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Classical Modular Forms

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1 Introduction

The simplest kind of automorphic forms (apart from Grossencharacters, which will also be discussed in this conference) are the "elliptic modular forms". We will study modular forms and their connection with automorphic forms on GL(2) , in the sense of representation theory.

Modular forms arise in many contexts in number theory , e.g. in questions involving representations of integers by quadratic forms, and in expressing elliptic curves over $\mathbb{Q}$ as quotients of Jacobians of modular curves , etc.

The simplest modular forms are those on the modular group $SL(2,\mathbb{Z})$ We will first define modular forms on $SL(2,Z)$. In section 2, we will describe a fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper half plane $\mathfrak{h}$. In section 3, we will define modular forms for $SL(2,\mathbb{Z})$ and construct some modular forms. In section 4, a representation theoretic interpretation of modular forms will be given, which will enable us to think of them as automorphic forms on $GL(2,\mathbb{R})$. In section 5, we will give an adelic version of modular forms, define Hecke operators and show the commutativity of the Hecke operators.

2 A Fundamental domain for $SL(2,\mathbb{Z})$

Notation 2.1 Denote by $\mathfrak{h}$ the "Poincare' upper half-plane" i.e. the space of complex numbers whose imaginary part is positive :

$$\mathfrak{h} = \{z \in \mathbb{C}; z = x + iy,\ x,\ y \in \mathbb{R},\ y > 0\}.$$

If $z \in \mathbb{C}$, denote respectively by $Re(z)$ and $\mathrm{Im}(z)$ the real and imaginary parts of z.

On the upper half plane $\mathfrak{h}$, the group $GL(2,\mathbb{R})^+$ of real 2×2 matrices with *positive* determinant operates as follows: let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})^+$, and let $z \in \mathfrak{h}$. Set $g(z) = (az+b)/(cz+d)$. Notice that if $cz + d = 0$ and $c \neq 0$ then, $z = -d/c$ is real, which is impossible since z has positive imaginary part. Thus, the formula for $g(z)$ makes sense. Observe that

$$\mathrm{Im}(g(z)) = \mathrm{Im}(z)(det(g))/ \mid cz + d \mid^2 . \tag{1}$$

The equation (1) shows that the map $(g,z) \mapsto g(z)$ takes $GL(2,\mathbb{R})^+ \times \mathfrak{h}$ into $\mathfrak{h}$. One checks immediately that this map gives an action of $GL(2,\mathbb{R})^+$ on the upper half plane $\mathfrak{h}$. Note also that

$$\mid cz + d \mid^2 = c^2y^2 + (cx+d)^2. \tag{2}$$

Therefore, $\mid cz + d \mid^2 \geq y^2$ or 1 according as $\mid c \mid = 0$ or nonzero. Therefore,

$$\mathrm{Im}(\gamma(z)) \leq y/\min\{1, y^2\} \quad \forall \gamma \in \Gamma_0 \subset SL(2,\mathbb{Z}), \tag{3}$$

where $\min\{1,y^2\}$ denotes the minimum of 1 and y^2 and $\Gamma_0 \subset SL(2,Z)$ is the group generated by the elements $T = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and $S = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Later we will see that Γ_0 is actually $SL(2,Z)$. The element T acts on the upper half plane $\mathfrak{h}$ by translation by 1:

$$T(z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(z) = z + 1 \quad \forall z \in \mathfrak{h}. \tag{4}$$

Similarly, the element S acts by inversion :

$$S(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(z) = -1/z \quad \forall z \in \mathfrak{h}. \tag{5}$$

Consider the set (see diagram 1)

$$F = \{z \in \mathfrak{h}; -1/2 < Re(z) \leq 1/2, \quad \mid z \mid \geq 1, \text{ and } 0 \leq Re(x)\} \text{ if } \mid z \mid = 1.$$

Theorem 2.2 *Given $z \in \mathfrak{h}$ there is a* **unique** *point $z_0 \in F$ and an element $\gamma \in SL(2,\mathbb{Z})$ such that $\gamma(z) = z_0$. Moreover, given $\gamma \in SL(2,\mathbb{Z})$, we have $\gamma(F) \cap F = \phi$ unless γ lies in a finite set (of elements of $SL(2,\mathbb{Z})$ which fix the point $\omega = 1/2 + i3^{1/2}/2 \in \mathfrak{h}$ or $i \in \mathfrak{h}$). [one then says that F is a* **fundamental domain** *for the action of $SL(2,\mathbb{Z})$ on the upper half plane $\mathfrak{h}$].*

Proof : We will first show that any point z on the upper half plane can be translated by an element of the subgroup Γ_0 of $SL(2,\mathbb{Z})$ (generated by $T(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and $S = (\begin{smallmatrix} & 0 & 1 \\ 0 & & 1 \end{smallmatrix})$) into a point in the "fundamental domain" F.

Now, given a real number x, there exists an integer k such that $-1/2 < x + k \leq 1/2$. Therefore, the equation (4) shows that given $z \in \mathfrak{h}$ there exists an integer k such that the real part x' of $T^k(z)$ satisfies the inequalities $-1/2 < x' \leq 1/2$.

Let y denote the imaginary part of z and denote by S_z the set

$$S_z = \{\gamma(z); \gamma \in \Gamma_0,\ \mathrm{Im}(\gamma(z)) \geq y\ , -1/2 < \Re(\gamma(z)) \leq 1/2\}\ .$$

We will first show that S_z is nonempty and finite. Let k be as in the previous paragraph. Then $-1/2 < Re(T^k(z)) \leq 1/2$ and $\mathrm{Im}(T^k(z) = \mathrm{Im}(z)$; therefore, $T^k(z)$ lies in S_z and S_z is nonempty.

Now, the equation (3) shows that the imaginary parts of elements of the set S_z are all bounded from above by $y/\min 1, y^2$. By definition, the imaginary parts of points on S_z are bounded from below by y . The definition of S_z shows that S_z is a relatively compact subset of $\mathfrak{h}$. We get from (3) that $\mid cz + d \mid^2 \leq 1$; now (2) shows that $\mid c \mid \leq 1/y^2$. Suppose $\gamma \in \gamma_0 = \begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ is such that $\gamma(z) \in S_z$ then, c is bounded by $1/y^2$ and is in a finite set. The fact that $cz + d$ is bounded now shows that d also lies in a finite set. Since S_z is relatively compact in $\mathfrak{h}$, it follows that $\gamma(z) = (az + b)/(cz + d)$ is bounded for all $\gamma(z) \in S_z$; therefore, $az + b$ is bounded as well, and hence a and b run through a finite set. We have therefore proved that S_z is finite.

Let y_0 be the supremum of the imaginary parts of the elements of the finite set S_z; let $S_1 = \{z' \in S_z; Im(z') = y_0\}$ and let $z_0 \in S_1$ be an element whose real part is maximal among elements of S_1. We claim that $z_0 \in F$. First observe that if $z' \in S_z$ then $S(z') = -1/z'$ has imaginary part $y_0/\mid z \mid^2 = Im(z')/\mid z' \mid^2 \leq y_0$ whence $\mid z' \mid^2 \geq 1$. If $\mid z_0 \mid > 1$, then it is immediate from the definitions of F and S_z that $z_0 \in F$. Suppose that $\mid z_0 \mid = 1$. Then, $S(z_0) = -1/z_0$ also has absolute value 1 , its imaginary part is y_0 and its real part is the negative of $Re(z_0)$; hence $S(z_0) \in S_1$. The maximality of the real part of z_0 among elements of S_1 now implies that $Re(z_0) \geq 0$. Therefore,

$z_0 \in F$. We have proved that every element z_0 may be translated by an element of Γ_0 into a point in the fundamental domain F.

Suppose now that $z \in \gamma^{-1}(F) \cap F$ for some $\gamma \in SL(2,\mathbb{Z})$. Write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Suppose that $Im(\gamma(z)) \geq Im(z) = y$ (otherwise, replace z by $\gamma(z)$). Then, by (3) one gets

$$(cx + d)^2 + c^2y^2 \leq 1. \tag{6}$$

Since $z \in F$, we have $x^2 + y^2 \geq 1$ and $0 \leq x \leq 1/2$. therefore $y^2 \geq 3/4$ and

$$(1 \geq) c^2y^2 \geq c^2 4/3. \tag{7}$$

This shows that $c^2 \leq 1$ since c is an integer.

Suppose $c = 0$. Then, $ad = 1$, $a, d \in Z$ and we may assume (by multiplying by the matrix $-Id$ [minus identity] if necessary) that $d = 1$. Hence $\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. Then, $\gamma(z) = z + b \in D$ which means that $0 \leq x + b \leq 1/2$ and $0 \leq x \leq 1/2$. Thus, $-1/2 \leq b \leq 1/2$, i.e. $b = 0$ and γ is the identity matrix.

The other possibility is $c^2 = 1$, and by multiplying by the matrix $-Id$ (minus identity) we may assume that $c = 1$. Suppose first that $d = 0$. Then , $bc = -1$ whence $b = -1$. Now, (7) shows that $x^2 + y^2 \leq 1$. Moreover,

$$\gamma(z) = az + b/z = a + b\overline{z}/ \mid z \mid^2 = a - \overline{z}$$

whence its real part is $a - x$ which lies between 0 and 1/2. Since $0 \leq x \leq 1$ it follows that $0 \leq a \leq 1$. If $a = 0$ then $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and lies in the isotropy of the point $i \in \mathfrak{h}$. If $a = 1$ then, $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ which lies in the isotropy of the point $\omega = 1/2 + i3^{1/2}/2$.

We now examine the remaining case of $c = 1$ and $d \neq 0$. From (6) we get $(x + d)^2 + y^2 \leq 1$. If $d \geq 1$ then the inequality $0 \leq x \leq 1/2$ shows that $1 \leq d \leq x + d$ which contradicts the inequality $(x + d)^2 + y^2 \leq 1$, which is impossible. Thus, $d \leq -1$; then the inequality $0 \leq x \leq 1/2$ implies that $x + d \leq 1/2 + (-1) = -1/2$ whence $(x + d)^2 \geq 1/4$. Since $y^2 \geq 3/4$ the inequality $(x+d)^2 + y^2 \leq 1$ implies that equalities hold everywhere: $y^2 = 3/4$, $x = 1/2$ and $d = -1$. Thus, $z = \omega$ and $z - 1 = z^2$. Since $1 = ad - bc = -a - b$ ($d = -1$ and $c = 1$), and

$$\gamma(z) = (az+b)/(z-1) = (az+b)/z^2 = -(az+b)z = a + (-a-b)z = a + z \in D,$$

the real part of $\gamma(z)$ is $a + x = a + 1/2$ and is between 0 and 1/2, i.e. $-1/2 \leq a \leq 0$ i.e. $a = 0$ and $b = -1$. Therefore, $\gamma = \begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}$ lies in the isotropy of ω. This completes the proof of Theorem (2.2).

Corollary 2.3 *The group $SL(2,\mathbb{Z})$ is generated by the matrices $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$.*

Proof In the proof of Theorem (2.2), a point on the upper half plane is brought into the fundamental domain F by applying only the transformations generated by S and T. The fact that the points oin the fundamental domain are inequivalent under the action of $SL(2,Z)$ now implies that $SL(2,Z)$ is generated by S and T .

(the Corollary can also be proved directly by observing that $ST^{-1}S^{-1} = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$. Now, the usual row-column reduction of matrices with integral entries implies that T and STS^{-1} generates $SL(2,Z)$).

Notation 2.4 Elliptic Functions. We recall briefly some facts on elliptic functions (for a reference to this subsection, see Ahlfors' book on Complex Analysis). Given a point τ on the upper half plane $\mathfrak{h}$, the space $\Gamma_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$ of integral linear combinations of 1 and τ forms a discrete subgroup of $\mathbb{C}$ with compact quotient. The quotient

$$E_\tau = \mathbb{C}/\Gamma_\tau$$

may be realised as the curve in $\mathbf{P}^2(\mathbb{C})$ whose intersection with the complement of the plane at infinity is given by

$$y^2 = 4x^3 - g_2 x - g_3 \tag{8}$$

The curve $E_\tau = \mathbb{C}/\Gamma_\tau$ is called an "elliptic curve".

The map of $\mathbb{C}/\Gamma_\tau$ to $\mathbf{P}^2$ is given by $z \mapsto (\wp'(z), \wp(z), 1)$ for $z \in \mathbb{C}$. Recall the definition of $\wp$: if $z \in \mathbb{C}$ and does not lie in the lattice Γ_τ, then write

$$\wp(z) = 1/z^2 + {\sum}'(1/(z+w)^2 - 1/w^2),$$

where $\sum'$ is the sum over all the **non-zero** points w in the lattice Γ_τ. The derivative $\wp'(z)$ of $\wp(z)$ is then given by

$$\wp'(z) = \sum 1/(z+w)^3,$$

where the sum is over all the points of the lattice Γ_τ. One has the equation (cf. equation (8))

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau). \tag{9}$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\tau \in SL(2,Z)$, then the elliptic curve $E_{\gamma(\tau)}$ is isomorphic as an algebraic group (which is also a projective variety) to the elliptic curve E_τ. The explicit isomorphism on $\mathbb{C}$ is given by $z \mapsto z/(c\tau+d)$. It is also possible to show that if E_τ and $E_{\tau'}$ are isomorphic elliptic curves, then τ' is a translate of τ by an element of $SL(2,\mathbb{Z})$.

Thus the fundamental domain F which was constructed in Theorem (2.2) parametrises isomorphism classes of elliptic curves .

In equation (9) , recall that the coefficients g_2 and g_3 are given by

$$g_2(\tau) = 60G_4(\tau) = 60\sum{}'(m\tau+n)^{-4}$$

and

$$g_3(\tau) = 140G_6(\tau) = 140\sum{}'(m\tau+n)^{-6}$$

where $\sum'$ is the sum over all the pairs of integers (m,n) such that not both m and n are zero. The discriminant of the cubic equation in (9) is given by $1/(16)\Delta(\tau)$ where

$$\Delta(\tau) = g_2^3 - 27g_3^2. \tag{10}$$

It is well known and easily proved that $\wp'(z)$ has a simple zero at all the **2-division points** $1/2$,$\tau/2$ and $(1+\tau)/2$ and that $\wp(1/2)$,$\wp(\tau/2)$ and $\wp((1+\tau)/2)$ are all distinct. Thus the discriminant of the cubic in equation (9) is non-zero and so we obtain that

$$\Delta(\tau) \neq 0 \tag{11}$$

for all $\tau \in \mathfrak{h}$.

3 Modular forms; Definition and examples

Notation 3.1 Given $z \in \mathfrak{h}$ ($\mathfrak{h}$ is the upper half plane) and an element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, write

$$j(g, z) = cz + d.$$

Note that if $j(g, z) = 0$, then by comparing the real part and imaginary parts we get $c = 0$ and $d = 0$ which is impossible since $ad - bc \neq 0$. Thus, $j(g, z)$ is never zero.

Definition 3.2 A function $f : \mathfrak{h} \to \mathbb{C}$ is **weakly modular of weight** w if the following two conditions hold.

(1) f is holomorphic on the upper half plane.

(2) for all $\gamma \in SL(2, \mathbb{Z})$, with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have the equation

$$f((az + b)/(cz + d)) = (cz + d)^w f(z). \tag{12}$$

Given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and a function f on the upper half plane $\mathfrak{h}$, define

$$g^{-1} * f(z) = (cz + d)^{-w} f(g(z)) \quad \forall z \in \mathfrak{h}.$$

Then, it is easily checked that the map $(g, f) \to g^{-1} * f$ defines an action of $GL(2, \mathbb{R})$ on the space of functions on $\mathfrak{h}$. Thus, the condition (2) above is that the function f there is invariant under this action by $SL(2, \mathbb{Z})$. Now by Corollary (2.3) , $SL(2, \mathbb{Z})$ is generated by the matrices $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus condition (2) is equivalent to saying that $\gamma^{-1} * f = f$ for $\gamma = S, T$. This amounts to saying that

$$f(-1/z) = z^w f(z) \tag{13}$$

and

$$f(z + 1) = f(z). \tag{14}$$

Note that the invariance of f under the action of -1 where 1 is the identity matrix in $SL(2, Z)$ implies that f is zero of w is odd: $f(z) = (-1)^w f(z)$. Therefore, we assume from now on (while considering modular forms for the group $SL(2, Z)$) that $w = 2k$ where k is an integer.

Definition 3.3 The map $exp : \mathfrak{h} \to D^*$ given by $z \mapsto e^{2\pi(z)} = q$ is easily seen to be a covering map of the upper half plane $\mathfrak{h}$ onto the set D^* of non-zero complex numbers of modulus less than one. The covering transformations are generated by $T(z) = z + 1$. A weakly modular function f is invariant under T and therefore yields a holomorphic map $f^* : D^* \to \mathbb{C}$ given by $f^*(q) = f(z)$ for all $z \in \mathfrak{h}$. We say that a weakly modular function of weight w is a **modular function of weight** w if f^* extends to a holomorphic function of D (the set of complex numbers of modulus less than one) i.e. f^* extends to $0 \in D$.

Let f be a weakly modular function on $\mathfrak{h}$. Then, f is a modular function if and only if the function f^* has the "Fourier expansion" (or the "**q-expansion**")

$$f^*(q) = \sum_{n \geq 0} a_n q^n, \tag{15}$$

where a_n are complex numbers and the summation is over all non-negative integers n. Observe that a weakly modular function is modular if and only if it is bounded in the fundamental domain F.

We will say that a modular form is a **cusp form** if the constant term of its q-expansion is zero: i.e. $a_0 = 0$ in the notation of equation (15).

Notation 3.4 Examples of modular forms.

First we note that if f and g are modular forms of weights w and w' then, the product function fg is a modular form of weight ww'.

We will first prove that for the modular group $SL(2, \mathbb{Z})$, there are no non-constant "weight zero" modular forms. First note that if f is a weight zero modular form, then the function f^* extends to 0 and hence is bounded in a disc of radius $r < 1$. Its inverse image under $exp : F \to D^*$ is precisely the set $A = \{z = x + iy \in F; y > -logr\}$ and f is bounded on the set A. The complement of the set A in the fundamental domain F is compact, and f is bounded there as well, whence f is bounded on all of the fundamental domain F as well as at "infinity". By the maximum principle, f is constant.

We will now show that there are no modular forms of weight two on $SL(2, \mathbb{Z})$. Suppose f is one and $F(z)$ be its integral from z_0 to z for some fixed $z_0 \in \mathfrak{h}$. The modularity of f shows that $\gamma \mapsto F(\gamma(z_0)$ gives a homomorphism from $SL(2, \mathbb{Z})$ to $\mathbb{C}$. But, $SL(2, \mathbb{Z})$ is generated by the finite order elements S and ST whence, this homomorphism is identically zero. This and the modularity of f shows that the integral F is invariant under $SL(2, \mathbb{Z})$. It is easy to show that F^* is holomorphic at 0 (integrate both sides of equation (15)), and use the invariance of F under T). Hence F is a modular form of weight zero. By the foregoing paragraph, f is a constant , i.e.$f = 0$.

Fix an even positive integer $2k$, with $k \geq 2$. We will construct a modular form of degree k as follows. Let $\tau \in \mathfrak{h}$ and write (compare the definition of g_2 and g_4 in section (2.4))

$$G_{2k}(\tau) = \sum{}' (m\tau + n)^{-2k}, \tag{16}$$

where $\sum'$ is the sum over all the pairs of integers (m, n) not both of which are zero. Then, G_{2k} is easily shown to be a weakly modular function of weight $2k$ on the upper-half plane. If τ is varying in the fundamental domain and its imaginary part tends to infinity, then it is clear from the formula for G_{2k} that $G_{2k}(\tau)$ tends to $\sum' n^{-2k} = 2\zeta(2k)$ where

$$\zeta(s) = \sum n^{-s}$$

is the Riemann zeta function (the sum is over all the positive integers n and in the sum, the real part of s exceeds 1). Consequently, G_{2k} is a modular form of weight $2k$. We will now outline a derivation of the q-expansion of G_{2k}. Start with the partial fraction expansion

$$\pi cot(\pi z) = z^{-1} + \sum (z+n)^{-1} + (z-n)^{-1} \tag{17}$$

where the sum is over all positive integers n. This series converges uniformly on compact subsets of the complement of $\mathbb{Z}$ in $\mathbb{C}$.

Write $q = e^{2\pi i z}$ (where $i \in \mathfrak{h}$ and $i^2 = -1$). Then one has the q-expansion

$$\pi cot(\pi z) = \pi i(q+1)/(q-1) = -\pi i - 2\pi i \sum_{n \geq 1} q^n \tag{18}$$

Differentiate $2k$- times , the right-hand sides of equations (17) and (18) with respect to z. We then get the equality

$$\sum_{n \in \mathbb{Z}} (z+n)^{-2k} = ((2k-1)!)^{-1} (2\pi i)^{2k} \sum_{n \geq 1} n^{2k-1} q^n \tag{19}$$

Fix m and in equation (19) take for z the complex number $m\tau$. Then sum over all m. We obtain by equations (16) and (18), the q-expansion

$$G_{2k}(\tau) = 2\zeta(2k) + ((2k-1)!)^{-1} (2\pi i)^{2k} \sum_{n \geq 1} \sigma_{2k-1} q^n \tag{20}$$

where for an integer r and $n \geq 1$, $\sigma_r(n)$ is defined to be the sum $\sum d^r$ where d runs over the positive divisors of n.

By using the power series expansion

$$(1+x)^{-2} = \sum_{n\geq 1} nx^{n-1}$$

and equation (17) one has the power series identity

$$\pi cot(\pi z) = z^{-1} + 2\sum_{n\geq 1}\sum_{j\geq 1} n^{2j} z^{2j-1} = z^{-1} + 2\sum_{j\geq 1}\zeta(2j)z^{2j-1} \tag{21}$$

By comparing the power series expansions $cos(x) = \sum_{m\geq 0}((2m)!)^{-1}x^{2m}$ and $sin(x) = \sum_{m\geq 0}((2m+1)!)^{-1}x^{2m+1}$ with the right hand side of equation (21) one obtains

$$\zeta(2) = \pi^2/6,\ \zeta(4) = \pi^4/90 \text{ and} \zeta(6) = \pi^6/(3^3.5.7). \tag{22}$$

Using (20) and (22) we get

$$g_2 = 60G_4 = (4/3)\pi^4 + 160\pi^4(q + \cdots) \tag{23}$$

where the expression $q + \cdots$ is a power series in q with *integral* coefficients with the coefficient of q being 1. Similarly, we get (again from (20) and (22))

$$g_3 = 140G_6 = (8/27)\pi^6 - 2^5.7\pi^6/3(q + \cdots) \tag{24}$$

Therefore, we get, after some calculation, that for all $z \in \mathfrak{h}$,

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^2 = 2^{11}\pi^{12}(q + \sum_{n\geq 2}\tau(n)q^n) \tag{25}$$

where $\tau(n)$ are integers. We recall that $\Delta(z)$ is never zero on the upper-half plane (section (2.4)). The equation (26) shows that the coefficient of q in q-expansion of Δ is non-zero, (and that its constant term is zero).

Lemma 3.5 *There are no modular forms of negative degree.*

Proof . Suppose that f is a modular form of degree $-l$ with $l > 0$. Form the product $g = f^{12}\Delta^l$. Since f and Δ are modular forms , so is the product. Since its degree is zero, g is a constant (see the beginning of this subsection). But, (26) shows that the q-expansion of g has no constant term. Hence $g = 0$ whence, $f = 0$.

Lemma 3.6 *Suppose that f is a cusp form. Then Δ divides f i.e., there is a modular form g such that $f = \Delta g$. In particular, the weight of f is at least 12.*

Proof Consider the quotient $g = f/\Delta$. Since Δ has no zero in $\mathfrak{h}$, it follows that g is holomorphic in $\mathfrak{h}$. Clearly, g is weakly modular of weight =weight of f -12. Now the q-expansion of f (and also Δ), has no constant term; and the coefficient of q in the q-expansion of Δ is non-zero. Therefore, g^*extends to a holomorphic function in a neighbourhood of 0. That is, g is a modular function.Since the weight of g is non-negative (by Lemma (3.)), it follows that the weight of f must be at least that of Δ, namely, 12.

Corollary 3.7 *The space of cusp forms of weight 12 (for $SL(2, \mathbb{Z})$,)is one dimensional.*

Proof If f is a cusp form of weight 12, then f/Δ is a modular form of weight zero, hence is a constant. That is, the space of cusp forms of weight 12 is spanned by Δ.

Theorem 3.8 *The space of modular forms of weight $2k$ with $k \geq 0$ is spanned by the modular forms $G_4^m G_6^n$ with $4m + 6n = 2k$.*

Proof Argue by induction on k. We have already excluded the possibilities $k < 0$ and $k = 0$ and $k = 1$.

Suppose that $k \geq 2$ and that f is modular of weight $2k$. First observe that any integer $k \geq 2$ may be written as $2m + 3n$ for non-negative integers m and n. Now, the q-expansion of G_4 and G_6 have non-zero constant term. Hence $h = f - \lambda G_4^m G_6^n)$ for a suitable constant λ, has no constant term in its q-expansion, and is a cusp form. Now, Lemma (3.) shows that $g = h/\Delta$ is a modular form of weight $2k-12$. By induction, g is a linear combination of the modular forms $G_4^a G_6^b$ with $k - 6 = 2a + 3b$ whence, h is a sum of monomials of the form $G_4^p G_6^q$ with $2p + 3q = k$ (recall that Δ is $(60G_4)^3 - 27(140G_6)^2)$. Therefore, so is f.

4 Modular Forms and Representation Theory

Notation 4.1 We will begin with some calculations on the Lie algebra $\mathfrak{g}$ of the group $GL(2,\mathbb{R})$. Write,

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \text{and } H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{26}$$

The complexified Lie algebra of $GL(2,\mathbb{R})$ is $M_2(\mathbb{C})$ the space of 2×2 matrices with complex entries ; the Lie algebra structure is given by $(a, b) \mapsto [a, b] = ab - ba$; $M_2(\mathbb{C})$ is spanned by X, Y, Z and A. Write $A = -iH$ (where $i \in \mathfrak{h}$ is the unique element whose square is -1). Then, A acts semisimply (under the adjoint action) on $\mathfrak{g}$ with real eigenvalues. Write

$$\mathfrak{g} = \mathbb{C}E^+ \oplus \mathbb{C}E^- \oplus \mathbb{C}Z \oplus \mathbb{C}A \tag{27}$$

where

$$E^- = X + iY - (i/2)A - (i/2)Z \text{ and } E^+ = X - iY - (i/2)A + (i/2)Z. \tag{28}$$

Then E^- and E^+ are eigenvectors for A with eigenvalues -2 and 2 respectively. Of course, on A and Z, A acts by 0. Thus, the complex Lie algebra spanned by E^+, E^- and A is isomorphic to $sl_2(\mathbb{C})$.

Definition 4.2 Fix the subgroup $K_\infty = O(2)$ of $GL(2,\mathbb{R})$. This is the group generated by

$$SO(2) = \{R_\theta = \begin{pmatrix} -cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} : \theta \in \mathbb{R}\} \tag{29}$$

and

$$\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{30}$$

Then, $O(2)$ is a maximal compact subgroup of $GL(2,\mathbb{R})$. Suppose that (π, V) is a module for $\mathfrak{g}$ as well as for $O(2)$ such that the module structures are *compatible*. That is, suppose that $v \in V$ and $\xi \in \mathfrak{g}$, and $\sigma \in O(2)$. Then,

$$\pi(\sigma)\pi(\xi)(v) = \pi((\sigma(\xi))(v)$$

where $\sigma(\xi)$ is the inner conjugation action of $O(2)$ on the Lie algebra $\mathfrak{g}$. One then says that (π, V) is a $(\mathfrak{g}, K_\infty)$-module. If , as a K_∞-module, (π, V) is a direct sum of irreducible representations of K_∞ with each irreducible representation occurring only finitely many times, then one says that the $(\mathfrak{g}, K_\infty)$-module is **admissible**. One then sees at once that a $(\mathfrak{g}, K_\infty)$-submodule (or a quotient module) of an admissible module is also admissible. One says that a vector $v \in V$ **generates** (π, V) as a $(\mathfrak{g}, O(2))$-module, if the smallest submodule of V containing v is all of V.

We will now prove the basic fact from representation theory which we will use.

Theorem 4.3 *Let (π, V) be a $(\mathfrak{g}, K_\infty)$-module. Suppose that $v \in V$ has the following properties:*

(1) v generates V.

(2) The connected component $SO(2)$ of $O(2)$ acts by the character determined by $R_\theta(v) = e^{2\pi i\theta m}v$, for some **positive integer** *m (i.e. A acts by the eigenvalue m) .*

(3) $E^-(v) = 0$ and $Z(v) = 0$.

Then the (π, V) is admissible and irreducible.

Proof Let $\mathfrak{u}(\mathfrak{g})$ denote the universal enveloping algebra of the Lie-algebra $\mathfrak{g}$. One has the decomposition (the Poincare'-Birkhoff-Witt Theorem)

$$\mathfrak{u}(\mathfrak{g}) = \mathfrak{u}(\mathfrak{g}).[E^-] + \mathfrak{u}(\mathfrak{g}).[Z] \oplus \mathbb{C}[E^+] \otimes \mathbb{C}[A] \tag{31}$$

where $\mathbb{C}[\xi]$ denotes the algebra generated by the operator ξ. Therefore, if (as in 30)

$$\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

then by assumptions (1) and (2) of the Lemma ,

$$V = \mathbb{C}[E^+](v) \oplus \iota\mathbb{C}[E^+](v). \tag{32}$$

On E^+ the element A acts by the eigenvalue 2. Therefore, for an integer $p \geq 0$, the element $(E^+)^p(v)$ is an eigenvector for A with eigenvalue $(2p+m)$, and $\iota(E^+)^p(v)$ is an eigenvector with eigenvalue $(-2p-m)$ (note that under the conjugation action of ι, the element A goes to $-A$, hence ι takes an r-eigenspace for A into the $-r$-eigenspace). Since all these weights are different, equation 32 shows that V is admissible as an $SO(2)$ module (A generates the complexified Lie algebra of $SO(2)$). In fact, equation 32 shows that the multiplicity of an irreducible representation of $SO(2)$ in V is at most one, i.e. V is admissible.

Suppose that $W \subset V$ is a submodule. In the last paragraph, we saw that the action of A on V is completely reducible; hence the same holds for W. Suppose that w is a weight vector in W of weight j, say. By replacing w by $\iota(w)$ if necessary, we may assume that $j > 0$. The last paragraph shows that $j = 2p + m$ for some $p \geq 0$ and also that $(E^+)^p(v) = w$ (upto scalar multiples). We may assume that p is the smallest non-negative integer such that W contains the eigenvector $(E^+)^p(v) = w$ with eigenvalue $2p + m$. The minimality of p implies that $E^-(w) = 0$. Let W' be the submodule of W generated by the vector w. To prove the irreducibility, it is enough to show that $W' = V$. We may assume then that $W = W'$.

Since v generates V and Z annihilates v, it follows that Z acts by zero on all of V. Therefore, the vector w satisfies all the properties that v does in the assumptions of the Theorem (except that in (2) the eigencharacter is $2p + m$). Therefore, cf. equation 32 , we have

$$W = \mathbb{C}[E^+](w) \oplus \iota\mathbb{C}[E^+](w) = \mathbb{C}[E^+](E^+)^p(v) \oplus \iota\mathbb{C}[E^+](E^+)^p(v). \tag{33}$$

Now the equations 32 and 33 show that the codimension of W in V is finite: $dim(V/W) < \infty$. Hence V/W also satisfies the assumptions of the Theorem, but is finite dimensional. This is impossible by the finite dimensional representation theory of $sl(2, \mathbb{C})$: a lowest weight vector (i.e. one killed by E^- of $sl(2)$) cannot have **positive** weight for A. This shows that $V/W = 0$ i.e. $W = V$.

Proposition 4.4 *Given $m > 0$, there is a* **unique** *irreducible $(\mathfrak{g}, K_\infty)$-module ρ_m which satisfies the properties of 4.3. The uniqueness follows easily from the above proof.*

Proof Let χ_m denote the one dimensional complex vector space on which the group $SO(2)$ acts by the character $R_\theta \mapsto e^{2\pi i m\theta}$ (where R_θ is, as in 29, the rotation by θ in 2-space). Consider the space

$$\mathfrak{u}(\mathfrak{g}) \otimes \chi_m.$$

This is a representation for $SO(2)$ (as well as for the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$). Let ρ_m be the $O(2)$-module induced from this $SO(2)$-module.

Then, ρ_m satisfies the properties of Theorem (4.) and is therefore irreducible. Moreover, it is clear that any module V of the type considered in 4.3 is a **quotient** of ρ_m. By irreducibility, $V = \rho_m$.

We are now in a position to state the precise relationship of modular forms with representation theory.

Notation 4.5 Let f be a modular form of weight $2k$ with $k > 0$. We will now construct a function on the group $G^+ = GL(2,\mathbb{R})^+$ as follows. Set

$$F_f(g) = j(g,i)^{-2k} f(g(i)) det(g)^k$$

where $i \in \mathfrak{h}$ is the point whose isotropy is the group $SO(2)$ as in equation 29. As before, $j(g,i) = cz + d$, where

$$g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

By using the modularity of f and the equation $j(gh,z) = j(g,h(z))j(h,z)$ for the "automorphy factor" $j(g,z)$, it is easy to see that F_f is invariant under left translation by elements of $SL(2,\mathbb{Z})$ and also under the centre Z_∞ of $GL(2,\mathbb{R})$.

We will now check that the $(\mathfrak{g}, O(2)$-module generated by F_f is isomorphic to ρ_{2k}, with ρ_{2k} as in 4.4. Note that F_f is contained in the space

$$C^\infty(Z_\infty GL(2,\mathbb{Z})\backslash GL(2,\mathbb{R}),$$

the space of smooth functions on the relevant space and that the latter is a $(\mathfrak{g}, O(2))$-module under right translation by elements of $GL(2,\mathbb{R})$.Moreover, for all $y > 0$ and $x \in \mathbb{R}$ we have

$$F_f(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = y^k f(x+iy) \tag{34}$$

The function $g \mapsto f(g(i))$ is right invariant under the action of $SO(2)$ since i is the isotropy of $SO(2)$. Using the fact that $j(R_\theta, i) = e^{-2i\theta}$ (where R_θ is as in 29) one checks that $j(gR_\theta, i) = j(g,i)(e^{-2i\theta})$. Therefore, it follows that

$$F_f(gR_\theta) = F_f(g)e^{4ik\theta}. \tag{35}$$

This equation implies that under the action of the element $A = i\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (A generates the Lie algebra of $SO(2)$), F_f is an eigenvector with eigenvalue $4k$.

Compute the action of E^- (E^- as in 28) on F_f. Using the invariance of of F_f under Z_∞ and that it is an eigenvector of A with eigenvalue $4k$, one sees that

$$E^-(F_f) = (X + iY)F_f - ikF_f.$$

Now use equation 34 to conclude that $E^-F_f = y^{2k}(\partial f/\partial \bar{z})$. Since f is **holomorphic**, we obtain that $E^-F_f = 0$.

The $(\mathfrak{g}, O(2))$ module generated by F_f satisfies the conditions of 4.3.

Consider now the growth properties of F_f. We have the quotient map

$$GL(2,\mathbb{R})^+ :\to \mathfrak{h} \supset F$$

where F is the fundamental domain constructed in section 2. Let S be the preimage of this quotient map. Then,

$$S \subset Z_\infty O(2)\{\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) : \ y^2 > 3/4 and -1/2 < x < 1/2\}$$

and the latter is a "Siegel set". Now,

$$F_f(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = y^k f(x+iy).$$

From the modularity property of f, it follows that f is "bounded at infinity" , which means that there exists a constant $C > 0$ such that on the fundamental domain F of $SL(2,\mathbb{Z})$, the function $z \mapsto f(z) = f(x+iy)$ is bounded by C:

$$| f(z) | \leq C \quad \forall z \in F.$$

Therefore, on the Siegel set S, we have

$$F_f(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) \leq Cy^k$$

, i.e. F_f has moderate growth on the Siegel Set.

The last three paragraphs imply the following

Theorem 4.6 *Let f be a modular form of weight $2k$. Let F_f be the associated function on $GL(2,\mathbb{Z})\backslash GL(2,\mathbb{R})$. Then, F_f is an* **automorphic form**. *Moreover, the $(\mathfrak{g}, O(2)$ module generated by F_f is isomorphic to ρ_{2k} with ρ_{2k} as in 4.4*

Proof We need only check that an ideal I of finite codimension in the centre $\mathfrak{z}$ of the universal enveloping algebra of $\mathfrak{g}$ annihilates F_f. But, the module generated by F_f is ρ_{2k} by 4.3 (and 4.4). Now, the $2k$ eigenspace of the operator A in the representation ρ_{2k} is one dimensional (and is generated by F_f), and $\mathfrak{z}$ commutes with the action of A (and in fact with all of $\mathfrak{u}(\mathfrak{g})$ as well). Therefore, the annihilator of F_f in $\mathfrak{z}$ is an ideal I of codimention **one**.

From 4.6, the following Theorem can be deduced.

Theorem 4.7 *The space of modular forms of weight $2k$ is isomorphic to the space $Hom(\rho_{2k}, \mathcal{A}(SL(2,\mathbb{Z})\backslash GL(2,\mathbb{R})^+)$ (the hom is the space of $(\mathfrak{g}, O(2)$-module maps and $\mathcal{A}$) is the space of automorphic forms.*

Given a modular form f of weight $2k$, consider the $(\mathfrak{g}, O(2)$-module generated by the automorphic form F_f. This is irreducible and isomorphic to ρ_{2k} by Theorem 5. Thus we get a homomorphism of ρ_{2k} into the space $\mathcal{A}$ of automorphic forms. Conversely, given a homomorphism ϕ of ρ_{2k} into $\mathcal{A}$, let F denote a nonzero vector in the image of ϕ on which is the group $SO(2)$ with character $R_\theta \mapsto e^{2\pi i(2k)\theta}$. This vector is unique upto scalar multiples. Define a function f on the upper half plane $\mathfrak{h}$ by setting $F = F_f$. Then , it is clear that f is a modular form of weight $2k$.

5 Modular Forms and Hecke Operators

Notation 5.1 Let $\mathbb{A}_f$ be the ring of finite adeles over $\mathbb{Q}$. Recall that this is the direct limit (the maps are inclusion maps) as the finite set S of primes varies, of the product

$$\mathbb{A}_S = \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin} \mathbb{Z}_p.$$

The group of units of $\mathbb{A}_f$ is the group $\mathbb{A}_f^*$ of **ideles** and is the direct limit as S varies, of

$$\mathbb{A}_S^* = \prod_{p \in S} \mathbb{Q}_p^* \times \prod_{p \notin} \mathbb{Z}_p^*,$$

(where $*$ denotes the group of units of the ring under consideration).

There is a natural inclusion of $\mathbb{Q}$ in $\mathbb{A}_f$ (and hence of $\mathbb{Q}^*$ in $\mathbb{A}_f^*$ and of $GL(2,\mathbb{Q})$ in $GL(2,\mathbb{A}_f)$). Denote by P the set of primes. The Strong Approximation Theorem (Chinese Remainder Theorem) implies that

$$\mathbb{A}_f = \mathbb{Q} + \prod_{p \in P} \mathbb{Z}_p. \qquad (36)$$

This, and the fact that $\mathbb{Z}$ is a principal ideal domain imply that

$$\mathbb{A}_f^* = \mathbb{Q}^* . \prod_{p \in P} \mathbb{Z}_p^*. \qquad (37)$$

From this it is not difficult to deduce that

$$GL(2,\mathbb{A}_f) = GL(2,\mathbb{Q}). \prod_{p \in P} GL(2,\mathbb{Z}_p) \qquad (38)$$

Note that the intersection of $GL(2,\mathbb{Q})$ with $K_f = \prod GL(2,\mathbb{Z}_p)$ is precisely $GL(2,\mathbb{Z})$.

Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the ring of adeles over $\mathbb{Q}$. Then, $\mathbb{Q}$ is diagonally imbedded in $\mathbb{A}$. Hence there is a diagonal imbedding of $GL(2,\mathbb{Q})$ in $GL(2,\mathbb{A}) = GL(2,\mathbb{R}) \times GL(2,\mathbb{A}_f)$. Then, $GL(2,\mathbb{Q})$ is a discrete subgroup of $GL(2,\mathbb{A})$. Now 38 (a consequence of strong approximation) implies that

$$GL(2,\mathbb{A}) = GL(2,\mathbb{Q})(GL(2,\mathbb{R}) \times \prod_{p \in P} GL(2,\mathbb{Z}_p)). \qquad (39)$$

Now, the equation 39 and the last sentence of the previous paragraph imply that the quotient

$$GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A}) = GL(2,\mathbb{Z})\backslash(GL(2,\mathbb{R}) \times \prod_{p\in P} GL(2,\mathbb{Z}_p)). \qquad (40)$$

Note that $GL(2,\mathbb{A})$ acts by right translations on the left hand side of the equation 40.

Notation 5.2 A representation (π, W) of $GL(2,\mathbb{A}_f)$ is said to be **smooth** if the isotropy of any vector in W is an open subgroup of $GL(2,\mathbb{A}_f)$. Define the "Hecke algebra" $\mathcal{H}$ of $GL(2,\mathbb{A}_f)$ as the space of compactly supported locally constant functions on $GL(2,\mathbb{A}_f)$. If W is a smooth representation of $GL(2,\mathbb{A}_f)$, then the Hecke Algebra $\mathcal{H}$ also operates on W by "convolutions": if μ is a Haar measure on $GL(2,\mathbb{A}_f)$, $\phi \in \mathcal{H}$, and $w \in W$ is a vector, then the W valued function $g \mapsto \phi(g)\pi(g)w$ is a locally constant *compactly supported function* and hence can be integrated with respect to the Haar measure μ. Define

$$\phi * w = \pi(\phi)(w) = \int \phi(g)\pi(g)(w)d\mu(g) \qquad (41)$$

This gives the $GL(2,\mathbb{A}_f)$-module π, the structure of an $\mathcal{H}$-module As is well known, the category of smooth representations of $GL(2,\mathbb{A}_f)$ is isomorphic to the category of representations of the Hecke algebra $\mathcal{H}$, the isomorphism arising from the foregoing action of the Hecke algebra on the smooth module π.

Notation 5.3 The group $K_0 = GL(2,\widehat{\mathbb{Z}})$ is an open compact subgroup of $GL(2,\mathbb{A}_f)$ and is the product opver all primes p of the groups $GL(2,\mathbb{Z}_p)$. Given $g \in GL(2,\mathbb{A}_f)$, consider the characterisic function χ_g of the double coset set $K_0 g K_0$. Then χ_g is an element of the Hecke algebra and elements of $\mathcal{H}$ which are bi-invariant under $\mathcal{H}$ are finite linear combinations of the functions χ_g as g varies. We will refer to the subalgebra generated by these elements as the '**unramified Hecke algebra** and denote it by $\mathcal{H}_0$.

Under convolution, $\mathcal{H}$ is an algebra and $\mathcal{H}_0$ is a commutative subalgebra. Fix a prime p. Let $\mathcal{H}_0(p)$ be the subalgebra generated by the elements χ_{M_p} and χ_{N_p} where $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $N_p = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$. It is easily proved that for varying p, the algebras $\mathcal{H}_0(p)$ generate the unramified Hecke algebra $\mathcal{H}_0$.

Notation 5.4 The equation 40 implies that the space of smooth functions on $Z_\infty SL(2,\mathbb{Z})\backslash GL(2,\mathbb{R})^+$ is isomorphic to the space V_0 of K_0-invariant smooth functions on the quotient $GL(2,\mathbb{Q})Z(\mathbb{A})\backslash GL(2,\mathbb{A})$. On V_0 the unramified Hecke algebra operates . Suppose S denotes the image of $F \times K_0$ in $Z(\mathbb{A})\backslash GL(2,\mathbb{A})$. Then, S is contained in a Siegel Set $S_0\times$ whose elements are of the form

$$z_\infty \begin{pmatrix} (y & x \\ 0 & 1) \end{pmatrix} \times k_0$$

where $z_\infty \in Z_\infty$, $k_0 \in K_0$, $\mid x \mid< 1/2$ and $y^2 > 3/4$. Suppose that f is a cusp form for $SL(2,\mathbb{Z})$ and F_f be as in section (4.). Given $g \in GL(2,\mathbb{A}) = GL(2,\mathbb{R}) \times GL(2,\mathbb{A}_f)$, write $g = (g_\infty, g_f)$ accordingly. Define the function Φ_f on $GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$ as follows. Set $\Phi_f(g_\infty, g_f) = F_f(g_\infty)$ if $g_f \in K_0$ and extend to $G(\mathbb{A})$ by demanding that Φ_f be $GL(2,\mathbb{Q})$-invariant. The $SL(2,\mathbb{Z})$-invariance of F_f implies that Φ_f is well defined. Now, 4.6 shows that Φ_f is an automorphic form on $GL(2,\mathbb{A})$.

By 4.7, F_f is rapidly decreasing on S_0; moreover, F_f is a **cuspidal automorphic form** in the sense that for all $g \in G(\mathbb{A})$, the following holds.

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \Phi_f(ng)dn = 0 \tag{42}$$

where U is the group of unipotent upper triangular matrices in GL(2) with ones on the diagonal and dn is the Haar measure on $U(\mathbb{A})$. This can be shown to imply that the function Φ_f is square summable on the quotient $Z(\mathbb{A})GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$ with respect to the Haar measure. Further, one has the L^2-metric $<\ ,\ >$ on the space of cuspidal automorphic forms which translates to the "Petersson" metric

$$< f, g >= \int_F f(z)\overline{g(z)}y^{2k}(y^{-2}dxdy)$$

for cusp forms f and g of weight k. As before, F is the fundamental domain for $SL(2,\mathbb{Z})$.

From now on, we will fix our attention on cusp forms. we have the natural inclusion of $GL(2,\mathbb{Q})$ in $GL(2,\mathbb{A}_f)$. Let $n > 0$ be an integer and let $g_n = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be thought of as an element in $GL(2,\mathbb{A}_f)$ under the foregoing inclusion. Let X_n denote the characteristic function of the double coset of K_0 through the element g_n.

If χ_n denotes the characteristic function X_n, and f is a cuspidal modular form of weight $2k$, then $\Phi' = \Phi_f * R(\chi_n)$ (where $R(\phi)$ denotes the right convolution by the function ϕ) is a smooth function on the quotient $GL(2,\mathbb{Q})Z(\mathbb{A})\backslash GL(2,\mathbb{A})$ whose "infinite" component is still ρ_{2k} (since χ_n commutes with the right action of $GL(2,\mathbb{R})$ on the above quotient). Since χ_n is K_0 invariant, it follows that Φ' is also right K_0 invariant. Therefore, it corresponds to a modular form g i.e. $\Phi' = \Phi_g$. It is easy to show that Φ' is cuspidal. Therefore, g is a cusp form of weight $2k$ as well. Denote $g = T(n)(f)$. Then, $T(n)$ is called the **Hecke operator** corresponding to n. By noting that convolution by χ_n is self adjoint for the L^2 metric on cuspidal automorphic functions on $GL(2)$ one immediately sees that $T(n)$ are self adjoint for the Petersson metric on the space of cusp forms of weight $2k$. The commutativity of the unramified Hecke algebra implies that the operators $T(n)$ commute as well.

Notation 5.5 We will now state without proof the computation of $T(p)$ for a prime p. Note that by strong approximation, the K_0 invariant function Φ_f on the quotient $Z(\mathbb{A})GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$ is completely determined by its values on elements of the form $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ with $y > 0$, in the quotient. We compute

$$\Phi_f * R(\chi_p)\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$$

and find that this is equal, to

$$p^{2k-1}\Phi_g\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$$

where

$$g(z) = (1/p) \sum_{0 \le m \le p-1} f((z+m)/p) + p^{2k-1} f(pz) = T(p)(f)(z).$$

The Fourier coefficients of g at infinity are given by

$$g(m) = a(mp)$$

if m is coprime to p and

$$g(m) = a(mp) + p^{2k-1} a(m/p)$$

if p divides m.

We write down the relations among the operators $T(n)$. If m and n are coprime, then $T(mn) = T(m)T(n)$. if p is a prime and $m \geq 1$ is an integer, then

$$T(p)T(p^m) = T(p^{m+1}) + p^{2k-1} T(p^{m-1})$$

. Thus, the $T(n)$ can be computed easily.

As we have observed before, the operators $T(n)$ are self adjoint for the Peterson metric and commute with each other. Therefore, they can be simultaneously diagonalised. In other words, every cusp form can be written as a sum of cusp forms each of which is a simultaneous eigenfunction for the Hecke operators $T(n)$.

Suppose now that f is a cusp form which is an eigenfunction of the Hecke operators. The associated function Φ_f is a function on $GL(2, \mathbb{A})$ and generates a unitary representation π of $GL(2, \mathbb{A})$. Its infinite component is isomorphic to ρ_{2k}. Its finite component π_f is generated by a K_0 invariant vector which is an eigenfunction for the unramified Hecke algebra $\mathcal{H}_0$ (recall that the $T(n)$'s were defined in terms of elements of $\mathcal{H}_0$).Moreover, π_f is unitary. Therefore, π is irreducible.

Conversely if π is an irreducible unitary cuspidal automorphic representation of $GL(2, \mathbb{A})$, whose infinite component is ρ_{2k}, and whose finite part is generated by a K_0 invariant vector, then, π is generated by a function Φ_f , where f is a cusp form which is an eigenfunction of the Hecke operators $T(n)$.