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SMR.1227 - 13

SUMMER SCHOOL ON ASTROPARTICLE PHYSICS AND COSMOLOGY

12 - 30 June 2000

DYNAMICS OF INFLATION

Lectures II & III

ROCKY KOLB
Fermi Lab. and Chicago
USA

Some Simple Questions:

1. Was inflation "normal" (*i.e.*, FRW)?
2. Can dynamics of inflation be described in terms of a single degree of freedom?
3. What was the expansion rate during inflation?
4. What was the *general* shape of the inflaton potential?
5. What was the (more or less) *exact* shape of the inflaton potential?
6. Can inflation tell us anything about physics at very high energies like the Planck scale?

The Proper Vibrations of the Expanding Universe

Erwin Schrödinger

Physica, **6** (1939) 899

Introduction:

... proper vibrations [positive and negative frequency modes] cannot be rigorously separated in the expanding universe.

Generally speaking, this is a phenomenon of outstanding importance [e.g., WIMPZILLAS!]. With particles it would mean production or annihilation of matter, merely by expansion,... Alarmed by these prospects, I have examined the matter in more detail.

Conclusion:

... There will be a mutual adulteration of positive and negative frequency terms in the course of time, giving rise to what in the introduction I called "the alarming phenomena".

The Proper Vibrations of the Expanding Universe

Applications:

Density Perturbations: Abbott, Allen, Bardeen, Fabbi, Guth, Hawking, Linde, Mukhanov, Pi, Pollack, Rubakov, Sazhin, Starobinski, Steinhardt, Turner, Veryashin, Wise

Vacuum quantum fluctuations of inflaton field during inflation
⟶ density perturbations

Vacuum quantum fluctuations of metric tensor during inflation
⟶ gravity waves

Dark Matter Production: Chung, Kolb, Kuzmin, Riotto, Tkachev

Vacuum quantum fluctuations of wimpzillas
⟶ dark matter

$$\delta(G_{\mu\nu}) = 8\pi G\, \delta(T_{\mu\nu})$$

Bardeen (1980)

Reference Spacetime: (Flat FRW)

$$ds^2 = a^2(\tau)\left\{d\tau^2 - \delta_{ij}dx^i dx^j\right\}$$

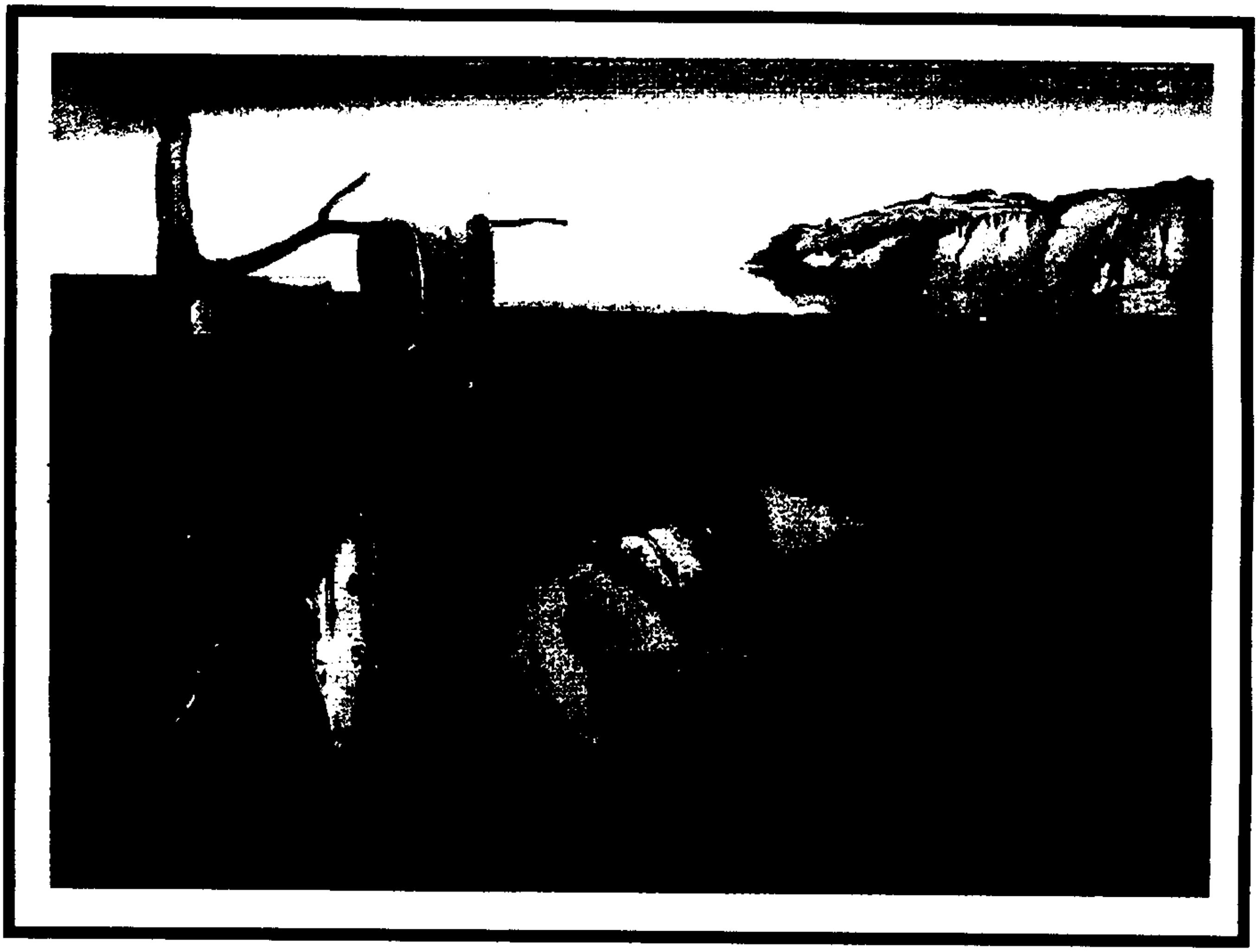
$\tau =$ conformal time $\qquad \mathcal{H} = \frac{1}{a}\frac{da}{d\tau}$

$$a^2(\tau)d\tau^2 = dt^2$$

Perturbed Spacetime:

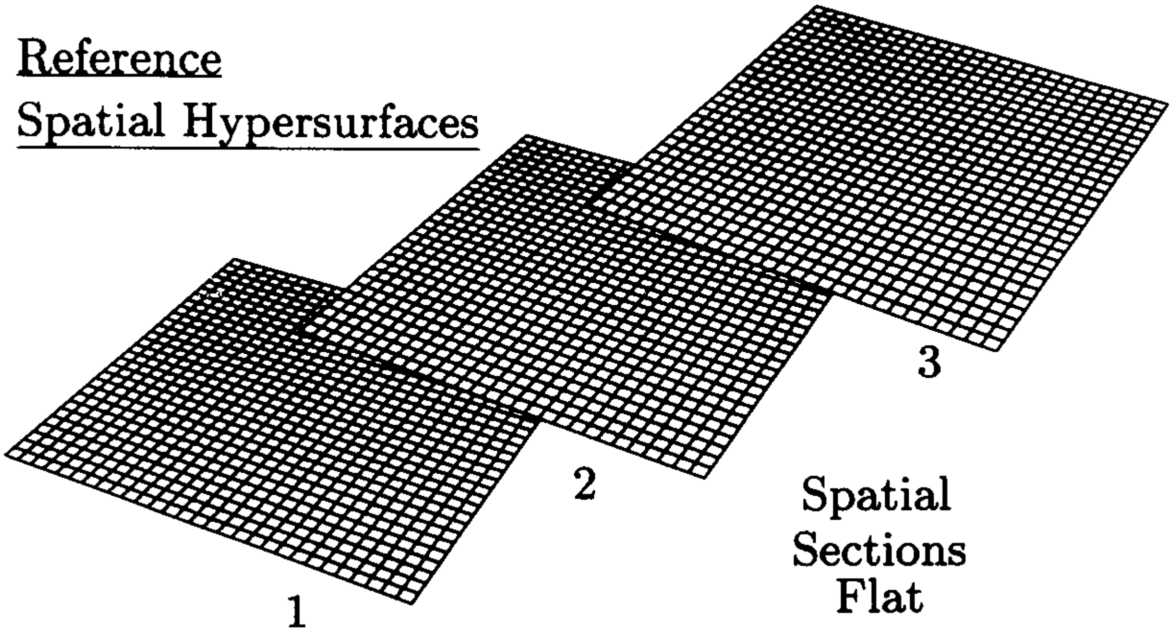
$$\begin{aligned} ds^2 &= a^2(\tau)\left\{(1+\delta g_{00})d\tau^2 \right.\\ &\left. -2g_{0i}dx^i d\tau - (\delta_{ij} + 2\delta g_{ij})dx^i dx^j\right\} \end{aligned}$$

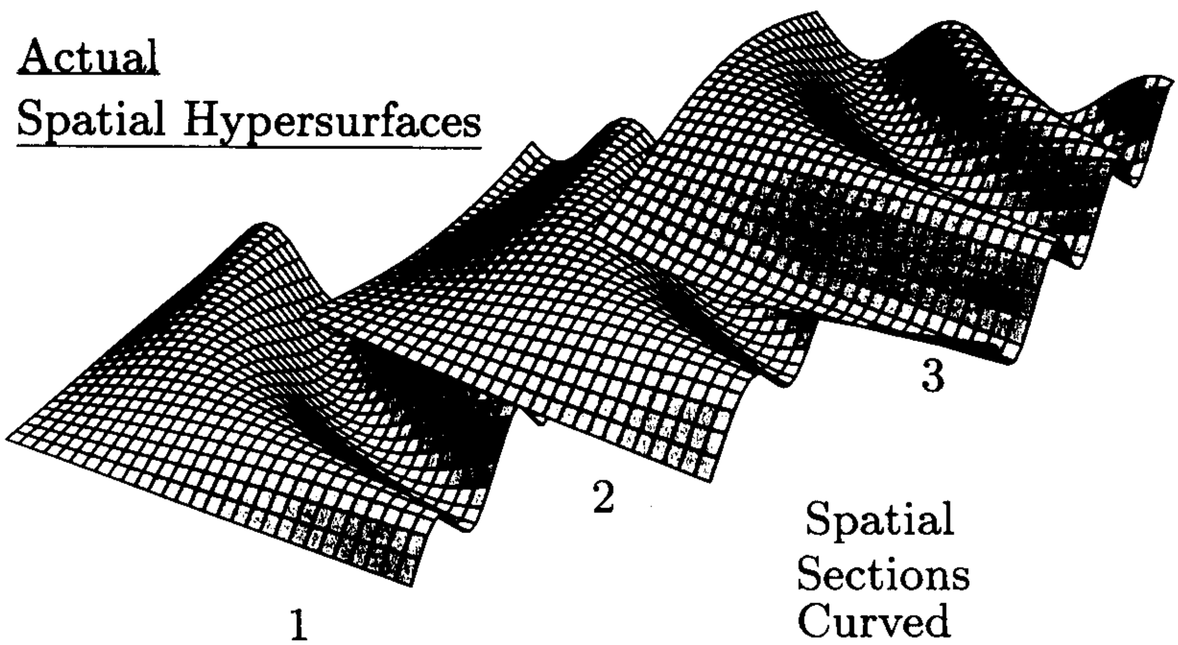
Conformal Time



$$ds^2 = a^2(\tau)\left[d\tau^2 - d\vec{x}^2\right]$$

$$a^2(\tau)d\tau^2 = dt^2$$


Reference
Spatial Hypersurfaces
1
2
3
Spatial
Sections
Flat


Actual
Spatial Hypersurfaces
1
2
3
Spatial
Sections
Curved

scalar, vector, tensor decomposition

$$
\begin{aligned}
\delta g_{00} &= 2A \\
\delta g_{0i} &= S_i + \partial_i B \\
\delta g_{ij} &= h_{ij} + \partial_i \partial_j E - \psi \delta_{ij} + \partial_i F_j + \partial_j F_i
\end{aligned}
$$

$$
\partial^i S_i = 0 \qquad \partial^i F_i = 0 \quad h^i_i = 0 \quad \partial^i h_{ij} = 0
$$

Perturbed Spacetime: (*Scalar*)

$$
\begin{aligned}
ds^2 &= a^2(\tau)\Big\{2Ad\tau^2 - 2\partial_i B dx^i d\tau \\
&\quad - \left[-2\psi\delta_{ij} + 2\partial_i\partial_j E\right] dx^i dx^j\Big\}
\end{aligned}
$$

Perturbed Spacetime: *(Tensor)*

$$ds^2 = a^2(\tau) h_{ij} dx^i dx^j$$

h_{ij} { transverse, traceless tensor (gravitational waves)

quantum perturbations of transverse, traceless component of metric tensor ---- gravitons!

scalar perturbations: couple to stress tensor
tensor perturbations: do not

10-4=6: Gauge freedom

A	1		
B	1		2 Vector
ψ	1		2 Scalar
E	1	→	2 Tensor
S_i	2		6
F_i	2		
h_{ij}	2		
	10		

Gauge Choices: (Scalar)

Synchronous gauge: $A = B = 0$

$$ds^2 = a^2(\tau)\left\{d\tau^2 - [1 - 2\psi\delta_{ij} + 2\partial_i\partial_j E]\, dx^i dx^j\right\}$$

Longitudinal gauge: $B = E = 0$

(conformal, Newtonian)

$$ds^2 = a^2(\tau)\left\{[1 + 2A]\, d\tau^2 - [1 - 2\psi]\, d\vec{x}^2\right\}$$

Gauge transformation

$$x^\mu \longrightarrow x^\mu + \xi^\mu$$

$$\begin{aligned}
A &\longrightarrow A - (a'/a)\xi^0 - \xi^{0\prime} \\
\psi &\longrightarrow \psi + \cdots \\
B &\longrightarrow B + \cdots \\
E &\longrightarrow E + \cdots \\
\delta\rho &\longrightarrow \delta\rho + \cdots \\
\delta\phi &\longrightarrow \delta\phi + \cdots
\end{aligned}$$

Gauge invariant combinations

$$\Phi = A + a^{-1}\left[(B - E')\,a\right]' \qquad \Psi = \psi - (a'/a)\,(B - E')$$

$$ds^2 = a^2(\tau)\left\{[1 + 2\Phi]\,d\tau^2 - [1 - 2\Psi]\,d\vec{x}^2\right\}$$

In absence of anisotropic stress $\Phi = \Psi$

Gauge invariant variable

$$\mathcal{R} = \psi - \frac{\mathcal{H}}{\phi'}\delta\phi$$

intrinsic curvature perturbations on comoving hypersurfaces

Variational Formalism for Quantization

Mukhanov

Action:

$$S = \int d^4x \sqrt{-g}\left[-\frac{M_{Pl}^2}{16\pi}R + \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]$$

$$g_{\mu\nu}(\vec{x},t) = g_{\mu\nu}^{FRW}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

$$\phi(\vec{x},t) = \phi_0(t) + \delta\phi(\vec{x},t)$$

...for scalar perturbations...

$$\delta_2 S = \int d^4x \left[\partial_\mu u \partial^\mu u - \frac{1}{2}m^2u^2\right]$$

Minkowski (in conformal time) scalar field u

$u = a\delta\phi + z\psi = -z\mathcal{R}$ [proportional to $\delta\rho/\rho$]

$z = a\phi'/\mathcal{H}$ [function of $V(\phi)$]

with mass^2 = $-z^{-1}d^2z/d\tau^2$

↗

$$u_k \longrightarrow \mathcal{R}_k \longrightarrow \Delta(k)$$

$$\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\tau^2}\right) u_k = 0$$

$$z = a\mathcal{H}^{-1} d\phi_0 / d\tau$$

- **"Mass" term complicated model-dependent function of *H* and how *H* changes during inflation.** [$V(\phi) \longleftrightarrow H(\phi)$]
- **Numerical integration**
  (Grivell & Liddle)
- **Expand about exact solution**
  (Stewart & Lyth)

Slow-Roll Parameters

- ***cosmological model [V(ϕ)]***

  $H(\phi), H'(\phi), H''(\phi), H'''(\phi)$

- **ε; η; ξ; ....** ***slow-roll parameters***

$$\epsilon(\phi) = \frac{m_{PL}^2}{4\pi}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \quad \eta(\phi) = \frac{m_{PL}^2}{4\pi}\frac{H''(\phi)}{H(\phi)}$$

$$\xi(\phi) = \frac{m_{PL}^2}{4\pi}\left(\frac{H'(\phi)H'''(\phi)}{H^2(\phi)}\right)^{1/2} + \ldots.$$

- ***gives mass as function of time***

$$-m_u^2 = 2a^2H^2\left[1+\epsilon-\frac{3}{2}\eta+\epsilon^2-2\epsilon\eta+\frac{1}{2}\eta^2+\frac{1}{2}\xi^2\right]$$

Amplitudes and Spectra

H~V ***ε~V′*** ***η~V″*** Stewart & Lyth

Scalar:

$$A_S(k) = \frac{2}{5\sqrt{\pi}}\frac{H(\phi)}{m_{Pl}}\frac{1}{\sqrt{\epsilon(\phi)}}\left[1 - 0.46\epsilon(\phi) - 0.73\eta(\phi)\right]$$

$$n(k) \equiv \frac{d\ln A_S^2}{d\ln k} = 1 - 4\epsilon(\phi) - 2\eta(\phi)$$
$$+2.16\epsilon^2(\phi) + 1.3\epsilon(\phi)\eta(\phi) - 1.46\xi^2(\phi)$$

Tensor:

$$A_T(k) = \frac{2}{5\sqrt{\pi}}\frac{H(\phi)}{m_{Pl}}\left[1 - 0.27\epsilon(\phi)\right]$$

$$n_T(k) \equiv \frac{d\ln A_T^2}{d\ln k} = -2\epsilon(\phi) - 3.08\epsilon^2(\phi) + 1.08\epsilon(\phi)\eta(\phi)$$

$$\text{CDM: } n = 1 \quad n_T = 0 \quad A_T = 0$$

Consistency Relation

Copeland, Kolb, Liddle, Lidsey

$$n_T = -2r\left[1 - r + (1 - n)\right]$$

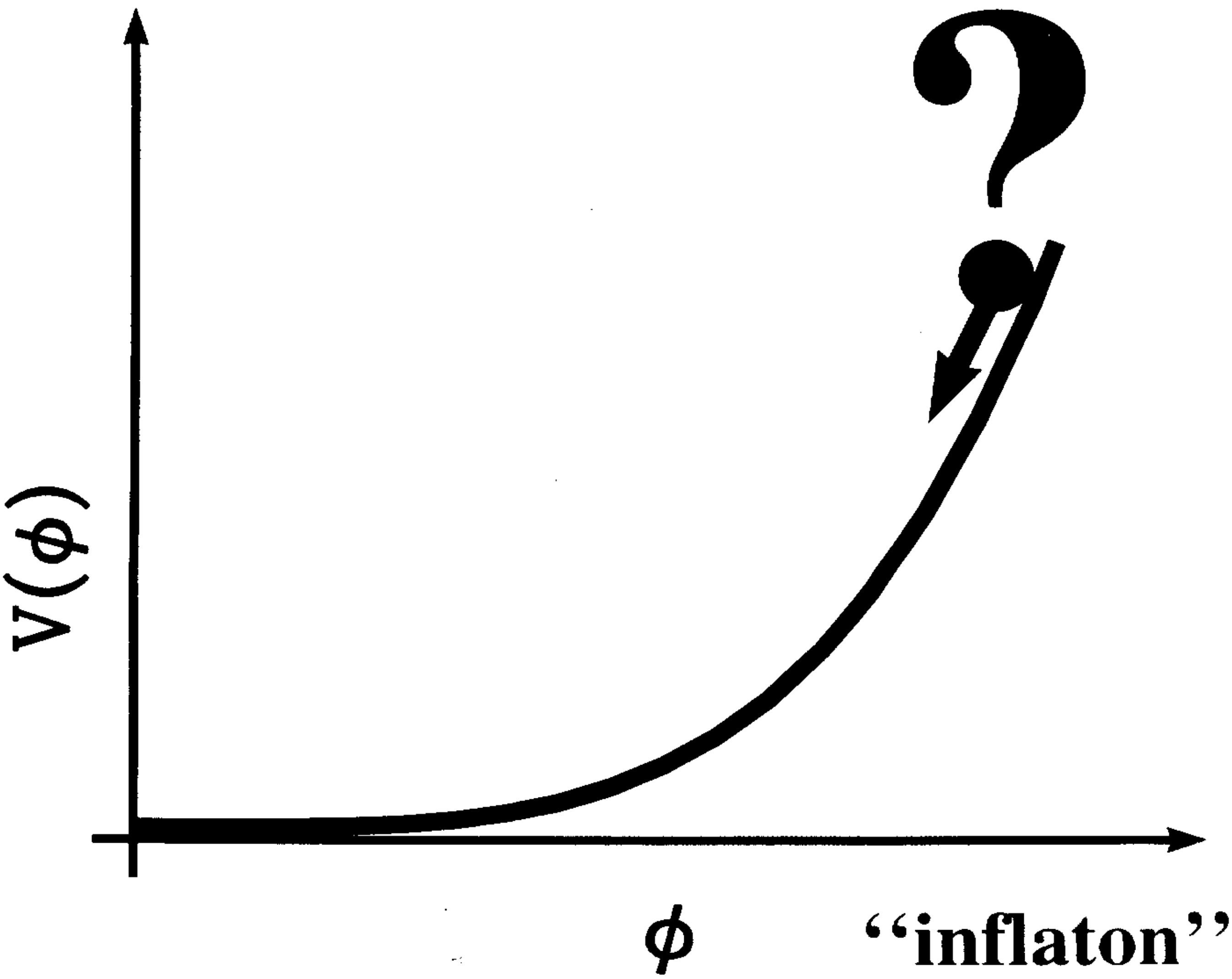
Consistency relation is a generic feature of Type I inflation models

There is NO general relation between *r* and *1-n* !!!!!!!!!!!!!

Model Space	**CBR Space**
$[\varepsilon, \eta]$	$[n, r]$
$\varepsilon \sim \dfrac{m_{PL}^{\,2}}{16\pi}\left(\dfrac{V'}{V}\right)^2$	$n =$ *scalar spectral index*
$\eta \sim \dfrac{m_{PL}^{\,2}}{8\pi}\dfrac{V''}{V} - \varepsilon$	$r = \left(\dfrac{\text{tensor}}{\text{scalar}}\right)_{l=2}$

$$V'(\phi)\ \&\ V''(\phi) \longleftrightarrow [\varepsilon, \eta] \longleftrightarrow [n, r]$$

Who is the inflaton?



Toy Models of Inflation

old, new, used (pre-owned),
chaotic, quixotic, ergodic,
exotic, heterotic, autoerotic,
eternal, internal, infernal,
natural, supernatural, *au natural,*
power-law, powerless, power-mad,
one-field, two-field, home-field,
modulus, modulo, moduli,
self-reproducing, self-promoting,
hybrid, low-bred, white-bread,
first-order, second-order, new-world order,
pre-big-bang, no-big-bang, post-big-bang,
***D*-term, *F*-term, winter-term,**
supersymmetric, superstring, superstitious,
extended, hyperextended, overextended,
***D*-brane, three-brane, *No*-brain,**
dilaton, dilettante, ...

Top down

Inflation

Bottom up

Classification*

Type I: *single-field, slow-roll models*
(or models that can be expressed as such)

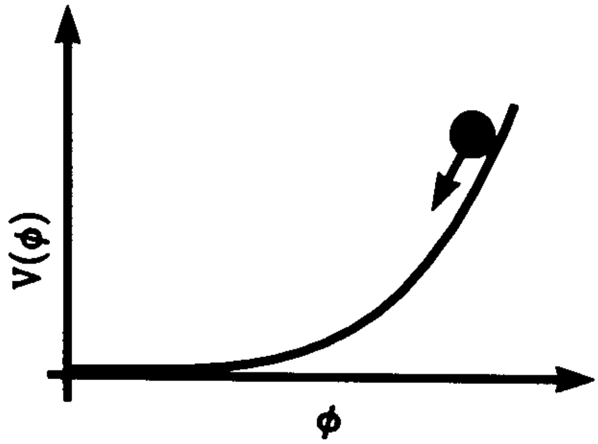
Ia: *large-field models*

Ib: *small-field models*

Ic: *hybrid models*

Type II: *anything else*
(pre big bang, strings, branes, M-theory, ...)

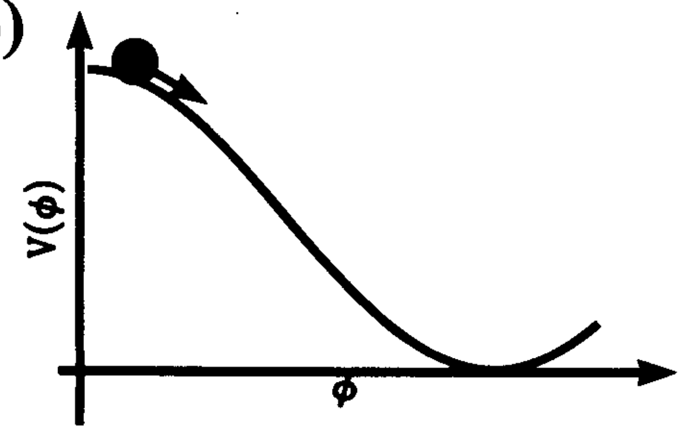
**Used for supernovae, superstrings, superconductors, ...*



Large-field models **(Ia)**

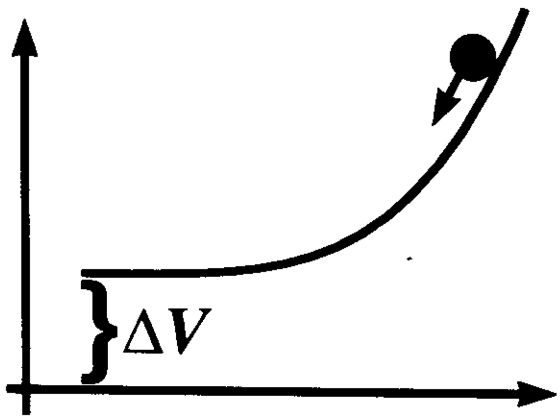
$H \sim 10^{13}$ **GeV**

Small-field models **(Ib)**



$H < 10^{13}$ **GeV**

(<< ?)

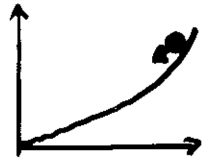
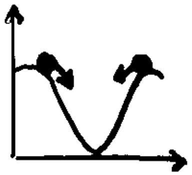
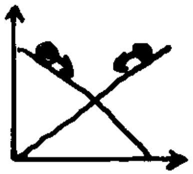
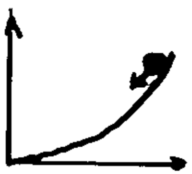


Hybrid models **(Ic)**

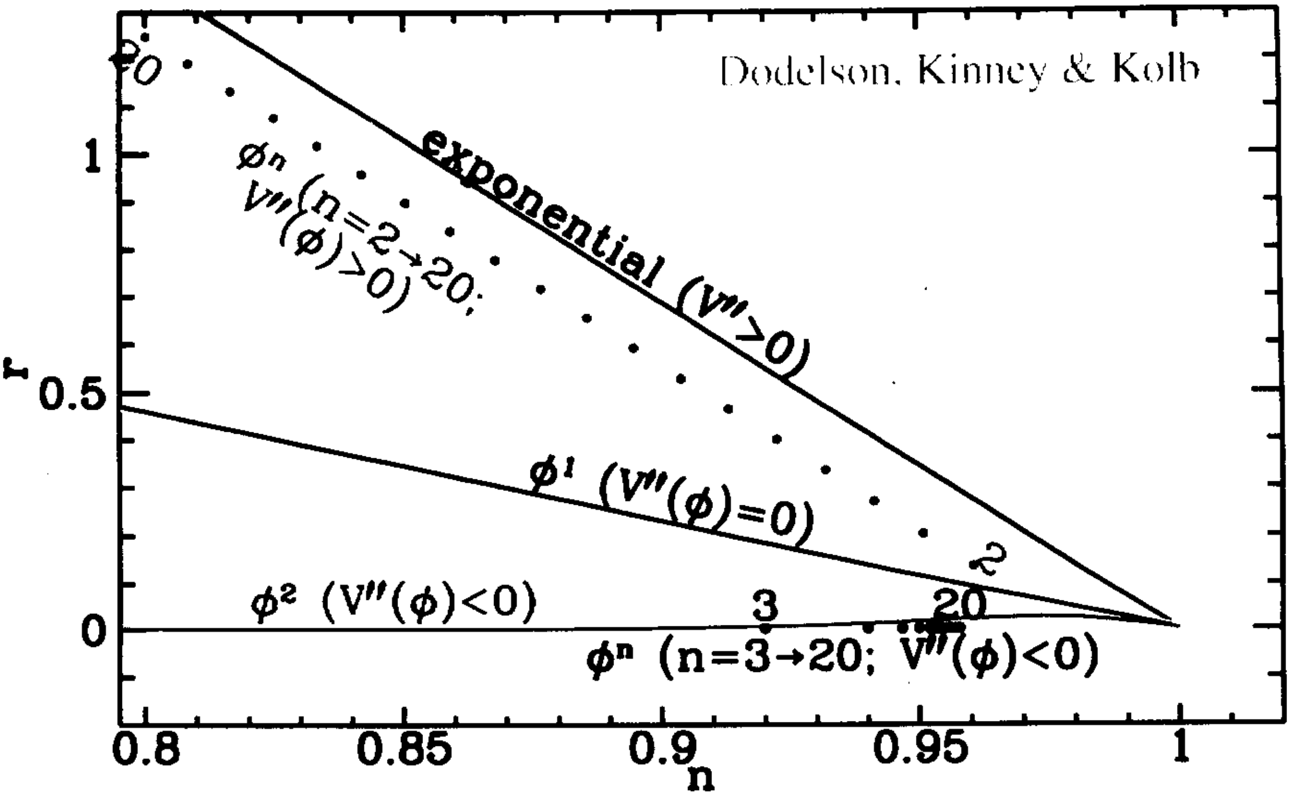
$H \sim ????$

Tensor pert's proportional to *H*

Well studied inflation models (to lowest order).

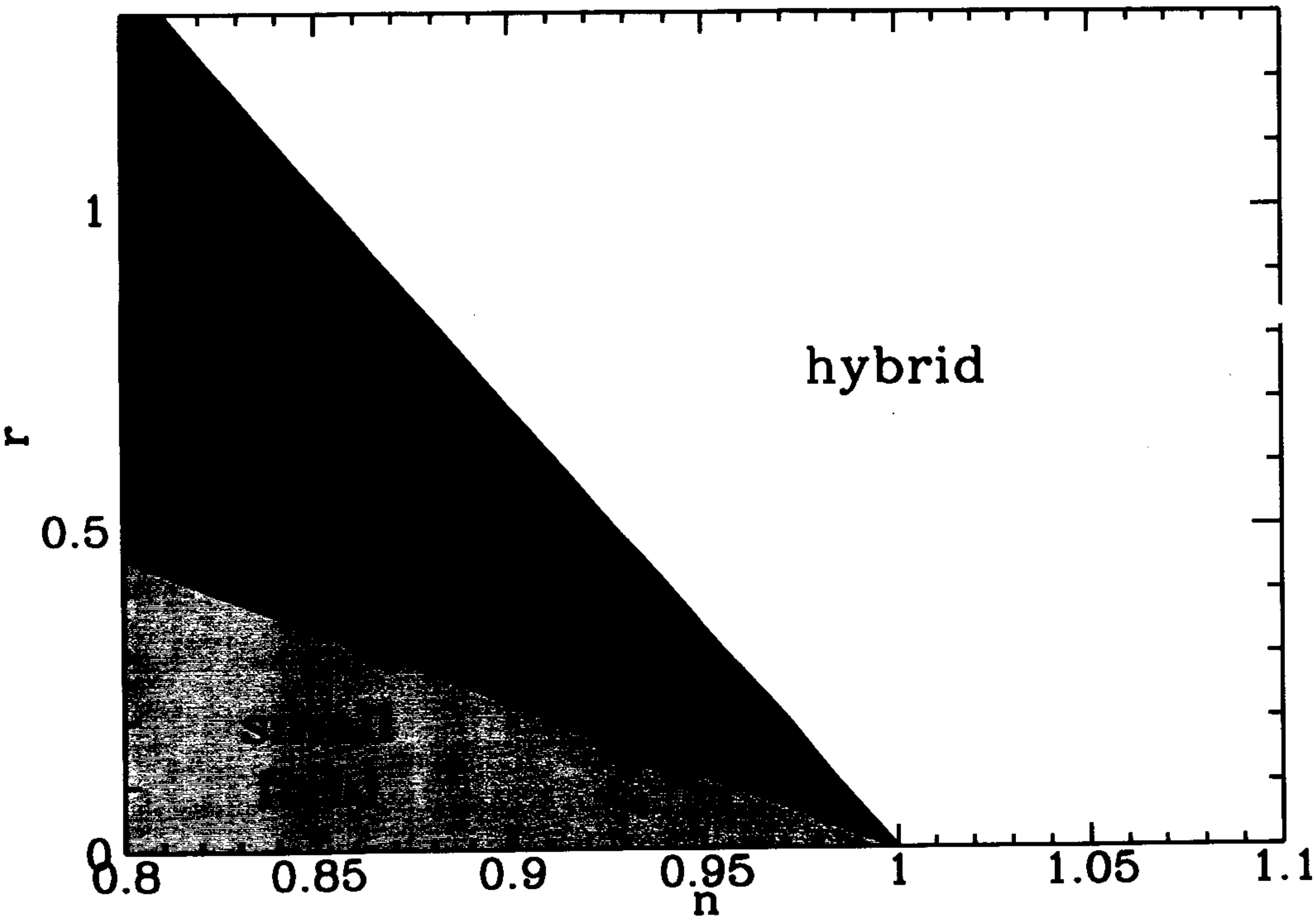
		model	$n-1$ $(2\eta - 4\epsilon)$	r (13.7ϵ)
	large field	$\Lambda^4(\phi/\mu)^p$	$-\frac{2p+4}{p+200}$	$\frac{13.7p}{p+200}$
	small field	$\Lambda^4[1-(\phi/\mu)^{p>2}]$	$-\frac{p-1}{25(p-2)}$	$\simeq 0$
	au natural	$\Lambda^4[1-(\phi/\mu)^2]$	$-\frac{m_{Pl}^2}{2\pi\mu^2}$	$\simeq 0$
	linear	$\Lambda^4(\phi/\mu);\ \Lambda^4[1-(\phi/\mu)]$	$-\frac{3m_{Pl}^2}{8\pi\mu^2}$	$\frac{13.7m_{Pl}^2}{16\pi\mu^2}$
	power law	$\Lambda^4 \exp\sqrt{16\pi\phi^2/pm_{Pl}^2}$	$-2p^{-1}$	$13.7p^{-1}$
	hybrid	$\Lambda^4[1+(\phi/\mu)^p]$	blue/red	????

Sorting the Toys

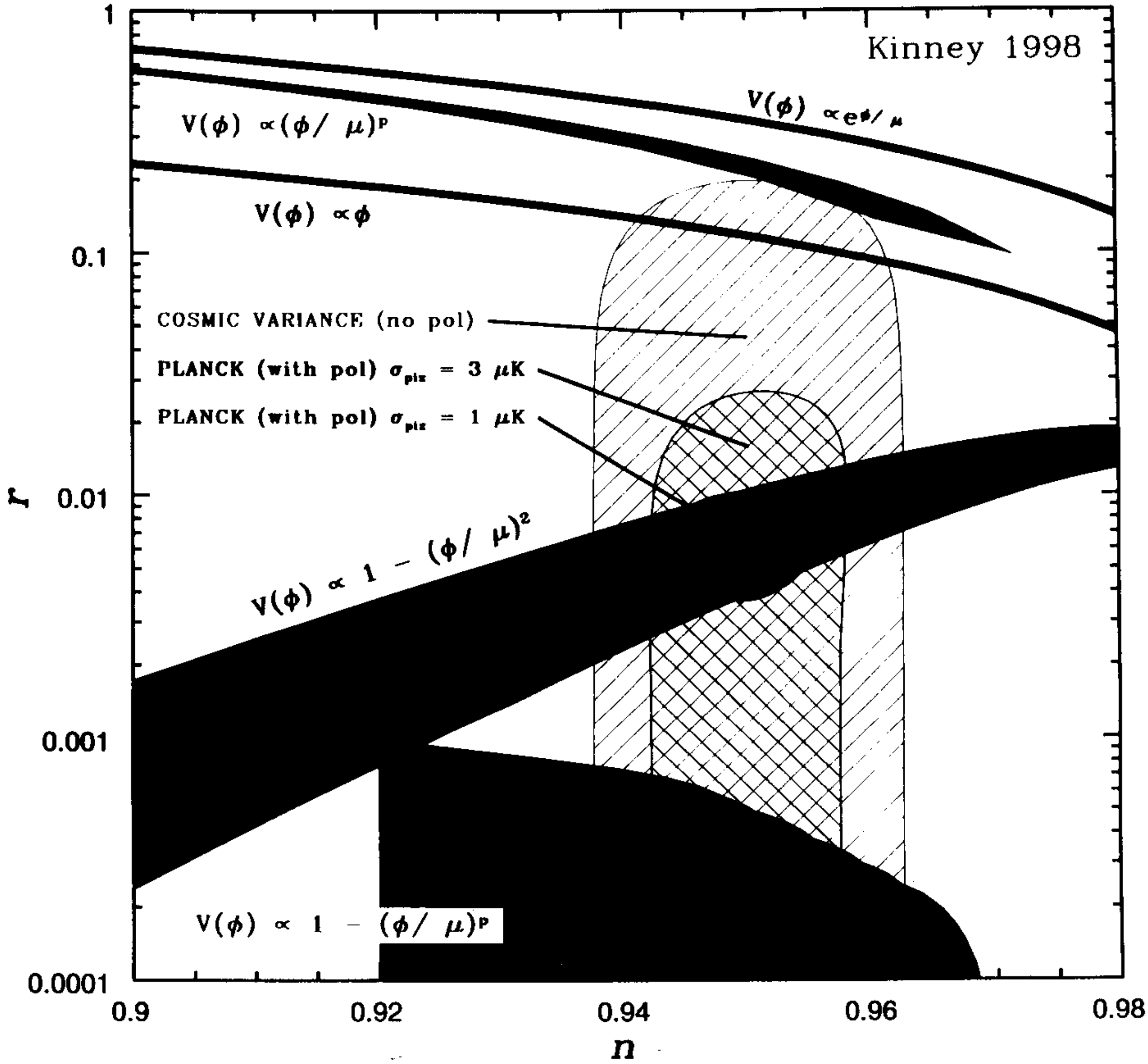


n = scalar spectral index

r = (tensor/scalar)$_{l=2}$



$r = (\text{tensor/scalar})_{l=2}$ n = scalar spectral index


Kinney 1998
V(φ) ∝ e^{φ/μ}
V(φ) ∝ (φ/ μ)^p
V(φ) ∝ φ
COSMIC VARIANCE (no pol)
PLANCK (with pol) σ_pix = 3 μK
PLANCK (with pol) σ_pix = 1 μK
V(φ) ∝ 1 − (φ/ μ)^2
V(φ) ∝ 1 − (φ/ μ)^p
r
n
1
0.1
0.01
0.001
0.0001
0.9
0.92
0.94
0.96
0.98

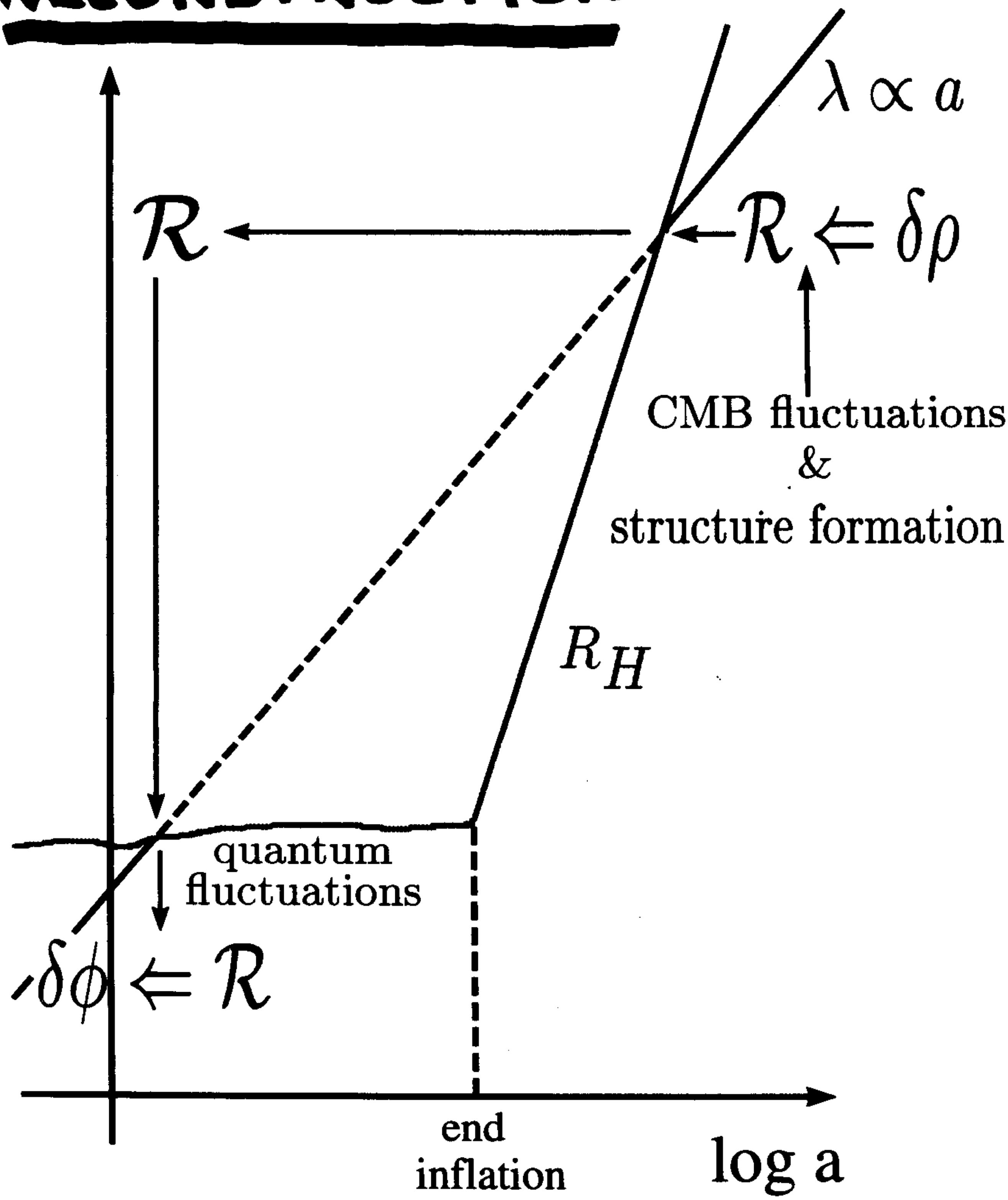
Type I Models Predict:*

1. a (nearly) exact ***power-law*** *(n)*
2. spectrum of ***gaussian***
3. ***scalar*** (density) perturbations and
4. ***tensor*** (gravity wave) perturbations *(r)*
5. in their ***growing mode***
6. in a ***spatially flat*** universe.

---

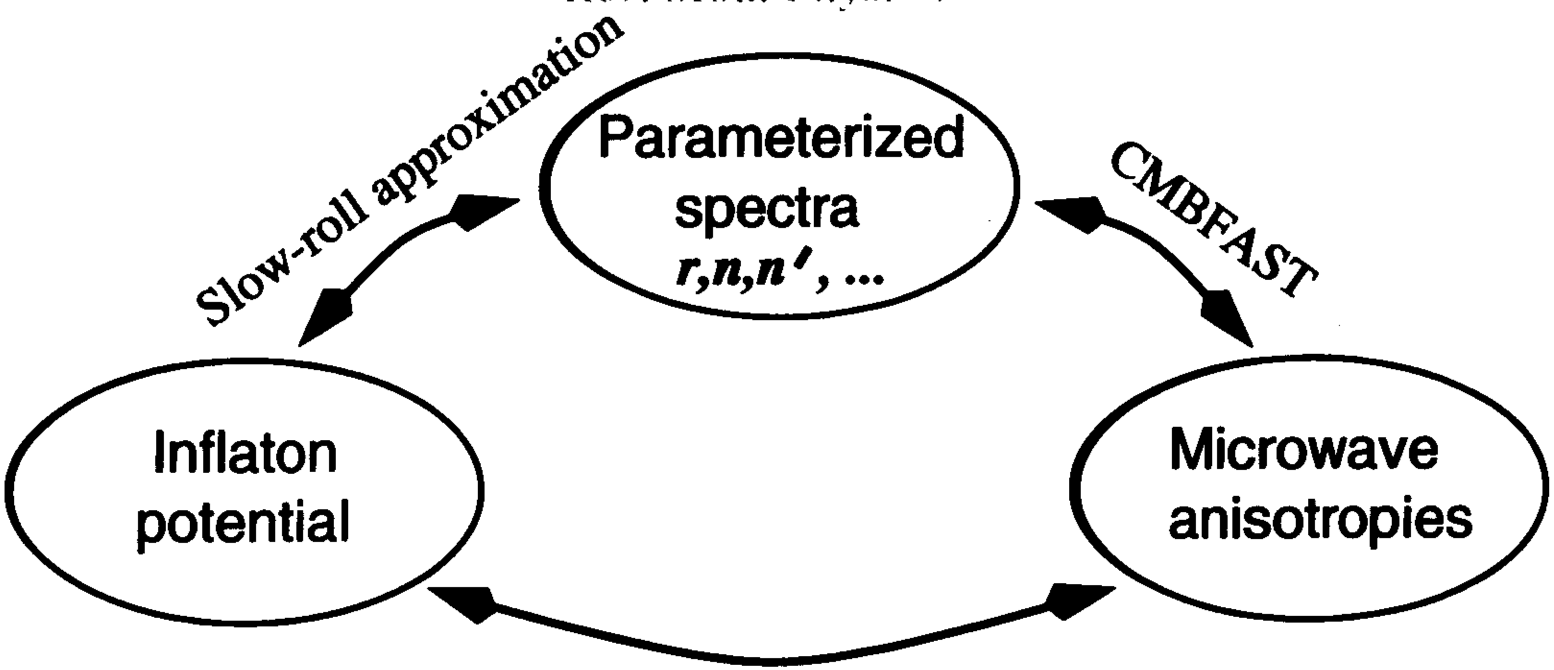
*At least the simplest models

RECONSTRUCTION



Reconstruction

Copeland, Kolb, Liddle, Lidsey
Rev. Mod. Phys. 97



Direct reconstruction
Mode equations + CMBFAST
Grivell and Liddle
astro-ph/9906327

Perturbative Reconstruction

Hamilton-Jacobi Approach: $\boldsymbol{H(\phi)}$

$\epsilon \propto H' \qquad \eta \propto H'' \qquad \xi \propto H''' \qquad \ldots$

$$\left(\frac{dH}{d\phi}\right)^2 - \frac{12\pi}{m_{Pl}^2}H^2 = -\frac{32\pi^2}{m_{Pl}^4}V(\phi)$$

$$V = \frac{m_{Pl}^2 H^2}{8\pi}(3 - \epsilon)$$

$$V' = -\frac{m_{Pl}^2}{\sqrt{4\pi}}H^2\epsilon^{1/2}(3 - \eta)$$

$$V'' = H^2\left(3\epsilon + 3\eta - \eta^2 + \xi^2\right)$$

Lidsey, Liddle, Kolb, Copeland, Barriero, Abney
Reviews of Modern Physics, 4/97

Reconstruction Strategy

I) From CBR anisotropy

1) Find tensor mode

$r = (tensor/scalar)_{l=2}$

2) Fit scalar spectrum

$$\left(A_S^2(k_*)\,,\, n_*\,,\, dn/dk\big|_*\,,\, ...\right)$$

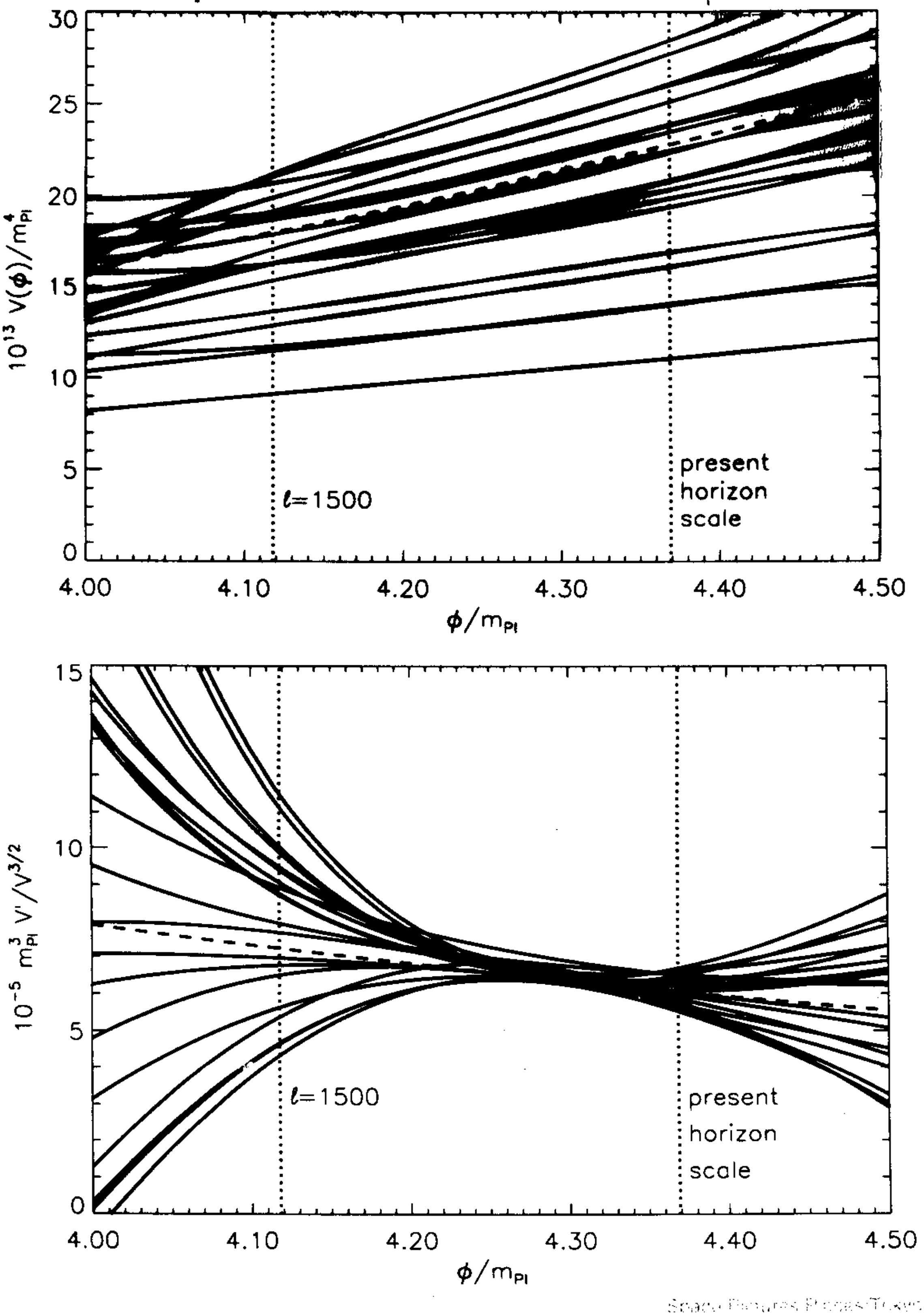
II) Express $V(\phi_*)$, $V'(\phi_*)$, ..., and $(\phi - \phi_*)$ in terms of observables

III) Taylor series for $V(\phi)$:

$$V(\phi) = V(\phi_*) + V'(\phi_*)\,(\phi-\phi_*) + ...$$

Grivell and Liddle
astro-ph/9906327

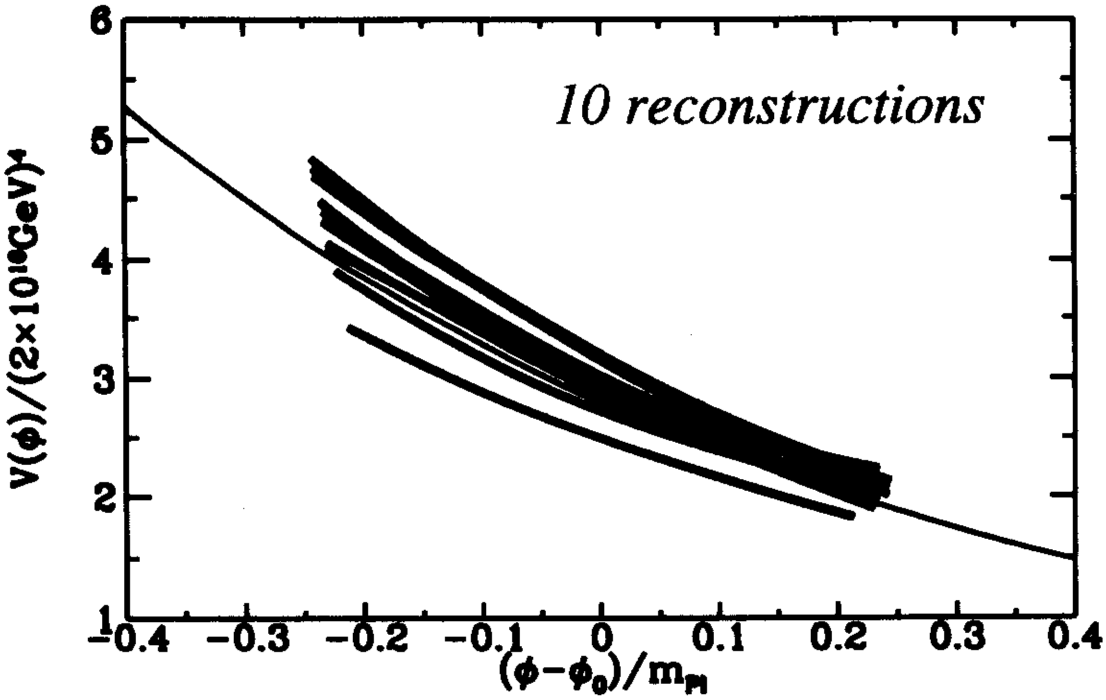
$\lambda\phi^4$

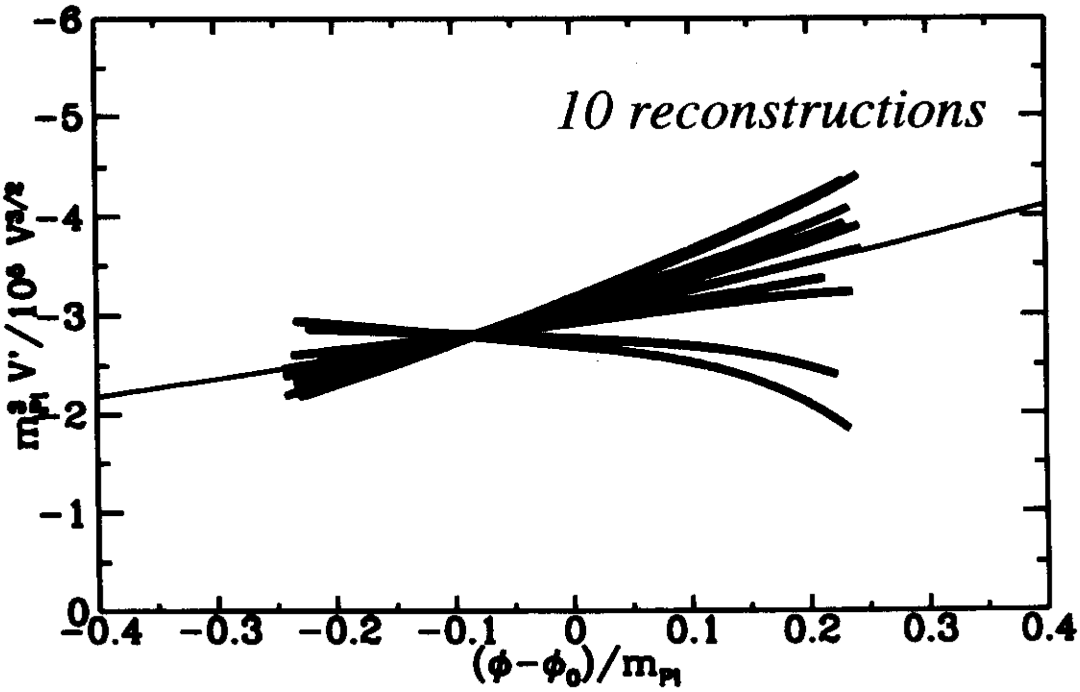


power-law spectrum

$$V(\phi) = V_0 \exp(-\alpha\phi/M_{Pl})$$

Copeland, Grivell, Kolb, Liddle





The potential expressed in terms of slow-roll parameters.

	lowest-order	next-order (exact)
$V(\phi_*)$	$H(\phi_*)$	$H(\phi_*),\ \epsilon(\phi_*)$
$V'(\phi_*)$	$H(\phi_*),\ \epsilon(\phi_*)$	$H(\phi_*),\ \epsilon(\phi_*),\ \eta(\phi_*)$
$V''(\phi_*)$	$H(\phi_*),\ \epsilon(\phi_*),\ \eta(\phi_*)$	$H(\phi_*),\ \epsilon(\phi_*),\ \eta(\phi_*),\ \xi(\phi_*)$
$V'''(\phi_*)$	$H,\ \epsilon(\phi_*),\ \eta(\phi_*),\ \xi(\phi_*)$	——

The slow-roll parameters in terms of observables.

parameter	lowest-order	next-order
$H(\phi_*)$	$A_T^2(k_*)$	$A_T^2(k_*)$, $A_S^2(k_*)$
$\epsilon(\phi_*)$	$A_T^2(k_*)$, $A_S^2(k_*)$	$A_T^2(k_*)$, $A_S^2(k_*)$, n_*
$\eta(\phi_*)$	$A_T^2(k_*)$, $A_S^2(k_*)$, n_*	$A_T^2(k_*)$, $A_S^2(k_*)$, n_*, $(dn/d\ln k)_*$
$\xi(\phi_*)$	$A_T^2(k_*)$, $A_S^2(k_*)$, n_*, $(dn/d\ln k)_*$	——

$A_T^2(k_*)$ and $A_S^2(k_*)$ are *primordial* scalar and tensor amplitudes (must be deduced from CMB)

Observables needed to reconstruct a given derivative of the potential to a given order.

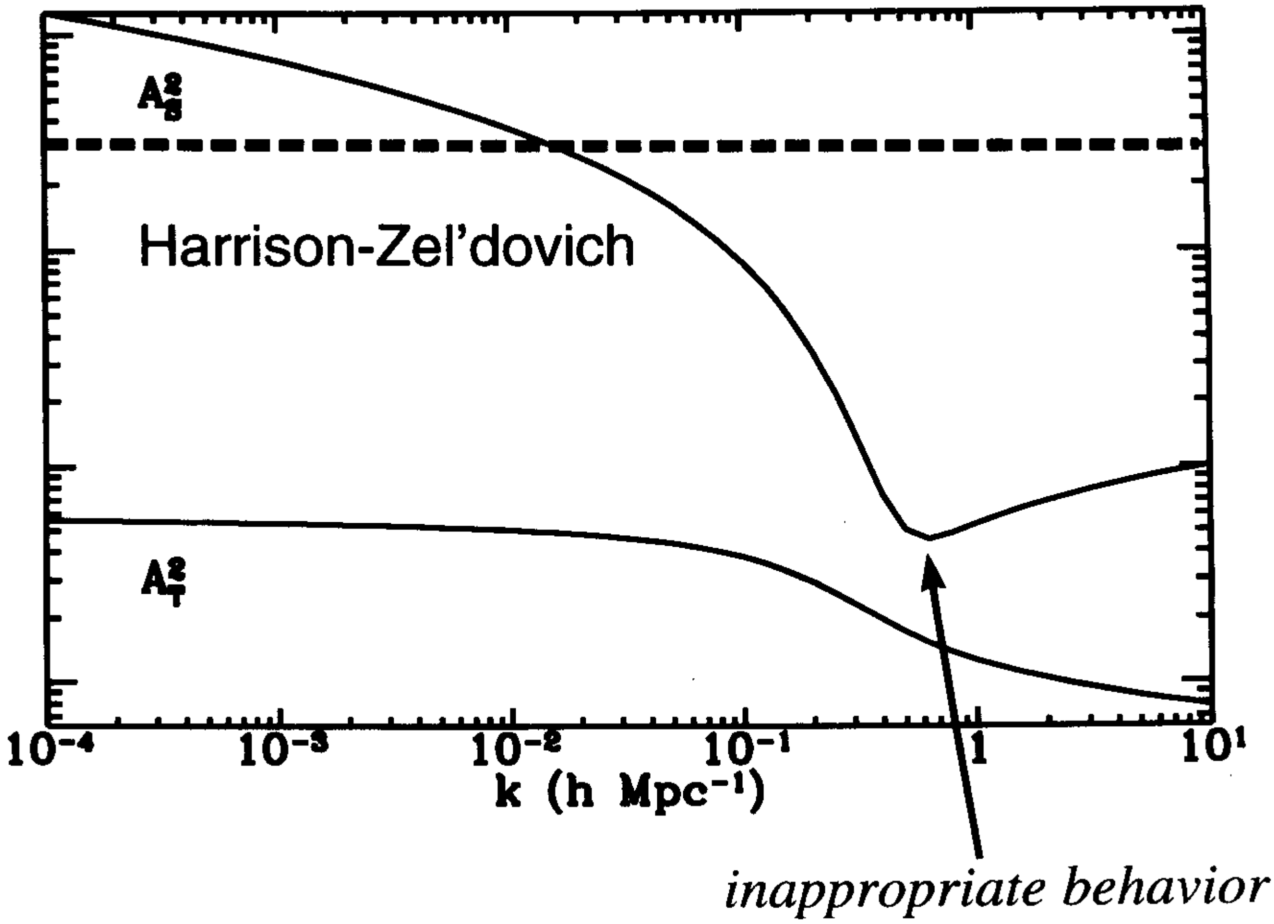
	lowest-order	next-order	next-to-next-order
V	A_T^2	A_T^2, A_S^2	A_T^2, A_S^2, n
V'	A_T^2, A_S^2	A_T^2, A_S^2, n	A_T^2, A_S^2, n, $dn/d\ln k$
V''	A_T^2, A_S^2, n	A_T^2, A_S^2, n, $dn/d\ln k$	
V'''	A_T^2, A_S^2, n, $dn/d\ln k$		

The potential in terms of observables.

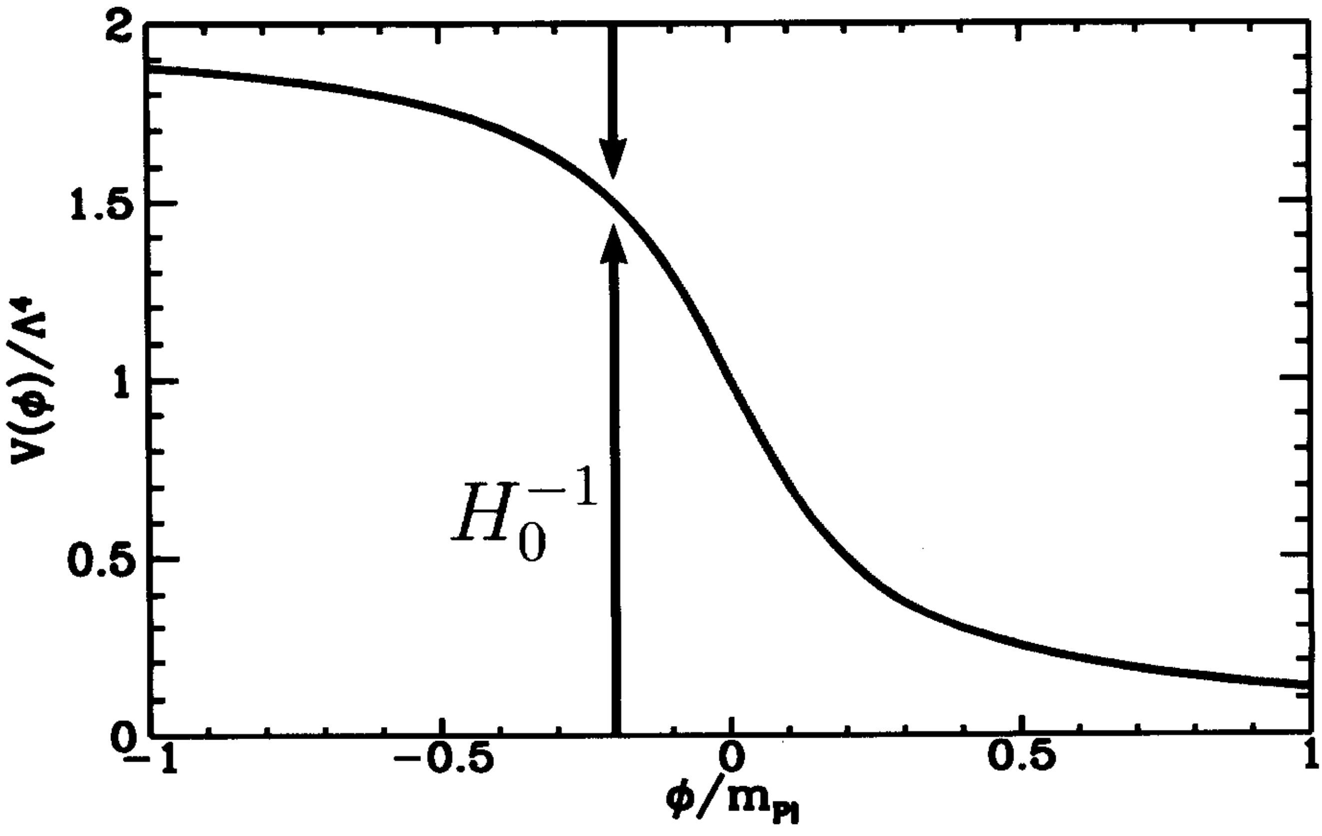
$\{Q_{RMS\text{-}PS}\,;\ r_*\}$

$+\,n_*$

	lowest-order	next-order	next-to-next-order
$V(\phi_*)$	✓	✓	✓
$V'(\phi_*)$	✓	✓	
$V''(\phi_*)$	✓		
$V'''(\phi_*)$			


A_S^2
Harrison-Zel'dovich
A_T^2
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
10^1
k (h Mpc^{-1})
inappropriate behavior

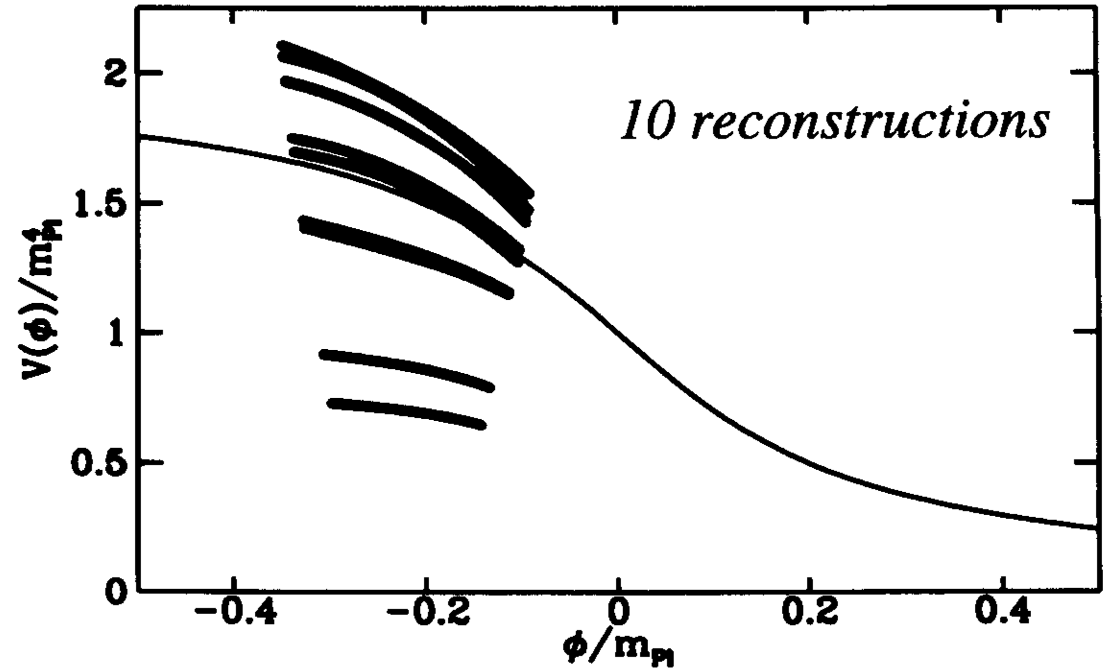
$$V(\phi) = \Lambda^4 \left[1 - \frac{2}{\pi} \tan^{-1} \frac{5\phi}{M_{Pl}} \right]$$

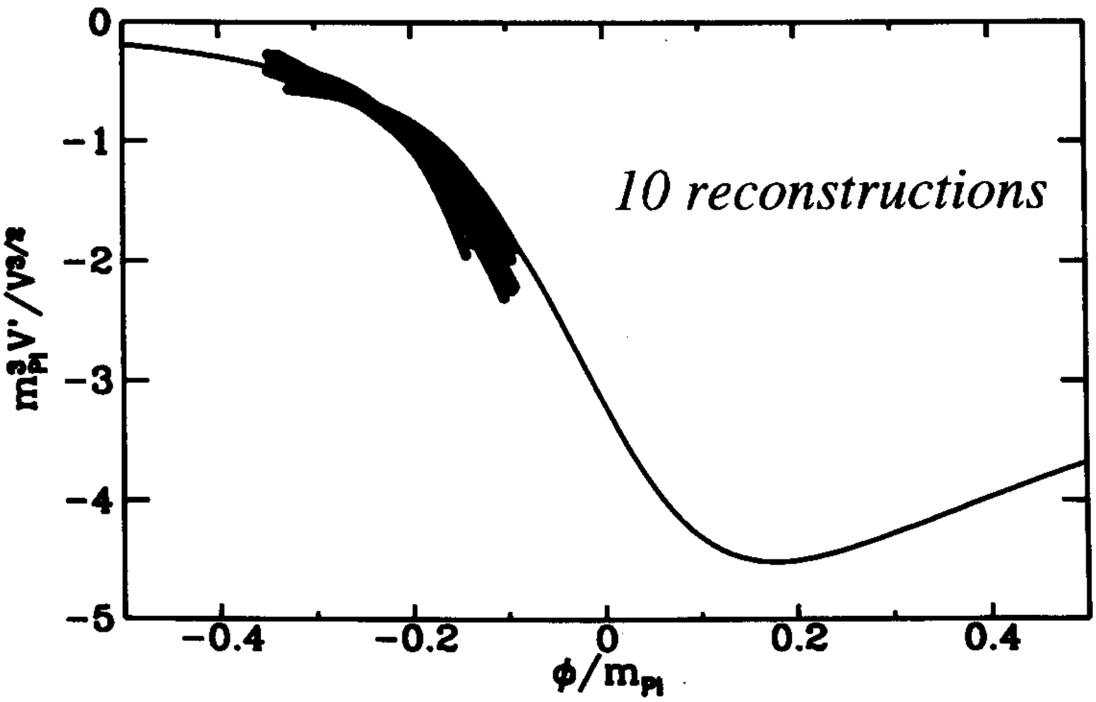


not power law spectrum

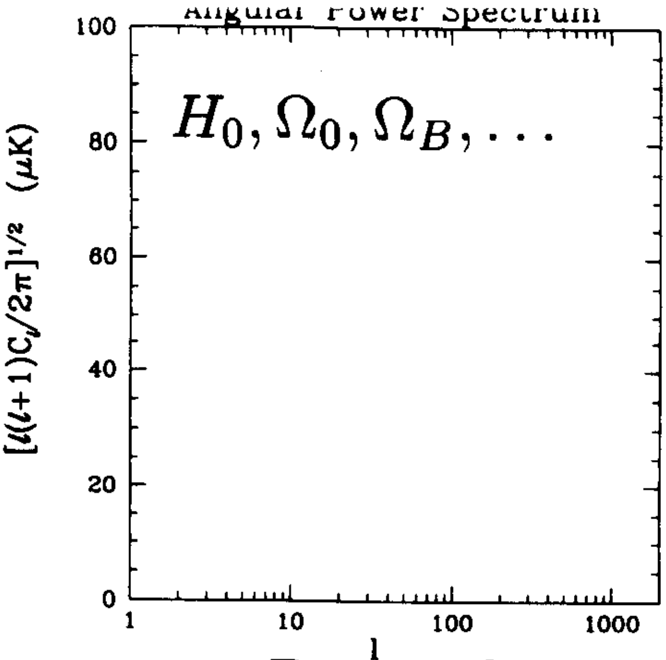
$$V(\phi) = \Lambda^4 \left[1 - \frac{2}{\pi} \tan^{-1} \frac{5\phi}{M_{Pl}}\right]$$

Copeland, Grivell, Kolb, Liddle



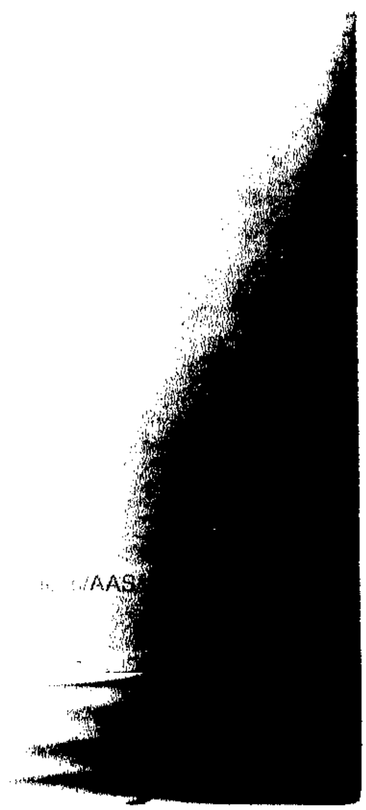


CMB *in the post-peak period search for*



Tensor Perturbations

- *determine expansion rate during inflation!*
- *discover gravitons!*
- *sort through models*



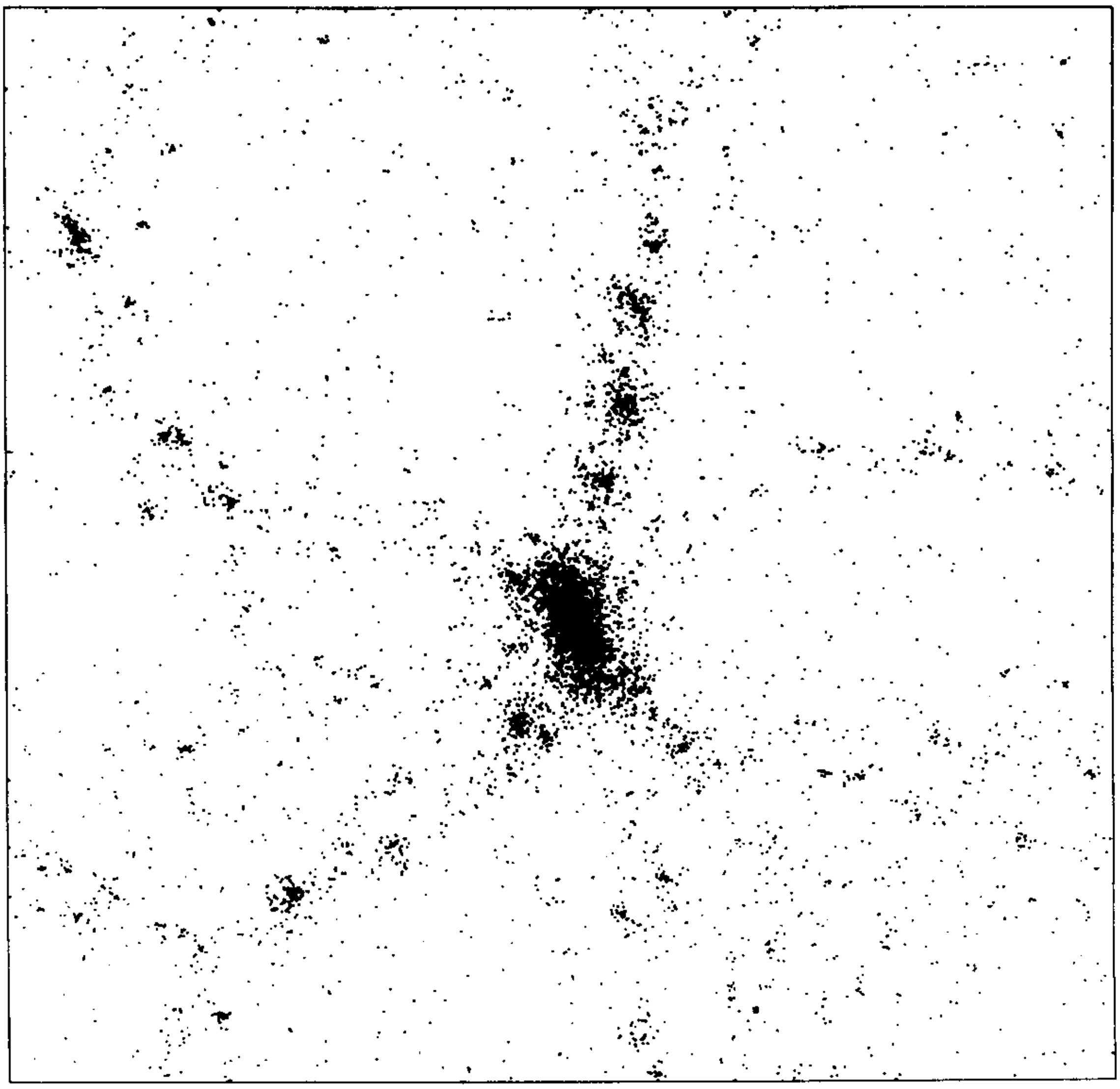
Some Conclusions:

1. If primordial perturbations consistent with
   - a) Harrison-Zel'dovich
   - b) No tensors
   - c) No features

   it will be very difficult to learn anything about the dynamics of inflation.
2. Even small departures from H-Z very interesting.
3. Curvature of scalar spectrum even more interesting.
4. Bumps in the primordial spectrum way interesting.
5. Wimpzillas could be a signal of inflation scale.
6. High priority should be given to gravity waves (polarization).

If you can look into the seeds of time
And say which grain will grow and which will not,
Speak then to me, who neither beg nor fear
Your favours nor your hate.

--- *MACBETH (Banquo)*



Late-night worries of a *Particle Cosmologist*

- ***Complete list of all known fundamental scalar fields:***

- ***Inflation is a theory in search of a model***