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Modularity of solvable Artin representations of GO(4)-type

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MODULARITY OF SOLVABLE ARTIN REPRESENTATIONS OF GO(4)-TYPE

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Introduction

Let F be a number field, and (ρ, V) a continuous, n-dimensional representation of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ on a finite-dimensional $\mathbb{C}$-vector space V. Denote by $L(s,\rho)$ the associated L-function, which is known to be meromorphic with a functional equation. Artin's conjecture predicts that $L(s,\rho)$ is holomorphic everywhere except possibly at $s=1$, where its order of pole is the multiplicity of the trivial representation in V. The modularity conjecture of Langlands for such representations, some times called the strong Artin conjecture, asserts that there should be an associated (isobaric) automorphic form $\pi = \pi_\infty \otimes \pi_f$ on $\mathrm{GL}(n)/F$ such that $L(s,\rho) = L(s,\pi_f)$. Since $L(s,\pi_f)$ possesses the requisite properties ([JS]), the modularity conjecture implies the Artin conjecture.

For any field k, let $GO(4,k)$ denote the subgroup of $GL(4,k)$ consisting of *orthogonal similitudes*, i.e., matrices g such that ${}^t gg = \lambda_g I$, with $\lambda_g \in k^*$.

We will say that a k-representation (ρ, V) with $\dim(V) = 4$ is *of $GO(4)$-type* iff it factors as

$$\rho = [\mathrm{Gal}(\overline{F}/F) \xrightarrow{\sigma} GO(4,k) \subset GL(V)].$$

In this article we prove

Theorem A *Let F be a number field and let (ρ, V) be a continuous, 4-dimensional $\mathbb{C}$-representation of $Gal(\overline{F}/F)$ whose image is solvable and lies in $GO(4,\mathbb{C})$. Then ρ is modular, i.e., $L(s,\rho) = L(s,\pi_f)$ for some isobaric automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $GL(4,\mathbb{A}_F)$. Moreover, π is cuspidal iff ρ is irreducible.*

One can ask if this helps furnish new examples of Artin's conjecture, and the answer is yes.

Corollary B *Let F be a number field, and let ρ, ρ' be continuous $\mathbb{C}$-representations of $Gal(\overline{F}/F)$ of solvable $GO(4)$-type. Then Artin's conjecture holds for $\rho \otimes \rho'$.*

We will show that in fact there is, for each F, a doubly *infinite* family of such examples where the representations $\rho \otimes \rho'$ are *irreducible* and *primitive* (i.e., not induced) of dimension 16 (see section 6). Primitivity is important because Artin L-functions are inductive, and one wants to make sure that these examples do *not* come by induction from known (solvable) cases in low dimensions. One can also show, given ρ, ρ' as in Corollary B with corresponding extnensions K, K' respectively of F, for any intermediate field E of KK'/F with $[E:F] = p^a$ with $p \in \{2,3\}$ and $a \leq 4$, the ratio $\zeta_E(s)/\zeta_F(s)$ is entire.

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It has been known for a long time, thanks to the results of Artin and Hecke, that monomial representations of $\mathrm{Gal}(\overline{F}/F)$, i.e., those induced by one-dimensional representations of $\mathrm{Gal}(\overline{F}/K)$ with K/F finite, satisfy Artin's conjecture. But the strong Artin conjecture is still open for these except when K/F is normal and solvable ([AC]) and when $[K:\mathbb{Q}]=3$ ([JPSS]).

The odd dimensional orthogonal representations are simpler than the even dimensional ones. Indeed we have

Proposition C *Let ρ be a continuous, irreducible $\mathbb{C}$-representation of $GO(n)$-type with n odd. Then ρ is monomial and hence satisfies the Artin conjecture. If ρ is in addition self-dual, it must be induced by a quadratic character.*

It may be helpful to recall the current status of the problem *beyond* the monomial case. By the groundbreaking work of Langlands in the seventies ([La1]), as completed by Tunnell in 1980 ([Tu]), one knows that the strong Artin conjecture holds for all two-dimensional representations with solvable image. For *odd*, 2-dimensionals of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of A_5-type, a very recent result of Buzzard, Dickinson, Sheppard-Barron and Taylor ([BDST]) establishes the modularity conjecture if certain ramification conditions are satisfied at 2, 3 and 5. If σ, σ' are both solvable and 2-dimensional, the Artin conjecture for $\sigma \otimes \sigma'$, also for $\mathrm{sym}^2(\sigma)\otimes \mathrm{sym}^2(\sigma')$ and $\mathrm{sym}^2(\sigma)\otimes\sigma'$, follows by the Rankin-Selberg theory, while the strong Artin conjecture for $\sigma\otimes\sigma'$ follows from the main theorem of [Ra]. Now let K/F be a quadratic extension with non-trivial automorphism θ. Given any irreducible, non-monomial 2-dimensional representation σ of $\mathrm{Gal}(\overline{F}/K)$ which is not isomorphic to any one-dimensional twist of $\sigma^{[\theta]}$ (see section 1), there is an irreducible 4-dimensional representation $As(\sigma)$ of $\mathrm{Gal}(\overline{F}/F)$ whose restriction to $\mathrm{Gal}(\overline{F}/K)$ is isomorphic to $\sigma\otimes\sigma^{[\theta]}$. When σ is of solvable type, one can combine Langlands-Tunnell with that of Asai ([HLR]) to deduce the Artin conjecture for $As(\sigma)$.

One of the main steps in our proof of Theorem A is that even modularity can be established for any Asai representation $As(\sigma)$ (in the solvable case). To begin, there exists, by Langlands-Tunnell, a cusp form π on $\mathrm{GL}(2)/K$ such that $L(s,\sigma)=L(s,\pi)$. It follows that $L(s,\sigma\otimes\sigma^{[\theta]})$ equals the Rankin-Selberg L-function $L(s,\pi\times(\pi\circ\theta))$. By [Ra], there exists an automorphic form $\pi\boxtimes(\pi\circ\theta)$ on $\mathrm{GL}(4)/K$ such that $L(s,\sigma\otimes\sigma^{[\theta]})=L(s,\pi\boxtimes(\pi\circ\theta))$. Since $\pi\boxtimes(\pi\circ\theta)$ is θ-invariant, one can now deduce the existence of a cusp form Π on $\mathrm{GL}(4)/F$ whose base change to K is $\pi\boxtimes(\pi\circ\theta)$, and which is unique up to twisting by the quadratic character δ, say, of the idele class group of F corresponding to K/F by class field theory. But it is not at all clear that Π, or $\Pi\otimes\delta$, should correspond to $As(\sigma)$, with an identity of the corresponding L-functions. (It is easy to see that the local factors agree at half the places.) Put another way, one can construct an irreducible admissible representation $As(\pi)$ of $\mathrm{GL}(4,\mathbb{A}_F)$ which has the same local factors as does $As(\sigma)$. But the problem is that there is no simple reason why $As(\pi)$ should be automorphic, *even* when π is of CM type, i.e., associated to a Hecke character of a quadratic extension. Anyhow we manage to solve this problem and establish the following

Theorem D *Let K/F be a quadratic extension of number fields with non-trivial automorphism θ, and let δ denote the quadratic idele class character of F corresponding to K/F. Let π be a cuspidal automorphic representation of $GL(2,\mathbb{A}_K)$ such that $\pi\otimes\chi$ is not isomorphic to $\pi\circ\theta$, for any idele class character χ of K. Then $As(\pi)$ is automorphic, which is moreover cuspidal if π is non-dihedral.*

The proof of this is similar to, but a bit subtler at places than, the proof in [Ra] of the existence of the Rankin-Selberg tensor product ⊠ on GL(2)×GL(2). The case when π is *distinguished*, i.e., when $\pi \circ \theta$ is an abelian twist of π, is treated separately. In the remaining *general type* situation, we use the converse theorem for GL(4) due to Cogdell and Piatetski-Shapiro ([COPS], which requires knowledge of the twisted L-functions $L(s, As(\pi) \times \pi')$ for automorphic forms π' of GL(2)/F. Many properties of such L-functions were established by Piatetski-Shapiro and Rallis ([PSR]) and by Ikeda ([Ik1,2]), which we use. A thorny problem arises however, with our inability to identify the bad and archimedean factors. Luckily, things simplify quite a bit under suitable, solvable base changes K/F, and after constructing the base-changed candidates $As(\pi)_K$ for an infinitude of such K, we descend to F as in [Ra]. We also have to control the intersection of the ramification loci of $As(\pi)$ and π'. As a consequence of the existence of $As(\pi)$ over F, one gets (after the fact) the correctness of the local factors vis-a-vis the local correspondence.

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1. Preliminaries on orthogonal similitude groups

Here we collect some basic facts, which we will need.

Let k be a field of characteristic different from 2, with separable algebraic closure $\overline{k}$. If V is a finite dimensional vector space with a non-degenerate, symmetric bilinear form B, the associated orthogonal similitude group is

(1.1)

$$GO(V,B) := \{g \in \mathrm{GL}(V) \mid B(gv, gw) = \lambda(g)B(v,w), \text{ with } \lambda(g) \in k^*, \forall v, w \in V\}.$$

The character $\lambda : GO(V,B) \to k^*$, $g \to \lambda(g)$, is the *similitude factor*. The kernel of λ is the *orthogonal group* $\mathrm{O}(V,B)$, whose elements necessarily have determinant ± 1, and the kernel of det is the *special orthogonal group* $\mathrm{SO}(V,B)$.

If $V = k^n$ with B the standard bilinear form $B_0 : (v,w) \to {}^t vw$, then one writes $\mathrm{GO}(n,k)$, $\mathrm{O}(n,k)$ and $\mathrm{SO}(n,k)$ instead of $\mathrm{GO}(V,B)$, $\mathrm{O}(V,B)$ and $\mathrm{SO}(V,B)$. Denote by $Z_n(k)$ the *center* of $\mathrm{GO}(n,k)$ consisting of all the scalar matrices cI_n, $c \in k^*$. Clearly, $\lambda(cI_n) = c^2$, so that k^{*2} is in the image of λ. The *odd dimensional case* is relatively simple. One has

Lemma 1.2 *If n is odd and $k = \overline{k}$, then we have the direct product decomposition*

$$GO(n,k) = SO(n,k) \times Z_n(k).$$

Indeed, as $k = \overline{k}$, $\lambda(Z_n(k))$ is all of k^*, and since $\mathrm{O}(n,k)$ is by definition the kernel of λ, $\mathrm{GO}(n,k)$ is generated by the normal subgroups $\mathrm{O}(n,k)$ and $Z_n(k)$. On the other hand, the intersection of these two groups is simply $\{\pm I_n\}$. Since

n is odd, the image of $det : \mathrm{O}(n,k) \to \{\pm 1\}$ is the same as that of $\{\pm I_n\}$. The assertion follows.

Note that $\mathrm{GO}(1,k) = Z_1(k) = k^*$. There is a useful description in the $n = 3$ case, which we will now recall. The *adjoint representation*

$$\mathrm{Ad} : PGL(2,k) \to GL(3,k)$$

is irreducible and self-dual with determinant 1. This identifies the image of Ad with $\mathrm{SO}(3,k)$, thanks to the simplicity of the latter. By abuse of notation, we will also write Ad for the composition with the canonical map from $\mathrm{GL}(2,k)$ onto $\mathrm{PGL}(2,k)$. This gives rise to the short exact sequence

$$(1.3) \qquad 1 \to k^* \to GL(2,k) \to SO(3,k) \to 1,$$

where the maps in the middle are $c \to cI_2$ and $g \to \mathrm{Ad}(g)$.

The *even dimensional case* $n = 2m$ is more interesting. Since for any g in $\mathrm{GO}(2m,k)$, the square of its determinant is $\lambda(g)^{2m}$, we can define a homomorphism, called the *similitude norm*

$$(1.4) \qquad \nu : GO(2m,k) \to \{\pm 1\},$$

by sending g to $\lambda(g)^{-m} det(g)$.

The kernel of ν, denoted $\mathrm{SGO}(2m,k)$, is called the *special orthogonal similitude group.* (Some people write $\mathrm{GSO}(2m,k)$ instead.) The map ν does not split. Since ν is just the determinant map on $\mathrm{O}(2m,k)$, the intersection of $\mathrm{SGO}(2m,k)$ with $\mathrm{O}(2m,k)$ is $\mathrm{SO}(2m,k)$. When $k = \mathbb{C}$, $\mathrm{SGO}(2m,k)$ (resp. $\mathrm{SO}(2m,k)$) is the connected component of $\mathrm{GO}(2m,k)$ (resp. $\mathrm{O}(2m,k)$). $\mathrm{SGO}(2,k)$ is abelian.

Note that $\nu(cI_{2m})$ is 1, and that $\mathrm{SGO}(2m,k)$ is generated by $\mathrm{SO}(2m,k)$ and $Z_{2m}(k)$; but their intersection is $\{\pm I_{2m}\}$.

We will conclude this section by recalling a low dimesional isomorphism, which we will need, between $\mathrm{GSO}(4,k)$ and a quotient of $\mathrm{GL}(2,k) \times GL(2,k)$.

Let W be k^2 with the standard symplectic form given by the determinant. Then the induced bilinear form B on the tensor product $W \otimes W$ is non-degenerate and symmetric. There is an isometry between $(W \otimes W, B)$ and (k^4, B_0). Since $\mathrm{GL}(2,k)$ is the symplectic similitude group of $(W, \det)$, we get an exact sequence

$$(1.5) \qquad 1 \to k^* \to GL(2,k) \times \mathrm{GL}(2,k) \to GO(4,k),$$

where the map on k^* is just given by $c \to cI_2$. The right map β, say, can be described explicitly as follows. The quadratic space (k^4, B_0) is also isometric to $(M_2(k), B_1)$, where B_1 is the symmetric bilinear map $(X,Y) \to {}^tXY$. Under this identification, $\beta(g,g')$ is, for all g, g' in $\mathrm{GL}(2,k)$, the automorphism of k^4 given by $X \to {}^tgXg'$. Clearly the kernel of β consists of pairs $(cI_2, c^{-1}I_2)$ with $c \in k^*$, proving the requisite exactness.

Note that $\lambda(\beta(g,g'))$ is $\det(g)\det(g')$, while the determinant of $\beta(g,g')$ is its square. Hence ν is trivial on the image of β. It is easy to see that $Z_4(k)$ lies in the image of β. Thus, by the simplicity of $\mathrm{SO}(4,k)$, we get

$$(1.6) \qquad \beta(GL(2,k) \times \mathrm{GL}(2,k)) = SGO(4,k).$$

2. The reducible case

Suppose we are given a representation ρ as in the statement of Theorem A, which is reducible. Thanks to Maschke's theorem we may write $\rho \simeq \oplus_j \rho_j$, with each ρ_j irreducible of dimension n_j, and $\sum_j n_j = 4$. Suppose we have found, for each j, a cuspidal automorphic representation $\pi_j = \pi_{j,\infty} \otimes \pi_{j,f}$ of $\mathrm{GL}(n_j, \mathbb{A}_F)$ such that $L(s, \rho_j) = L(s, \pi_{j,f})$. Then we can consider the *isobaric sum* of Langlands ([La2], [JS])

$$\pi = \boxplus_j \pi_j, \tag{2.1}$$

which is automorphic and satisfies

$$L(s, \pi) = \prod_j L(s, \pi_j).$$

Since the L-functions of Artin are also additive, we get $L(s, \rho) = L(s, \pi_f)$ as desired. So it remains to find the π_j.

Note that cuspidal automorphic representations of $\mathrm{GL}(1, \mathbb{A}_F)$ are just idele class characters of F. So when $n_j = 1$, the existence of π_j follows from class field theory.

Since the image of ρ is by hypothesis solvable, the same will be true for each ρ_j. So if $n_j = 2$, we may apply the celebrated theorem of Langlands ([La1]) and Tunnell ([Tu]) to conclude the existence of π_j.

It remains to consider the case when n_j is 3 for some j, say for $j = 1$. Then we must have a decomposition

$$\rho \simeq \rho_1 \oplus \rho_2,$$

with ρ_1 (resp. ρ_2) irreducible of dimension 3 (resp. 1). Since by hypothesis, the image of ρ lands is $\mathrm{GO}(4, \mathbb{C})$, and since there can be no intertwining between ρ_1 and ρ_j, we must have

$$im(\rho_1) \subset GO(3, \mathbb{C}).$$

Thanks to Lemma 1.2, $\mathrm{GO}(3, \mathbb{C})$ is $\mathrm{SO}(3, \mathbb{C}) \times \mathbb{C}^*$. So we may write

$$\rho_1 \simeq \rho' \otimes \chi, \tag{2.2}$$

where χ is a character $\mathfrak{G}_F \to \mathbb{C}^*$, and ρ' is a 3-dimensional representation of $\mathfrak{G}_F$ with image in $\mathrm{SO}(3, \mathbb{C})$.

Moreover, the exact sequence (1.3), which can be viewed as an exact sequence of trivial modules under $\mathfrak{G}_F = \mathrm{Gal}(\overline{F}/F)$, furnishes the cohomology exact sequence

$$\mathrm{H}om(\mathfrak{G}_F, GL(2, \mathbb{C})) \to \mathrm{H}om(\mathfrak{G}_F, SO(3, \mathbb{C})) \to H^2(\mathfrak{G}_F, \mathbb{C}^*), \tag{2.3}$$

with ρ' belonging to the middle group. On the other hand, a theorem of Tate (see [Se], for a proof) asserts that the group on the right hand side of (2.3) is trivial as F is a number field. Thus we may lift ρ' to an element of the left hand side group of (2.3). In other words, we can find a (non-unique) 2-dimensional representation τ_1 of $\mathfrak{G}_F$ such that

$$\rho_1 \simeq Ad(\tau_1) \otimes \chi. \tag{2.4}$$

Since ρ_1 has solvable image, τ_1 is also forced to have the same property. Applying Langlands-Tunnell once again, we get an isobaric automorphic representation η_1, which must in fact be cuspidal as ρ_1 and hence τ_1 are irreducible, such that $L(s, \tau_1)$ equals $L(s, \eta_{1,f})$.

By a theorem of Gelbart and Jacquet ([GeJ]) one knows that, given any cuspidal automorphic representation η of $\mathrm{GL}(2, \mathbb{A}_F)$, there exists a functorially associated (isobaric) automorphic representation $\mathrm{Ad}(\eta)$ such that

$$L(s, \mathrm{Ad}(\eta)) = \prod_v L(s, \mathrm{Ad}(\sigma_v(\eta))), \tag{2.5}$$

where the product is over all the places v of F, and $\sigma_v(\eta)$ is the 3-dimensional representation of the Weil group (resp. Weil-Deligne group) W_{F_v} (resp. W'_{F_v}) when v is archimedean (resp. non-archimedean), associated to η_v by the local Langlands correspondence for GL(3)([He]).

Then it follows that

$$L(s, \rho_1) = L(s, (\mathrm{Ad}(\eta_1) \otimes \chi)_f).$$

So we are done by setting $\pi_1 = \mathrm{Ad}(\eta_1) \otimes \chi$. Done. □

3. Modularity modulo Theorem D

In this section we will show how to prove Theorem A if we admit the truth of Theorem D. Thanks to the discussion in the previous section, we may assume that ρ is irreducible.

By hypothesis, the image of ρ lies in $\mathrm{GO}(4, \mathbb{C})$. Recall from section 1 the definion of the subgroup $\mathrm{SGO}(4, \mathbb{C})$, which is the kernel of the *similitude norm*

$$\nu : GO(4, \mathbb{C}) \to \{\pm 1\}.$$

Let K be the extension of F defined by the kernel of $\nu \circ \rho$. Then $[K : F] = 2$. Write ρ_K for the restriction of ρ to $\mathfrak{G}_K = \mathrm{Gal}(\overline{F}/K)$. Thanks to (1.5) and (1.6), one has the following short equence of trivial Galois modules:

$$1 \to \mathbb{C}^* \to GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \to SGO(4, \mathbb{C}) \to 1, \tag{3.1}$$

where the maps in the middle are $c \to (cI_2, c^{-1}I_2)$ and $(g, g') \to (X \to {}^t gXg')$. The associated (continuous) cohomology exact sequence gives

(3.2)

$$Hom(\mathfrak{G}_K, \mathbb{C}^*) \to Hom(\mathfrak{G}_K, GL(2, \mathbb{C}) \times GL(2, \mathbb{C})) \to Hom(\mathfrak{G}_K, SGO(4, \mathbb{C})) \to H^2(\mathfrak{G}_K, \mathbb{C}^*),$$

with ρ_K belonging to the second group from the right. Recall Tate's theorem which says that the first group on the right is trivial. So we may find σ, σ' in $Hom(\mathfrak{G}_K, GL(2, \mathbb{C}))$ such that

$$\rho \simeq \sigma \otimes \sigma'. \tag{3.3}$$

Note that this lifting is unique only up to changing (σ, σ') by $(\sigma \otimes \mu, \sigma' \otimes \mu^{-1})$, for any character $\mu \in Hom(\mathfrak{G}_K, \mathbb{C}^*)$.

Since the image of ρ was assumed to be solvable, we see easily that the images of σ, σ' should also be solvable. And since ρ is irreducible, the same should hold for σ and σ'. So we may apply the theorem of Langlands and Tunnell to deduce the existence of cuspidal automorphic representations π, π' of $\mathrm{GL}(2, \mathbb{A}_F)$, respectively associated to σ, σ'.

Now suppose the image of ρ lands in $\mathrm{SGO}(4,\mathbb{C})$ itself, in which case $K = F$. Then by the main theorem of [Ra], we know the existence of an isobaric automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}(4, \mathbb{A}_F)$ such that

$$L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi'), \tag{3.4}$$

where the L-function on the right is the Rankin-Selberg L-function associated to the pair (π, π'). In addition, we have at any place v, the local factors of $\pi \boxtimes \pi'$ identify functorially with those of the tensor product $\sigma_v(\pi) \otimes \sigma_v(\pi')$ of the local Langlands parameters $\sigma(\pi), \sigma(\pi')$ ([Ku]). Since π (resp. π' is associated to σ (resp. σ'), the local representations $\sigma_v(\pi)$ (resp. $\sigma_v(\pi')$) are isomorphic to the ones defined by the restriction at v of σ (resp. σ'). Thus the automorphic representation Π of $\mathrm{GL}(4, \mathbb{A}_F)$, whose existence is predicted by Theorem A, is none other than $\pi \boxtimes \pi'$. The cuspidality criterion of [Ra] shows easily that, since ρ is irreducible, Π must be cuspidal. Indeed, the irreducibility of ρ implies that σ, σ' cannot be dihedral, and moreover, σ' cannot be of the form $\sigma \otimes \chi$ for any character χ of $\mathfrak{G}_K$; consequently, π, π' are non-dihedral and are not related by a character twist, which is precisely what one needs to ensure the cuspidality of Π (see [Ra] for details). We are done in this case.

We may henceforth assume that $[K : F] = 2$, which is the subtler case. Denote by θ the non-trivial automorphism of K over F. We can again find cuspidal automorphic representations π, π' of $\mathrm{GL}(2, \mathbb{A}_K)$ such that

$$L(s, \rho_K) = L(s, \pi_f \boxtimes \pi'_f), \tag{3.5}$$

which proves that the restriction ρ_K of ρ to $\mathfrak{G}_K$ is modular.

We will now explain why this case is difficult. The identity (3.5) implies that $\pi \boxtimes \pi'$ is θ-invariant, so by the base change theorem of Arthur and Clozel ([AC]), we can find an isobaric automorphic representation Π of $\mathrm{GL}(4, \mathbb{A}_F)$ such that its base change Π_K is isomorphic to $\pi \boxtimes \pi'$. One can also see easily that the local factors of Π and ρ agree at all the places of F which split in K. But one is stuck at this point and and cannot deduce the requisite identity at the inert places, except when ρ_K is no longer simple.

Suppose ρ_K is *reducible*. Pick any irreducible summand τ of ρ_K. Then by Frobenius reciprocity, ρ should intertwine with the induction $\mathrm{Ind}_K^F(\tau)$ of τ to $\mathfrak{G}_F$. As ρ is irreducible, we are forced to have

$$\rho \simeq \mathrm{Ind}_K^F(\tau),$$

with $\dim(\tau) = 2$. The solvability of the image of ρ implies the same about that of τ, and so we may apply Langlands-Tunnell to get a cuspidal automorphic representation η of $\mathrm{GL}(2, \mathbb{A}_K)$ associated to τ, and we are done by taking Π to be the automorphically induced representation $I_K^F(\tau)$ (see [AC], and also [Ra], sec. 2).

So we may, and we will, assume that ρ_K is irreducible. Since it is the restriction of ρ, we have

$$(\sigma \otimes \sigma')^{[\theta]} \simeq \rho_K^{[\theta]} \simeq \rho_K \simeq \sigma \otimes \sigma^{[\theta]}, \tag{3.6}$$

where θ is the nontrivial automorphism of K over F. The irreducibility of ρ_K implies that σ, σ' are non-dihedral and are not related by a character twist. So

(3.3) forces the identities $\sigma^{[\theta]} \simeq \sigma' \otimes \mu$ and $(\sigma')^{[\theta]} \simeq \sigma \otimes \mu^{-1}$, for some character μ of $\mathfrak{G}_K$. This gives

$$(3.7) \qquad \sigma \simeq (\sigma^{[\theta]})^{[\theta]} \simeq (\sigma')^{[\theta]} \otimes \mu^{\theta]} \simeq \sigma \otimes (\mu^\theta/\mu).$$

Since σ is not dihedral, it does not admit any non-trivial self-twist by a character, and so we must have $\mu = \mu^\theta$. So there exists a character ν of $\mathfrak{G}_F$ such that μ is the restriction ν_K of ν to $\mathfrak{G}_K$. Thus we get (from (3.4),

$$(\rho \otimes \nu)_K \simeq \sigma \otimes (\sigma' \otimes \mu) \simeq \sigma \otimes \sigma^{[\theta]}.$$

It suffices to show that some character twist of ρ is modular. So we may, after replacing ρ by its twist by ν^{-1}, that

$$(3.8) \qquad \rho_K \simeq \sigma \otimes \sigma^{[\theta]}.$$

If δ denotes the quadratic character of $\mathfrak{G}_F$ corresponding to K/F, then ρ and $\rho \otimes \delta$ are the only representations for which (3.8) holds.

It is easy to see that the induction (to $\mathfrak{G}_F$) of the exterior square, i.e., the determinant, of σ is a summand of the exterior square of the induction of σ. Thanks to semisimplicity, we may then define the *Asai representation* of σ, denoted $As(\sigma)$, by the decomposition

$$(3.9) \qquad \Lambda^2(\mathrm{Ind}_K^F(\sigma)) \simeq As(\sigma) \oplus \mathrm{Ind}_K^F(det(\sigma)).$$

Lemma 3.10 *ρ is isomorphic to $As(\sigma)$ or $As(\sigma) \otimes \delta$.*

Proof of Lemma. Write β denote the tensor square representation of $\mathrm{Ind}_K^F(\sigma)$, so that

$$(3.10) \qquad \beta = \Lambda^2(\mathrm{Ind}_K^F(\sigma)) \oplus Sym^2(\mathrm{Ind}_K^F(\sigma)).$$

We can also write

$$(3.11) \qquad \beta \simeq \mathrm{Ind}_K^F(\sigma \otimes \mathrm{Ind}_K^F(\sigma)_K),$$

which implies that

$$(3.12) \qquad \beta \simeq \beta \otimes \delta.$$

So $As(\sigma) \otimes \delta$ must also occur in β.

On the other hand, since the restriction to $\mathfrak{G}_K$ of $\mathrm{Ind}_K^F(\sigma)$ is $\sigma \oplus \sigma^{[\theta]}$, we get from (3.11),

$$(3.12) \qquad \beta \simeq \mathrm{Ind}_K^F(Sym^2(\sigma) \oplus \Lambda^2(\sigma) \oplus (\sigma \otimes \sigma^{[\theta]}).$$

Since ρ_K is (by (3.8)) isomorphic to $\sigma \otimes \sigma^{[\theta]}$, it must occur in the induction of the latter to $\mathfrak{G}_F$; ditto for the twist of ρ by δ. Hence, by the additivity of induction, the representation on the right of (3.12) is forced to be

$$\mathrm{Ind}_K^F(Sym^2(\sigma)) \oplus \mathrm{Ind}_K^F(\Lambda^2(\sigma)) \oplus \rho \oplus (\rho \otimes \delta).$$

The lemma now follows in view of (3.9). □

So we may, after possibly replacing ρ by $\rho \otimes \delta$, assume that

$$(3.13) \qquad \rho \simeq As(\sigma),$$

for an irreducible 2-dimensional, continuous $\mathbb{C}$-representation σ of $\mathfrak{G}_K$ with solvable image.

For any cuspidal automorphic representation η of $\mathrm{GL}(2, \mathbb{A}_K)$, one may associate the following *Asai L-function:*

$$L(s,\eta,r) = \prod_v L(s, As(\sigma_v(\eta))), \tag{3.14}$$

where v runs over the set Σ_F of all places of F, and $As(\sigma_v(\eta))$ denotes the Asai representation associated to $\sigma_w(\eta)$; when v splits in K, $K_v = K_w \times K_{\theta w}$, and $As(\sigma_v(\eta))$ simply means the tensor product $\sigma_w(\eta) \otimes \sigma_{\theta w}(\eta)$.

The L-function on the left of (3.14) looks like a *Langlands L-function*, and we need to explain why we are justified in adopting such a notation. For this recall that the L-group of the restriction of scalars of $\mathrm{GL}(2)/K$ to F is the semidirect product

$$^L(R_{K/F}\mathrm{GL}(2)/K) = (\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})) \times Gal(K/F), \tag{3.15}$$

where θ acts by interchanging the two factors. One defines a representation

$$r : {}^L(R_{K/F}\mathrm{GL}(2)/K) \to \mathrm{GL}(\mathbb{C}^2 \otimes \mathbb{C}^2) \simeq \mathrm{GL}(4,\mathbb{C}) \tag{3.16}$$

by setting, for all x, y in $\mathbb{C}^2$,

$$r(g,g';1)(x \otimes y) = g(x) \otimes g(y)$$

and

$$r(1,1;\theta)(x \otimes y) = y \otimes x.$$

At any finite place w of K where η is unramified, there is a diagonal matrix $[\alpha_w, \beta_w]$ in $\mathrm{GL}(2,\mathbb{C})$ such that

$$L(s,\eta_w) = \frac{1}{(1-\alpha_w q_w^{-s})(1-\beta_w q_w^{-s})}, \tag{3.17}$$

where q_w is the norm of w. If v is any finite place of F which is unramified for $(K/F, \eta)$, i.e., if v is unramified in K and if η is unramified at any place w of K above v, then one may associate, as in [HLR], a (Langlands) conjugacy class $A_v(\eta)$ in $^L(R_{K/F}\mathrm{GL}(2)/K)$. When composed with r, one gets

$$r(A_v(\eta)) = [\alpha_w\alpha_{\theta w}, \alpha_w\beta_{\theta w}, \beta_w\alpha_{\theta w}, \beta_w\alpha_{\theta w}] \tag{3.18}$$

if v *splits* into $(w, \theta w)$ in K, and

$$r(A_v(\eta)) = \begin{pmatrix} \alpha_v & 0 & 0 & 0 \\ 0 & 0 & \alpha_v & 0 \\ 0 & \beta_v & 0 & 0 \\ 0 & 0 & 0 & \beta_v \end{pmatrix}$$

if v remains prime. Since $L(s,\eta_w)$ is $L(s,\sigma_w(\eta))$, we get easily the identity

$$L(s, As_v(\sigma(\eta))) = L(s, r(A_v(\eta))) \tag{3.19}$$

at any finite place v unramified for $(K/F, \pi)$. This shows the appropriateness of the notation of (3.14). It is also important because the automorphic results we will need later will use the Langlands formalism.

If we admit Theorem D, we then have a unique isobaric automorphic representation Π of $\mathrm{GL}(4, \mathbb{A}_F)$ such that

$$L(s,\Pi) = L(s,\pi,r).$$

In view of the discussion above, it is clear that

$$L(s,\Pi_v) = L(s,\rho_v) \tag{3.20}$$

at almost all places v. By a standard argument comparing the functional equations of $L(s,\Pi)$ and $L(s,\rho)$, we also get such an equality of L-factors at *every* place v. Since ρ_K is by construction associated to $\pi \boxtimes (\pi \circ \theta)$, we get the identity

$$L(s,\Pi_K) = L(s,\pi \boxtimes (\pi \circ \theta)). \tag{3.21}$$

Since $\rho_K \simeq (\sigma \otimes (\sigma^{[\theta]})$ is irreducible, the main result of [Ra] implies that $\pi \boxtimes (\pi \circ \theta)$ is cuspidal. Since Π base changes to a cuspidal representation, it must itself be cuspidal by [AC].

This finishes the proof of Theorem A modulo Theorem D. □

4. Distinguished representations

Let K/F be a quadratic extension of number fields with $\mathrm{Gal}(K/F) = \{1,\theta\}$. The object of this section is to establish Theorem E for the nice subclass of *distinguished* cusp forms π ([HLR]) on GL(2)/K. It is necessary to treat this case separately as character twists of the Asai L-function of π will in such a case admit poles, making inoperative the argument using the converse theorem, which we utilize for π of *general type* in the next section.

We will use the following notation. If χ is an idele class character of K, we will write χ_0 for its restriction to F. (This corrsponds to taking the *transfer* of the corresponding Galois character.) Moreover, if μ is a character of F, then we will write μ' to signify any character of K such that $\mu = \mu'_0$. (This corresponds to the restriction of the corresponding Galois character.) If μ'_1 is another extension of μ, then there exists a character ν of K such that

$$\mu'_1 = \mu'(\nu/(\nu \circ \theta)). \tag{4.1}$$

This is because any character of K whose restriction to F is trivial lies in $\mathrm{Ker}(\theta - 1)$.

Let π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_K)$ with space $\mathcal{V}_\pi$. If μ is a unitary character of F, then π is said to be *μ-distinguished* ([HLR]) iff the following *μ-period integral* is non-zero for some function f in $\mathcal{V}_\pi$:

$$\mathcal{P}_\mu(f) := \int_{H(F)Z_H(F_\infty)^+\backslash H(\mathbb{A}_F)} \mu(det(h)f(h)dh, \tag{4.2}$$

where H denotes $\mathrm{GL}(2)/F$ with center Z_H, and dh is the quotient measure induced by the Haar measure on $H(\mathbb{A}_F)$. It may be useful to note for the uninitiated that when $F = \mathbb{Q}, K$ real quadratic, $f \in \pi$ a holomorphic newform of weight $(2,2)$, $\mathcal{P}_\mu(f)$ is the μ-twisted integral of the $(1,1)$ differential form $(2\pi i)^2 f(z_1,z_2)dz_1 \wedge \overline{dz_2}$ on the associated Hilbert modular surface over (the homology class of) the modular curve; so one is justified in calling this a period integral.

A basic result of [HLR], section 2, asserts that, once we have fixed an extension μ' of μ, the necessary and sufficient condition for π to be μ-distinguished is that there exists a cuspidal automorphic representation π_0 of $H(\mathbb{A}_F)$ with central character $\nu\delta$ such that

$$\pi_{0,K} \simeq \pi \otimes \nu'\mu', \tag{4.3}$$

for a suitable extension ν' of ν.

Fix such a μ-distinguished π with (π_0, ν) as above. Since $\pi \boxtimes (\pi \circ \theta)$ is θ-invaraint, it descends (by [AC]) to an isobaric automorphic representation of $\mathrm{GL}(4, \mathbb{A}_F)$. We can give an explicit candidate for this descent by setting

$$\Pi := Sym^2(\pi_0) \otimes \delta(\mu\nu)^{-1} \boxplus \delta\mu^{-1}. \tag{4.5}$$

That the base change Π_K is $\pi \boxtimes (\pi \circ \theta)$ is easily deduced from (4.3). There are at least four possible descents, namely by leaving in or removing the character δ at the places where it appears in (4.5), and this is why we needed to make a specific choice. Note also that the automorphic induction of π to F satisfies

$$I_K^F(\pi) \simeq \pi_0 \boxtimes I_K^F((\mu'\nu')^{-1}). \tag{4.6}$$

It suffices to prove that the local factors of $L(s, \Pi)$ and $L(s, \pi; r)$ agree almost everywhere, because the remaining factors must also agree by a standard argument comparing their respective functional equations ([J], [HLR]) and using a modest estimate on the cefficients, which we have here. (See [Ra], page 14, for the details of such an argument.)

Let v be a finite place where π and K/F are unramified. If v splits in K, the desired identity is immediate. So assume v is inert, and denote the unique place of K above it by w. Recall that the exterior square of a tensor product $V \otimes W$ is the direct sum of $\mathrm{Sym}^2(V) \otimes \Lambda^2(W)$ and $\mathrm{Sym}^2(W) \otimes \Lambda^2(V)$. Using this conjunction with (4.6), and by the compatibility of local and global automorphic induction, we have

(4.7)

$$\Lambda^2(\sigma_v(I_K^F((\pi))) \simeq Sym^2(\sigma_v(\pi_0)) \otimes \delta_v(\mu_v\nu_v)^{-1} \oplus Sym^2(\mathrm{Ind}_{K_w}^{F_v}((\mu_w'\nu_w')^{-1})) \otimes \nu\delta.$$

We also have

$$Sym^2(\mathrm{Ind}_{K_w}^{F_v}((\mu_w'\nu_w')^{-1})) \simeq \mathrm{Ind}_{K_w}^{F_v}((\mu_w'\nu_w')^{-2}) \oplus (\mu\nu)^{-1}. \tag{4.8}$$

Combining these two identities with the fact that the induced module on the right of (4.8) is simply the induction of the determinant of $\sigma_w(\pi)$, we get, from the definition of the Asai representation

$$As_v(\sigma(\pi)) \simeq Sym^2(\sigma_v(\pi_0)) \otimes \delta_v(\mu_v\nu_v)^{-1} \oplus \delta\mu^{-1}. \tag{4.8}$$

Its L-factor, in view of (4.5), coincides with that of $\sigma_v(\Pi)$. Done.

□

5. Asai L-functions and GL(4)

Fix K/F quadratic above with non-trivial automorphism θ as above, and a cuspidal automorphic representation π of $\mathrm{GL}(2, \mathbb{A}_K)$. Let $m \in \{1, 2\}$. Then for any cuspidal automorphic representation η of $\mathrm{GL}(m, \mathbb{A}_F)$, one may define the *$\eta$-twisted Asai L-function* of π by setting

$$L(s, \pi; r \otimes \eta) = \prod_v L(s, r(A_v(\pi)) \otimes \sigma_v(\eta)), \tag{5.1}$$

which converges normally in a right half plane and defines a holomorphic function there. It is also known (cf. [PS-R] for m=2, and [HLR] for m=1) that this L-function admits a meromorphic continuation to the whole s-plane, and satisfies a standard functional equation relating it to $L(1-s, \pi^\vee; r \otimes \eta^\vee)$. Of course, when $m = 1$, η is just an idele class character of F with $\eta^\vee = \eta^{-1}$.

Suppose χ is an idele class character of K with restriction χ_0 to F. Unwinding the definition, one gets

$$L(s, \pi \otimes \chi; r \otimes \eta) = L(s, \pi; r \otimes (\eta \otimes \chi_0)). \tag{5.2}$$

It also follows from the definition that

$$L(s, \pi \boxtimes (\pi \circ \theta) \otimes \eta_K) = L(s, \pi; r \otimes \eta) L(s, \pi; r \otimes (\eta \otimes \delta)), \tag{5.3}$$

where again δ denotes the quadratic character of F defined by K/F.

The first step in proving Theorem D is to analyze when the poles of these η-twisted Asai L-functions could occur. This is easy for $m = 1$ (using [HLR]), while for $m = 2$ we make use of Ikeda's work [Ik1], plus a supplementary local hypothesis. Eventually, after we prove the existence of Π, the best possible result will be a consequence.

Proposition 5.4 *Let $(\pi, K/F, \theta, \delta)$ be as above, and let η be a cusp form on $GL(m)/F$, with $m = 1, 2$. Suppose π is not distinguished, and that if $m = 2$, η_v is unramified at any finite place v which is ramified for $(K/F, \pi)$. Then*

(a) *If $m = 1$, $L(s, \pi; r \otimes \eta)$ is entire;*
(b) *For $m = 2$, the same holds if π is either non-dihedral or automorphically induced by a ramified character of a quadratic extension.*

When we say that π is not distinguished, we mean that it is not μ-distinguished for *any* character μ of F; so being distinguished is a property shared by all the character twists. And when we say that v is ramified for $(K/F, \pi)$, we mean that either v ramifies in K or π is ramified at a place w above v; clearly, the set of such "bad" places is finite.

Proof. (a) Suppose $L(s, \pi; r \otimes \mu)$ has a pole for an idele class character μ of F. Then, up to replacing π with $\pi \otimes |.|^{s_0}$ for some s_0, we may assume the pole to be at $s = 1$. Let S be the finite set of places containing the archimedean and ramified places for π. Since the local factors have no zeros, the incomplete L-function $L^S(s, \pi; r \otimes \mu)$ has also has a pole at $s = 1$. It is known that the pole must be simple. Moreover, by Asai's integral representation ([HLR]), the residue at $s = 1$ of this incomplete L-function is a non-zero multiple of the μ-period $\mathcal{P}_\mu(f)$ (see (4.2)), which means π is distinguished. Done.

(b) Let S be as in (a), and η a cusp form on $\mathrm{GL}(2)/F$ whose conductor has no intersection with the finite places v ramified for $(K/F, \pi)$. Again we may assume that the pole is at $s = 1$. Denote by $L_1(s, \pi, r \otimes \eta)$ the L-function defined as the gdc of the integral representation of Piatetski-Shapiro and Rallis ([PS-R]). Then it is well known that

$$L^S(s, \pi; r \otimes \mu) = L_1^S(s, \pi; r \otimes \mu). \tag{5.5}$$

Applying Theorem 2.8 of [Ik1], and taking note of the remarks in the proof of Theorem 3.3.11 of [Ra] about Ikeda's theorem, we see that the (incomplete) L-function on the right, hence left, of (5.5) admits no pole only in the following special case: There exists a quadratic extension E/F, and a character χ of KE such that

(i) $\pi \simeq I_K E^K(\chi)$, and
(ii) $\eta \simeq I_E^F(\chi_1)$, where χ_1 is the inverse of the restriction of χ to the idele class group of E.

In particular, there is no pole when π is non-dihedral. Further, in the dihedral case, since we assumed that π is induced by a ramified character, any pole here will imply that there is a finite place where π and η are both ramified. But this contradicts the choice of η and we are done.

□

Theorem 5.6 *Let F be a totally imaginary number field, and K a quadratic extension with non-trivial automorphism θ over F. Let π be a non-distinguished, cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$ which is either non-dihedral or automorphically induced by a ramified character of a quadratic extension of K, and η a cuspidal automorphic representation of $GL(m, \mathbb{A}_F)$, $m = 1, 2$, such that if $m = 2$, the following are satisfied:*

- *the conductors of π and η have disjoint support;*
- *If v is a finite place of F which is ramified for either π or η, then v splits in K.*

Then $L(s, \pi; r\otimes\eta)$ is entire and bounded in vertical strips, and moreover, it admits a meromorphic continuation to the whole plane and satisfies the following functional equation

$$L(1-s, \pi^\vee; r\otimes\eta^\vee) = \varepsilon(s, \pi; r\otimes\eta)L(s, \pi; r\otimes\eta), \tag{5.7}$$

where the everywhere invertible ε-factor is defined by

$$\varepsilon(s, \pi; r\otimes\eta) = \prod_v \varepsilon(s, As(\sigma_v(\pi))\otimes\sigma_v(\eta)). \tag{5.8}$$

The local factors on the right of (5.8) are the ones defined by the theory of Langlands and Deligne.

Proof. The hypotheses assure us, thanks to Proposition 5.4, that $L(s, \pi; r\otimes\eta)$ is entire. Since the L-function defined by the integral representation, namely $L_1(s, \pi; r\otimes\eta)$ satisfies the functional equation of the requisite type ([PS-R]), it suffices, for this part, to check the following

Lemma 5.9 *Preserve the hypotheses of Theorem 5.6. Then for any character ν of F, the local factors of $L(s, \pi; r\otimes\eta\otimes\nu)$ and $L_1(s, \pi; r\otimes\eta\otimes\nu)$ agree at every place.*

Proof of Lemma. First consider the case when v is archimedean. Since F is assumed to be totally imaginary, and this is important, the group of F_v-points of $(R_{K/F}GL(2)/K\times GL(2)$ identifies with $\mathrm{GL}(2,\mathbb{C})^3$, and the assertion of the Lemma is a consequence of Proposition 3.3.2 of [Ra], which relies in turn on Ikeda's results ([Ik1,2]).

So we may assume that v is a finite place. Again, if v splits in K, say into $w, \overline{w}$, then the group of F_v-rational points of $R_{K/F}GL(2)/K\times GL(2)/F$ identifies with $\mathrm{GL}(2, F_v)\times\mathrm{GL}(2, F_v)\times\mathrm{GL}(2, F_v)$. So we are done by the triple product result of [Ra], Prop. 3.3.2. So assume that v is inert or ramified in K, and denote by w the unique place above v. By hypothesis, both π_w and η_v are unramified, the two local factors coincide by [PS-R]; in fact, this holds for any character twist. Hence the lemma.

Proof of Theorem 5.6 (contd.): Let S be the (finite) set of places of F containing the archimedean ones and those finite ones ramifiying in K or for π or η. Then the integral representation of [PS-R] implies the following:

$$(5.10) \qquad L_1(s,\pi;r\otimes\eta)\prod_{v\in S}\frac{\Psi(f_{v,s};W_v)}{L_1(s,\pi_v;r\otimes\eta_v)} = \langle E(f_s),\varphi\otimes\varphi'\rangle_H,$$

where φ (resp. φ') is a cusp form in the space of π (resp. η), $E(f_s)$ is the Sigel Eisenstein series on GSP(6)/F (see [Ra], sec.3.4, and [Ik1]) associated to a good section $f_s = \otimes_v f_{v,s}$, $\Psi(f_{v,s};W_v)$ is, for each v, a local integral having $L_1(s,\pi_v;r\otimes\eta_v)$ as its gcd for a suitable $f_{v,s}$ and Whittaker function $W)v$,

$$H := \{(g,g')\in R_{K/F}GL(2)/K\times GL(2)/F \mid \det(g)=det(g')\}$$

with center C, and

$$\langle E(f_s),\varphi\rangle_H := \int_{C(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} E(h,f_s)\varphi(h)dh) = \prod_v \Psi(f_{v,s};W_v).$$

Using Lemma 3.4.5 of [Ra] and the hypotheses of this Theorem, we get

Lemma 5.11. *For each v, the function $\frac{\Psi(f_{v,s};W_v)}{L_1(s,\pi_v;r\otimes\eta_v)}$ is entire and of bounded order, for a suitable choice of $f_{v,s}$.*

By Proposition 3.4.6 of [Ra], we also know that $E(f_s)$ is a function of bounded order. Since $\varphi\otimes\varphi'$ vanishes rapidly at infinity, we deduce, using (5.10), that $L_1(s,\pi;r\otimes\eta)$ is of bounded order in vertical strips of finite width. The same holds then for $L(s,\pi;r\otimes\eta)$ by Lemma 5.9. On the other hand, since this L-function has an Euler product, is bounded for large positive $\Re(s)$, and hence for large negative $\Re(s)$. Applying the Phragman-Lindelöf theorem, we then conclude the boundedness in vertical strips of $L(s,\pi;r\otimes\eta)$ as asserted.

Done with the proof of Theorem 5.6. □

The next object is to establish the following

Proposition 5.12 Let K/F be a quadratic extension of number fields with K totally imaginary, and let π be a cuspidal automorphic representation of $GL(2,\mathbb{A}_K)$ such that the finite number of finite places w of K where π_w is ramified are all of degree 1 over F. Then there exists an irreducible, admissible, generic representation $\Pi=\otimes_v\Pi_v$ of $GL(4,\mathbb{A}_F)$ such that, for any place v of F, we have

$$L(s,\Pi_v\times\eta_v) = L(s,\pi_v;r\otimes\eta_v)$$

and

$$\varepsilon(s,\Pi_v\times\eta_v) = \varepsilon(s,\pi_v;r\otimes\eta_v).$$

Proof. If v is an archimedean or a ramified place of F, or if v is split in K, the ramification hypothesis allows us to reduce the definition of Π_v to the Rankin-Selberg product case, as was done on page 31 of [Ra]. So let v be finite and unramified with unique divisor w in K. Then we may write π_w as the isobaric sum $\mu_1\boxplus\mu_2$, so that $\sigma_w(\pi_w)\simeq\mu_1\oplus\mu_2$ and

$$(5.13) \qquad \Lambda^2(Ind_{K_w}^{F_v}(\sigma_w(\pi_w)) \simeq Ind_{K_w}^{F_v}(\mu_1)\otimes Ind_{K_w}^{F_v}(\mu_2)\oplus\mu_{1,0}\delta_v\oplus\mu_{2,0}\delta_v.$$

Here we have used the fact that the determinant of $\mathrm{Ind}_{K_w}^{F_v}(\mu_j)$ is the product of the restriction $\mu_{j,0}$ of μ_j to F_v^* times the quadratic character δ_v associated to K_w/F_v.

Since the determinant of $\mu_1 \oplus \mu_2$ is $\mu_1\mu_2$, which occurs in $Ind_{K_w}^{F_v}(\mu_1) \otimes Ind_{K_w}^{F_v}(\mu_2)$, we get

$$As(\sigma_v(I_{K_w}^{F_v}(\pi_w))) \simeq Ind_{K_w}^{F_v}(\mu_1(\mu_2 \circ \theta)) \oplus \mu_{1,0}\delta_v \oplus \mu_{2,0}\delta_v.$$

Consequently, if we set

$$(5.14) \qquad \Pi_v = I_{K_w}^{F_v}(\mu_1(\mu_2 \circ \theta)) \boxplus \mu_{1,0}\delta_v \boxplus \mu_{2,0}\delta_v,$$

we will have the properties asserted in the Proposition.

□

Proposition 5.15 *Let $(K/F, \pi, \eta)$ be as in Theorem 5.6, and let Π be as in Proposition 5.12. Then Π is a cuspidal automorphic representation of $GL(4, A_K)$.*

Proof. We may assume that π is not distinguished. We will need to appeal to the following

Theorem 5.16 (Cogdell - Piatetski-Shapiro [CoPS]) *Let T be a fixed finite set of finite places of F. Let β be an irreducible unitary, admissible, generic representation of $GL(4, \mathbb{A}_F)$ which satisfies the following:*
For every cuspidal η on $GL(m)/F$, $m \leq 2$, with η_v unramified at every v in T, we have:

(MC) $\qquad L(s, \beta \times \eta)$ and $L(s, \beta^\vee \times \eta^\vee)$ converge absolutely in large $\Re(s)$,

and they admit meromorphic continuations to the whole s-plane.

(E) $\qquad L(s, \beta \times \eta)$ and $L(s, \beta^\vee \times \eta^\vee)$ are entire.

There is a functional equation

$$(FE) \qquad L(1 - s, \beta^\vee \times \eta^\vee) = \varepsilon(s, \beta \times \eta)L(s, \beta \times \eta).$$

(BV) $\qquad L(s, \beta \times \eta)$ is bounded in vertical strips.

Then β is nearly automorphic, i.e., there exists a generic, isobaric automorphic representation β_1 of $GL(4, \mathbb{A}_F)$ such that $\beta_v \simeq \beta_{1,v}$ for almost all v.

Thanks to our hypotheses, Theorem 5.6, Lemma 5.9 and Proposiion 5.12, our Π staisfies the (MC), (E), (FE) and (BV). So we may apply Theorem 5.16 to conclude that there exists a generic, isobaric automorphic representation Π_1 which is almost everywhere equivalent to Π. Now Π and Π_1 are both generic and they both have their own functional equations. Comparing $L(s, \Pi \times \eta)$ and $L(s, \Pi_1 \times \eta)$, for all η as above, we conclude that the local factors of pairs agree at every place. Since this determines the local components by a result of Jeff Chen, we see that Π is itself automorphic.

It remains to show that Π is cuspidal. Suppose not, and pick a finite set S of places containing the archimedean and ramified places. We can decompose Π, for some $r > 1$, as an isobaric sum:

$$\Pi \simeq \boxplus_j^r \beta_j,$$

where β_j cuspidal on $\mathrm{GL}(n_j)/F$. Since $\sum_j^r n_j = 4$ and $r > 1$, at least one of the indices, say n_j must be ≤ 2. Then we have

$$L^S(s, \Pi \times \beta_k^\vee) = \prod_j^r L^S(s, \beta_j \times \beta_k^\vee).$$

By the Rankin-Selberg theory ([JS]) one knows that $L^S(s, \beta_k \times \beta_k^\vee)$ has a simple pole at $s = 1$, while for for any j, $L^S(s, \pi_j \times \pi_k^\vee)$ has no zero at $s = 1$. Cosequently, $L^S(s, \Pi \times \beta_k^\vee)$ has a pole at $s = 1$. But this incomplete L-function is the same as $L_1^S(s, \Pi \times \beta_k)$, which is entire. So we get a contradiction, proving that Π needed to be cuspidal. □

It remains to prove Theorem D in the general case. Let $(K/F, \pi)$ be arbitrary, but with π not distinguished. Note that we can always find, using Lemma 3.7.1 of [Ra], a finite, solvable, Galois extension E/F, disjoint from K/F such that E is totally imaginary, with $(EK/E, \pi_E)$ satisfying the hypotheses of Theorem 5.6. So by Proposition 5.15, the associated representation Π^E is cuspidal, automorphic on $\mathrm{GL}(4)/E$. Now by using the descent argument of section 3.7 (and Proposition 6.1) of [Ra], we get a unique descent Π, which is cuspidal automorphic and satisfies

$$L(s, \Pi) = L(s, \pi; r),$$

as desired. Theorem D is now proved. □

6. New cases of Artin's conjecture

Let ρ, ρ' be continuous $\mathbb{C}$-representations of solvable GO(4)-type. By Theorem A, they are modular, associated to isobaric automorphic representations π, π' of $\mathrm{GL}(4, \mathbb{A}_F)$. Then

$$L(s, \sigma \otimes \sigma') = L(s, \pi_f \times \pi'_f).$$

Since the Rankin-Selberg theory of Jacquet, Piatetski-Shapiro and Shalika, and of Shahidi, says that the L-function on the right is entire (see [MW] and the references therein), Corollary B follows immediately.

Now we show how this gives new examples where Artin's conjecture holds. Fix any quadratic extension E/F with non-trivial automorphism θ, and choose an irreducible quartic polynomial f in $\mathfrak{O}_E[X]$ whose discriminant D_f is square-free. Then it is easy to see that the Galois group of the splitting field K, say, of f over E is the symmetric group S_4. Since S_4 is a subgroup of $\mathrm{PGL}(2, \mathbb{C})$, we get a projective representation $\overline{\sigma}$ of $\mathrm{Gal}(\overline{F}/F)$. Fix a lifting σ into $\mathrm{GL}(2, \mathbb{C})$ (by using Tate, as in section 2), and denote by L the quadratic extension of K corresponding to $\ker(\sigma)$. Then $\mathrm{Gal}(L/F)$ will be $\tilde{S}_4$, a double cover of S_4. Now restrict the choice of f so that the discriminant D_{f^θ} of the conjugate polynomial f^θ is not a square in $E(\sqrt{D_f})$. Then the splitting field K^θ of f^θ is linearly disjoint from K over E. The corresponding L^θ will also be linearly disjoint from L. There are clearly an infinite number of choices for f satisfying these conditions. Note that the Galois group of LL^θ over F is a non-trivial extension of $\mathbb{Z}/2$ by $\tilde{S}_4 \times \tilde{S}_4$, and the resulting representation ρ of $\mathrm{Gal}(\overline{F}/F)$ is irreducible and of GO(4)-type. Now choose a disjoint quadratic extension E'/F and a quartic polynomial g satisfying analogous properties over E'. We can arrange, in an infinite number of ways, for the resulting extension of F to be linearly disjoint from LL^θ. Denoting by ρ' the corresponding representation of $\mathrm{Gal}(\overline{F}/F)$, we see that $\rho \otimes \rho'$ is irreducible and satisfies the Artin conjecture by Corollary B. It remains to check that this is not covered by known cases in lower dimensions. For this it suffices to show that $\rho \otimes \rho'$ is primitive, i.e., it is not induced from a proper subgroup. By construction, ρ and ρ' are primitive. So applying a result of Aschbacher ([A], Theorem 1), we see that the tensor product remains primitive. (In fact this can be verified by direct computation, which is what we did originally leading us to pose the general question to Aschbacher, but now that this primitivity question has been solved in general, we can do no better than to refer to [A].)

Now we will prove Proposition C.

We are given a continuous irreducible representation of $\mathrm{Gal}(\overline{F}/F)$ of $\mathrm{GO}(n)$-type, wth n odd. By Lemma 1.2, we may, up to replacing ρ by a one-dimensional twist, which does not affect the conclusion of the Proposition, assume that the image of ρ lies in $\mathrm{O}(n, \mathbb{C})$. Let K be the number field cut out by the kernel of ρ with (finite) Galois group G over F, so that ρ can be viewed as a faithful representation of G. From the derived series we may extract, by the solvability of G, an elementary abelian p-group A which is characteristic in G. Applying Clifford's theorem, we see that

$$\rho|_A \simeq m(\chi_1 \oplus \cdots \oplus \chi_r),$$

for some $m, r > 0$ with $mr = n$, and 1-dimesnional representations $\chi_1, \cdots, \chi_r$ of A such that $\chi_i \neq \chi_j$ if $i \neq j$. Moreover, for every j there exists $g_j \in G$ such that $\chi_j(a) = \chi_1(g_j a g_j^{-1})$ for all $a \in A$; hence each χ_j has the same order, which must be p as ρ is injective. If p is odd, then no χ_j is self-dual, while ρ is itself self-dual, giving a contradiction as n is odd. So $p = 2$. Let $\rho_1 = m\chi_1$ and $G_1 = \mathrm{Stab}_G(\rho_1)$. Then $\rho \simeq ind_{G_1}^G(\rho_1)$ by Clifford. We are done if $m = 1$, so we may assume that $m > 1$. If $r = 1$, $A \simeq \mathbb{Z}/2$ (by the faithfulness of ρ) and $\rho(A) = \pm I$. But by construction $A \subset (G, G)$, which forces $\det(\rho)$ to be trivial on A. On the other hand, since n is odd, $\det\rho(A) = -1$, a contradiction. So $r > 1$. Then G_1 is a proper subgroup of G and (ρ_1, G_1) satisfies the same hypotheses as (ρ, G), since ρ_1 is self-dual by virtue of χ_1 being quadratic. Since induction is natural in stages, we are done by infinite descent.

□

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