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Very ample line bundles on quasi-Abelian varieties

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VERY AMPLE LINE BUNDLES ON QUASI-ABELIAN VARIETIES

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1. Introduction

This article is a continuation of my previous works concerning adjoint bundles on weakly 1-complete Kähler manifolds [T2] [T3]. A complex manifold X is said to be *weakly* 1-*complete* if there exists a smooth function $\Phi : X \longrightarrow \mathbb{R}$ which is plurisubharmonic and exhaustive. Here we consider a concrete application of the following abstract, but effective, vanishing and existence theorem.

Theorem 1.1. [T3, Theorem 4.1.A-B] *Let X be a weakly 1-complete manifold with a positive line bundle L. Assume that X has no compact complex subspace of positive dimension. Then $H^q(X, K_X \otimes L \otimes \mathcal{J}) = 0$ for any analytic coherent ideal sheaf $\mathcal{J}$ which defines a zero-dimensional complex subspace of X and for any $q > 0$. In particular the restriction map $H^0(X, K_X \otimes L) \longrightarrow K_X \otimes L \otimes \mathcal{O}_X/\mathcal{J}$ is surjective.*

Here K_X is the canonical bundle of X. The assumption: the absence of positive dimensional compact subspaces seems to be restrictive. However this theorem can be seen as a variation of Cartan-Serre's fundamental theorem for Stein manifolds. Moreover, as we will discuss, such weakly 1-complete manifolds make an important class of open manifolds.

We let Γ be a discrete subgroup of $\mathbb{C}^n$. The quotient complex Lie group $\mathbb{C}^n/\Gamma$, which is called a *quasi-torus*, is a weakly 1-complete Kähler manifold with trivial canonical bundle [K1]. The existence of a positive line bundle is described by the so-called generalized Riemann relations [AG] [CC2, §2]. If a quasi-torus $\mathbb{C}^n/\Gamma$ has a positive line bundle L, there exist an abelian variety A, a quasi-torus Y without a compact subvariety of positive dimension, and a surjective holomorphic group homomorphism $g : A \times Y \longrightarrow \mathbb{C}^n/\Gamma$ with finite kernel (see 2A). Theorem 1.1 for Y says that there exists few obstruction to construct global sections of $g^*L|_{a\times Y}$ for $a \in A$. Thus if every $g^*L|_{A\times y}$ with $y \in Y$ has sections (by virtue of Lefschetz' theorem on A), we can construct sections of g^*L, as well as that of L. By using this principle, we shall show the following very ampleness criteria of positive line bundles over quasi-tori.

Lefschetz-type Theorem 1.2. *Let X be a quasi-torus with a positive line bundle L.*

(1) *Assume that X has no positive dimensional compact subtorus. Then L is very ample.*

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(2) $L^{\otimes 2}$ *is very ample if and only if there is no positive dimensional compact subtorus* A *of* X *such that* $(A, L|_A)$ *is a principally polarized abelian variety.*

(3) $L^{\otimes 3}$ *is very ample.*

The second assertion for abelian varieties is proved by Ohbuchi [Ob]. In a previous work [T2, §4C], we obtained satisfactory results on the distinct points separation by global sections. It depended on existence theorems in [T1]. We also showed Theorem 1.2(3) in [T2, Theorem 1.3], for which the so-called Lefschetz' trick for non-compact quasi-abelian varieties (cf. [CC1, Corollary 3.7]) was needed to separate infinitesimally near points. Unfortunately we were not able to show Theorem 1.2(1) and (2) in [T2], because Lefschetz' trick did not work well for lower tensor powers, like very ampleness of L and of $L^{\otimes 2}$. Theorem 1.1 (or its variation: Theorem 2.4) enables us to show Lefschetz-type theorem 1.2 for general polarized quasi-abelian varieties without using Lefschetz' trick for non-compact quasi-abelian varieties.

There are several earlier works on Lefschetz-type theorems. The author was influenced by works of Abe [A1] [A2] [A3], and of Capocasa-Catanese [CC1] [CC2]. Although we will not use them explicitly here, their works of Kazama-Umeno [K2] [KU], and of Vogt [V1] [V2] are also important as foundations in this field. Refer [T2, §1] about relations with results in [CC2].

We show Theorem 1.2 in the next section. In the last section, we will give two extremal examples of the pair X and L as in Theorem 1.2(1).

2. Lefschetz-type theorem

2A Polarized quasi-abelian variety. We recall basic notions and properties. A quasi-torus $\mathbb{C}^n/\Gamma$ is said to be a *toroidal group* (also a *Cousin* quasi-torus), if there is no non-constant holomorphic function on it. There exists a decomposition: *Morimoto's decomposition* [M, Theorem 3.2], unique up to isomorphism, as a product of complex Lie groups $\mathbb{C}^n/\Gamma \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X$, where X is toroidal. Hence the study of quasi-tori is almost reduced to that of toroidal groups.

Definition 2.1. A toroidal group $X = \mathbb{C}^n/\Gamma$ is said to be a *quasi-abelian variety* if the following *generalized Riemann relations* are satisfied: there exists a Hermitian form H on $\mathbb{C}^n$ such that

(i) H is positive definite, and that

(ii) the imaginary part $\operatorname{Im} H$ of H takes integral values on $\Gamma \times \Gamma$.

A Hermitian form H satisfying (i) and (ii) is said to be a *polarization* of X; in addition such a pair (X, H) is said to be a *polarized quasi-abelian variety.*

The condition (i) can be replaced by the following: H is positive definite on the maximal complex subspace contained in the real span of Γ (cf. [AG]). A Hermitian form H satisfying (ii) defines a cohomology class $(\operatorname{Im} H)|_{\Gamma\times\Gamma} \in H^2(X, \mathbb{Z})$. This definition is slightly different from that of [CC2, Definition 2.1] [T2, §4], in which we consider a pair (X, L) of a quasi-abelian variety X and a holomorphic line bundle L whose first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ is obtained as the imaginary part of a positive definite Hermitian form on $\mathbb{C}^n$. Here we stress on the topological object $c_1(L) = (\operatorname{Im} H)|_{\Gamma\times\Gamma} \in H^2(X, \mathbb{Z})$.

By a partial analogy of Poincaré's reducibility theorem (cf. [CC2, Proposition 2.3]) and a theorem of Morimoto [M, Theorem 6.4, Remark 6.6], we can decompose quasi-abelian varieties after taking finite coverings as follows (cf. [T2, Proposition 4.8]):

Lemma 2.2. *Let X be a quasi-abelian variety. Then*

(1) *there exists a compact subtorus A (: the maximum abelian subvariety) of X which contains any compact connected complex subspace V of X passing through the unit element;*

(2) *there exists a closed quasi-abelian subvariety Y of X without a compact complex subspace of positive dimension, such that the natural homomorphism $A \times Y \longrightarrow X$ is surjective with finite kernel;*

(3) *moreover Y in (2) is uniquely determined from a given polarization of X.*

2B Positive line bundle and canonical reduction. We will discuss properties of positive line bundles on quasi-abelian varieties. The following result gives some equivalent formulations of positive line bundles on quasi-abelian varieties.

Theorem 2.3. [T2, Theorem 1.2] *Let L be a holomorphic line bundle over a quasi-torus $\mathbb{C}^n/\Gamma$. Then the following four conditions are equivalent to one another:*

(1) *L is ample.*

(2) *L is positive.*

(3) *There exists a Kähler form in the first Chern class $c_1(L) \in H^2(\mathbb{C}^n/\Gamma, \mathbb{R})$.*

(4) *The alternating form $c_1(L) : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ given by the first Chern class is obtained as the imaginary part of a positive definite Hermitian form H on $\mathbb{C}^n$.*

By using Morimoto's decomposition and Lemma 2.2, we see the following theorem as a corollary of our general theory: Theorem 1.1.

Theorem 2.4. *Let X be a quasi-torus. Assume that X admitts a positive line bundle L and that there is no positive dimensional compact subtorus of X. Then $H^q(X, L \otimes \mathcal{J}) = 0$ for any analytic coherent ideal sheaf $\mathcal{J}$ which defines a zero-dimensional complex subspace of X and for any $q > 0$. In particular the restriction map $H^0(X, L) \longrightarrow L \otimes \mathcal{O}_X/\mathcal{J}$ is surjective.*

Hereafer we focus on a polarized quasi-abelian variety (X, H) with a non-trivial abelian subvariety. Let $g : A \times Y \longrightarrow X$ be *the* finite covering homomorphism as in Lemma 2.2. We regard the kernel G of g as the covering transformation group. We take a holomorphic line bundle L on X such that $c_1(L) = (\operatorname{Im} H)|_{\Gamma\times\Gamma}$ in $H^2(X, \mathbb{Z})$. By Theorem 2.3, L is a positive line bundle.

Let us decompose the line bundle L after taking the finite covering g. We let $i_a : Y \longrightarrow a \times Y \subset A \times Y$ for $a \in A$ and $j_y : A \longrightarrow A \times y \subset A \times Y$ for $y \in Y$ be the inclusions, and let $p_A : A \times Y \longrightarrow A$ and $p_Y : A \times Y \longrightarrow Y$ be the projections. We set

$$\mathcal{L}_A := j_0^* g^* L \quad \text{and} \quad \mathcal{L}_Y := i_0^* g^* L$$

positive line bundles over A and Y respectively. We let $\widehat{A} := \mathrm{Pic}^0 A$ be the dual torus of A which is the group of line bundles over A with the null first Chern class, and let

$$\mathcal{P} \longrightarrow A \times \widehat{A} \text{ be } \textit{the Poincaré line bundle}$$

[GH, Chapter 2 §6] [LB, Chapter 2 §5]. We let $q_A : A \times \widehat{A} \longrightarrow A$ be the projection. We consider a holomorphic map

$$\beta : Y \longrightarrow \widehat{A} \quad \text{defined by} \quad \beta(y) := j_y^* g^* L \otimes j_0^* g^* L^{\otimes(-1)}.$$

Since Y is toroidal and $\beta(0) = 0$, the map β must be a group homomorphism (cf. [AG]). Then by the universal property of the Poincaré line bundle [LB, Chapter 2 Proposition 5.2], the following isomorphism follows:

$$g^* L \cong (1_A \times \beta)^* (q_A^* \mathcal{L}_A \otimes \mathcal{P}) \otimes p_Y^* \mathcal{L}_Y \cong p_A^* \mathcal{L}_A \otimes (1_A \times \beta)^* \mathcal{P} \otimes p_Y^* \mathcal{L}_Y,$$

where $1_A : A \longrightarrow A$ is the identity map. We will use the following general result for the compact factor A (cf. [GH, p. 329]).

Lemma 2.5. *The dimension $d := \dim H^0(A, \mathcal{L}_A)$ is positive. Moreover there exist d-sections $\{\theta_i\}_{i=1}^d \subset H^0(A \times \widehat{A}, q_A^* \mathcal{L}_A \otimes \mathcal{P})$ such that their restrictions $\{\theta_i|_{A\times\xi}\}_{i=1}^d$ generate $H^0(A \times \xi, \mathcal{L}_A \otimes \mathcal{P}_\xi)$ for every $\xi \in \widehat{A}$, where $\mathcal{P}_\xi := \mathcal{P}|_{A\times\xi}$ the restriction.*

2C Lefschetz-type theorem. We let (X, H) be a polarized quasi-abelian variety, and let L be a positive line bundle on X with $c_1(L) = (\mathrm{Im}\, H)|_{\Gamma\times\Gamma}$. Let $g : A \times Y \longrightarrow X$ be the finite covering homomorphism as in Lemma 2.2. We assume that both factors A and Y are non-trivial. We use these notations throughout this subsection.

We would like to show a counter part of the following theorem (cf. [LB, Chapter 4 §5]):

Ohbuchi's Lefschetz-type Theorem 2.6. [Ob] *Let $\mathcal{L}$ be an ample line bundle on an abelian variety $\mathcal{A}$. Then $|\mathcal{L}^{\otimes 2}|$ is not very ample if and only if $(\mathcal{A}, \mathcal{L})$ is reducible with a principally polarized abelian variety factor.*

Our main technical result is as follows:

Proposition 2.7. (1) *L is generated by its global sections if and only if $(g^*L)|_{A\times y}$ is, for every $y \in Y$.* (2) *L is very ample if and only if $(g^*L)|_{A\times y}$ is, for every $y \in Y$.*

Using [T2, 2B], by virtue of this proposition, Lefschetz' theorem (cf. [GH, p. 317]), Ohbuchi's theorem 2.6 and of Morimoto's decomposition, we can state our results as follows:

Theorem 2.8. *Let $X := \mathbb{C}^n/\Gamma$ be a quasi-torus with a positive line bundle L. Then*

(1) *$L^{\otimes m}$ is generated by its global sections for every $m \geq 2$;*

(2) *$L^{\otimes m}$ is very ample for every $m \geq 3$;*

(3) *$L^{\otimes 2}$ is very ample if and only if there exists no positive dimensional compact subtorus A of X such tha $(A, L|_A)$ is a principally polarized abelian variety.*

As we will see below, our claim follows from elementary combinatorial constructions by virtue of Theorem 2.4 and Lefschetz' theorem for abelian varieties. We can paraphrase Theorem 2.8(3), by using [CC2, Proposition 2.5], as in [LB, Chapter 4 Theorem 5.5].

Proof of Proposition 2.7

The proof consists of two parts: Sublemma 2.9: the separation of distinct points; Sublemma 2.10: the separation of infinitesimally near points.

Sublemma 2.9. *Assume that the linear system $|(g^*L)|_{A\times y}|$ over $A \times y$ separates any set of r distinct points on $A \times y$ for every $y \in Y$, then $|L|$ also separates any set of r distinct points on X.*

Sublemma 2.10. *Let k be a non-negative integer. Assume that for every $y \in Y$, the restriction map $H^0(A \times y, g^*L) \longrightarrow g^*L \otimes \mathcal{O}_{A\times y}/\mathcal{M}^{k+1}_{A\times y, a\times y}$ is surjective for every $a \in A$. Then the restriction map $H^0(X, L) \longrightarrow L \otimes \mathcal{O}_X/\mathcal{M}^{k+1}_{X,x}$ is also surjective for every $x \in X$.*

Proof of Sublemma 2.9. This is proved in [T2, Lemma 4.12(3)]. However we give the proof for the readers conveniences.

Let $x_1, \dots, x_r$ be r distinct points on X. We set $g^{-1}(x_i) = \{a_{ij} \times y_{ij}\}_{j=1}^{|G|}$ for every i. For example we construct a section $s \in H^0(X, L)$ such that $s(x_1) \neq 0$ and $s(x_i) = 0$ for any $i > 1$. We note that

(i) for every x_i, a set $g^{-1}(x_i) \cap (A \times y_{11})$ consists of at most one point;

(ii) for every pair (i, j) with $i \neq j$, $g^{-1}(x_i) \cap g^{-1}(x_j) \cap (A \times y_{11})$ is an empty set. Hence a set $(\bigcup_{i=1}^r g^{-1}(x_i)) \cap (A \times y_{11})$ consists of at most r distinct points.

Then by the assumption and Lemma 2.5, there exists a section $\theta \in H^0(A \times \widehat{A}, q_A^*\mathcal{L}_A \otimes \mathcal{P})$ such that $((1_A \times \beta)^*\theta)(a_{11} \times y_{11}) \neq 0$ and that $((1_A \times \beta)^*\theta)(a_{ij} \times y_{11}) = 0$ for any $a_{ij} \times y_{11} \in g^{-1}(x_i) \cap (A \times y_{11})$ with $i \neq 1$.

By Theorem 2.4 for Y and $\mathcal{L}_Y$, there exists a section $f \in H^0(Y, \mathcal{L}_Y)$ such that $f(y_{11}) \neq 0$ and that $f(y_{ij}) = 0$ for any y_{ij} with $y_{ij} \neq y_{11}$.

Then the trace of the product

$$s := |G|^{-1} \sum_{g_i \in G} g_i^*((1_A \times \beta)^*\theta \otimes p_Y^* f)$$

is a section with the desired properties, where $|G|^{-1}$ is the degree of the covering $g : A \times Y \longrightarrow X$. □

Proof of Sublemma 2.10. We take a point $x \in X$, and set $g^{-1}(x) := \{a_j \times y_j\}_{j=1}^{|G|}$. We consider a fibre $A \times y_1$. By our assumption and Lemma 2.5, there exists a finite number of sections $\{\theta_\lambda\}_{\lambda\in\Lambda} \subset H^0(A \times \widehat{A}, q_A^*\mathcal{L}_A \otimes \mathcal{P})$ such that their restrictions $\{((1_A \times \beta)^*\theta_\lambda)|_{A\times y_1}\}_{\lambda\in\Lambda}$ on $A \times y_1$ generate $g^*L \otimes \mathcal{O}_{A\times y_1}/\mathcal{M}^{k+1}_{A\times y_1, a_1\times y_1}$.

On the other hand, by Theorem 2.4 for Y and $\mathcal{L}_Y$, we can take a finite number of sections $\{f_\mu\}_{\mu\in M}$ of $\mathcal{L}_Y$ such that they generate $\mathcal{O}_Y/\mathcal{M}^{k+1}_{Y,y_1}$ at y_1, and $\{f_\mu\}_{\mu\in M} \subset H^0(Y, \mathcal{L}_Y \otimes \bigotimes_{y_j\neq y_1} \mathcal{M}^{k+1}_{Y,y_j})$. Let us consider their products

$$S := \{(1_A \times \beta)^*\theta_\lambda \otimes p_Y^* f_\mu\}_{\lambda\in\Lambda, \mu\in M} \subset H^0(A \times Y, g^*L).$$

We just note that the differentials of the maps $1_A \times \beta$ and p_Y do not cause any troubles, since they are group homomorphisms, i.e., they are locally linear. Then it is clear that S generates $\mathcal{O}_{A\times Y}/\mathcal{M}^{k+1}_{A\times Y, a_1\times y_1}$ at $a_1 \times y_1$ and vanishes in $\mathcal{O}_{A\times Y}/\mathcal{M}^{k+1}_{A\times Y, a_j\times y_j}$ at any other $a_j \times y_j$. Then their traces

$$s := \left\{ |G|^{-1} \sum_{g_i \in G} g_i^*((1_A \times \beta)^*\theta_\lambda \otimes p_Y^* f_\mu) \right\}_{\lambda\in\Lambda, \mu\in M} \subset H^0(X, L)$$

generate $L \otimes \mathcal{O}_X/\mathcal{M}^{k+1}_{X,x}$ □

Let us complete the proof of Proposition 2.7. It is easy to see that the "only if" parts of (1) and (2). Sublemma 2.9 with $r = 1$ shows the "if" part of (1). We also see that Sublemma 2.9 with $r = 2$ and Sublemma 2.10 with $k = 1$ shows the "if" part of (2).

3. Example

We present two extremal examples of quasi-abelian varieties without a non-trivial compact subtorus. The first one is a good example to explain our Lefschetz-type theorem 1.2(1).

Let Γ be a discrete subgroup of $\mathbb{C}^n$ of rank $\Gamma = n + q$ for some $0 < q < n$, and set $X = \mathbb{C}^n/\Gamma$. We (may) assume that Γ is generated by $(n+q)$-row vectors of the following period matrix; Cousin second normal form (cf. [V1, Proposition 2]):

$$P = (p_1 \cdots p_{n+q}) = \begin{pmatrix} 0 & Q \\ I_{n-q} & R \end{pmatrix},$$

where $Q = (I_q, S)$ is a period matrix of a q-dimensional compact complex torus, and R is a real matrix. We let $\mathbb{C}^q_\Gamma := \mathbb{R}\Gamma \cap \sqrt{-1}\mathbb{R}\Gamma$ be the maximal complex linear subspace contained in the real span of Γ. We take a holomorphic coordinate $(z_1, \ldots, z_q, w_1, \ldots, w_{n-q})$ of $\mathbb{C}^n = \mathbb{C}^q_\Gamma \times \mathbb{C}^{n-q}$, and let $\{e_1, \ldots, e_n\}$ be the unit vectors. We set $\Gamma_T := Q\mathbb{Z}^{\oplus 2q}$ and $T := \mathbb{C}^q_\Gamma/\Gamma_T$. The projection $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^q_\Gamma$ induces a group extension:

$$0 \longrightarrow (\mathbb{C}^*)^{n-q} \longrightarrow X \longrightarrow T \longrightarrow 0.$$

We see that X is a quasi-abelian variety without a non-trivial compact subtorus, if $\Gamma \cap \mathbb{C}^q_\Gamma = \{0\}$ and if T is abelian. We set

$$\mathbb{H}\Gamma := \left\{ H; \begin{array}{c} \text{a Hermitian form on } \mathbb{C}^n \text{ such that} \\ \operatorname{Im} H \text{ takes integral values on } \Gamma \times \Gamma \end{array} \right\};$$

$$\mathbb{H}\Gamma_T := \left\{ H; \begin{array}{c} \text{a Hermitian form on } \mathbb{C}^q_\Gamma \text{ such that} \\ \operatorname{Im} H \text{ takes integral values on } \Gamma_T \times \Gamma_T \end{array} \right\}.$$

The projection π induces a map $\pi^* : \mathbb{H}\Gamma_T \longrightarrow \mathbb{H}\Gamma$. The principal part of $H \in \mathbb{H}\Gamma$ is a Hermitian matrix $(H(e_i, e_j))_{1\le i,j\le q}$, because the following relation holds: $(H(e_i, e_j))_{1\le i,j\le q}$ is positive definite if and only if there exists a positive line bundle L on X such that $c_1(L) = (\operatorname{Im} H)|_{\Gamma\times\Gamma}$. We would like to understand the difference $\mathbb{H}\Gamma \setminus \pi^*\mathbb{H}\Gamma_T$. We set

$$\mathbb{M}\Gamma := \{(H(e_i, e_j))_{1\le i,j\le q};\ H \in \mathbb{H}\Gamma\};$$

$$\mathbb{M}\Gamma_T := \{(\pi^* H(e_i, e_j))_{1\le i,j\le q};\ H \in \mathbb{H}\Gamma_T\}.$$

Example 1 [A1, §4]

We assume that rank $\Gamma = n+1$ and that $X = \mathbb{C}^n/\Gamma$ is toroidal. Then X is a quasi-abelian variety without a non-trivial compact subtorus. In this case, Abe showed

Proposition 3.1. [A1, Proposition 4.1] *The set* $\mathbb{M}\Gamma$ *is dense in* $\mathbb{R}$. *In particular, for any given holomorphic line bundle* L *on* X *satisfying the condition in Theorem* 2.3(4) *and for any given positive integer* q, *there exist holomorphic line bundles* L_1 *and* L_2 *on* X, *both satisfying the condition in Theorem* 2.3(4), *such that* $L \cong L_1^{\otimes q} \otimes L_2$.

For example we take

$$P = (p_1\; p_2\; p_3) = \begin{pmatrix} 0 & 1 & \sqrt{-1} \\ 1 & a & b \end{pmatrix},$$

where a and b are real numbers. The quotient $\mathbb{C}^2/\Gamma$ is toroidal if and only if a or b is irrational. Then

$$\mathbb{M}\Gamma = \{a\ell + bm + n;\ \ell, m, n \in \mathbb{Z}\};$$
$$\mathbb{M}\Gamma_T = \mathbb{Z}.$$

Since we can take the positive integer q in Proposition 3.1 arbitrary large, if we know Theorem 2.3, it sounds reasonable that L would be very ample, and so on. This example was a supporting fact, in October 1996, that we would have a chance to prove Theorem 1.2(1). However the following example shows that the situation is not as simple as we expected.

Example 2

We take real numbers $r_0 = 1, r_1, \dots, r_4$ which are linearly independent over $\mathbb{Q}$. We let Γ be a discrete subgroup of $\mathbb{C}^3$ which is generated by five row vectors $\{p_1, \dots, p_5\}$ of the following period matrix:

$$P = (p_1 \cdots p_5) = \begin{pmatrix} 0 & 1 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 1 & 0 & \sqrt{-1} \\ 1 & r_1 & r_2 & r_3 & r_4 \end{pmatrix}.$$

Then $X = \mathbb{C}^3/\Gamma$ is a quasi-abelian variety without a non-trivial compact subtorus. We take a Hermitian form $H \in \mathbb{H}\Gamma$. We set $A := \operatorname{Im} H$ and $k_{ij} := -A(p_i, p_j) \in \mathbb{Z}$. We can compute relations among r_i's, k_{ij}'s, $A(e_i, e_j)$'s and $A(e_i, \sqrt{-1}e_j)$'s, by using

$$p_1 = e_3,\quad p_2 = e_1 + r_1 e_3,\quad p_4 = \sqrt{-1}e_1 + r_3 e_3,$$
$$p_3 = e_2 + r_2 e_3,\quad p_5 = \sqrt{-1}e_2 + r_4 e_3.$$

Then it follows, from our assumption: $1, r_1, \dots, r_4$ are linearly independent over $\mathbb{Q}$, that

$$k_{12} = k_{13} = k_{14} = k_{15} = 0,\quad k_{23} = k_{45},\quad k_{25} = k_{34}.$$

Then we have

$$\Big(H(e_i, e_j)\Big)_{1 \le i,j \le 3} = \begin{pmatrix} k_{24} & k_{25} - \sqrt{-1}k_{23} & 0 \\ k_{25} + \sqrt{-1}k_{23} & k_{35} & 0 \\ 0 & 0 & H(e_3, e_3) \end{pmatrix}.$$

This implies that

Proposition 3.2. (1) *For any given* $H \in \mathbb{H}\Gamma$*, there exist* $H_T \in \mathbb{H}\Gamma_T$ *and* $H_0 \in \mathbb{H}\Gamma$ *with* $\operatorname{Im} H_0 = 0$*, such that* $H = \pi^* H_T \oplus H_0$*; in other words,*

(2) *For any given holomorphic line bundle* L *on* X*, there exist a holomorphic line bundle* L_T *on* T *and a topologically trivial holomorphic line bundle* L_0 *on* X *such that* $L \cong \pi^* L_T \otimes L_0$*. Moreover* L *is positive if and only if* L_T *is.*

In spite of this situation, every positive line bundle on X is very ample. These phenomena: Example 1-2 and Theorem 1.2(1) are still mysterious to the author.

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