}_X$, we obtain $H^1(X, \mathcal{I}_W(D)) = 0$. Hence the homomorphism $H^0(X, D) \to H^0(W, D|_W)$ is surjective. We have $H^0(W, D|_W) \neq 0$ by Theorem 3.3. It follows that W is not contained in the base locus of $|D|$, a contradiction. □

3.4 Weak log Fano varieties

The following is proved by Ambro [1]. We shall give a shorter proof of the second part as an application of Theorem 3.3.

Theorem 3.7. *Let X, B and D be as in Conjecture 3.1. Assume that there exists a positive rational number r such that $r > \dim X - 3 \geq 0$ and $-(K_X + B) \sim_{\mathbb{Q}} rD$. Then the following hold.*

(1) $H^0(X, D) \neq 0$.

(2) Let $Y \in |D|$ be a general member. Then the pair $(X, B + Y)$ is PLT.

Proof. (1) is proved in [1] Lemma 2. We recall the proof for the convenience of the reader. We set $n = \dim X$, $d = D^n \geq 1$, $\beta = BD^{n-1} \geq 0$, and $p(t) = \chi(X, tD)$ for $t \in \mathbb{Z}$. Since $p(0) = 1$ and $p(-1) = p(-2) = \cdots = p(-n+3) = 0$, we can write

$$\begin{aligned} p(t) &= \frac{d}{n!}t^n + \frac{\beta + dr}{2(n-1)!}t^{n-1} + \dots \\ &= \frac{d}{n!}(t+1)(t+2)\cdots(t+n-3)(t^3 + at^2 + bt + \frac{n(n-1)(n-2)}{d}) \end{aligned}$$

for some numbers a, b. Hence

$$a = \frac{n(r-n+5)}{2} - 3 + \frac{\beta n}{2d}.$$

On the other hand, we have

$$0 \leq (-1)^n p(-n+2) = \frac{d}{n(n-1)}((n-2)^2 - a(n-2) + b) - 1.$$

Therefore

$$h^0(X,D) = p(1) = n - 1 + (-1)^n p(-n+2) + \frac{\beta + d(r-n+3)}{2} > 0.$$

(2) We may assume that D is ample by Theorem 3.2. Note that the morphism ϕ in this case is birational and the condition $-(K_X + B) \sim_{\mathbb{Q}} rD$ is preserved. Assume that $(X, B+Y)$ is not PLT, and let $c \le 1$ be the LC threshold so that $(X, B+cY)$ is properly LC. Let W be a minimal center. By [15], for any positive rational number ϵ, there exists an effective $\mathbb{Q}$-divisor B' on W such that $(K_X + B + cY + \epsilon D)|_W \sim_{\mathbb{Q}} K_W + B'$ and (W, B') is KLT. Then $-(K_W + B') \sim_{\mathbb{Q}} r'D|_W$ for $r' = r - c - \epsilon$. Since ϵ can be arbitrarily small, we have $r' > \dim W - 3$ and $r' > -1$.

If $\dim W \ge 3$, then $H^0(W, D|_W) \ne 0$ by (1). Otherwise, we have also $H^0(W, D|_W) \ne 0$ by Theorem 3.3. By the vanishing theorem, we have $H^1(X, \mathcal{I}_W(D)) = 0$ as in the proof of Proposition 3.6. From an exact sequence

$$0 \to \mathcal{I}_W(D) \to \mathcal{O}_X(D) \to \mathcal{O}_W(D|_W) \to 0,$$

it follows that W is not contained in the base locus of $|D|$, a contradiction. □

The following result deals with the case which is just beyond the scope of Theorem 3.7.

Theorem 3.8. *Let X be a complete variety of dimension 4 with at most Gorenstein canonical singularities. Assume that $D \sim -K_X$ is ample. Then the following hold.*

(1) $H^0(X, D) \ne 0$.

(2) Let $Y \in |D|$ be a general member. Then (X, Y) is PLT, hence $K_Y \sim 0$ and Y has only Gorenstein canonical singularities.

Proof. We shall prove (1) and (2) simultaneously. Let m be the smallest positive integer such that $H^0(X, mD) \ne 0$. We shall derive a contradiction from $m > 1$. We take a general member $Y \in |mD|$.

Assume first that (X, Y) is PLT and $m > 1$. Then Y is Gorenstein canonical. We have an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_Y(K_Y) \to 0.$$

We have $\chi(X, \mathcal{O}_X(-D)) = 1$ and $\chi(X, \mathcal{O}_X((m-1)D)) = 0$. On the other hand, we have $\chi(Y, K_Y) = \frac{1}{24} K_Y c_2 \ge 0$ by [23], a contradiction.

Next assume that (X, Y) is not PLT and $m \ge 1$. Let c be the LC threshold so that $c \le 1$ and (X, cY) is properly LC. Let W be a minimal center. If $\dim W = 3$, then we have $c \le \frac{1}{2}$. By [15], for any positive rational number ϵ, there exists an effective $\mathbb{Q}$-divisor B' on W such that $(K_X + cY + \epsilon D)|_W \sim_{\mathbb{Q}} K_W + B'$ and (W, B') is KLT. By the perturbation technique, we may assume that W is the only center if we replace c and ϵ suitably.

We consider an exact sequence

$$0 \to \mathcal{I}_W(mD) \to \mathcal{O}_X(mD) \to \mathcal{O}_W(mD|_W) \to 0.$$

Since $mD - (K_X + cY + \epsilon D)$ is ample, we have $H^1(X, \mathcal{I}_W(mD)) = 0$ by the vanishing theorem. Hence the homomorphism $H^0(X, mD) \to H^0(W, mD|_W)$ is surjective. If $\dim W \le 2$, then we have $H^0(W, mD|_W) \ne 0$ by Theorem 3.3. We shall also prove that $H^0(W, mD|_W) \ne 0$ in the case $\dim W = 3$. Then it follows that W is not contained in the base locus of $|mD|$, a contradiction, and (1) and (2) are proved.

Assume that $\dim W = 3$. We set $r' = cm - 1 + \epsilon$ so that $K_W + B' \sim_{\mathbb{Q}} r'D|_W$. Let $p(t) = \chi(W, tD|_W)$. If we set $d = (D|_W)^3 > 0$ and $\delta = B'(D|_W)^2 \ge 0$, then

$$p(t) = \frac{d}{6}t^3 + \frac{-r'd + \delta}{4}t^2 + bt + c$$

for some numbers b and c by the Riemann-Roch theorem. By the vanishing theorem, we have $-p(-1) \ge 0$ and $p(m-1) \ge 0$, because $r' < m - 1$. Then

$$p(m) = \frac{(m-1)(m+1)d}{3} + \frac{(m+1)(-r'd + \delta)}{4} + \frac{-p(-1) + (m+1)p(m-1)}{m} > 0.$$

Thus we have $H^0(W, mD|_W) \ne 0$. □

4 Length of extremal ray

Theorem 4.1. *Let (X, B) be a log terminal pair of dimension n and let $\phi : X \to Y$ be a contraction morphism associated to an extremal ray. Then the exceptional locus E is covered by a family of rational curves C such that $(-(K_X + B) \cdot C) \le 2n$. Moreover, if ϕ is birational, then the strict inequality holds.*

Theorem 4.2. *Let X be a normal projective variety of dimension n with an ample Cartier divisor H, and let C be a curve contained in the smooth locus of X such that $(K_X \cdot C) < 0$. Then for any point $x \in C$ there exists a rational curve L containing x such that*

$$(H \cdot L) \leq \max\{\frac{2n(H \cdot C)}{-(K_X \cdot C)}, (H \cdot C)\}$$

Theorem 4.3. *Let (X, B) be a log terminal pair and $f : X \to S$ a projective morphism with an f-ample Cartier divisor H. Then*

$$\lambda = \sup\{t \in \mathbb{Q}; H + t(K_X + B) \text{ is } f\text{-ample}\}$$

is eith $+\infty$ or a rational number. In the latter case, let r be the smallest positive integer such that $r(K_X + B)$ becomes a Cartier divisor and let d be the maximum of the dimension of fibers of f. Express $\lambda/r = p/q$ for coprime positive integers p, q. Then $q \leq r(d+1)$.

5 Relative version of Fujita's freeness conjecture

The following is Fujita's freeness conjecture:

Conjecture 5.1. *Let X be a smooth projective variety of dimension n and L an ample divisor. Then the invertible sheaf $\mathcal{O}_X(K_X + mL)$ is generated by global sections if $m \geq n+1$, or $m = n$ and $L^n \geq 2$.*

We have a stronger local version of Conjecture 5.1 (cf. [13]):

Conjecture 5.2. *Let X be a smooth projective variety of dimension n, L a nef and big invertible sheaf on X, and $x \in X$ a point. Assume that $L^n > n^n$ and $L^d Z \geq n^d$ for any ireducible subvariety Z of X of dimension d which contains x. Then the natural homomorphism*

$$H^0(X, \omega_X \otimes L) \to \omega_X \otimes L \otimes \kappa(x)$$

is surjective.

We shall extend the above conjecture to a relative setting.

Let $f : Y \to X$ be a surjective morphism of smooth projective varieties. We note that the geometric fibers of f are not necessarily connected. Assume that there exists a normal crossing divisor $B = \sum_{i=1}^{h} B_i$ on X such that f is smooth over $X_0 = X \setminus B$. Then the sheaves $R^q f_* \omega_{Y/X}$ are locally free for $q \geq 0$ ([6] for $q = 0$ and [19] in general). We note that even if we change the birational model of Y, the sheaf $R^q f_* \omega_{Y/X}$ does not change.

The relative version is the following:

Conjecture 5.3. *Let $f : Y \to X$ be a surjective morphism from a smooth projective variety to a smooth proective variety of dimension n such that f is smooth over $X_0 = X \setminus B$ for a normal crossing divisor B on X. Let L be an ample divisor on X. Then the locally free sheaf $R^q f_* \omega_Y \otimes \mathcal{O}_X(mL)$ is generated by global sections if $m \geq n+1$, or $m = n$ and $L^n \geq 2$.*

We have again a stronger local version:

Conjecture 5.4. *Let $f : Y \to X$ be a surjective morphism from a smooth projective variety to a smooth proective variety of dimension n, L a nef and big invertible sheaf on X, and $x \in X$ a point. Assume the following conditions:*

(1) There is a normal crossing divisor B on X such that f is smooth over $X_0 = X \setminus B$.

(2) $L^n > n^n$ and $L^d Z \geq n^d$ for any ireducible subvariety Z of X of dimension d which contains x.

Then the natural homomorphism

$$H^0(X, R^q f_* \omega_Y \otimes L) \to R^q f_* \omega_Y \otimes L \otimes \kappa(x)$$

is surjective for any $q \geq 0$.

In [13], the following strategy toward Conjectures 5.1 and 5.2 was developed:

Theorem 5.5. *Assume the followig condition: for any effective $\mathbb{Q}$-divisor D_0 on X such that (X, D_0) is KLT, there exists an effective $\mathbb{Q}$-divisor D on X such that*

(1) $D \equiv \lambda L$ for some $0 < \lambda < 1$,

(2) The pair $(X, D_0 + D)$ is properly log canonical at x, and

(3) $\{x\}$ is a log canonical center for $(X, D_0 + D)$.

Then the conclusion of Conjecture 5.2 holds. In this way, Conjecture 5.2 for $\dim X \leq 3$ *and Conjecture 5.1 for* $\dim X = 4$ *hold.*

Our main result is the relative version of the above theorem:

Theorem 5.6. *Assume the followig condition: for any effective $\mathbb{Q}$-divisor D_0 on X such that (X, D_0) is KLT, there exists an effective $\mathbb{Q}$-divisor D on X such that*

(1) $D \equiv \lambda L$ for some $0 < \lambda < 1$,

(2) The pair $(X, D_0 + D)$ is properly log canonical at x, and

(3) $\{x\}$ is a log canonical center for $(X, D_0 + D)$.

Then the conclusion of Conjecture 5.4 holds. In particular, Conjecture 5.4 for $\dim X \leq 3$ and Conjecture 5.3 for $\dim X = 4$ hold.

Remark 5.7. It is easy to prove Conjecture 5.1 in the case L is very ample. [19] proved Conjecture 5.3 in the case L is very ample.

5.1 Parabolic structure and vanishing theorem

We generalize the notion of parabolic structures of vector bundles by [21] over higher dimensional base space:

Definition 5.8. Let $f : Y \to X$ be a surjective morphism of smooth projective varieties. Assume that there exists a normal crossing divisor $B = \sum_{i=1}^{h} B_i$ on X such that f is smooth over $X_0 = X \setminus B$. We fix a nonnegative integer q. We define a *parabolic structure* on the sheaf $R^q f_* \omega_{Y/X}$. It is a decreasing filtration of subsheaves $F^{t_1,\dots,t_h} = F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X})$ of $R^q f_* \omega_{Y/X}$ with multi-indices $t = (t_1, \dots, t_h)$ $(0 \leq t_i)$ defined by

$$\Gamma(U, F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X})) = \{s \in \Gamma(U, R^q f_* \omega_{Y/X}) \mid (\prod_i z_i^{-t_i}) s \text{ is } L^2 \text{ with respect to the Hodge metric}\},$$

where z_i is a local equation of the branch B_i on an open subset $U \subset X$.

Lemma 5.9. *(1) $F^t \supset F^{t'}$ for $t < t'$.*

(2) $F^{t_1,\dots,t_i+\epsilon,\dots,t_h} = F^{t_1,\dots,t_i,\dots,t_h}$ for $0 < \epsilon \ll 1$.

(3) $F^{t_1,\dots,t_i+1,\dots,t_h} = F^{t_1,\dots,t_i,\dots,t_h} \otimes \mathcal{O}_X(-B_i)$.

(4) Let $Y_0 = f^{-1}(X_0)$, $f_0 = f|_{Y_0}$ and $d = \dim Y - \dim X$. If all the local monodromies of $R^{d+q} f_{0*} \mathbb{Q}_{Y_0}$ around the branches of B are unipotent, then $F^t = F^0$ for any $t = (t_1, \dots, t_h)$ with $0 \leq t_i < 1$.

Proof. (1) through (3) are obvious. (4) It folows from the fact that the growth of the Hodge metric is logarithmic in this case ([6]). □

Remark 5.10. For negative values of the t_i, we can also define F^t as subsheaves of $R^q f_* \omega_{Y/X} \otimes \mathcal{O}_X(mB)$ for sufficiently large m by using the formula Lemma 5.9 (3). We also write $F^{t_1,\dots,t_h} = F^{\sum_i t_i B_i}$.

Definition 5.11. For a local section $s \in \Gamma(U, R^q f_* \omega_{Y/X})$, we define its *order of growth* along B by

$$\operatorname{ord}(s) = \sum_i \operatorname{ord}_i(s) B_i = \inf\{\sum_i (1 - t_i) B_i \mid s \in \Gamma(U, F^{t_1,\dots,t_h})\}.$$

We note that $s \notin \Gamma(U, F^{B-\operatorname{ord}(s)})$, and

$$\Gamma(U, F^{t_1,\dots,t_h}) = \{s \in \Gamma(U, R^q f_* \omega_{Y/X}) \mid \operatorname{ord}(s) + \sum_i t_i B_i < B\}.$$

There is a nice local basis of the sheaf $R^q f_* \omega_{Y/X}$:

Lemma 5.12. *At any point $x \in X$, there exists an open neighborhood U and a basis $\{s_1, \dots, s_k\}$ of $\Gamma(U, R^q f_* \omega_{Y/X})$ such that*

$$(\prod_i z_i^{\llcorner t_i + \operatorname{ord}_i(s_1) \lrcorner}) s_1, \dots, (\prod_i z_i^{\llcorner t_i + \operatorname{ord}_i(s_k) \lrcorner}) s_k$$

generates $\Gamma(U, F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X}))$ for any t, where the z_i are local equations of the B_i on U. In particular, the sheaf $F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X})$ is locally free for any t.

Proof. It is known that the local monodromies around the branches of B acting on the cohomology sheaf $R^{d+q} f_{0*} \mathbb{Z}_{Y_0}$ are quasi-unipotent, where $d = \dim Y - \dim X$. We shall prove that the filtration F^t is determined by the eigenvalues of this action.

Since our assertion is local, we replace X by an open neighborhood U of $x \in X$ in the classical topology which is isomorphic to a polydisk with coordinates $\{z_1, \dots, z_n\}$ such that $B \cap U$ is defined by $z_1 \cdots z_r = 0$. To simplify the notation, we write X instead of U. There exists a finite Galois and surjective morphism $\pi : X' \to X$ which is etale over X_0 such that, for the induced morphism $f' : Y' \to X'$ from a desingularization Y' of the fiber product $Y \times_X X'$, the local system $R^{d+q} f'_{0*} \mathbb{Z}_{Y'_0}$ has unipotent local monodromies around the branches of $B' = \pi^{-1}(B)$, where we set $X'_0 = \pi^{-1}(X_0)$, $Y'_0 = f'^{-1}(X'_0)$ and $f'_0 = f'|_{Y'_0}$. Let $\sigma : Y' \to Y$ be the induced morphism.

We may assume that X' is isomorphic to a polydisk centered at a point $x' = \pi^{-1}(x)$ with coordinates $\{z'_1, \dots, z'_n\}$, and the morphism $\pi : X' \to X$ is given by $\pi^* z_i = z_i'^{m_i}$ for some positive integers m_i, where $m_i = 1$ for $i > r$. The Galois group $G = \mathrm{Gal}(X'/X)$ is isomorphic to $\prod_i \mathbb{Z}/(m_i)$. Let $g_1, \dots, g_r$ be generators of G such that $g_i^* z'_j = \zeta_{m_i}^{\delta_{ij}} z'_j$ for some roots of unity ζ_{m_i} of order m_i.

The group G acts on the sheaves $R^q f'_* \omega_{Y'/X'}$ and $\omega_{X'}$ equivariantly such that the invariant part $(\pi_*(R^q f'_* \omega_{Y'/X'} \otimes \omega_{X'}))^G$ is isomorphic to $R^q f_* \omega_{Y/X} \otimes \omega_X$, because $(\sigma_* \omega_{Y'})^G = \omega_Y$ and $R^p \sigma_* \omega_{Y'} = 0$ for $p > 0$.

The vector space $R^q f'_* \omega_{Y'/X'} \otimes \kappa(x')$ is decomposed into simultaneous eigenspaces with respect to the action of G. Let $s_{x'}$ be a simultaneous eigenvector such that $g_i^* s_{x'} = \zeta_{m_i}^{a_i} s_{x'}$ for some a_i with $0 \le a_i < m_i$. Let $\bar{s}'$ be a section of $R^q f'_* \omega_{Y'/X'}$ which extends $s_{x'}$. Then the section

$$s' = \frac{1}{\prod_i m_i} \sum_{i=1}^{r} \sum_{k_i=0}^{m_i-1} \frac{(\prod_i g_i^{k_i})^* \bar{s}'}{\prod_i \zeta_{m_i}^{a_i k_i}}$$

satisfies that $g_i^* s' = \zeta_{m_i}^{a_i} s'$.

On the other hand, $dz'_1 \wedge \dots \wedge dz'_n = (\prod_i m_i^{-1} z_i'^{1-m_i}) dz_1 \wedge \dots \wedge dz_n$ is a generating section of $\omega_{X'}$. Therefore, $(\prod_i z_i'^{-a_i}) s'$ descends to a section s of $R^q f_* \omega_{Y/X}$. If the $s_{x'}$ varies among a basis of $R^q f'_* \omega_{Y'/X'} \otimes \kappa(x')$, then the corresponding sections s make a basis of the locally free sheaf $R^q f_* \omega_{Y/X}$.

We have $\mathrm{ord}_i(s) = a_i/m_i$, since the Hodge metric on the sheaf $R^q f'_* \omega_{Y'/X'}$ has logarithmic growth along B'. Therefore, the sections $(\prod_i z_i^{\llcorner t_i + a_i/m_i \lrcorner}) s$ form a basis of a locally free sheaf $F^{t_1, \dots, t_h}(R^q f_* \omega_{Y/X})$. □

Remark 5.13. The Hodge metric and the flat metric on the canonical extension of the variation of Hodge structures $R^{d+q} f_{0*} \mathbb{Z}_{Y_0} \otimes \mathcal{O}_{X_0}$ coincide when restricted to the subsheaf $R^q f_* \omega_{Y/X}$. Therefore, the statement that the Hodge metric on the canonical extension has logarithmic growth is easily proved for $R^q f_* \omega_{Y/X}$.

By using the basis obtained in Lemma 5.12, we can study the base change of the sheaf $R^q f_* \omega_{Y/X}$:

Lemma 5.14. *Let $\pi : X' \to X$ be a generically finite and surjective morphism from a smooth projective variety such that $B' = (\pi^* B)_{\mathrm{red}} = \sum_{i'=1}^{h'} B'_{i'}$ is a normal crossing divisor. Let $\mu : Y' \to Y \times_X X'$ be a birational morphism from a smooth projective variety such that the induced morphism $f' : Y' \to X'$*

is smooth over $X'_0 = X' \setminus B'$. Let $\sigma : Y' \to Y$ be the induced morphism. Then the following hold.

(1) Let $\{s_1, \dots, s_k\}$ be the basis of $\Gamma(U, R^q f_* \omega_{Y/X})$ in Lemma 5.12, and let U' be an open subset of X' in the classical topology such that $\pi(U') \subset U$. Then the equality $\mathrm{ord}(\pi^* s_j) = \pi^* \mathrm{ord}(s_j)$ holds, and the basis $\{\pi^* s_1, \dots, \pi^* s_k\}$ of $\Gamma(U', \pi^* R^q f_* \omega_{Y/X})$ satisfies the conclusion of Lemma 5.12 in the sense that sections

$$(\prod_{i'} z_{i'}^{\llcorner t_{i'} + \mathrm{ord}_{i'}(\pi^* s_1) \lrcorner}) \pi^* s_1, \dots, (\prod_{i'} z_{i'}^{\llcorner t_{i'} + \mathrm{ord}_{i'}(\pi^* s_k) \lrcorner}) \pi^* s_k$$

form a basis of $\Gamma(U', F^{t'_1, \dots, t'_{h'}}(R^q f'_* \omega_{Y'/X'}))$ for any $t' = (t'_1, \dots, t'_{h'})$, where the $z'_{i'}$ are local equations of the $B'_{i'}$ on U'. In particular, the sections

$$(\prod_{i'} z_{i'}^{\llcorner \mathrm{ord}_{i'}(\pi^* s_1) \lrcorner}) \pi^* s_1, \dots, (\prod_{i'} z_{i'}^{\llcorner \mathrm{ord}_{i'}(\pi^* s_k) \lrcorner}) \pi^* s_k.$$

form a basis of $\Gamma(U', R^q f'_* \omega_{Y'/X'})$.

(2)

$$\begin{aligned} &F^{t'_1, \dots, t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \\ &= \sum_t \pi^* F^{t_1, \dots, t_h}(R^q f_* \omega_{Y/X}) \otimes \mathcal{O}_{X'}(-\llcorner \sum_{i'} t'_{i'} B'_{i'} + \sum_i (1 - t_i) \pi^* B_i \lrcorner), \end{aligned}$$

where the sum is taken inside the sheaf $\pi^* R^q f_* \omega_{Y/X}$. In particular,

$$R^q f'_* \omega_{Y'/X'} = \sum_t \pi^* F^{t_1, \dots, t_h}(R^q f_* \omega_{Y/X}) \otimes \mathcal{O}_{X'}(-\llcorner \sum_i (1 - t_i) \pi^* B_i \lrcorner).$$

Proof. (1) Since the s_j are derived from the basis in the case of unipotent monodromies, we obtain our assertion by Lemma 1.2.

(2) We can check the assertion locally. We write $\pi^* B_i = \sum_{i'} m_{ii'} B_{i'}$ for some nonnegative integers $m_{ii'}$. Then the left hand side is generated by the sections

$$(\prod_{i'} z_{i'}^{\llcorner t'_{i'} + \mathrm{ord}_{i'}(\pi^* s_j) \lrcorner}) \pi^* s_j$$

for $1 \le j \le k$, while the right hand side is by

$$(\prod_{i'} z_{i'}^{\prime \sum_i (\llcorner t_i + \mathrm{ord}_i(s_j) \lrcorner) m_{ii'} + \llcorner t'_{i'} + \sum_i (1 - t_i) m_{ii'} \lrcorner}) \pi^* s_j$$

for $1 \le j \le k$.

Since $\mathrm{ord}_{i'}(\pi^* s_j) = \sum_i \mathrm{ord}_i(s_j) m_{ii'}$, we should compare

$$\llcorner t'_{i'} + \sum_i \mathrm{ord}_i(s_j) m_{ii'} \lrcorner$$

and

$$\min_t \{ \sum_i (\llcorner t_i + \mathrm{ord}_i(s_j) \lrcorner) m_{ii'} + \llcorner t'_{i'} + \sum_i (1 - t_i) m_{ii'} \lrcorner \}.$$

Since

$$\begin{aligned} &\sum_i (\llcorner t_i + \mathrm{ord}_i(s_j) \lrcorner) m_{ii'} + \llcorner t'_{i'} + \sum_i (1 - t_i) m_{ii'} \lrcorner - \llcorner t'_{i'} + \sum_i \mathrm{ord}_i(s_j) m_{ii'} \lrcorner \\ > &\sum_i (\llcorner t_i + \mathrm{ord}_i(s_j) \lrcorner) m_{ii'} + \sum_i (1 - t_i - \mathrm{ord}_i(s_j)) m_{ii'} - 1 > -1, \end{aligned}$$

we observe that they are equal, where the minimum is attained when $t_i = 1 - \mathrm{ord}_i(s_j) - \epsilon_i$ for $0 < \epsilon_i \ll 1$. Therefore, we obtain the equality. □

Theorem 5.15. *Let L be a nef and big $\mathbb{Q}$-divisor whose fractional part is supported on B. Then*

$$H^p(X, \sum_t F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X}) \otimes \omega_X(\ulcorner L - \sum_i (1 - t_i) B_i \urcorner)) = 0$$

for $p > 0$, where the sum is taken inside the sheaf $R^q f_* \omega_Y \otimes \mathcal{O}_X(\ulcorner L \urcorner)$.

Proof. By [6], if we replace B by another normal crossing divisor $\bar{B}$ such that $B \le \bar{B}$, then there exists a finite Galois and surjective morphism $\pi : X' \to X$ from a smooth projective variety which is etale over X_0 and such that $\pi^* L$ has integral coefficients and all the local monodromies for $R^{d+q} f'_{0*} \mathbb{Z}_{X'_0}$ are unipotent. Let G be the Galois group for π. By [19] Theorem 2.1 and [8] Theorem 3.3, we have

$$H^p(X', R^q f'_* \omega_{Y'} \otimes \mathcal{O}_{X'}(\pi^* L)) = 0$$

for $p > 0$. Let $B' = \pi^{-1}(B)$. Then $\pi^*(K_X + B) = K_{X'} + B'$. By Lemma 5.14, we have

$$\begin{aligned} H^p(X', &\sum_t \pi^* F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X}) \otimes \mathcal{O}_{X'}(-\llcorner \sum_{i'} (1 + t'_{i'}) B'_{i'} \\ &+ \sum_i (1 - t_i) \pi^* B_i \lrcorner + \pi^*(K_X + B + L))) = 0 \end{aligned}$$

for any $0 \le t'_{i'} < 1$.

We want to calculate the G-invariant part of the locally free sheaf

$$\begin{aligned} F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X}) \otimes \pi_*(\mathcal{O}_{X'}(-\llcorner &\sum_{i'} (1 + t'_{i'}) B'_{i'} \\ &+ \sum_i (1 - t_i) \pi^* B_i \lrcorner + \pi^*(K_X + B + L))). \end{aligned}$$

For this purpose, let A be the largest divisor on X such that

$$\pi^* A \le \ulcorner - \sum_{i'} (1 + t'_{i'}) B'_{i'} - \sum_i (1 - t_i) \pi^* B_i \urcorner + \pi^*(B + L).$$

If we set $t'_{i'} = 0$, then this is equivalent to the condition

$$\pi^* A < \pi^*(L - \sum_i (1 - t_i) B_i) + \pi^* B.$$

Hence we obtain

$$A = \ulcorner L - \sum_i (1 - t_i) B_i \urcorner$$

and our assertion is proved. □

Remark 5.16. We note that the sum

$$\sum_t F^{t_1,\dots,t_h}(R^q f_* \omega_{Y/X}) \otimes \omega_X(\ulcorner L - \sum_i (1 - t_i) B_i \urcorner)$$

is a locally free sheaf because it is the G-invariant part of a locally free sheaf as shown in the above proof, though it looks complicated. It is the subsheaf of L^2 section. We can extend the non-vanishing problem to this sheaf.

Proof. Let $\{s_1, \dots, s_k\}$ be the basis of $R^q f_* \omega_{Y/X}$ in a neighborhood of x which is obtained in Lemma 5.12. We shall prove that the image of the homomorphism

$$H^0(X, R^q f_* \omega_Y \otimes L) \to R^q f_* \omega_Y \otimes L \otimes \kappa(x)$$

contains $s_j \otimes \omega_X \otimes L \otimes \kappa(x)$. Let us consider $\mathrm{ord}(s_j)$ as an effective divisor on X. By the assumption of the theorem, there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} \lambda L$ with $0 < \lambda < 1$, $(X, \mathrm{ord}(s_j) + D)$ is properly log canonical at x, and that $\{x\}$ is a minimal log canonical center. By the perturbation of D, we may assume that $\{x\}$ is the only log canonical center which contains x, and there exists only one log canonical place E above the center $\{x\}$. Let $\mu : X' \to X$ be a birational morphism from a smooth projective variety such that E appears as a smooth divisor on X'. We write

$$\mu^*(K_X + \mathrm{ord}(s_j) + D) = K_{X'} + E + F,$$

where the coefficients of F are less than 1. We may assume that the union of the exceptional locus of μ and the support of $\mu^{-1}(B+D)$ is a normal crossing divisor. Let $B' = \mu^* B_{\mathrm{red}} = \sum_{i'} B'_{i'}$. For any $t' = (t_1, \cdots, t'_{h'})$, we have

$$\begin{aligned} &K_{X'} + (1-\lambda)\mu^* L - \sum_{i'}(1 - t'_{i'})B'_{i'} \\ &= \mu^*(K_X + L) - E - F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'}. \end{aligned}$$

By Theorem 5.15, we obtain

$$\begin{aligned} H^1(X', \sum_{t'} &F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \otimes \mathcal{O}_{X'}(\mu^*(K_X + L) \\ &- E + \ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner)) = 0. \end{aligned}$$

Since

$$\begin{aligned} \sum_{t'} &F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \otimes \mathcal{O}_{X'}(\mu^*(K_X + L) \\ &+ \ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner)) \end{aligned}$$

is a locally free sheaf on X', we have a surjective homomorphism

$$\begin{aligned} H^0(X', \sum_{t'} &F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \otimes \mathcal{O}_{X'}(\mu^*(K_X + L) \\ &+ \ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner)) \\ \to H^0(E, \sum_{t'} &F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \otimes \mathcal{O}_{E}(\mu^*(K_X + L) \\ &+ \ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner)). \end{aligned}$$

We have

$$\mu_*(\ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner) \le 0$$

if $0 \le t'_{i'} < 1$. Hence

$$\begin{aligned} H^0(X', \sum_{t'} &F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \otimes \mathcal{O}_{X'}(\mu^*(K_X + L) \\ &+ \ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner)) \\ &\subset H^0(X, R^q f_* \omega_{Y/X} \otimes \mathcal{O}_X(K_X + L)). \end{aligned}$$

We note that the $t'_{i'}$ need not be contained in the interval $[0, 1)$ in the above sum. On the other hand, if we define the $t'_{i'}$ by

$$\sum_{i'} t'_{i'} B_{i'} = \sum_{i'}(1 - \epsilon'_{i'})B_{i'} - \mu^*\mathrm{ord}(s_j)$$

for sufficiently small and positive numbers $\epsilon'_{i'}$, then the divisor

$$\ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner$$

is effective and its support does not contain E, even if E is contained in B'. Since $\mathrm{ord}(\mu^* s_j) = \mu^*\mathrm{ord}(s_j)$, we have $\mu^* s_j \in F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'})$ for such $t'_{i'}$. Hence

$$\begin{aligned} \mu^* s_j &\otimes \mathcal{O}_E(\mu^*(K_X + L)) \\ &\in H^0(E, \sum_{t'} F^{t'_1,\dots,t'_{h'}}(R^q f'_* \omega_{Y'/X'}) \otimes \mathcal{O}_E(\mu^*(K_X + L) \\ &\quad + \ulcorner -F + \mu^*\mathrm{ord}(s_j) - \sum_{i'}(1 - t'_{i'})B'_{i'} \urcorner)). \end{aligned}$$

Therefore, $s_j \otimes \omega_X \otimes L \otimes \kappa(x)$ is contained in the image of the set of global sections $H^0(X, R^q f_* \omega_Y \otimes L)$. □

6 The extension problem of pluricanonical forms

The purpose of this paper is to review some recent development on the extension problem of pluricanonical forms from a divisor to the ambient space.

The main tools of the proofs are the multiplier ideal sheaves and the vanishing theorems for them.

Let X be a compact complex manifold. The *m-genus* $P_m(X)$ of X for a positive integer m is defined by $P_m(X) = \dim H^0(X, mK_X)$. The growth order of the plurigenera for large m is called the *Kodaira dimension* $\kappa(X)$: we have $P_m(X) \sim m^{\kappa(X)}$ for any sufficiently large and divisible m. We have the following possibilities: $\kappa(X) = -\infty, 0, 1, \cdots$, or $\dim X$. In particular, if $\kappa(X) = \dim X$, then X is said to be of *general type*. It is important to note that these invariants are independent of the birational models of X.

The plurigenera are fundamental discrete invariants for the classification of algebraic varieties. But they are by definition not topological invariants. However, in order that such classification is reasonable, the following conjecture due to Iitaka should be true:

Conjecture 6.1. *Let S be an algebraic variety, and let $f : \mathcal{O}_X \to S$ be a smooth algebraic morphism. Then the plurigenera $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

A morphism between complex varieties which is birationally equivalent to a projective morphism will be called an *algebraic morphism* in this paper. The algebraicity assumption in the conjecture is slightly weaker than the projectivity.

This conjecture is confirmed by Iitaka [4] in the case in which $\dim X_t = 2$ by using the classification theory of surfaces. Nakayama [26] proved that the conjecture follows if the minimal model exists for the family and the abundance conjecture holds for the generic fiber. Thus the conjecture is true if $\dim X_t = 3$ by [12] and [20].

On the other hand, Nakamura [25] provided a counterexample for the generalization of the conjecture in the case where the morphism f is not algebraic. In his example, the central fiber X_0 is a quotient of a 3-dimensional simply connected solvable Lie group by a discrete subgroup. We note that X_0 is a non-Kähler manifold which has non-closed holomorphic 1-forms. So we only consider algebraic morphisms in this paper. It is interesting to extend our results to the case in which the fibers are in Fujiki's class $\mathcal{C}$.

The following theorem of Siu was the starting point of the recent progress on this conjecture which we shall review.

Theorem 6.2. *(Siu) Let S be a complex variety, and let $f : \mathcal{X} \to S$ be a smooth projective morphism. Assume that the generic fiber X_η of f is a variety of general type. Then the plurigenera $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

We have also a slightly stronger version:

Theorem 6.3. *Let S be an algebraic variety, let $\mathcal{X}$ be a complex variety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities for any $t \in S$ and that the generic fiber X_η is a variety of general type. Then the plurigenera $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

According to Nakayama, we define the *numerical Kodaira dimension* $\nu(X)$ as follows (this is $\kappa_\sigma(X)$ in [27]; there is another version $\kappa_\nu(X)$ of numerical Kodaira dimension in [27] which we do not use). Let X be a compact complex manifold and let k be a nonnegative integer. We define $\nu(X) \geq k$ if there exist a divisor H on X and a positive number c such that $\dim H^0(X, mK_X + H) \geq cm^k$ for any sufficiently large and divisible m. If there is no such k, then we put $\nu(X) = -\infty$. It is easy to see that $\kappa(X) \leq \nu(X) \leq \dim X$. By the Kodaira lemma, $\kappa(X) = \dim X$ if and only if $\nu(X) = \dim X$. The *abundance conjecture* states that the equality $\kappa(X) = \nu(X)$ always holds. Nakayama confirmed this conjecture in the case $\nu(X) = 0$ ([27]).

By considering $mK_X + H$ instead of mK_X, Nakayama obtained the following:

Theorem 6.4. *(Nakayama) Let S be an algebraic variety, let $\mathcal{X}$ be a complex variety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities for any $t \in S$. Then the numerical Kodaira dimension $\nu(X_t)$ is constant on $t \in S$. In particular, if one fiber X_0 is of general type, then so are all the fibers.*

For the finer classification of algebraic varieties, it is useful to consider not only the discrete invariants $P_m(X)$ but also the infinite sum of vector spaces

$$R(X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$$

which has a natural graded ring structure over $\mathbb{C} = H^0(X, \mathcal{O}_X)$. This continuous invariant $R(X)$, called the *canonical ring* of X, is also independent of the birational models of X. It is conjectured that R(X) is always finitely generated as a graded $\mathbb{C}$-algebra. If this is the case, then Proj $R(X)$ is called a *canonical model* of X.

A *canonical singularity* (resp. *terminal singularity*) is defined as a singularity which may appear on a canonical model of a variety of general type whose canonical ring is finitely generated (resp. on a minimal model on an algebraic variety). The formal definition by Reid is as follows: a normal variety X is said to have only canonical singularities (resp. terminal singularities) if the canonical divisor K_X is $\mathbb{Q}$-Cartier and, for a resolution of singularities $\mu : Y \to X$ which has exceptional divisors F_j, if we write $\mu^* K_X = K_Y + \sum_j a_j F_j$, then $a_j \le 0$ (resp. $a_j < 0$) for all j.

For example, the canonical singularities in dimension 2 have been studied extensively. They are called in many names such as du Val singularities, rational double points, simple singularities, or A-D-E singularities. The terminal singularity in dimension 2 is smooth, and the terminal singularities in dimension 3 are classified by Mori and others (cf. [31]).

Let us consider the subset of a Hilbert scheme with a given Hilbert polynomial which consists of points corresponding to the canonical models of varieties of general type. This set should be open from the view point of the moduli problem of varieties (cf. [38]). The following is a local version of Theorem 1.2' and says that this is the case (this result was previously known up to dimension 2):

Theorem 6.5. *Let $f : \mathcal{X} \to B$ be a flat morphism from a germ of an algebraic variety to a germ of a smooth curve. Assume that the central fiber $X_0 = f^{-1}(P)$ has only canonical singularities. Then so has the total space $\mathcal{X}$ as well as any fiber X_t of f. Moreover, if $\mu : V \to \mathcal{X}$ is a birational morphism from a normal variety with the strict transform X of X_0, then $K_V + X \ge \mu^*(K_{\mathcal{X}} + X_0)$.*

The folloing theorem answers a similar question for the deformations of minimal models (this was previously known up to dimension 3):

Theorem 6.6. *[28] Let $f : \mathcal{X} \to B$ be a flat morphism from a germ of an algebraic variety to a germ of a smooth curve. Assume that the central fiber $X_0 = f^{-1}(P)$ has only terminal singularities. Then so has the total space $\mathcal{X}$ as well as any fiber X_t of f. Moreover, if $\mu : V \to \mathcal{X}$ is a birational morphism from a normal variety with the strict transform X of X_0, then the support of $K_V + X - \mu^*(K_{\mathcal{X}} + X_0)$ contains all the exceptional divisors of μ.*

The following theorem, which is stronger than Theorem 6.3, says that only the abundance conjecture for the generic fiber implies the deformation invariance of the plurigenera:

Theorem 6.7. *[28] Let S be an algebraic variety, let $\mathcal{X}$ be a complex variety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities and that $\kappa(X_\eta) = \nu(X_\eta)$ for the generic fiber X_η of f. Then the plurigenera $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

Now we explain the idea of the proofs. Since we assumed the algebraicity of varieties, there exist divisors which are big. Hence we can use the vanishing theorems of Kodaira type as in [7] and [36]. Indeed, if K_{X_0} is nef and big for the central fiber X_0 in Theorem 1.2, then the extendability of pluricanonical forms follows immediately from the vanishing theorem.

Thus the problem is to extract the nef part from the big divisor K_{X_0}. This is similar to the *Zariski decomposition* problem (cf. [11]): Let X be a smooth projective variety of general type. If we fix a positive integer m, then there exists a projective birational morphism $\mu_m : Y_m \to X$ such that $\mu_m^*(mK_X)$ is decomposed into the sum of the free part and the fixed part: $\mu_m^*(mK_X) = P_m + M_m$. If there exists one $\mu : Y \to X$ which serves as the μ_m simultaneously for all m, then $P = \sup_{m>0} P_m/m$ is the desired nef part, and the decomposition $\mu^* K_X = P + N$ in $\mathrm{Div}(Y) \otimes \mathbb{R}$ for $N = \inf_{m>0} N_m/m$ gives the Zariski decomposition of K_X. The difficulty arises when we have an infinite tower of blow-ups. It is known that if the Zariski decomposition of the canonical divisor exists, then the canonical ring $R(X)$ is finitely generated ([11]).

So we use instead the concept of *multiplier ideal sheaf* which was first introduced by Nadel. We consider the series of ideal sheaves on X_0 instead of the decompositions on the series of blow-ups. Since the structure sheaf of X_0 is noetherian, we do not have the difficulty of the infitity in this case; we take just the union of the ideals.

The remaining thing to be proved is the compatibility of the multiplier ideal sheaves on X_0 and on the total space $\mathcal{X}$ constructed similarly for $K_{\mathcal{X}}$. This is proved by a tricky induction on m discovered by Siu.

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