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SUMMER SCHOOL ON ASTROPARTICLE PHYSICS AND COSMOLOGY

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TOPOLOGICAL DEFECTS

Part I

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1. Introduction:

(i) FRW cosmology - homogeneous, isotropic.

$$ds^2 = a^2(\tau)\left(d\tau^2 - dl^2\right) \qquad \tau = \text{conformal time}$$

$$= dt^2 - a^2(t)\,dl^2 \qquad t = \text{cosmic time}.$$

$$dl^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

$k = 0, \pm 1$ $\rightarrow$ flat, closed ($k=+1$) and open universe.

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho(t). \qquad \text{Einstein eq.}$$

$$\dot\rho + 3H(\rho + p) = 0 \qquad \text{conservation of energy-momentum}$$

$$p = p(\rho) \qquad \text{eq. of state.}$$

$p = \frac{1}{3}\rho$ radiation; $p = 0$ matter; $p = -\rho$ cosmological constant

Thermodynamics $\longrightarrow$ $T = \dfrac{\text{const.}}{a(t)}$.

$t_0 \simeq 10^{17}\,\text{s}$, $T_0 = 2.7°\text{K}$

(ii) Cosmological phase transitions:

$$L = L_B + L_F$$

$$L_B = \frac{1}{2} D_\mu \Phi_i D^\mu \Phi_i - V(\Phi) - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

$$L_F = i\bar{\Psi} \gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma_i \Psi \Phi_i$$

$$D_\mu \equiv \partial_\mu - ie A^a_\mu T^a \qquad \text{acting on fundamental rep.}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e f^{abc} A^b_\mu A^c_\nu$$

If $\langle \Phi \rangle = \Phi_0$, mass matrices are:

$$\mu^2_{ij} = \left. \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \right|_{\Phi_0} \qquad \text{scalar}$$

$$m = \Gamma_i \Phi_i \big|_{\Phi_0} \qquad \text{spinor}$$

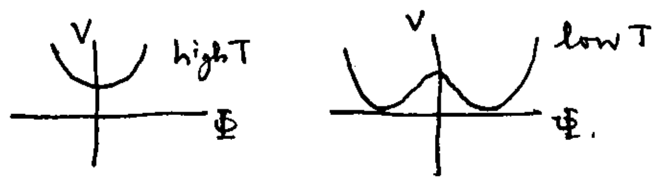
$$M^2_{ab} = e^2 (T_a T_b)_{ij} \Phi_i \Phi_j \qquad \text{vector}$$

$$V_{eff}(\Phi, T) = V(\Phi) + \frac{\mathcal{M}^2}{24} T^2 - \frac{\pi^2}{90} \mathcal{N} T^4.$$

$$\mathcal{N} = \mathcal{N}_B + \frac{7}{8} \mathcal{N}_F \quad = \# \text{ of degs. of freedom}$$

$$\mathcal{M}^2 = \mathrm{Tr}\,\mu^2 + 3\,\mathrm{Tr}\,M^2 + \frac{1}{2} \mathrm{Tr}(\gamma^0 m \gamma^0 m)$$

– depends on Φ_0



(iii) Vaccuum manifold:

Symmetry group G.

Then: $L(\Phi, A_\mu, \Psi) = L(D(g).\Phi, D(g)A_\mu, D(g)\Psi)$ + total derivs.

for $g \in G$.

$\therefore$ if $\langle \Phi \rangle = \Phi_0$, ~~so~~ then $D(g)\Phi_0$ is also valid VEV.

However: $D(h)\Phi_0 = \Phi_0$ for $h \in H$

— this is the defining eq. for the unbroken symmetry group H.

"Vacuum manifold" = manifold of Φ at minimum (global) of ~~[illegible]~~ $V(\Phi)$.

Since $D(gh)\Phi_0 = D(g)\Phi_0$, ~~every point~~ distinct points on the vacuum manifold are labeled by ~~points~~ elements of $G/H = \{gH\}$.

$\therefore$ vacuum manifold $= G/H$.

Note: Vacuum manifold determined by symmetries but not by whether they are local or global.

2. Domain Walls:

(i) Topology: π_0.

If vacuum manifold contains disconnected component, domain walls can be formed. eg. Z_2.

Topology characterized by $\pi_n(G/H)$. This is the group formed by S^n (n-spheres) in G/H. Each element of the group consists of those S^n that can be deformed into one another. Group multiplication involves "joining" S^n's.

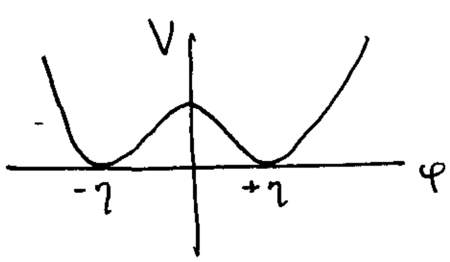
$n = 0 \Rightarrow \pi_0(G/H)$ = group formed by points on G/H that cannot be deformed into one another

= {group of discrete elements}.

(ii) Examples:

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda}{4}(\varphi^2 - \eta^2)^2. \qquad \boxed{\lambda\eta^2 \equiv m^2}$$

$G = Z_2 : \varphi \to -\varphi.$; $H = \mathbb{1}$



$\{+\eta, -\eta\}$. = φ @ $V(\varphi) = \min$

$\pi_0(G/H) = Z_2$.

Domain wall solution:

$$E = \int dx \left[\tfrac{1}{2}(\partial_t\varphi)^2 + \tfrac{1}{2}(\partial_x\varphi)^2 + V(\varphi)\right]$$

$$= \int dx \left[\tfrac{1}{2}(\partial_t\varphi)^2 + \left(\tfrac{1}{\sqrt{2}}\partial_x\varphi + \sqrt{V(\varphi)}\right)^2 - \tfrac{2}{\sqrt{2}}\sqrt{V}\,\partial_x\varphi\right]$$

$$= \int dx \left[\tfrac{1}{2}(\partial_t\varphi)^2 + \left(\tfrac{1}{\sqrt{2}}\partial_x\varphi + \sqrt{V}\right)^2\right] - \sqrt{2}\int_{\varphi=\varphi(-\infty)}^{\varphi(+\infty)} d\varphi\,\sqrt{V(\varphi)}$$

$\therefore\ E = E_{min}$ if $\partial_t\varphi = 0$

$$\partial_x\varphi + \sqrt{2V} = 0.$$

And then:

$$E_{min} = -\sqrt{2}\int_{\varphi(-\infty)}^{\varphi(+\infty)} d\varphi\,\sqrt{V}$$

$$= -\sqrt{2}\,\frac{\sqrt{\lambda}}{2}\int_{-\eta}^{+\eta} d\varphi\,(\varphi^2-\eta^2)$$

$$= -\sqrt{\frac{\lambda}{2}}\cdot 2\left[\frac{\varphi^3}{3} - \eta^2\varphi\right]_0^{\eta}$$

$$= \sqrt{\frac{\lambda}{2}}\cdot 2\cdot\frac{2}{3}\eta^3 = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\,\eta^3. = \frac{2\sqrt{2}}{3}\frac{m^3}{\sqrt{\lambda}}$$

$$\partial_x\varphi = -\sqrt{\frac{\lambda}{2}}(\varphi^2-\eta^2). \qquad \text{— Bogomolnyi eq.}$$

$$\varphi = \eta\tanh(\sigma x) \Rightarrow \eta\sigma\,\mathrm{Sech}^2(\sigma x) = -\sqrt{\frac{\lambda}{2}}\,\eta^2\,\mathrm{Sech}^2(\sigma x).$$

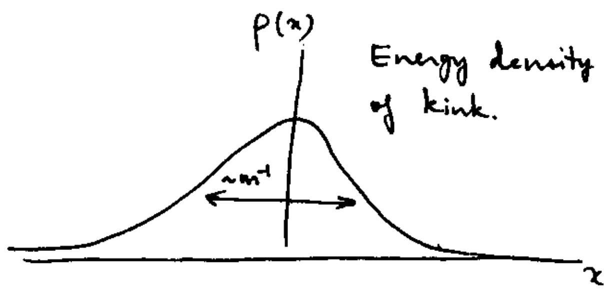
$$\Rightarrow \sigma = \eta\sqrt{\frac{\lambda}{2}} = \frac{m}{\sqrt{2}} \qquad \therefore\ \varphi = \eta\tanh\left(\frac{m}{\sqrt{2}}x\right)$$

In 3-D, same analysis, same solution.

But ∞ energy since $E_{min} = \int dy\,dz \left(\frac{2\sqrt{2}}{3} \frac{m^3}{\sqrt{\lambda}} \right)$

This is ok, since soln. is ∞ in extent.

$\therefore$ should consider energy per unit area (i.e. wall tension): $\sigma = \frac{2\sqrt{2}}{3} \frac{m^3}{\sqrt{\lambda}}$.



Note: $\varphi(x=0) = 0 \Rightarrow Z_2$ is restored inside kink.

(iii) SU(5).

$$L = \mathrm{Tr}(D_\mu \Phi)^2 - \frac{1}{2} \mathrm{Tr}(X_{\mu\nu} X^{\mu\nu}) - V(\Phi).$$

$$V(\Phi) = -m^2 \mathrm{Tr}(\Phi^2) + h\left[\mathrm{Tr}(\Phi^2)\right]^2 + \lambda \mathrm{Tr}(\Phi^4) + \gamma \mathrm{Tr}(\Phi^3) - V_0.$$

Φ (= 5x5 matrix) is SU(5) adjoint. ; $\Phi = \Phi^a T^a$

Note: $\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$.

$$\Phi_0 = \frac{\eta}{2\sqrt{15}} \operatorname{diag}(2,2,2,-3,-3)$$

$$\eta = \frac{m}{\sqrt{\lambda'}} \quad , \quad \lambda' \equiv h + \frac{7}{30}\lambda.$$

$V(\Phi_0)$ = minimum if $\lambda \geq 0$, $\lambda' \geq 0$.

Set $V(\Phi_0) = 0$ by choosing $V_0 = -\frac{\lambda'}{4}\eta^4$.

Note: $\Phi \longrightarrow -\Phi$ is not ~~an~~ an $SU(5)$ transformation.

If γ is small, L has approximate Z_2 symmetry $\Phi \to -\Phi$.

$\therefore$ consider $\gamma = 0$. $G = SU(5) \times Z_2$.

Then we get "domain walls".

What is ~~this~~ ~~so~~ the domain wall solution?

First guess: $\Phi_k = \tanh(\sigma z)\,\Phi_0$ (kink).

$(\sigma = m/\sqrt{2}.)$

Check: solves eq. of motion, satisfies correct boundary conditions.

But – ~~stability~~ ~~needs~~ solution guarantees extremum of energy, not minimum.

$\therefore$ perturb solution $\Phi = \Phi_k + \Psi$ $\quad \Psi = \psi^a T^a$.

Consider: $\Psi = 4 \cdot \frac{1}{2} \begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}.$

Linearized eq. for $e^{i\omega t}\psi(z)$.

$$\left[-\partial_z^2 + m^2\{\tanh^2(\sigma z) - 1\}\right]\psi = \omega^2 \psi.$$

with $\psi(\pm\infty) = 0$.

Solution: $\psi(z) = \mathrm{sech}(\sigma z)$ with $\omega^2 = -m^2/2$.

$\Rightarrow$ kink is unstable.

Then what is the stable domain wall solution?

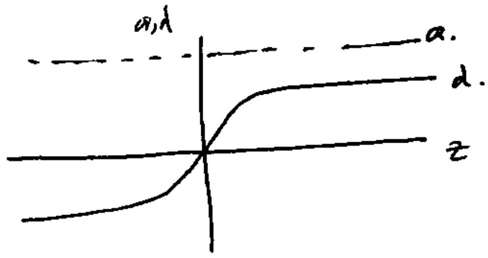
$$\Phi(-\infty) = \frac{\eta}{2\sqrt{15}}\,\mathrm{diag}(3,-2,-2,3,-2) = \eta\sqrt{\frac{5}{12}}(\lambda_3+\tau_3) + \frac{\eta}{6}(Y-\sqrt{5}\lambda_8)$$

$$\Phi(+\infty) = \frac{\eta}{2\sqrt{15}}\,\mathrm{diag}(2,-3,2,2,-3) = \eta\sqrt{\frac{5}{12}}(\lambda_3+\tau_3) - \frac{\eta}{6}(Y-\sqrt{5}\lambda_8)$$

$$\Phi_{DW}(z) = a(z)(\lambda_3+\tau_3) + d(z)(Y-\sqrt{5}\lambda_8).$$

with $a(z) \simeq \eta\sqrt{\frac{5}{12}}$

$d(z) = \frac{\eta}{6}\tanh\left(\sqrt{\frac{p}{2}}\,mz\right)$ $\qquad p = \frac{1}{6}\left[1 + \frac{5\lambda}{12\lambda'}\right].$



$$\Phi_{DW}(0) \simeq \eta\sqrt{\frac{5}{12}}\cdot\frac{1}{2}(1,-1,0,1,-1).$$

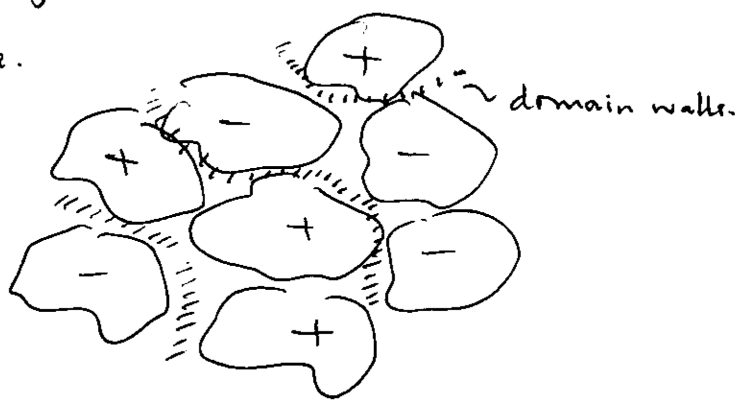
$\therefore$ symmetry inside wall = $[SU(2)\times U(1)] \times [SU(2)\times U(1)]$

— smaller than $SU(3)\times SU(2)\times U(1)$.

(IV) Formation:

Φ_0 acquires random values on the vacuum manifold in regions of space separated by ξ = correlation length at phase transition. (Determination of ξ being experimentally tested. We only require $\xi \lesssim t \simeq$ horizon size.) which is guaranteed by causality.)

Z_2 case.



Domain wall network?

Cluster size:	1	2	3	4	6	10	31082
Number:	462	84	14	13	1	1	1.

$\Rightarrow$ one infinite domain wall.

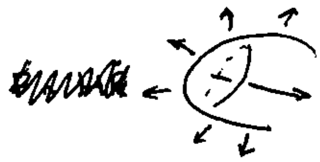
(V) Evolution

Factors –

a) Tension:

$\sigma(\ddot{R} - R'') \sim 0$

b) Hubble expansion:

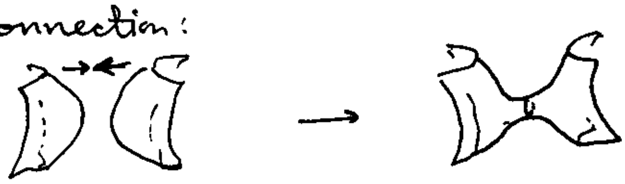


$\ddot{z} + 3\frac{\alpha}{\tau}\dot{z} - z''_{xx} - z''_{yy} = 0.$

τ = conformal time, $a \sim \tau^{\alpha}$.

c). Friction

d) Reconnection:



Network evolution quite complicated.

However for cosmology, only need causal constraint on evolution i.e. at least one domain wall per horizon.

$$\therefore \quad \rho_{DW} \gtrsim \frac{\text{energy in 1 DW in horizon}}{\text{horizon volume}} \simeq \frac{\sigma t^2}{t^3} = \frac{\sigma}{t}.$$

$$\rho_{cr} = \frac{3H^2}{8\pi G} \qquad H \simeq 70\ \text{km/s/Mpc}.$$

$\left.\frac{\rho_{DW}}{\rho_{cr}}\right|_{eq.} \lesssim 10^{-5} = \frac{\delta T}{T}$

$$\Longrightarrow \quad \sigma < \frac{3H^2 t_0}{8\pi G} \qquad t_0 \sim 10^{17}\,\text{s}.$$

$$\Longrightarrow \quad \eta \lesssim 1\ \text{MeV}.$$

$\therefore$ particle physics models that predict domains at $\eta >$ 1MeV are ruled out.

(vi) Complexities:

(a) Inflation.

(b) ~~Eg~~ Discrete symmetry restoration at low temperatures

(c) Bias. eg SU(5) with small γ.