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Representation theory of GL(n) over non-Archimedean local fields

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1 Introduction

In these lectures we will survey representation theory of the group $GL(n)$ over non-Archimedean local fields. Since in the short time devoted to this topic, it is not possible to give all the proofs, all but the most elementary will be omitted. There are several excellent sources where we can refer the reader for a fuller treatment. We give these references at the end.

We denote by k a non-Archimedean local field. The maximal compact subring of k will be denoted by $\mathcal{O}_k$, and we will use π_k to denote a uniformising element of $\mathcal{O}_k$; $\mathcal{O}_k/\pi_k$ is a finite field, to be denoted by $\mathbb{F}_q$. The absolute value of k will be denoted by $|x|$ with the normalisation $|\pi_k| = q^{-1}$.

In these notes we will be looking at complex representations of the group $GL_n(k)$. These vector spaces will be treated in these notes as algebraic objects, and not with any topology; hence the theory could as well be developed over any algebraically closed field of characteristic zero.

We recall that inside $GL_n(k)$, we have the open compact subgroup $GL_n(\mathcal{O}_k) = \{X \in M_n(\mathcal{O}_k) | \det X \in \mathcal{O}_k^*\}$. In fact any open compact subgroup of $GL_n(k)$ can be conjugated to be inside $GL_n(\mathcal{O}_k)$, but we will not use this fact in these notes.

The compact open subgroup $GL_n(\mathcal{O}_k)$ has many other compact open subgroups such as the principal congruence subgroups defined for $m \geq 1$ by

$$\mathcal{K}_m = \{g \in GL_n(\mathcal{O}_k) | (g - I_n) \in \pi_k^m M_n(\mathcal{O}_k)\}.$$

It is easy to see that

$$\begin{aligned} GL_n(\mathcal{O}_k)/\mathcal{K}_m &\cong GL_n(\mathcal{O}_k/\pi_k^m) \\ \mathcal{K}_m/\mathcal{K}_{m+1} &\cong M_n(\mathcal{O}_k/\pi_k). \end{aligned}$$

We note that any neighbouhood of the identity element of $GL_n(k)$ contains $\mathcal{K}_m$ for some $m \geq 1$.

Definition : A representation π of $GL_n(k)$ on a complex vector space V is called smooth if for every vector v in V, there exists a compact open subgroup K_v of $GL_n(k)$ such that $g \cdot v = v$ for all $g \in K_v$.

In these notes we will be studying *admissible* representations defined as follows.

Definition : A representation π of $GL_n(k)$ on a complex vector space V is called admissible if it is

1. smooth
2. If K is any compact open subgroup of $GL_n(k)$ then the space of K fixed vectors in V, to be denoted by V^K, is finite dimensional.

Exercise: A finite dimensional smooth irreducible representation of $GL_n(k)$ is 1 dimensional.

It may be noted that the condition of admissibility of a representation is not strong enough to make the representation one of finite legth. For instance the sum of all characters of k^* on which π_k (the uniformising parameter of k fixed earlier) operates via a given scalar is an infinite dimensional admissible representation of k^*. (In these notes, by a character of a group, we mean a 1 dimensional smooth representation of the group.) To force representations to have finite length (i.e., to have a finite Jordan-Holder sequence of $GL_n(k)$ sub-representations with irreducible sub-quotients), the concept of *finitely generated* representations seems useful.

Definition : A smooth representation π of $GL_n(k)$ is said to be finitely generated if there exists vectors $v_1, v_2, \cdots v_l$ such that the smallest $GL_n(k)$ submodule of V generated by the vectors v_i is V.

As an example, we note that a finitely generated admissible representation of k^* is finite dimensional. (An elementary exercise for the reader!) The following is however a non-trivial theorem.

Theorem 1 *A finitely generated admissible representation of $GL_n(k)$ is of finite length.*

The aim of these lectures is to classify irreducible admissible representations of $GL_n(k)$ in terms of what is called the Langlands quotient theorem. It was one of the conjectures of Langlands formulated in the late 60's which related irreducible admissible representations of $GL_n(k)$ to n-dimensional representations of the Galois group $\mathrm{Gal}(\bar{k}/k)$ of the separable closure $\bar{k}$ of k, or rather a variant of it in which the Galois group is replaced by a closely related group which is called the Weil-Deligne group which will be introduced later. The conjecture of Langlands was proved by Jacquet and Langlans for $n = 2$ for non-Archimedean local fields of residue characteristic not 2, by Kutzko in general for $n = 2$, and very recently by Harris and Taylor for all n. The proof of Harris and Taylor has been simplified by Henniart. These recent developments are the topic of the talks by Wedhorn.

We note that Schur's lemma holds for irreducible admissible representations.

Lemma 1 *If V is an irreducible admissible representation of $GL_n(k)$, and $A : V \to V$ is an endomorphism of V such that $A(gv) = gA(v)$ for all $v \in V$, and $g \in GL_n(k)$, then A is multiplication by a scalar.*

Proof : Since A is a $GL_n(k)$-equivariant endomorphism, it takes the space of K-fixed vectors V^K to itself for any compact open subgroup K of $GL_n(k)$. Choose a compact open subgroup K such that $V^K \neq 0$. Since V^K is a finite dimensional complex vector space, there exists an eigenvector for the action of A on V^K, say $v \in V^K$ with $Av = \lambda v$. It follows that the kernel of $(A - \lambda)$ is a nonzero $GL_n(k)$invariant subspace of V, and hence must be all of V, i.e., $A = \lambda$.

Corollary 1 *Any irreducible admissible representation of $GL_1(k) = k^*$ is one dimensional.*

Corollary 2 *On an irreducible admissible representation of $GL_n(k)$, the centre of $GL_n(k)$, which can be identified to k^*, operates via a character, called the central character of the representation.*

Twisting by characters. It is known that the commutator subgroup of $GL_n(k)$ is $SL_n(k)$. It follows that any character of $GL_n(k)$ is obtained from a character of k^* by composition with the determinant map:

$$GL_n(k) \stackrel{\det}{\to} k^* \stackrel{\chi}{\to} \mathbb{C}^*.$$

Given any representation π of $GL_n(k)$ and a character $k^* \stackrel{\chi}{\to} \mathbb{C}^*$, one can construct another representation of $GL_n(k)$, denoted by $\pi \otimes \chi$, and called π twisted by χ, which is $\pi(x)\chi(\det x)$ for $x \in GL_n(k)$.

2 Parabolic Induction

Parabolic induction provides a very important means of constructing representations of bigger groups from representations of smaller groups. It takes irreducible admissible representations of finite length to irreducible admissible representations of finite length. Usually it takes irreducible representations to irreducible representations, but might also take irreducible representations to reducible representations. The question about reducibility of these induced representations is rather subtle, where there are even no general conjectures about the reduciblity points, or the possible irreducible subquotients, but happily for $GL(n)$, these questions can be answered completely because of works of Bernstein-Zelevinsky, and of Zelevinsky. We will however not be describing these works in any detail.

The process of parabolic induction induces representations of parabolic subgroups which are trivial on its unipotent radical to the full group. We define these terms in some detail here.

Define a flag in k^n to be a *strictly* increasing sequence of subspaces

$$W_. = \{W_0 \subset W_1 \subset \cdots W_m = k^n\}.$$

The subgroup of $GL_n(k)$ which stabilises the flag $W_.$, i.e. with the property that $gW_i = W_i$ for all i is defined to be a parabolic (associated to the flag $W_.$).

The stabiliser of the flags of the form

$$W_. = (v_1) \subset (v_1, v_2) \subset \cdots (v_1, v_2, \cdots, v_n) = k^n$$

is called a Borel subgroup. It can be seen that $GL_n(k)$ operates transitively on the set of such flags, and hence the stabiliser of any two such (maximal) flags are conjugate under $GL_n(k)$.

If

$$W_. = \{W_0 \subset W_1 \subset \cdots W_m = k^n\},$$

then inside the associated parabolic subgroup P, there exists the normal subgroup N consisting of those elements which operate trivially on W_{i+1}/W_i for $0 \le i \le k-1$. The subgroup N is called the unipotent radical of P. It can be seen that there is a semi-direct product decomposition $P = MN$ with

$$M = \prod_{i=0}^{k-1} GL(W_{i+1}/W_i).$$

The decomposition $P = MN$ is called a Levi decomposition of P with N the unipotent radical, and M a Levi subgroup.

Since P sits in the exact sequence of groups,

$$\{e\} \to N \to P \to M \to \{e\},$$

any representation of M can also be considered to be a representation of P (said to be obtained by extending trivially across N).

If ρ is a smooth representation of M on a vector space V, define a representation, $\mathrm{Ind}_P^{GL_n(k)}\rho$, called the representation of $GL_n(k)$ obtained by parabolically inducing the representation ρ of M as follows:

$$\mathrm{Ind}_P^{GL_n(k)}\rho = \left\{ f : G \to V \;\middle|\; \begin{array}{l} f(mng) = \rho(m)\delta_M^{1/2} f(g), \text{ and} \\ f(gh) = f(g) \text{ for all } h \in K_f, \\ \qquad\qquad \text{a compact open subgroup of } GL_n(k) \end{array} \right\}$$

Here δ_M is a certain character of M, called the modulus function of M, which is the character of M by which it operates (by the inner conjugation action) on the 1 dimensional space of Haar measures on N. The group $GL_n(k)$ operates on $\mathrm{Ind}_P^{GL_n(k)}\rho$ by right translation. It is customary to put the character δ_M in the definition of induced representations which has several simplifying aspects:

1. Induction takes unitary representations to unitary representations.

2. The contragredient of an induced representation is the induction of the contragredient. (The definition of the contragredient is recalled later.)

Exercise : Prove that $\delta_B(x_1, x_2, \cdots, x_n) = |x_1^{\frac{n-1}{2}} x_2^{\frac{n-3}{2}} \cdots x_n^{-\frac{n-1}{2}}|$

Example : For any parabolic P, the space of locally constant functions on G/P, denoted by $\mathcal{S}(G/P)$, is an induced representation (from the character $\delta_P^{-1/2}$). It contains $\mathcal{S}(G/Q)$, for Q any parabolic subgroup of G containing P, as a G-invariant subspace.

It is clear from the definition that $\mathrm{Ind}_P^{GL_n(k)}\rho$, is a smooth representation of $GL_n(k)$. To prove that if ρ is an admissible representation of M, then so is $\mathrm{Ind}_P^{GL_n(k)}\rho$, we will need the Iwasawa decomposition:

$$GL_n(k) = GL_n(\mathcal{O}_k) \cdot P,$$

where P is any parabolic subgroup of $GL_n(k)$.

It follows from the Iwasawa decomposition that the action of any compact open subgroup on any flag variety $GL_n(k)/P$ has only finitely many orbits from which the admissibility of $\mathrm{Ind}_P^{GL_n(k)}\rho$ comes out quite easily, and will be left as an exercise to the reader.

The parabolic induction takes finitely generated representations to finitely generated representations. We give a proof of this fact.

Proposition 1 *If ρ is a finitely generated representation of M, then $\mathrm{Ind}_P^{GL_n(k)}\rho$ is also a finitely generated representation of $GL_n(k)$.*

Proof : We give the argument to show that the space of locally constant functions on $GL_n(k)/P$ is a finitely generated representation of $GL_n(k)$. The argument for general parabolically induced representations is similar. To prove that the space of locally constant functions on $GL_n(k)/P$ is finitely generated, it is sufficient to treat the case when P is the Borel subgroup B of the group of upper triangular matrices.

The proof of the finite generation of the space of locally constant functions on $GL_n(k)/B$ depends on what is called the Iwahori factorisation which plays a fundamental role in many questions in representation theory of p adic groups. It says that certain maximal compact subgroups K of $GL_n(k)$, such as the principal congruence subgroups $\mathcal{K}_m$, $m \geq 1$, can be written as a product

$$K = (K \cap N^-) \cdot (K \cap A) \cdot (K \cap N)$$

where $B = AN$ is the Borel subgroup of upper triangular matrices with A as the diagonal subgroup, and N (resp. N^-) is the group of strictly upper (resp. lower) triangular matrices.

We note that there are elements in A which shrink $K_- = K \cap N^-$ to the identity. For example if we take the matrix

$$\mu = \begin{pmatrix} 1 & & & & & \\ & \pi & & & & \\ & & \pi^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \pi^{n-1} \end{pmatrix}$$

then the powers of μ have the property that they shrink $K_- = K \cap N^-$ to the identity.

Let $\chi(X)$ denote the characteristic function of a subset X of a certain ambient space. Look at the translates of $\chi(B \cdot K_-)$ by the powers μ^{-i}. This will give us,

$$\begin{aligned} \chi(B \cdot K_- \mu^{-i}) &= \chi(B \cdot \mu^{-i} \mu^i K_- \mu^{-i}) \\ &= \chi(B \cdot \mu^i K_- \mu^{-i}) \end{aligned}$$

Therefore translating $\chi(BK_-)$ by μ^{-i}, we get characteristic function of $B \cdot \mathcal{K}$ for $\mathcal{K}$ a compact open subgroup as small as we like. These characteristic functions together with their $GL_n(k)$ translates clearly span all the locally constant functions on $GL_n(k)/B$, completing the proof of the proposition.

3 Jacquet Functors

Parabolic induction constructs representations of $GL_n(k)$ from representations of its Levi subgroups. There is a dual procedure (or, more correctly, adjoint functor) which constructs representations of Levi subgroups from representations of $GL_n(k)$. The importance and basic properties of this construction was noted by Jacquet for the $GL_n(k)$ case, which was generalised to all reductive groups by Harish-Chandra.

Definition :Let $P = MN$ be the Levi decomposition of a parabolic P of $GL_n(k)$. For a smooth representation ρ of P on a vector space V, define ρ_N to be the largest quotient of ρ on which N operates trivially, or

$$\rho_N = \frac{V}{\rho(N) = \{n \cdot v - v | n \in N, v \in V\}}.$$

Lemma 2 *For a smooth representation V of N, $V(N) = \{n \cdot v - v | n \in N, v \in V\}$ is exactly the space of vectors $v \in V$ such that*

$$\int_{K_N} n \cdot v dn = 0,$$

where K_N is a compact open subgroup of N and dn is a Haar measure on N. (The integral is actually a finite sum.)

Proof : Clearly the integral of the vectors of the form $n \cdot v - v$ on a compact open subgroup of N containing n is zero.

We note that N is a union of compact open subgroups. For instance, if μ is the element

$$\mu = \begin{pmatrix} 1 & & & & & \\ & \pi & & & & \\ & & \pi^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \pi^{n-1} \end{pmatrix}$$

then it is easy to see that

$$\bigcup_{m \geq 1} \mu^m N(\mathcal{O}) \mu^{-m} = N(k).$$

It follows that for any vector v in V, and $n \in N$, there exists a compact open subgroup K_N such that the integral of $n \cdot v - v$ on K_N is zero. Conversely, suppose

$$\int_{K_N} n \cdot v dn = 0.$$

Since V is completely reducible as a K_N module, v lies in a finite dimensional K_N subrepresentation of W of V. Write $W = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ with W_1 the subspace of W on which K_N operates trivially and non-trivially on all the other $W_i, i > 1$.

Since $\int_{K_N} n \cdot v dn = 0$, the component of v in W_1 is zero. Vectors in $W_i, i > 1$ can be written as a sum of $n_i w_i - w_i$ for $n_i \in K_N$, $w_i \in W_i$. This follows as the largest quotient of $W_i, i > 1$ on which K_N operates trivially is zero.

Lemma 3 *The Jacquet functor $V \to V_N$ is an exact functor, i.e., if*

$$0 \to V_1 \to V \to V_2 \to 0$$

is an exact sequence of P modules, then

$$0 \to (V_1)_N \to (V)_N \to (V_2)_N \to 0$$

is an exact sequence of M modules.

Proof : For the proof of the lemma it suffices to prove that $V_1 \cap V(N) = V_1(N)$ which is clear from the previous lemma.

The importance of the Jacquet functor stems from the following version of the Frobenius reciprocity.

Lemma 4 *For a smooth representation ρ of $GL_n(k)$ and μ of a Levi subgroup M of a parabolic $P = MN$, we have,*

$$\mathrm{Hom}_{GL_n(k)}(\rho, \mathrm{Ind}_P^{GL_n(k)} \mu) \cong \mathrm{Hom}_M(\rho_N, \mu\delta_P^{1/2}).$$

Proof : We first define a map from $\mathrm{Hom}_{GL_n(k)}(\rho, \mathrm{Ind}_P^{GL_n(k)} \mu)$ to $\mathrm{Hom}_M(\rho_N, \mu\delta_P^{1/2})$ by sending $\phi \in \mathrm{Hom}_{GL_n(k)}(\rho, \mathrm{Ind}_P^{GL_n(k)} \mu)$ to the homomorphism from ρ to $\mu\delta_P^{1/2}$ which sends a vector $v \in \rho$ to $\phi_v(1) \in \mu\delta_P^{1/2}$ where $\phi_v \in \mathrm{Ind}_P^{GL_n(k)} \rho$ is the image of the vector v under ϕ. The mapping $v \to \phi_v(1)$ from ρ to $\mu\delta_P^{1/2}$ clearly factors through ρ_N.

We next define a mapping from $\mathrm{Hom}_M(\rho_N, \mu\delta_P^{1/2})$ to $\mathrm{Hom}_{GL_n(k)}(\rho, \mathrm{Ind}_P^{GL_n(k)}\mu)$ by sending $\psi \in \mathrm{Hom}_M(\rho_N, \mu\delta_P^{1/2})$ to the homomorphism in $\mathrm{Hom}_{GL_n(k)}(\rho, \mathrm{Ind}_P^{GL_n(k)}\mu)$ which sends a vector w in ρ to the function F_w on G with values in the vector space underlying μ defined by $F_w(g) = \psi(g.v)$.

It can be checked that the two maps are inverse to each other, and hence define an isomorphism from $\mathrm{Hom}_{GL_n(k)}(\rho, \mathrm{Ind}_P^{GL_n(k)}\mu)$ to $\mathrm{Hom}_M(\rho_N, \mu\delta_P^{1/2})$.

3.1 Jacquet functor for principal series

Theorem 2 *If $\pi = \mathrm{Ind}_B^{GL_n(k)}\chi$, then*

$$\pi_N \cong \sum \chi^w \delta_B^{1/2} \quad \text{(up to semi − simplification)},$$

where for a character χ of the torus, χ^w denotes the character of the torus obtained by using the automorphism of the torus given by the action of w which is an element in the Weyl group of the torus, i.e., in the group $N(T)/T$ for $N(T)$ the normaliser of the torus.

Proof : The representation π can be thought of as a certain space of "functions on $GL_n(k)/B$ twisted by the character χ"; more precisely, as the space of locally constant functions on $GL_n(k)/B$ with values in a sheaf $\mathcal{E}_\chi$ obtained from the character χ of the Borel subgroup B.

If Y is a closed subspace of a topological space X "of the kind that we are considering here", e.g. locally closed subspaces of the flag variety, then there is an exact sequence,

$$0 \to C_c^\infty(X - Y, \mathcal{E}_\chi|_{X-Y}) \to C_c^\infty(X, \mathcal{E}_\chi) \to C_c^\infty(Y, \mathcal{E}_\chi|_Y) \to 0.$$

It follows that Mackey's theory (originally for finite groups) about restriction of an induced representation to a subgroup holds good for p-adic groups too. Hence,

$$\begin{aligned} \mathrm{Res}_B \mathrm{Ind}_B^{GL_n(k)}\chi &= \sum_{w\in W} \mathrm{ind}_{B\cap wBw^{-1}}^B (\chi\delta_B^{1/2})^w \quad \text{(up to semi − simplification)} \\ &= \sum_{w\in W} \mathrm{ind}_{T\cdot N_w}^B (\chi\delta_B^{1/2})^w \quad \text{(up to semi − simplification)} \\ &= \sum_{w\in W} C_c^\infty(N/N_w, (\chi\delta_B^{1/2})^w) \quad \text{(up to semi − simplification)} \end{aligned}$$

We now note that the largest quotient of $C_c^\infty(N/N_w)$ on which N operates trivially is one dimensional (obtained by integrating a function with respect to a Haar measure on N/N_w) on which the action of the torus T is by the sum of positive roots which are not in N_w which can be seen to be $[\delta_B \cdot (\delta_B)^{-w}]^{1/2}$. (Here δ_B^{-w} stands for the w translate of δ_B^{-1}!) This implies that the largest quotient of $C_c^\infty(N/N_w, (\chi\delta_B^{1/2})^w)$ on which N operates trivially is the character $\chi^w\delta_B^{1/2}$. Hence,

$$\left[\mathrm{Ind}_B^{GL_n(k)}\chi\right]_N \cong \sum_{w\in W} \chi^w\delta_B^{1/2} \quad \text{(up to semi}-\text{simplification)},$$

proving the theorem.

Corollary 3 *If* $\mathrm{Hom}_{GL_n(k)}(\mathrm{Ind}_B^{GL_n(k)}\chi_1, \mathrm{Ind}_B^{GL_n(k)}\chi_2)$ *is nonzero, then* $\chi_1 = \chi_2^w$ *for some* w *in the Weyl group.*

Proof : This is a simple consequence of the Frobenius reciprocity combined with the calculation of the Jacquet functor.

Example : Denote the principal series representation of $GL_2(k)$ induced from a character χ of the diagonal torus by $Ps(\chi)$. Then,

1. If $\chi \neq \chi^w$, then $Ps(\chi)_N \cong \chi\delta_B^{1/2} \oplus \chi^w\delta_B^{1/2}$.
2. If $\chi = \chi^w$, then $Ps(\chi)_N$ is a non-trivial extension of k^* modules:

$$0 \to \chi\delta_B^{1/2} \to Ps(\chi)_N \to \chi\delta_B^{1/2} \to 0.$$

Exercise: With notation as in the previous exercise, prove that a principal series representation of $GL_2(k)$ induced from a unitary character is irreducible.

Exercise : Let G be an abelian group with characters χ_1 and χ_2. Prove that if $\chi_1 \neq \chi_2$, then any exact sequence of G-modules,

$$0 \to \chi_1 \to V \to \chi_2 \to 0,$$

splits.

4 Supercuspidal representations

A very important and novel feature of p-adic groups (compared to real groups) is the existence of supercuspidal representations. These representations were first noticed for GL_2 by Mautner in the early 60's. We will see that these representations are the building blocks of all irreducible admissible representations of p-adic groups. A complete set of supercuspidals for $GL_n(k)$ was constructed by Bushnell and Kutzko in their book. The local Langlands correspondence proved by Harris-Taylor-Henniart interprets supercuspidals of $GL_n(k)$ in terms of irreducible representations of the Galois group of k of dimension n. Before we come to the definition of a cuspidal representation, we need to define matrix coefficient of a representation.

For a smooth representation π of $GL_n(k)$, let π^* denote the space of all linear forms on π. Let $\pi^\vee$ denote the subspace of π^* consisting of K-finite vectors, i.e., vectors in π^* whose translates by a (and hence any) compact open subgroup of $GL_n(k)$ is contained in a finite dimensional subspace of π^*. It is easy to see that $\pi^\vee$ is a $GL_n(k)$ invariant subspace of π^*, and is an irreducible admissible representation of $GL_n(k)$, if π is. Moreover, $\pi^{\vee\vee} \cong \pi$. The representation $\pi^\vee$ is called the contragredient, or the smooth dual of π.

For a smooth representation π of $GL_n(k)$, vectors v in π, v' in $\pi^\vee$, define the matrix coefficient $f_{v,v'}$ to be the function on $GL_n(k)$ given by $f_{v,v'}(g) = v'(gv)$.

Definition : An irreducible admissible representation V of $GL_n(k)$ is called supercuspidal if it satisfies one of the following equivalent conditions.

1. A matrix coefficient of V is compactly supported modulo the center.
2. All the Jacquet functors (for all proper parabolics!) are zero.
3. The representation V does not occur as subquotient of any principal series representation induced from a proper parabolic subgroup.

We will not prove that the various conditions occurring in the definition above are equivalent.

We note that any 1 dimensional representation of $GL_1(k) = k^*$ is supercuspidal. But since parabolic induction is of no use for constructing supercuspidal representation, a totally new approach is needed. One way to construct supercuspidal representations is via induction from *certain* finite dimensional representations of compact open subgroups. This has been a big programme in recent times which has been completed for the case of GL_n by Bushnell and Kutzko in their book. They prove that any supercuspidal representation of $GL_n(K)$ is obtained by induction from a finite dimensional representation of certain open compact modulo center subgroup of $GL_n(k)$.

Here is a sample of such a construction in the simplest possible situation. In this theorem, by a cuspidal representation of the finite group $GL_n(\mathbb{F}_q)$ we mean an irreducible representation for which all the Jacquet functors are zero. Cuspidal representations of $GL_n(\mathbb{F}_q)$ are completely known, and there is a simple classification of them.

Proposition 2 *Consider a representation of $GL_n(\mathbb{F}_q)$ to be a representation of $GL_n(\mathcal{O}_k)$ via the natural projection from $GL_n(\mathcal{O}_k)$ to $GL_n(\mathbb{F}_q)$. If π is an irreducible cuspidal representation of $GL_n(\mathbb{F}_q)$ thought of as a representation of $GL_n(\mathcal{O}_k)$, and χ is a character of k^* whose restriction to $\mathcal{O}_k^*$ is the same as the central character of π (thought of as a character of $\mathcal{O}_k^*$), then $\chi \cdot \pi$ is a representation of $k^* GL_n(\mathcal{O}_k)$. This when induced to $GL_n(k)$ gives rise to an irreducible admissible supercuspidal representation of $GL_n(k)$. Moreover,*

$$\mathrm{ind}_{k^* GL_n(\mathcal{O}_k)}^{GL_n(k)}(\chi \cdot \pi) = \mathrm{Ind}_{k^* GL_n(\mathcal{O}_k)}^{GL_n(k)}(\chi \cdot \pi),$$

where as is conventional, we are using ind to denote induction where the functions are taken with compact support in $GL_n(k)/[k^* GL_n(\mathcal{O}_k)]$, a restriction which is not imposed for Ind.

We next have the following basic theorem.

Theorem 3 *For any irreducible admissible representation π of $GL_n(k)$, there exists a Levi subgroup M and a cuspidal representation ρ of M such that π is contained in the representation of $GL_n(k)$ obtained from ρ by the process of parabolic induction. For a given π, the pair (M, ρ) is unique up to conjugation inside $GL_n(k)$.*

Proof : Let P be the parabolic subgroup of $GL_n(k)$ smallest for the property that the Jacquet functor with respect to P is non-zero. (So $P = G$ is a possibility which occurs if and only if the representation is cuspidal.) If $P = MN$, it is clear that the Jacquet functor π_N is a cuspidal representation of M of finite length. Let ρ be an irreducible quotient of π_N as an M module. Since $\mathrm{Hom}_M(\pi_N, \rho) \neq 0$, it follows from the Frobenius reciprocity that

$$\mathrm{Hom}_{GL_n(k)}(\pi, \mathrm{Ind}_P^{GL_n(k)} \rho) \neq 0,$$

which gives a realisation of π inside a principal series induced from a cuspidal representation.

5 Examples for GL_2

In this section we summarise the broad classification of the representations of $GL_2(k)$.

1. Characters of k^* thought of as 1 dimensional representations of $GL_2(k)$ via composition with the determinant map.

2. Principal series representations $Ps(\chi_1, \chi_2)$ where χ_1 and χ_2 are characters of k^*. This is irreducible if and only if $\chi_1\chi_2^{-1} \neq |x|^{\pm 1}$.

3. Steinberg representation of $GL_2(k)$: The group $GL_2(k)$ operates on the projective line $\mathbf{P}^1(k) = GL_2(k)/B$, and hence on the space of locally constant functions on $\mathbf{P}^1(k)$. (This gives the reducible principal series representation $Ps(|x|^{1/2}, |x|^{-1/2})$.) The constant functions form an invariant subspace the quotient by which is an infinite dimensional irreducible representation of $GL_2(k)$, called the Steinberg representation, and denoted by St_2. The twists of the Steinberg, $St_2 \otimes \chi$ by characters χ of k^* are called twisted Steinberg.

4. The rest which is exactly the set of supercuspidal representations of $GL_2(k)$. When the characteristic of the residue field of k is not 2, then irreducible admissible supercuspidal representations of $GL_2(k)$ are obtained from a pair (K, χ) where K is a separable quadratic extension of k and χ is a character of K^* which does not factor through the norm mapping. It can however happen that a cuspidal representation comes from more than 1 quadratic extension though this happens for a very small number of cases.

6 The Steinberg representation

The following theorem due to Casselman defines the Steinberg representation.

Theorem 4 *Let St_n denote the virtual representation (in the Grothendieck group of representations of $GL_n(k)$ of finite length)*

$$St_n = \sum_{B \subset P} (-1)^{\mathrm{rank}(P)} \mathcal{S}(GL_n(k)/P),$$

where the sum runs over all the parabolics P which contain a fixed Borel subgroup B of $GL_n(k)$; $\mathrm{rank}(P)$ denotes the semisimple rank of a Levi subgroup of P. Then St_n is an irreducible admissible representation of $GL_n(k)$ of trivial central character. This is a unitary square-integrable representation of $GL_n(k)/k^*$ in the sense that each of its matrix coefficients is square-integrable on $GL_n(k)/k^*$.

This theorem of Casselman (proved for general quasi-split reductive groups) was generalised to $GL_n(K)$ as follows by Bernstein and Zelevinsky.

Theorem 5 *For a cuspidal representation μ of $GL_m(k)$, and $n = mr, r \geq 1$, the principal series representation of $GL_n(k)$,*

$$Ps(\mu|\det|^{\frac{r-1}{2}}, \mu|\det|^{\frac{r-3}{2}}, \cdots, \mu|\det|^{-\frac{r-1}{2}}),$$

induced from the $(m, m, \cdots, m)$ parabolic subgroup is not irreducible, but has a unique irreducible quotient, denoted by $St_r(\mu)$, and called the generalised Steinberg representation. The generalised Steinberg representation is a discrete series representation of $GL_n(k)$, and every irreducible discrete series representation of $GL_n(k)$ arises in this way.

7 Representations of the Galois group

The Galois group $G = \mathrm{Gal}(\bar{k}/k)$ has distinguished normal subgroups I, the Inertia subgroup, and P the wild Inertia subgroup which is contained in I.

The Inertia subgroup sits in the following exact sequence,

$$1 \to I \to \mathrm{Gal}(\bar{k}/k) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1.$$

Here the mapping from $\mathrm{Gal}(\bar{k}/k)$ to $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is given by the natural action of the Galois group of a local field on its residue field.

The Inertia group can be thought of as the Galois group of $\bar{k}$ over the maximal unramified extension k^{un} of k. Let $k^t = \cup_{(d,q)=1} k^{un}(\pi^{1/d})$. The field k^t is known to be the maximal extension of k^{un} which is tamely ramified. (An extension is called tamely ramified if the index of ramification is coprime to the characteristic of the residue field.) We have,

$$\begin{aligned} \mathrm{Gal}(\bar{k}/k)/I &\cong \hat{\mathbb{Z}} \\ I/P &\cong \prod_{\ell \neq q} \mathbb{Z}_l^*. \end{aligned}$$

One defines the Weil group W_k of k to be the inverse image in $\mathrm{Gal}(\bar{k}/k)$ of the subgroup $\mathbb{Z}$ inside $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ generated by the Frobenius automorphism $x \to x^q$ on the residue field. One considers representations of the Weil group which are continuous on the Inertia subgroup which boils down to considering only those representations of the Weil group for which the image of the Inertia subgroup is finite.

The Weil group is a dense subgroup of the Galois group hence an irreducible representation of the Galois group defines an irreducible representation of the Weil group. It is easy to see that a representation of the Weil group can, after twisting by a character, be extended to a representation of the Galois group.

The Local class field theory implies that the maximal abelian quotient of the Weil group of k is naturally isomorphic to k^*, and hence 1 dimensional representations of W_k are in bijective correspondence with characters of $GL_1(k) = k^*$. It is this statement of abelian class field theory which is generalised by the local Langlands correspondence. However, there is a slight amount of change one needs to make, and instead of taking the Weil group, one needs to take what is called the Weil-Deligne group whose representations are the same as representations of W_k on a vector space V together with a nilpotent endomorphism N such that

$$wNw^{-1} = |w|N$$

where $|w| = q^{-i}$ if the image of w in $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the i power of the Frobenius.

One can identify representations of the Weil-Deligne group to representations of $W_k \times SL_2(\mathbb{C})$ via the Jacobson-Morozov theorem. It is usually much easier to work with $W_k \times SL_2(\mathbb{C})$ but the formulation with the nilpotent operators appears more naturally in considerations of ℓ-adic cohomology of Shimura varieties where the nilpotent operator appears as the 'monodromy' operator.

Theorem 6 *(Local Langlands Conjecture) There exists a bijective correspondence between irreducible admissible representations of $GL_n(k)$ and n-dimensional representations of the Weil-Deligne group.*

We end with the following proposition for which we first note that any field extension K of degree n of a local field k gives rise to an inclusion of the Weil group W_K inside W_k as a subgroup of index n. Since characters on W_K are, by local class field theory, identified to characters of K^*, a character of K^* gives by induction a representation of W_k of dimension n.

Proposition 3 *If $(n, q) = 1$, then any irreducible representation of W_k of dimension n is induced from a character χ of K^* for a field extension K of degree n.*

Proof : Since representations of the Weil group and Galois groups are the same, perhaps after a twist, we will instead work with the Galois group. Let ρ be an irreducible representation of the Galois group $\mathrm{Gal}(\bar{k}/k)$ of dimension $n > 1$. We will prove that there exists an extension L of k of degree greater than 1, and an irreducible representation of $\mathrm{Gal}(\bar{k}/L)$ which induces to ρ. This will complete the proof of the proposition by induction on n.

The proof will be by contradiction. We will assume that ρ is not induced from any proper subgroup.

We recall that there is a filtration on the Galois group $P \subset I \subset G = \mathrm{Gal}(\bar{k}/k)$. Since P is a pro-p group, all its irreducible representations of dimension greater than 1 are powers of p. Since n is prime to p, this implies that there is a character of P which appears in ρ restricted to P. Since P is a normal subgroup of the Galois group, it implies that the restriction of ρ to P is a sum of characters. All the characters must be the same by the Clifford theory. (Otherwise, the representation ρ is induced from the stabiliser of any χ-isotypical component.) So, under the hypothesis that ρ is not induced from any proper subgroup, P acts via scalars on ρ. Since the exact sequence,

$$1 \to P \to I \to I/P \to 1$$

is a split exact sequence, let M be a subgroup in I which goes isomorphically to I/P. Take a character of M appearing in ρ. As P operates via scalars, the corresponding 1 dimensional space is invariant under I. It follows that ρ restricted to I contains a character. Again I being normal, ρ restricted to I is a sum of characters which must be all the same under the assumption that ρ is not induced from any proper subgroup. Since G/I is pro-cyclic, this is not possible.

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