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Term Structure

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Term Structure of Interest Rates

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These class notes were prepared to help the students in the course. Most of this material can be found in the references in the end. The next section contains some background material. Section 2 discusses term structure models. There are surely many misprints.

1 Background material

In what follows $(\Omega, \mathcal{F}, P)$ is a probability space, W a d-dimensional Brownian motion and $\mathcal{F}_t$, the standard filtration generated by the Brownian motion (i.e. the filtration generated by the union of $\sigma(W_u : u \leq t)$, with all the sets in Ω, with P measure 0). All processes indexed by t will be adapted to $\mathcal{F}_t$. All equalities and inequalities hold except perhaps for a set F with $P(F) = 0$. We write E for the expected value using the measure P.

1.1 Ito processes, Ito's lemma

An R^n valued process x_t is called an *Ito Process*, if it can be witten as:

$$x_t = x_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \tag{1}$$

where μ_t is an (adapted)R^n valued process with $\int_0^t |\mu_s| ds < \infty$ $(a.s.)$ and, σ_t is an $n \times d$ matrix valued process with $\int_0^t ||\sigma_s||^2 ds < \infty$ $(a.s.)$. If x_t is an

Ito process we write

$$dx_t = \mu_t dt + \sigma_t dW_t \tag{2}$$

If x is an Ito process given by (2) let $\mathcal{L}(x)$ denote the set of R^n valued adapted processes θ such that $\int_0^t |\theta_s \mu_s| ds < \infty$ (w.p.1) and $\int_0^t ||\theta_s \sigma_s||^2 ds < \infty$ (w.p.1). For such processes one can construct a stochastic integral

$$\int_0^t \theta_s dx_s = \int_0^t \theta_s \mu_s ds + \int_0^t \sigma_s \theta_s dW_s.$$

Let $A_t = \sigma_t \sigma_t^T$. If $V : R \times R^n \to R$ is C^1 on t and C^2 on x let:

$$LV(t,x) \equiv V_t(t,x) + V_x(t,x)\mu(t,x) + \frac{1}{2}\,\text{trace}(A(t,x)V_{xx}(t,x)). \tag{3}$$

Ito's Lemma states that the process $y_t = V(t, x_t)$ is also an Ito process and

$$dy_t = LV(t,x_t)dt + V_x(t,x_t)\sigma_t dW_t \tag{4}$$

1.2 Martingale equivalent measures

There are $n+1$ assets. The zero-th asset is a "money market account," that is its price satisfies: $X_t^0 = e^{[\int_0^t r_s ds]}$, where r is a process (the short or instantaneous rate of interest) that is bounded bellow. Associated with each of the other n assets there is a Gains process

$$G_t^i = X_t^i + D_t^i. \tag{5}$$

The D_t^i are to be interpreted as the cumulative dividends paid up to, and including, t, and the X_t^i as the *ex dividend* price process. We choose a normalization such that $D_0^i = 0$, for each i. In general one can treat D's such that:

$$D_t = Z_t + V_t - U_t, \tag{6}$$

where Z is an Ito process and V and U are non-decreasing and right continuous. In what follows we will be mostly interested in the case where $U \equiv 0$ and there exists a t_0 such that $V_s = 0$ for $s < t_0$ and $V_s = 1$ for $s \geq t_0$.

If $x = z + v - u$, where z is an Ito process, and u and v are non-decreasing and right continuous, we will say that $\theta \in \mathcal{L}(x)$ if $\theta \in \mathcal{L}(z)$, and if $\int_0^t \theta dv$ and $\int_0^t \theta du$ are well defined as Stieltjes integrals.

A self financing trading strategy θ is a process in $\mathcal{L}(G)$ such that:

$$\theta_t(X_t + \Delta D_t) = \theta_0 X_0 + \int_0^t \theta_s dG_s \tag{7}$$

A self financing strategy θ is an *arbitrage* if $\theta_0 X_0 < 0$ and $\theta_T X_T \geq 0$, or $\theta_0 X_0 \leq 0$ and $\theta_T X_T > 0$.

We will consider the deflated gains process $\bar{G}_t = \frac{X_t}{X_t^0} + \bar{D}_t$ where $d\bar{D}_t = \frac{dD_t}{X_t^0}$. We write $\bar{X}_t = \frac{X_t}{X_t^0}$ for the deflated price process. It is easy to show that self financing and arbitrage are properties that that preserved by deflating.

A measure Q is a martingale equivalent measure (also called a risk-neutral measure) if $Q \sim P$ (that is P and Q have exactly the same measure zero sets) and Q is a martingale measure for $\bar{G}_t$, that is:

1. $\bar{G}_t$ is an $\mathcal{F}_t$ martingale in $(\Omega, \mathcal{F}, Q)$, that is, $E^Q|\bar{G}_t| < \infty$,and

$$E^Q[\bar{G}_t \mid \mathcal{F}_s] = \bar{G}_s,$$

if $s \leq t < \infty$,

2. $\frac{dQ}{dP}$ has finite variance.

In particular, under the risk neutral measure all assets have a zero (deflated) expected rate of return.

If Q is a martingale equivalent measure, and $t > s$, then:

$$\bar{X}_s = E^Q[\bar{X}_t + \bar{D}_t - \bar{D}_s \mid \mathcal{F}_s]. \tag{8}$$

Suppose now that $D_t = \int_0^t \delta_s ds + \chi_{\{t=T\}}\Delta_T$, Then it is a consequence of (8) that:

$$X_s = E^Q[\frac{X_t}{X_t^0} + \int_s^t \frac{\delta_s}{X_s^0} ds \mid \mathcal{F}_s] = E^Q[\frac{\Delta_T}{X_T^0} + \int_s^T \frac{\delta_s}{X_s^0} ds \mid \mathcal{F}_s].$$

That is the price of an asset equal the expected (under Q) dividend flow, where each dividend is discounted by the value of the money market account (which is also a random variable).

The existence of a martingale equivalent measure is "equivalent" to the absence of arbitrage. In fact, the existence of a martingale equivalent measure is equivalent to the absence of an "approximate arbitrage" (see Duffie[1996], Chapter 6 section M.)

1.3 Martingale representation and Girsanov's theory

As we just seen martingales, under appropriately transformed measures, have an important role in asset pricing. In this subsection we summarize some results concerning martingales and characterizing equivalent measures.

Recall that the filtration $\mathcal{F}_t$ is the tsndard filtration generated by a Brownian motion. It can be shown that every $\mathcal{F}_t$-martingale is an Ito process with zero drift. In other words if M_t is a martingale, there exists a d dimensional vector of adapted processes θ_t with $\int_0^t ||\theta_s||^2 dt < \infty$ for each $t \geq 0$, and such that for each t:

$$M_t = M_0 + \int_0^t \theta_s dW_S.$$

Precise statements can be found in *e.g.* Karatzas and Shreve, theorem 4.15 in section 3.4 and problem 4.16.

Girsanov's theory treats the set of equivalent measures to P. One way to construct such an equivalent measure is to start with a d-vector of measurable processes λ such that

$$\int_0^T \mid \lambda_t \mid^2 dt < \infty$$

Let:

$$\xi_t = e^{[-\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t |\lambda|^2 dt]} \tag{9}$$

Note that $\xi_0 = 1$. Suppose that ξ is an $\mathcal{F}_t$-martingale for $0 \leq t \leq T$. Then Q, defined, for each $f \in \mathcal{F}_T$ by

$$Q(F) = \int_F \xi_T dP, \tag{10}$$

is a probability measure in $\mathcal{F}_T$. Furthermore, Girsanov's theorem (*e.g.* Karatzas and Shreve section 3.5) states that the process

$$\tilde{W}_t = W_t + \int_0^t \lambda_s ds \tag{11}$$

is a d-dimensional $\mathcal{F}_t$-Brownian motion in $(\Omega, \mathcal{F}_T, Q)$, that is, W_t starts at zero, has increments between s and t that are independent of $\mathcal{F}_s$, and normally distributed with mean zero and variance $t - s$.

There is also a "converse". If $Q \sim P$, then Q can be constructed using a function λ via equations (9) and (10), and the process $\tilde{W}_t$ given by (11) is a Brownian motion.

As a consequence suppose that X_t is an n-dimensional Ito process:

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and that $Q \sim P$. Then:

$$dX_t = [\mu_t - \sigma_t \lambda_t]dt + \sigma_t d\tilde{W}_t,$$

where $\tilde{W}_t$ is constructed using (9), (10), and (11). In particular, there exits a Q under which X is a martingale, then there must exist a solution to the equations:

$$\sigma_t \lambda_t = \mu_t.$$

Also, if σ has rank d, there exists at most one such martingale measure for X.

1.4 S.D.E.'s

Let $\mu^i : R \times R^n \to R$, $i = 1, \ldots, n$ and $\sigma^{ij} : R \times R^n \to R$ $i = 1, \ldots, n$, $j = 1, \ldots, d$ be Borel measurable functions. A process X_t is a (strong) solution to

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad (12)$$

with initial condition ξ if:

1. $X_0 = \xi$,

2. $$P[\int_0^t \mid \mu(s, X_s) \mid + \parallel \sigma(s, X_s) \parallel^2 ds < \infty] = 1,$$

3. $$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dWs,$$

   for each $0 \le t < \infty$

Remark 1 *The solution X_t is a Markov process. More precisely write $X(t, x)$ for the solution of (12) with initial condition x. Given a bounded Borel measurable function $g : R^n \to R$, and $s \le t$ then:*

$$E[g(X(t, x)) \mid \mathcal{F}_s] = E[g(X(t, x)) \mid X(s, x)]$$

Conditions for the existence and uniqueness of solutions can be found in *e.g.* Karatzas and Shreve page 289.

1.4.1 Feynman-Kac

Consider functions $r : [0,T]\times R^n \to R$, $h : [0,T]\times R^n \to R$, $\mu : [0,T]\times R^n \to R^n$, $\sigma : [0,T] \times R^n \to M(n \times d)$ and $g : R^n \to R$. Consider the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{13}$$

W a d-dimensional Brownian. Let:

$$\phi_t^T = e^{-\int_t^T r(s,X_s)ds}$$

and consider the function

$$f(t,x) = E[\int_t^T \phi_t^s h(s, X_s)ds + \phi_t^T g(X_T) \mid X_t = x] \tag{14}$$

Notice that

$$f(t,x)\phi_0^t + \int_0^t \phi_0^s h(s, X_s)ds = E[Z_T \mid X_t = x], \tag{15}$$

where $Z_T = \int_0^T \phi_0^s h(s, X_s)ds + \phi_0^T g(X_T)$. Since conditioning Z_T on X_t is the same as conditioning on $\mathcal{F}_t$, the right hand side of equation (15) is a martingale. If f satisfies the assumptions of Ito's lemma, then, using the notation defined in expression (3) above,

$$Lf(t,x)\phi_0^t - f(t,x)r(t,x)\phi_0^t + h(t,x)\phi_0^t = 0, \text{or,}$$

$$Lf(t,x) - f(t,x)r(t,x) + h(t,x) = 0, \text{ with } f(T,x) = g(x) \tag{16}$$

Equation (16) defines a *Cauchy problem.* It turns out that under some regularity conditions (see Karatzas and Shreve page 366, for example), the only solution to (16) that satisfies some growth conditions is the one given by (14).

2 Term structure of interest rates

2.1 Interest rate contracts

A *zero-coupon bond* pays one unit at the maturity date T and pays no other dividends. The *discount function* at t is the function $B(t,T)$ of prices of zero-coupon bonds as a function of the maturity.

The (continuously compounded) *yield to maturity* is (all logs are on the natural basis):

$$Y_{t,T} = -\frac{\log B_{t,T}}{T-t}. \tag{17}$$

The *yield curve* at time t is defined by $y_t(\tau) = Y_{t,t+\tau}$, that is it expresses the yields of zero-coupon bond that as a function of the time to maturity.

A *forward contract* struck at t for delivery of an asset at time $s \geq t$ involves the payment of an amount $F_{t,s}$ at s in exchange for the asset. No other payments are received. The amount $F_{t,s}$ is called the forward price at t for delivery of the asset at s (it is **not** the price of an asset.)

A *forward rate agreement* written at t for the period $[s,T]$, $s \geq t$, is an agreement in which, during the period $[s,T]$ one party pays a rate fixed at t (the forward rate) and the other party pays the rate prevailing at s for a loan that matures at T. No other cash flows occur.

2.2 Pricing zero-coupon bonds

We will assume that P is a martingale equivalent measure. In particular there is no arbitrage. Given a short rate process r_t, $t \geq 0$, the price at s of a zero-coupon bond maturing at T is:

$$B_{t,T} = E[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t]. \tag{18}$$

No arbitrage implies that the forward price at t for delivery at time $s \geq t$ of a zero coupon bond with maturity $T \geq s$ must be given by:

$$F_{t,s}^T = \frac{B_{t,T}}{B_{t,s}}, \tag{19}$$

and the associated forward rate is given by:

$$\Phi_{t,s}^T = \frac{\log B_{t,s} - \log B_{t,T}}{T-s}, \tag{20}$$

The instantaneous forward-rate, if it exists is defined for each t and delivery date $s \geq t$, by:

$$f(t,s) = \lim_{T \to s} \Phi_{t,s}^T. \tag{21}$$

This instantaneous forward rate is the forward rate that prevails at t for a loan in the interval $[s, s+\Delta s]$. From equations (20) and (21), we see that the instantaneous forward rate equals minus the derivative of log bond prices with respect to maturity. Hence:

$$B_{t,T} = e^{-\int_t^T f(t,s)ds}. \tag{22}$$

Hence we can compute all bond prices if we know the instantaneous forward rates.

2.3 One factor models

We start by taking $d = 1$ and assuming that r solves the following SDE:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \tag{23}$$

We further assume that the bond prices defined by equation (18) are finite.

We already know that under certain technical conditions the bond prices $B_{t,T} = h_T(t, r_t)$, where h_T solves the PDE,

$$Lh_T(t, x) - xh_T(t, x) = 0, \tag{24}$$

and $h_T(T, x) = 1$. Here L is the second order operator associated with (23).

2.3.1 Time homogeneous models

The Vasicek model specifies:

$$dr_t = (a - br_t)dt + \sigma dW_t \tag{25}$$

The solutions to this linear SDE are Gaussian, that is, given any collection of times $t_1, t_2, \ldots, t_n$, the joint distribution of the short rates $r_{t_1}, r_{t_2}, \ldots, r_{t_n}$ is normal. Also, $B_{t,T} = h_T(t, r_t)$, where:

$$\frac{\partial h_T}{\partial t}(t, r) + \frac{1}{2}\sigma^2 \frac{\partial^2 h_T}{\partial r^2}(t, r) + (a - br)\frac{\partial h_T}{\partial r}(t, r) - rh_T(t, r) = 0. \tag{26}$$

One can show that this PDE and boundary condition $h_T(T, r) \equiv 1$ has a solution of the form:

$$h_T(t, r_t) = e^{m(t,T) - \ell(t,T)r_t} \tag{27}$$

where, $\ell(t, T) = \frac{1}{b}(1 - exp(-b(T - t)))$ and,

$$m(t, T) = \frac{\sigma^2}{2}\int_t^T \ell^2(u, T)du - a\int_t^T \ell(u, T)du.$$

To demonstrate this fact, you should try a solution of the form (27) on equation (26). Notice that, after cancelling the exponential and collecting terms, you are left with a term in r and a constant. Hence the coefficient of r and the constant must each be zero. This will give a linear differential equation for ℓ and, after solving the differential equation using the fact that $\ell(T,T) = 0$, you can obtain m. Notice that, in particular, the yield is an affine function of the short rate.

The Cox-Ingersoll-Ross model specifies:

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t \tag{28}$$

Again, it can be shown that the bond prices satisfy equation (27), but with different functions ℓ and m. In any case the yield is an affine function of interest rates.

2.3.2 Time inhomogeneous models

There are many ways to introduce time explicitly in the SDE governing the short rate. A natural generalization of the Vasicek and CIR modles is to assume that: $\mu(t,r) = \alpha_1(t) + \alpha_2(t)r$ and, $\sigma^2(t,r) = \beta_1(t) + \beta_2(t)r$. We want to show that equation (24), has a solution

$$h_T(t,r) = e^{a(T-t)+b(T-t)r}, \tag{29}$$

with $a(0) = b(0) = 0$. To do this, try a solution of the form given by equation (29) to the PDE that bond prices must satisfy. It is easy to show, that a solution of this form exists, if and only if:

$$-a'(t) - b'(t)r + \frac{1}{2}(\beta_1 + \beta_2 r)b^2 + (\alpha_1 + \alpha_2 r)b - r = 0. \tag{30}$$

Hence the terms in r must add up to zero, and the same must happen to terms independent of r. Hence

$$b' = \alpha_2 b + \frac{1}{2}\beta_2 b^2 - 1, \tag{31}$$

and $a(t) = \int_t^T [\alpha_1(s)b(s) + \frac{1}{2}\beta_1(s)b^2(s)]ds$. Except for technical assumptions to guarantee that the ODE (31), has a solution with $b(T) = 0$ that is well defined for each $t \leq T$ and that yields a finite value for a, we would obtain a term structure with yields that are affine on the short rate (an affine term structure).

Remark 2 *It can also be shown that an affine term structure implies affine drift and variance*

2.4 Multi factor models

You will show in an exercise that in a one factor model any claim can be hedged by any other "invertible" claim. To avoid this result one must use multi factor models. Many of these models are written as:

$$dX_t^i = \mu^i(X_t^i)dt + \sigma^i(X_t^i)dW_t^i \tag{32}$$

where $i = 1, \ldots, n$, W is an n-dimensional Brownian motion, and $r_t = \sum_{i=1}^n X_t^i$. Since the W^i's are independent, so are the X^i's, and hence:

$$\begin{aligned} B_{t,T} &= E[e^{-\int_t^T r_u du} \mid r_t] = E[e^{-\int_t^T \sum_{i=1}^n X_u^i du} \mid r_t] = \\ &= E[\Pi_{i=1}^n e^{-\int_t^T X_u^i du} \mid r_t] = \Pi_{i=1}^n E[e^{-\int_t^T X_u^i du} \mid r_t] \end{aligned}$$

In particular, if each factor X^i generates an affine yield curve, this last expression equals:

$$\Pi_{i=1}^n e^{a^i(T-t)+c^i(T-t)X_t^i} = e^{A(T-t)+C(T-t)X_t},$$

where $A = \sum a_i$ and $C = (c^1, \ldots, c^n)$. As we observed before, the factors generate affine yield curves if either the coefficients of (32) are linear, or if each μ^i is linear and $\sigma^i(X_t^i) = \sqrt{X_t^i}$. For these affine models it is typically possible to choose n maturity dates $\tau_1, \ldots, \tau_n$ such that

$$X = h + \Gamma Y,$$

where $Y = (Y_{\tau_1}, \ldots, Y_{\tau_n})$. In this case we can use the yields Y_{τ_i}, $i = 1, \ldots, n$, as the factors. Such models are called yield-factor models.

Remark 3 *Suppose that we do not assume that P is a martingale equivalent measure, but instead we assume that there is no arbitrage. Hence there exists a $Q \sim P$, which is an equivalent martingale measure. It is a consequence of Girsanov's theorem that there exists an adapted process λ such that $dQ/dP = exp(-\int_0^t \lambda_s dW_s - 1/2(\int_0^t \mid \lambda_s \mid^2 ds)$ and $W^\lambda = W + \int_0^t \lambda_t dt$ is an n-dimensional Brownian under Q. Further since $M_t = E^Q[e^{-\int_0^T r_s ds} \mid \mathcal{F}_t]$ is a martingale it is an Ito process in $(\Omega, \mathcal{F}, Q^\lambda)$ with zero drift. Hence, $B_{t,T} = e^{\int_0^t r_s ds} M_t$ is an Ito process and,*

$$dB_{t,T} = B_{t,T}(r_t dt + \sigma_t^{T,\lambda} dW_t^\lambda) \tag{33}$$

Hence:

$$dB_{t,T} = B_{t,T}\left[(r_t - \lambda_t \sigma^{T,\lambda})dt + \sigma_t^{T,\lambda} dW_t\right] \tag{34}$$

In particular, $\sigma^{T,\lambda}$ is independent of λ. The n-vector λ_t is referred to as the market price of risk.

2.5 Forward rate models (Heath-Jarrow-Morton)

For each $s \leq T$, $t \leq s$ suppose, the evolution of the forward rate satisfies

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s)du + \int_0^t \sigma(u,s)dW_u \tag{35}$$

In this equation for fixed s, the processes $\mu(\cdot, s)$ and $\sigma(\cdot, s)$ are adapted and such that the relevant integrals are well defined. If this is an equation for the evolution of forward rates, and P is a martingale equivalent measure, then

$$M_t = E[e^{-\int_0^s r_u du} \mid \mathcal{F}_t] = e^{-\int_0^t r_u du} B_{t,s},$$

is a non-negative martingale. From equation (22), $\log M_t = -\int_0^t r_u du - \int_t^s f(t,u)du$. Here $\log M$ is being written as an integral of Ito processes, and except for technicalities, it can be shown that the drift of $\log M$ is $-\int_t^s \mu(t,u)du$ and the volatility is $-\int_t^s \sigma(t,u)du$

Since M_t is a martingale,

$$M_t = M_0 + \int_0^t \eta_s(u)dW_u.$$

From Ito's, $\log M_t$ is an Ito process with volatility vector $H(t,s)$ and drift $-\frac{1}{2}H(t,s)H'(t,s)$, where $H(t,s) = \eta_s(t)/M_t$.

We have thus obtained two formulas for the drift and volatility of $\log M_t$. In particular,

$$H(t,s) = -\int_t^s \sigma(t,u)du.$$

Differentiating with respect to s, we obtain:

$$\frac{\partial H(t,s)}{\partial s} = -\sigma(t,s). \tag{36}$$

Equating the two formulae for the drift and differentiating w.r.t. s, we get

$$H(t,s)(\frac{\partial H(t,s)}{\partial s})^T = \mu(t,s). \tag{37}$$

Combining equations (36) and (37), we obtain that:

$$\mu(t,s) = \sigma(t,s)\int_t^s \sigma^T(t,u)du. \tag{38}$$

Remark 4 *If we had instead assumed that P is the actual measure, but there is no arbitrage, we could deduct a more general relationship between the drift and volatilities of the forward rates.(See e.g. Musiela and Rutkowsky p. 308)*

Using the fact that $r_t = f(t,t)$ and equation (38), we obtain:

$$r_t = f(0,t) + \int_0^t \sigma(v,t) \int_v^t \sigma^T(v,u) du dv + \int_0^t \sigma(v,t) dW_v. \tag{39}$$

2.6 Models of random fields

In analogy to a stochastic progress we define a random field as a set of random variables on a common $(\Omega, \mathcal{F}, P)$, indexed by an index set T. In the example we are interested in $T \subset R_+^2$. In addition we will assume that X is a Gaussian random field: the joint distribution of any finite set $(X_t^1, \ldots, X_t^n)$ is normal. A Gaussian random field is characterized by its first two moments.

Example 1 *The Standard Brownian sheet $W_{u,v}$, $(u,v) \in R_+^2$ is the centered (that is, mean zero) Gaussian random field with*

$$Cov(W_{u_1,v_1}, W_{u_2,v_2}) = \min(u_1,u_2)\min(v_1,v_2).$$

In what follows we will always take our field to be defined in $\{(u,v) \in R^2 : v \geq u \geq 0\}$, have mean zero and with $\text{Cov}(X_{u_1,v_1}, X_{u_2,v_2}) = c(\min(u_1,u_2), v_1, v_2)$, where $c(0,v_1,v_2) = 0$, and c is symmetric on (v_1,v_2) and nonnegative definite in (u_1,v_1) and (u_2,v_2). The fact that the covariance function is specified as a function of $\min(u_1,u_2)$, implies that, for any $0 \leq u \leq u' \leq v$, the random variable $X_{u',v} - X_{u,v}$ is independent of the $\sigma-$ field $\mathcal{F}_u = \sigma\{X_{p,q} : p \leq u,\ p \leq q\}$. To show this notice that, since the variables are centered and $\min(u',p) = \min(u,p) = p$:

$$\text{Cov}(X_{u',v} - X_{u,v}, X_{p,q}) = c(\min(u',p), v, q) - c(\min(u,p), v, q) = 0.$$

Since the random variables are Gaussian, independence follows. Notice that this independence says that X has independent increments in the u direction.

In addition let $\mu : R_+^2 \to R$ be a deterministic function, and let the instantaneous forward rate for s as of t, be given by

$$F_{t,s} = \mu_{t,s} + X_{t,s} \tag{40}$$

In Kennedy [1994] it is shown that the following restriction is necessary and sufficient for P to be a martingale equivalent measure.

$$\mu_{t,s} = \mu_{0,s} + \int_0^s c(\min(t,v), v, s)dv, \tag{41}$$

what shows that, once we specify the initial yield curve $\mu_{0,s}$ and the covariance structure of forward rates, the full distribution of the forward rate surface, and hence of bond-prices, via (22) is determined.

Example 2 *Suppose that c and τ are differentiable functions from R_+ into R_+, with $\sigma' > 0$, $\tau' < 0$, and $\sigma(0) = 0$. Set: $c(r,t,s) = \sigma(r)\tau(\max(t,s))$. Then:*

$$X_{t,s} = W_{\sigma(t),\tau(s)},$$

a time-changed Brownian sheet. The forward rates are the sum of the deterministic drift and this time changed Brownian sheet. The assumption that τ is decreasing amounts to assuming that the volatility of forward rates decreases with the maturity time. One useful parametrization is $\tau(s) = e^{-\lambda s}$. In this example, if we set $\sigma(t) = \sigma^2 t$, then from (41) we get a typically non-linear formula for μ.

2.7 Foreign exchange

Let $\tilde{W}_t$, $0 \le t \le T$ be a standard 2-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_t$, $0 \le t \le T$, be the (completed) filtration generated by the Brownian motion. Consider two currencies dollars ($) and euros (€). The short rate in dollars (the instantaneous interest rate paid on a dollar money market account) is given by a process r_t, while the short rate in euros (the instantaneous interest rate paid on an euro money market account) is given by a process r_t^e. The exchange rate, the price of an euro in dollars, is a positive Ito process X_t, satisfying $dX_t/X_t = \mu_t^X dt + \sigma_t^X d\tilde{W}_t$. We assume that no (approximate) arbitrage exists, and hence that there exists an equivalent martingale measure Q, and a two dimensional vector of processes λ_t such that $W_t = \tilde{W}_t + \int_0^t \lambda_s ds$ is a Brownian motion under Q. Notice that the value in dollars of an euro money market account, with initial value of €Y is given by:

$$Y e^{\int_0^t r_s^e ds} X_t.$$

This is a dollar asset, that has to have an expected rate of return (under Q) that equals r_t. Hence from Ito's we get that:

$$dX_t/X_t = (r_t - r_t^f)dt + \sigma_t^X dW_t. \tag{42}$$

In particular, the expected change of the exchange rate under the risk-neutral measure is the difference between the two short rates.

3 Main References for lectures

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